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On *CW* cospectra

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Introduction

E. L. Lima [3] defined a direct spectrum $\{E_i, \phi_i: E_i \to E_{i+1}\}$ and an inverse spectrum $\{F_i, \psi_i: F_{i+1} \to F_i\}$. The former has been developed by many authors into the theory of CW spectra, which is now the basic notion in the cohomology theory ([1], [6], [7], [8]). In this paper, we shall define the notion of CW cospectra corresponding to the latter, and argue the homotopy category of CW cospectra by treating it as dual to that of CW spectra.

In this paper, a CW complex is called a nice complex if each cell is a subcomplex, and a map between nice complexes is called a nice map if each cell is mapped onto a subcomplex. By using the category NCW of nice complexes and nice maps, we define a *CW cospectrum E* as a collection

$$E = \{E_n, \varepsilon_n: E_{n+1} \longrightarrow SE_n \mid n \in Z\}$$

in NCW where S denotes the suspension and ε_n is the projection shrinking a subcomplex of E_{n+1} to $*$, and a *map*

$$f: E = \{E_n, \varepsilon_n\} \longrightarrow F = \{F_n, \varepsilon'_n\}$$

between CW cospectra is a collection of $f_n: E_n \to F_n/F'_n$ in NCW commuting with ε_n and ε'_n, where $F' = \{F'_n\}$ is a null subcollection of F, (see Definitions 1.1, 1.4 and 1.10). Further, a homotopy is a map $h: E \wedge I^+ \to F$ where $I^+ = \{*\} \cup [0, 1]$ (disjoint union) and $(E_n \wedge I^+)_n = E_n \wedge I^+$ (see Definition 1.14).

Thus, we obtain the *homotopy category of CW cospectra.* Furthermore, by considering the notion of cells in a CW cospectrum, we define a CW cospectrum E of finite type and the *cohomotopy group*

$$\pi^n(E) = [E, \Sigma^n S^0] \quad \text{(homotopy set)} \qquad \text{for any} \quad n \in Z$$

where $(\Sigma^n S^0)_i = *$ $(i < -n)$, $= S^{n+i}$ $(i \geq -n)$, (see Definitions 2.1, 2.4 and 3.3). Then, we have the following

THEOREM 3.5. *Assume that a CW cospectrum E of finite type satisfies $\pi^n(E) = 0$ for any n. Then, E is contractible in the homotopy category of CW cospectra.*

COROLLARY 3.8. *Let E be a CW cospectrum of finite type. Then, there is* *a natural homotopy equivalence* $E \wedge S^1 \simeq \Sigma^1 E$, *where* $(E \wedge S^1)_n = E_n \wedge S^1$ *and* $(\Sigma^* E)_n = E_{n+*}$.

Moreover, any *CW* cospectrum E defines a *homology theory*

$$E_*(X) = [\Sigma^* E, E(X)] \quad \text{(homotopy set)} \qquad \text{for} \quad X \in NCW,$$

where $E(X)$ is the *CW* cospectrum of X given by $E(X)_n = *(n<0)$, $= S^n X (n \geq 0)$. Conversely, we have the following representation theorem of homology theories by *CW* cospectra.

THEOREM 4.5. *Let* h_* *be a given reduced homology theory satisfying the following condition*:
(4.6) $h_m(S^n)$ *is finitely generated for any* $n \geq 0$ *and* m, *and there is an integer* N *with* $h_m(S^0) = 0$ *for* $m \leq N$.
Then, there exist a CW cospectrum E of finite type and a natural equivalence

$$T\colon E_*(X) \cong h_*(X) \qquad \textit{for any finite nice complex} \quad X.$$

This paper is organized as follows. In § 1, we introduce the notion of nice complexes and nice maps, and define the homotopy category of *CW* cospectra and maps. In § 2, we consider the notion of cells in a *CW* cospectrum and of locally finite maps, and prove the homotopy extension property and the homotopy excision theorem. By using the results in § 2, we prove Theorem 3.5 in § 3. Finally, Theorem 4.5 is proved in § 4.

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§ 1. *CW* cospectra

In this paper, we are concerned with nice complexes defined as follows:

DEFINITION 1.1 A *CW* complex X with base point $*$ is called a *nice complex*, if each cell e in X is a subcomplex of X ("cell" means "closed cell" in this paper). For nice complexes X and Y, a continuous map $f\colon X \to Y$ preserving base points is called a *nice map*, if the image $f(A)$ of each subcomplex A of X is a subcomplex of Y. Two nice complexes X and Y are said to be *isomorphic*, written frequently $X = Y$, if there is a homeomorphism $f\colon X \to Y$ preserving the cell structures and base points.

Then, we can consider the category *NCW* of nice complexes and nice maps; and we see easily the following

EXAMPLE 1.2. Let X and Y be nice complexes.
(1) Any subcomplex $A(\ni *)$ of X and the *CW* complex X/A obtained from

X by shrinking A to $*$ are nice complexes, and the inclusion $i: A \subset X$ and the projection $\mathrm{pr}: X \to X/A$ are nice maps. Moreover, cells $e'(\neq *)$ in X/A are in 1-1 correspondence with cells $e(\not\subset A)$ in X by $e' = \mathrm{pr}(e)$, and $\mathrm{pr}^{-1}(B')$ is a subcomplex of X for any subcomplex B' of X/A.

(2) The product complex $X \times Y$ (with the weak topology), the wedge sum $X \vee Y$ and the smash product $X \wedge Y = X \times Y / X \vee Y$ are nice complexes.

(3) We consider the n-sphere S^n as the nice complex $S^n = \{*\} \cup e^n$ and $(n+1)$-disc D^{n+1} as that $D^{n+1} = S^n \cup e^{n+1} = \{*\} \cup e^n \cup e^{n+1}$. Then, the suspension $S^n X = S^n \wedge X$ of X is a nice complex, and $S^i \wedge S^j = S^{i+j}$ for any integers $i, j \geq 0$. Moreover, $S^n X (n \geq 1)$ has a single 0-cell $*$, and cells $e'(\neq *)$ in $S^n X$ are in 1-1 correspondence with cells $e(\neq *)$ in X by $e' = S^n e$ (where $S^n e$ denotes $\mathrm{pr}(S^n \times e)$ for the projection $\mathrm{pr}: S^n \times X \to S^n X$).

(4) We consider the interval $I = [0, 1]$ as the nice complex $I = \{0\} \cup \{1\} \cup I$ with base point 0. Then, the mapping cone $C_f = Y \cup_f X \wedge I$ of a nice map $f: X \to Y$ is a nice complex.

(5) For any nice complex A without considering base point, we can consider the nice complex $A^+ = \{*\} \cup A$ (disjoint union). Then, the mapping cylinder $M_f = Y \cup_f X \wedge I^+$ of a nice map $f: X \to Y$ is a nice complex. Furthermore, the *nice homotopy* of X to Y is defined to be a nice map $h: X \wedge I^+ \to Y$.

We notice that the inclusion of NCW to the category CW of pointed CW complexes and continuous maps induces an equivalence of associated homotopy categories; i.e., we have the following

PROPOSITION 1.3. (1) *Any continuous map $f: X \to Y$ between nice complexes X and Y such that $f|A$ is nice for a subcomplex $A \subset X$ is homotopic* rel A *to a nice map $f': X \to Y$.*

(2) *Any CW complex is homotopy equivalent to a nice complex.*

(3) *Two maps between nice complexes are homotopic* (rel $*$) *if and only if they are nicely homotopic* (rel $*$).

PROOF. (1) for the case $X = D^n$ and $A = S^{n-1}$: Since $f(D^n)$ is contained in a finite subcomplex of Y, we may assume that Y is finite. If $f(D^n) = Y$, then f is clearly nice. If $f(D^n) \subsetneq Y$, then Y contains a cell $e \not\subset f(D^n)$, and we take e of maximum dimension among such cells. Thus, $Y = Y_0 \cup e$ where $e \not\subset f(D^n)$ and Y_0 is a subcomplex of Y with $Y_0 \not\supset e$. We notice that $f(S^{n-1}) \subset Y_0$ because $f|S^{n-1}$ is nice by the assumption. Now, we can find a point x in the open cell of e such that $x \notin f(D^n)$, i.e., $f(D^n) \subset Y - \{x\}$. Since Y_0 is a deformation retract of $Y - \{x\}$, we see that f is homotopic rel S^{n-1} to a map $f_0: D^n \to Y_0 \subset Y$; and we obtain the desired homotopy by induction on the number of cells in Y.

(1) for the general case can be proved inductively by using the above result on cells in X.

(2) The desired nice complex is obtained by deforming inductively the attaching maps of a given complex to make them nice.

(3) This follows immediately from (1). q.e.d.

Now, we define the notion of *CW* cospectra.

DEFINITION 1.4. (1) A *CW cospectrum* is a collection $\{E_n, \varepsilon_n | n \in Z\}$ of nice complexes E_n and maps $\varepsilon_n: E_{n+1} \to SE_n$ where $SE_n = S^1 \wedge E_n$, satisfying the following condition:

(1.5) *There is a subcomplex $E'_{n+1}(\ni *)$ of E_{n+1} such that $\varepsilon_n: E_{n+1} \to E_{n+1}/E'_{n+1} = SE_n$ is the composition of the projection and an isomorphism between nice complexes.*

(2) A *subcollection* $F = \{F_n\}$ of E consists of subcomplexes $F_n(\in *)$ of E_n such that $\varepsilon_n(F_{n+1}) \subset SF_n$ for each n. Moreover, if $\varepsilon_n(F_{n+1}) = SF_n$ for each n, then F is called a *subcospectrum* of E; and (E, F) is called a *pair* of *CW* cospectra.

(3) Let $E = \{E_n, \varepsilon_n\}$ and $F = \{F_n, \varepsilon'_n\}$ be *CW* cospectra. Then, a *function* $f = \{f_n\}: E \to F$ is a collection of nice maps $f_n: E_n \to F_n$ with $\varepsilon'_n \circ f_{n+1} = Sf_n \circ \varepsilon_n$ for each n. A *function* $f: (E, E') \to (F, F')$ between pairs of *CW* cospectra is a function $f: E \to F$ with $f(E') \subset F'$.

EXAMPLE 1.6. (1) Let $\{(E_n, E'_n),\ \eta_n: E_n \to E'_{n+1} | n \in Z\}$ be a collection of pairs (E_n, E'_n) of nice complexes and nice maps η_n such that $E_{n+1} = C_{\eta_n}$, the mapping cone of η_n. Then, by taking $\varepsilon_n: E_{n+1} = C_{\eta_n} \to C_{\eta_n}/E'_{n+1} = E_n \wedge S^1 = S^1 \wedge E_n = SE_n$ to be the projection shrinking E'_{n+1} to $*$, we have a *CW* cospectrum $\{E_n, \varepsilon_n\}$.

(2) If X is a nice complex, then we can define a *CW cospectrum* $E(X)$ of X by taking

$$E(X)_n = \begin{cases} * & (n < 0) \\ S^n X & (n \geq 0), \end{cases} \qquad \varepsilon_n = \begin{cases} * & (n < 0) \\ \mathrm{id} & (n \geq 0). \end{cases}$$

We notice that $E(X)$ is a *CW* cospectrum given by taking $E_n = *(n<0)$, $E_0 = X(n=0)$, $E'_n = *$ and $\eta_n = *$ in (1).

(3) For any *CW* cospectrum $E = \{E_n\}$ and integer k, we can define a *CW* cospectrum $\Sigma^k E$ by taking $(\Sigma^k E)_n = E_{n+k}$.

(4) Given a function $f: E \to F$ between *CW* cospectra, we can define new *CW* cospectra M_f (the *mapping cylinder* of f) and C_f (the *mapping cone* of f), by taking $(M_f)_n = M_{f_n}$ and $(C_f)_n = C_{f_n}$, respectively. The *cone* CE of E is the mapping cone C_{id} of the identity function id: $E \to E$.

LEMMA 1.7. *Let E be a CW cospectrum.*

(1) *For subcollections $F = \{F_n\}$ and $G = \{G_n\}$ of E, $F \cup G = \{F_n \cup G_n\}$ is a subcollection of E. Moreover, if F and G are subcospectra of E, then so is $F \cup G$.*

(2) *For any subcospectrum* $F=\{F_n\}$ *of* E, $F=\{F_n, \varepsilon_n|F_{n+1}\}$ *is a* CW *cospectrum and the inclusion* $F\subset E$ *consisting of the inclusions* $F_n\subset E_n$ *is a function.*

(3) *For any subcollection* $F=\{F_n\}$ *of* E, $E/F=\{E_n/F_n, \varepsilon'_n\}$ *is a* CW *cospectrum where* $\varepsilon'_n\colon E_{n+1}/F_{n+1}\to SE_n/SF_n=S(E_n/F_n)$ *is the map induced from* $\varepsilon_n\colon E_{n+1}\to SE_n$ $(\varepsilon_n(F_{n+1})\subset SF_n)$, *and the projection* $E\to E/F$ *consisting of the projections* $E_n\to E_n/F_n$ *is a function.*

(4) *In addition to* (3), *let* G *be a subcollection of* E/F. *Then, there is a subcollection* G' *of* E *with* $G'\supset F$ *and* $G'/F=G$.

(5) *Let* $f\colon E\to F$ *be a function between* CW *cospectra. Then, for any subcollection* G *of* E, $f(G)=\{f_n(G)_n\}$ *is a subcollection of* F. *Moreover, if* G *is a subcospectrum of* E, *then* $f(G)$ *is a subcospectrum of* F.

PROOF. The lemma can be proved easily by definition and by Example 1.2. For example, (3) is shown by noticing that $\varepsilon_n^{-1}(SF_n)$ is a subcomplex of E_{n+1} containing $E'_{n+1}\cup F_{n+1}$ and that $E_{n+1}/E'_{n+1}=SF_n$ induces $E_{n+1}/\varepsilon_n^{-1}(SF_n)=SE_n/SF_n=S(E_n/F_n)$; and $G'=\{G'_n\}$ in (4) is given by $G'_n=\mathrm{pr}_n^{-1}(G_n)$ where $\mathrm{pr}_n\colon E_n\to E_n/F_n$ is the projection. q. e. d.

In order to construct the category of CW cospectra, we need to specify the morphisms, called maps.

DEFINITION 1.8. We say that a subcollection $F=\{F_n\}$ of a CW cospectrum $E=\{E_n, \varepsilon_n\}$ is *null in* E, if for each n, there is an integer $k>0$ such that $\varepsilon(F_{n+k+1})=*$, where ε denotes the composition $\varepsilon_{n+k}\circ S\varepsilon_{n+k-1}\circ\cdots\circ S^k\varepsilon_n$.

By the definition, we see easily the following

LEMMA 1.9. (1) *In Lemma* 1.7 (1), *if* F *and* G *are null in* E, *then so is* $F\cup G$.

(2) *In Lemma* 1.7 (4), *if* F *is null in* E *and* G *is null in* E/F, *then so is* G' *in* E.

(3) *In Lemma* 1.7 (5), *if* G *is null in* E, *then so is* $f(G)$ *in* F.

(4) *For any* CW *cospectrum* E *and* $r\in Z$, *the subcollection* $E^{(r)}$, *given by*

$$(E^{(r)})_n = E_n \quad \textit{for} \quad n<r, \; = * \quad \textit{for} \quad n\geq r,$$

is null in E. *Moreover, for the* CW *cospectrum* $E(X)$ *of a nice complex* X *given in Example* 1.6 (2), F *is null in* $E(X)$ *if and only if* F *is contained in* $E(X)^{(r)}$ *for some* r.

We define a map as follows.

DEFINITION 1.10. (1) A *map* $f\colon E\to F$ between CW cospectra E and F is an equivalence class of a function $f_{F'}\colon E\to F/F'$ such that F' is a null subcollection of F, where

(1.11) *two functions* $f_{F_i}: E \to F/F_i$ (F_i: *null in* F; $i=1, 2$) *are equivalent if and only if there is a function* $f_3: E \to F/F_3$ (F_3: *null in* F) *with* $F_1 \cup F_2 \subset F_3$ *and* $\mathrm{pr}_1 \circ f_{F_1} = \mathrm{pr}_2 \circ f_{F_2} = f_3$($\mathrm{pr}_i$: $F/F_i \to F/F_3$ *are the projections*). (We see that (1.11) defines an equivalence relation by Lemma 1.9.)

(2) A *map* $f: (E, G) \to (F, H)$ between pairs of CW cospectra is a map f: $E \to F$ with $f(G) \subset H$ which means that for some representative $f_{F'}: E \to F/F'$ (F': null in F) of f, $f_{F'}(G) \subset H/H \cap F'$. Then, the *restriction* $f|G: G \to H$ is the map represented by $f_{F'}|G: G \to H/H \cap F'$.

DEFINITION 1.12. The *composition* $g \circ f: E \to G$ of maps $f: E \to F$ and g: $F \to G$ between CW cospectra is defined as follows: Let f and g be represented by functions $f: E \to F/F'$ (F': null in F) and $g: F \to G/G'$(G': null in G), respectively. Then, by Lemma 1.9 (3) and (2), $g(F')$ is null in G/G' and there is a null subcollection G'' of G with $G'' \supset G'$ and $G''/G' = g(F')$. Thus, we obtain the composed function

$$g' \circ f: E \xrightarrow{f} F/F' \xrightarrow{g'} (G/G')/g(F') = G/G'',$$

where g' is the function induced from g; and we can define the composition $g \circ f$ as the map represented by $g' \circ f$.

By the definition of maps, we see easily the following

LEMMA 1.13. *In the category of CW cospectra and maps, the projection* $E \to E/E'$ *is an equivalence for any null subcollection* E' *of* E. *In particular,* E *is equivalent to* $E/E^{(r)}$ *where* $E^{(r)}$ *is that given in Lemma* 1.9 (4) *and* $(E/E^{(r)})_n = E_n$ *for* $n \geq r$, $= *$ *for* $n < r$.

Now, we define a homotopy of functions and maps.

DEFINITION 1.14. (1) If $E = \{E_n, \varepsilon_n\}$ is a CW cospectrum and X is a nice complex, then we can define a new CW cospectrum $E \wedge X$, the *smash product* of E and X, by taking $(E \wedge X)_n = E_n \wedge X$ (with the weak topology) and $\varepsilon_n \wedge \mathrm{id}$: $E_{n+1} \wedge X \to SE_n \wedge X = S(E_n \wedge X)$. Given a function or a map $f: E \to F$ of CW cospectra and a nice map $g: X \to Y$, we get a function or a map $f \wedge g: E \wedge X \to F \wedge Y$ of CW cospectra naturally.

(2) A *homotopy* of functions or maps between CW cospectra E and F is a function or a map $h: E \wedge I^+ \to F$, where $I^+ = I \cup \{*\}$. There are two functions i_t: $E = E \wedge \{t\}^+ \to E \wedge I^+ (t=0, 1)$ induced by $\{t\}^+ \subset I^+$, and we write h_t for $h \circ i_t (t=0, 1)$. Then, we say that two functions or maps $f_0, f_1: E \to F$ are *homotopic* if there is a homotopy $h: E \wedge I^+ \to F$ with $h_t = f_t (t=0, 1)$.

(3) It is easy to see that the homotopy defines an equivalence relation, and we denote by $[E, F]$ the set of all homotopy classes of maps from E to F.

(4) A *homotopy* of functions or maps between pairs (E, G) and (F, H) of

CW cospectra and the homotopy set $[E, G; F, H]$ are defined in the same way.

PROPOSITION 1.15. *For any CW cospectrum $E=\{E_n, \varepsilon_n\}$ and a nice complex X, consider the direct system $\{[E_n, S^nX], \varepsilon_n^*\circ\Sigma | n\geq 0\}$ of the (pointed) homotopy set $[E_n, S^nX]$ and the compositions*

$$\varepsilon_n^*\circ\Sigma\colon [E_n, S^nX] \xrightarrow{\Sigma} [SE_n, S^{n+1}X] \xrightarrow{\varepsilon_n^*} [E_{n+1}, S^{n+1}X]$$

(Σ: the homotopy suspension). Then, there is a natural bijection

$$[E, E(X)] = \operatorname{dir\,lim}\,[E_n, S^nX]$$

of the homotopy set of maps of E to the CW cospectrum $E(X)$ of X in Example 1.6 (2) *onto the direct limit of* $\{[E_n, S^nX]\}$.

PROOF. By Lemma 1.9 (4) and the definition of functions and maps, we see that any map $f\colon E\to E(X)$ is represented by a function $f\colon E\to E(X)/E(X)^{(r)}$ for some $r\geq 0$, and that the latter is just a nice map $f_r\colon E_r\to E(X)_r=S^rX$ together with all $f_n=S^{n-r}f_r\circ\varepsilon\colon E_n\to S^{n-r}E_r\to S^nX$, and it determines the element $\{[f_n] | n\geq r\}$ in $\operatorname{dir\,lim}[E_n, S^nX]$. We see easily that this correspondence gives the desired bijection by using Proposition 1.3. q.e.d.

We can see easily the following lemma (cf. [7, Lemma 2.36]), and this is used to prove Theorem 3.6.

LEMMA 1.16. *Let $f\colon E\to F$ be a function of CW cospectra. For any CW cospectrum G, a sequence*

$$[C_f, G] \xrightarrow{i^*} [F, G] \xrightarrow{f^*} [E, G]$$

is exact, where $i\colon F\subset F\cup_f E\wedge I=C_f$ is the inclusion.

§2. Cells in CW cospectra and locally finite maps

We introduce the notion of cells of CW cospectra.

DEFINITION 2.1. Let $E=\{E_n, \varepsilon_n\}$ be a CW cospectrum. Then, there is the subcospectrum F of E such that $F_n=*$ for all n; we denote F by $*$ also and call a *cell of dimension* $-\infty$. If e_n is any d-cell of E_n (other than $*$, if $d=0$), then there exists a unique $(d+1)$-cell e_{n+1} of E_{n+1} such that $\varepsilon_n(e_{n+1})=Se_n$, and in general there exists a unique $(d+m)$-cell e_{n+m} of E_{n+m} such that $\varepsilon(e_{n+m})=S^me_n$. The d-cell e_n cannot be "desuspended" more than d times, since $\varepsilon(e_n)\subset S^jE_{n-j}$ is a d-cell S^je_{n-j} or $*$, where e_{n-j} is a $(d-j)$-cell of E_{n-j}. Therefore, we obtain a collection $e=\{\cdots, e_k, e_{k+1},\dots, e_n,\dots\}$ such that

(2.2) $e_m=*$ *for* $m<k$ *and* e_m *is a* $(d-n+m)$*-cell in* E_m *with* $\varepsilon_m(e_{m+1})=Se_m$ *for* $m\geq k$ *and* $\varepsilon_{k-1}(e_k)=*$.

We call this subcospectrum e of E a *cell of dimension* $d-n$ in E. Thus, we see that each cell ($\neq *$) in each complex E_n is a member of exactly one cell ($\neq *$) of E, and that E is the union of all cells of E.

LEMMA 2.3. *Let E be a CW cospectrum.*

(1) *For a subcollection F of E such that each F_n is finite, F is null in E if and only if any cell $e \neq *$ of E is not contained in F.*

(2) *Assume that each E_n is finite and that there is given an integer n_e for each cell e of E. Then, $F=\bigcup_{e\subset E} e^{(n_e)}$ is null in E, where $e^{(r)}$ is the subcollection with $(e^{(r)})_n = e_n$ for $n<r$, and $=*$ for $n \geq r$.*

PROOF. (1) The necessity is clear. Now, for a subcollection F of E, assume that each F_n is finite and any cell $e \neq *$ of E is not contained in F. For any fixed n, let J'_n be the set of cells e in E with $e_n \neq *$ and $e_n \subset F_n$. Then, J'_n is finite since F_n is finite and each cell ($\neq *$) in E_n is a member of exactly one cell in E. Therefore, there is $k>0$ such that $e_{n+k} \not\subset F_{n+k}$ for any $e \in J'_n$ since $e \not\subset F$. Thus, $\varepsilon(F_{n+k}) = * \subset F_n$ by the definition of cells; hence F is null in E.

(2) For any fixed n, the set J_n of cells e in E with $e_n \neq *$ is finite. Therefore, there is an integer $k>0$ such that $n+k>n_e$ for any $e \in J_n$, and $(e^{(n_e)})_{n+k} = *$ for $e \in J_n$. Also $\varepsilon(e^{(n_e)})_{n+k} = * \subset E_n$ for any $e \notin J_n$ by the definition of J_n. Thus, $\varepsilon(F_{n+k}) = * \subset E_n$, and F is null in E. q.e.d.

DEFINITION 2.4. (1) Let E be a CW cospectrum. We say that E is *finite* if the number of cells in E is finite, and that E is *locally finite* if any cell in E is finite as a CW cospectrum. Moreover, we say that E is *of finite type* if each E_n is finite and E is locally finite.

(2) A function $f: E \to F$ is said to be *locally finite* if $f(e)$ is finite for any cell e of E, and a map f is said to be *locally finite* if some representative of f is locally finite. We denote by $[E, F]_L$ the set of all locally finite homotopy classes of locally finite maps of E to F.

EXAMPLE 2.5. (1) For any nice complex X, the CW cospectrum $E(X)$ of X is locally finite, and for any CW cospectrum E any function $f: E \to E(X)$ is locally finite by the definition. Thus, we have $[E, E(X)]_L = [E, E(X)]$.

(2) Let $C_n = *$ for $n<0$, $C_0 = S^0$, $C_{n+1} = CC_n$ for $n \geq 0$ and $\varepsilon_n: C_{n+1} = CC_n \to C_n \wedge S^1 = SC_n$ be the projection shrinking C_n to $*$ for $n \geq 0$. Then, $\{C_n, \varepsilon_n\}$ is a CW cospectrum which is not locally finite. In fact, the cell e in $\{C_n, \varepsilon_n\}$ with $e_0 \neq * (\in S^0)$ contains infinitely many cells.

By (2) of the above example, a CW cospectrum E is not necessarily locally finite even if each E_n is finite. But, we have the following

EXAMPLE 2.6. Let $\{(E_n, E'_n), \eta_n: E_n \to E'_{n+1}\}$ be a collection with $E_{n+1} = C_{\eta_n}$

and $E=\{E_n, \varepsilon_n\}$ be a CW cospectrum with the projection $\varepsilon_n\colon E_{n+1}=C_{\eta_n}\to C_{\eta_n}/E'_{n+1}=SE_n$ given in Example 1.6 (1). Then, we set for $n\leq 0$,

$$(\bar{E}_n, \bar{E}'_n) = (E_n, E'_n),\ \bar{\eta}_{n-1} = \eta_{n-1},\ p_n = \mathrm{id}\colon \bar{E}_n\to E_n \quad \text{and} \quad C_n = *,$$

and for $n\geq 0$, we set by induction as follows:

$\bar{E}'_{n+1} = M_{\eta_n}$ (the mapping cylinder of $\eta_n\colon E_n\to E'_{n+1}$), $i_n\colon E_n \subset \bar{E}'_{n+1}$,

$C_{n+1} = C_{p_n}$ (the mapping cone of $p_n\colon \bar{E}_n\to E_n$), $E_n\subset C_{n+1}$,

$\bar{\eta}_n = i_n\circ p_n\colon \bar{E}_n \longrightarrow E_n \subset \bar{E}'_{n+1},\ \bar{E}_{n+1} = C\bar{\eta}_n = \bar{E}'_{n+1} \cup C_{n+1},\ E_n = \bar{E}'_{n+1} \cap C_{n+1},$

$\bar{\varepsilon}_n\colon \bar{E}_{n+1} \longrightarrow \bar{E}_{n+1}/\bar{E}'_{n+1} = C_{n+1}/E_n = C_{p_n}/E_n = S\bar{E}_n$ (the projection),

$p_{n+1} = \bar{E}_{n+1} \longrightarrow \bar{E}_{n+1}/C_{n+1} = \bar{E}'_{n+1}/E_n = M_{\eta_n}/E_n = C_{\eta_n} = E_{n+1}$ (the projection).

Now, *we have a locally finite CW cospectrum* $\bar{E}=\{\bar{E}_n, \bar{\varepsilon}_n\}$, *which is of finite type if each* E_n *is finite, and a homotopy commutative diagram*

$$\begin{array}{ccc} \bar{E}_{n+1} & \xrightarrow{\bar{\varepsilon}_n} & S\bar{E}_n \\ {\scriptstyle p_{n+1}}\downarrow & & \downarrow{\scriptstyle Sp_n} \\ E_{n+1} & \xrightarrow{\varepsilon_n} & SE_n \end{array}$$

where each p_n *is a homotopy equivalence.*

In fact, we see easily that $\bar{E}$ is locally finite, because $\bar{\varepsilon}_n(C_{n+1})=SE_n(n\geq 0)$ and $\bar{\varepsilon}_k$ maps $SC_k(\subset C_{k+1}\subset \bar{E}_{k+1})$ identically onto $SC_k(\subset S\bar{E}_k)$ for $k\geq n$ by the construction. Further, we see immediately that C_n is contractible and p_n is a homotopy equivalence by induction and that the above diagram is homotopy commutative.

We see easily the following lemma by definition.

LEMMA 2.7. *Any finite CW cospectrum E is equivalent to the CW cospectrum* $\Sigma^{-i} E(X)$ *for some nice complex X and some integer i. More precisely,* $E/E^{(i)}=\Sigma^{-i} E(E_i)$ *for some i.*

Locally finite CW cospectra can be built up cell by cell just as CW complexes and CW spectra can be built up.

DEFINITION 2.8. (1) For a CW cospectrum $E=\{E_n, \varepsilon_n\}$, let $e=\{\dots, *, e_m, e_{m+1},\dots\}$ be a d-cell of E and $f_k\colon S^{k+d-1}\to E_k(k\geq m)$ be the attaching map *of* e_k. Then, $\varepsilon_k\circ f_{k+1}=Sf_k$ by Definition 2.1, and the collection $\{f_k\}$ (with $f_k=*$ for $k<m$) defines a function $f\colon \Sigma^{-m} S^{m+d-1}\to E$ and a map $\Sigma^{d-1} S^0\to E$, which are called the *attaching function* and *map* of e, respectively, (where $\Sigma^i S^j=\Sigma^i E(S^j)$ is the CW cospectrum such that $(\Sigma^i S^j)_n=*$ for $n<-i$, $=S^{j+n+i}$ for $n\geq -i$).

(2) For a locally finite CW cospectrum E, we can define a filtration

$$* = E^0 \subset E^1 \subset \cdots \subset E^n \subset \cdots \subset E = \cup_{n \geq 0} E^n$$

as follows: For any cell $e \neq *$ in E, let $u(e)$ be the number of cells contained in e, which is finite because E is locally finite; and define E^n to be the union of $*$ and cells $e \neq *$ in E with $u(e) \leq n$.

Then, we can see easily the following

LEMMA 2.9. *For any $n>0$, let $\{e_\alpha | \alpha \in J_n\}$ be the set of cells of $E^n - E^{n-1}$, and for each $\alpha \in J_n$ let f_α: $\Sigma^{-m_\alpha} S^{m_\alpha + d_\alpha - 1} \to E$ be the attaching function of e_α. Then, the function*

$$g: \vee_{\alpha \in J_n} \Sigma^{-m_\alpha} S^{m_\alpha + d_\alpha - 1} \longrightarrow E, \; g \,|\, \Sigma^{-m_\alpha} S^{m_\alpha + d_\alpha - 1} = f_\alpha,$$

factors through E^{n-1}, and E^n is the mapping cone

$$C_g = E^{n-1} \cup_g (\vee_{\alpha \in J_n} \Sigma^{-m_\alpha} S^{m_\alpha + d_\alpha - 1}) \wedge I.$$

For any locally finite CW cospectrum, we have the following homotopy extension property.

PROPOSITION 2.10. *Let E be a locally finite CW cospectrum and F be its subcospectrum. Then, for any function $f: E \to G$ and any homotopy $h: F \wedge I^+ \to G$ of $f|F$, there is a homotopy $H: E \wedge I^+ \to G$ of f with $H|F \wedge I^+ = h$. Moreover, if f and h are locally finite, then we can take a locally finite homotopy H.*

PROOF. At first, we prove the proposition for the case that E is the mapping cone $C_g = F \cup_g F' \wedge I$ of a function $g: F' \to F$. Let $r: I \wedge I^+ \to I \wedge \{0\}^+ \cup \dot{I} \wedge I^+$ be a retraction ($\dot{I} = \{0, 1\}$). Define a function

$$h': F' \wedge I \wedge \{0\}^+ \cup F' \wedge \dot{I} \wedge I^+ \longrightarrow G$$

by $h'|F' \wedge I \wedge \{0\}^+ = f|F' \wedge I$ and $h'|F' \wedge \dot{I} \wedge I^+ = h \circ (g \wedge \mathrm{id})$, and a homotopy

$$H: (F \cup_g F' \wedge I) \wedge I^+ \longrightarrow G$$

by $H|F \wedge I^+ = h$ and $H_n([x, s], t) = h'_n([x, p_1(r(s, t))], p_2(r(s, t)))$ for $[x, s] \in F'_n \wedge I$ and $t \in I^+$, where $p_i(s_1, s_2) = s_i (i=1, 2)$ for $[s_1, s_2] \in I \wedge \{0\}^+ \cup \dot{I} \wedge I^+$. Then, H is well defined and is the desired homotopy. The latter half is clear.

For the general case, we can construct inductively the desired homotopy by using Lemma 2.9 and the above case. q.e.d.

Now, we see the following excision isomorphism theorem for the locally finite homotopy sets.

THEOREM 2.11. *Let G be a CW cospectrum, and H and K be subcospectra* *of G with $G=H \cup K$. Then, for any CW cospectrum E of finite type, the induced map*

$$i_*: [CE, E; K, K \cap H]_L \longrightarrow [CE, E; G, H]_L$$

of the inclusion i: $(K, K \cap H) \subset (G, H)$ is bijective, where $CE=E \wedge I$ is the cone of E.

Before proving this theorem, we state some applications. The following is the suspension isomorphism theorem.

THEOREM 2.12. *Let E be a CW cospectrum of finite type and F be a CW cospectrum. Then, the homotopy suspension*

$$\Sigma: [E, F]_L \longrightarrow [E \wedge S^1, F \wedge S^1]_L,$$

given by $\Sigma f=f \wedge \mathrm{id}$ for f: $E \to F$, is bijective.

PROOF. It is clear that Σ is well defined (without assuming that E is of finite type). Consider a commutative diagram

$$\begin{array}{ccc}
[E, F]_L & \xrightarrow{\text{cone}} & [CE, E; CF, F]_L \\
\Big\downarrow{\scriptstyle \Sigma} & & \big\downarrow{\scriptstyle i_*} \\
 & & [CE, E; CF \cup C_-F, C_-F]_L \\
 & & \big\downarrow{\scriptstyle p_*} \\
[E \wedge S^1, F \wedge S^1]_L & \xrightarrow{p'^*} & [CE, E; F \wedge S^1, *]_L
\end{array}$$

where cone is given in the same way as Σ, C_-F is another cone of F, i is the inclusion, p: $CF \cup C_-F \to (CF \cup C_-F)/C_-F=F \wedge S^1$ and p': $CE \to E \wedge S^1$ are the projections. Then, p'^* is bijective by the definition, and so is i_* by Theorem 2.11. Furthermore, we can show that cone and p_* are bijective as follows; hence so is Σ.

(1) cone *is bijective*: The restriction defines

$$\partial: [CE, E; CF, F]_L \longrightarrow [E, F]_L$$

with $\partial \circ \mathrm{cone}=\mathrm{id}$. For any locally finite map f: $(CE, E) \to (CF, F)$, we define a map g: $CE \wedge \dot{I}^+ \cup E \wedge I^+ \to CF$ by $g|CE \wedge \{0\}^+=f$, $g|CE \wedge \{1\}^+=\mathrm{cone} \circ \partial(f)$ and $g_n(x, t)=f_n(x)$ for $[x, t] \in E_n \wedge I^+$. Then, g is 0-homotopic since CF is contractible, and we have a locally finite map h: $CE \wedge I^+ \to CF$ with $h|CE \wedge \dot{I}^+ \cup E \wedge I^+=g$ by Proposition 2.10, which is a locally finite homotopy between f and $\mathrm{cone} \circ \partial(f)$. Thus $\mathrm{cone} \circ \partial=\mathrm{id}$.

(2) p_* *is bijective*: By using the fact that C_-F is contractible, we can prove the bijectivity of p_* by the formally identical proof to that for CW complexes (cf. [7, Prop. 6.6]). q.e.d.

Corollary 2.13. *Let E be a CW cospectrum of finite type. Then,*

(1) *for any CW cospectrum F, $[E, F]_L$ has the structure of an abelian group so that composition is bilinear, and*

(2) *for any locally finite map $f: F \to G$ between CW cospectra, the sequence*

$$[E, F]_L \xrightarrow{f_*} [E, G]_L \xrightarrow{i_*} [E, C_f]_L$$

is exact, where $i: G \subset C_f$ is the inclusion.

Proof. (1) The homotopy commutative comultiplication $S^2 \to S^2 \vee S^2$ gives us a function

$$E \wedge S^2 \longrightarrow E \wedge (S^2 \vee S^2) = (E \wedge S^2) \vee (E \wedge S^2),$$

which induces the structure of an abelian group on $[E \wedge S^2, F \wedge S^2]_L$. Thus, by using the bijection $\Sigma^2: [E, F]_L \approx [E \wedge S^2, F \wedge S^2]_L$ of the above theorem, we see (1).

(2) The equality $i_* \circ f_* = 0$ is clear.

Let $g: E \to G$ be a locally finite map with $i_*(g) = 0$. Then, by dividing $E \wedge S^1$ to the union of two cones C_-E and CE and by taking a locally finite map h: $C_-E \to C_f$ which is a 0-homotopy of $i \circ g$, we have a homotopy commutative diagram

$$\begin{array}{ccccc} E \wedge S^1 & \xrightarrow{g \wedge \mathrm{id}} & G \wedge S^1 & \xleftarrow{f \wedge \mathrm{id}} & F \wedge S^1 \\ \mathrm{id} \downarrow & & \uparrow p & & \uparrow \mathrm{id} \wedge \nu \\ C_-E \cup CE & \xrightarrow{h \circ (g \wedge \mathrm{id})} & C_f \cup CG & \xrightarrow{p'} & F \wedge S^1 \end{array}$$

where p and p' are the projections shrinking C_f and CG to $*$, respectively, and $\nu(t) = 1 - t$ for $t \in S^1$. Thus, $g \wedge \mathrm{id}$ is homotopic to $(f \wedge \mathrm{id}) \circ g'$ for some locally finite map $g': E \wedge S^1 \to F \wedge S^1$. By the above theorem, take a locally finite map $\bar{g}: E \to F$ such that $\bar{g} \wedge \mathrm{id}$ is homotopic to g'. Then, $g \wedge \mathrm{id}$ is homotopic to $(f \wedge \bar{g}) \circ \mathrm{id}$; hence g is homotopic to $f \circ \bar{g}$ by the above theorem. Therefore, $g \in \mathrm{Im}\, f_*$ and (2) is proved. q.e.d.

In the rest of this section, we shall prove Theorem 2.11, by showing the following

Lemma 2.14. *Let G, H, K and E be as in Theorem 2.11, F be a subcospectrum of E, and $f: CE \to G$ be a locally finite map with $f(E) \subset H$ and $f(CF) \subset K$. Then, there is a locally finite homotopy*

$$h: CE \wedge I^+ \longrightarrow G$$

of maps such that $h(CE \wedge I^+) = f(CE)$, $h(E \wedge I^+) \subset H$, $h(CF \wedge I^+) \subset K$, $h_0 = f$, $h_1(E) \subset H \cap K$ and h is stationary on CF.

PROOF OF THEOREM 2.11. By taking $F=*$ in Lemma 2.14, we see easily that the induced map i_* in Theorem 2.11 is surjective. Furthermore, by considering $E\wedge I^+$ instead E and by taking $F=E\wedge \dot{I}^+$ in Lemma 2.14, we can see that i_* is injective. Thus, i_* is bijective. q.e.d.

PROOF OF LEMMA 2.14. We may assume f is a function.

(1) In the first place, we prove the lemma for the case that $E=E(D^n)$, $F=E(S^{n-1})$. Consider the image $L=f(E(D^n)\wedge I)$ of f, which is a finite subcospectrum of G. Then, by Lemma 2.7, there is k such that

$$L_r = S^{r-k}L_k,\quad L_r\cap H_r = S^{r-k}(L_k\cap H_k),\quad L_r\cap K_r = S^{r-k}(L_k\cap K_k),$$
$$\text{and}\quad f_r = S^{r-k}f_k \quad\text{for}\quad r\geq k;$$

and these are $(r-k-1)$-connected and $L_r=(L_r\cap H_r)\cup(L_r\cap K_r)$. Therefore, by the homotopy excision theorem (cf. e. g. [7, Th. 6.21]), there is $r(>n+2k+3)$ such that the map

$$f_r\colon (CD^{n+r}, D^{n+r}) \longrightarrow (L_r, L_r\cap H_r),\quad f_r(CS^{n+r-1})\subset L_r\cap K_r,$$

is homotopic rel CS^{n+r-1} to a map $(CD^{n+r}, D^{n+r})\to(L_r\cap K_r, L_r\cap H_r\cap K_r)$. Denoting this homotopy by h_r and setting $h_m=S^{m-r}h_r(m\geq r)$, $h_m=*(m<r)$, we obtain a function

$$h=\{h_m\}\colon E(D^n)\wedge I\wedge I^+ \longrightarrow L/L^{(r)}\subset G/L^{(r)}$$

which is the desired homotopy.

(2) For the general case, consider the filtration $\{E^n\cup F|n\geq 0\}$ where E^n is the one in Definition 2.8 (2). Assume inductively that there exist some integer r_e for each cell e in $E^n\cup F$ and a function

$$h^n\colon C(E^n\cup F)\wedge I^+ \longrightarrow G/A^n,$$

where $A^n=\cup\{f(e^{(r_e)})|e\colon \text{cell in } E^n\cup F\}$ is null by Lemma 2.3 (2), with the following conditions:

1) $h^n(C(E^n\cup F)\wedge I^+)=f(C(E^n\cup F))/A^n$, $h^n((E^n\cup F)\wedge I^+)\subset H/H\cap A^n$, $h^n(CF\wedge I^+)\subset K/K\cap A^n$,
2) $h^n_0=f|C(E^n\cup F)$, $h^n_1(C(E^n\cup F), E^n\cup F)\subset(K/K\cap A^n, H\cap K/H\cap K\cap A^n)$,
3) h^n is stationary on CF.

Then, by Lemma 2.9 and by applying (1) for each cell e in $E^{n+1}\cup F-E^n\cup F$, we obtain r_e for each e and a function $h^{n+1}\colon C(E^{n+1}\cup F)\wedge I^+\to G/A^{n+1}$ with the above 1)~3) for $n+1$ instead of n. Thus, we have a desired homotopy $h\colon CE\wedge I^+\to G/\cup_n A^n$, because $E=\cup_n E^n$, where $\cup_n A^n$ is null in G by Lemma 2.3 (2). This completes the proof of Lemma 2.14 and hence that of Theorem 2.11. q.e.d.

§ 3. Cohomotopy properties

In this section, we study the cohomotopy groups of CW cospectra. For any CW cospectrum E and any integer n, consider the cohomotopy set

$$\pi^n(E) = [E, \Sigma^n S^0] = [E, \Sigma^n S^0]_L, \tag{3.1}$$

where $\Sigma^n S^0$ is the sphere CW cospectrum given by

$$(\Sigma^n S^0)_i = * \quad \text{for} \quad i < -n, = S^{n+i} \quad \text{for} \quad i \geq -n.$$

we notice that Proposition 1.15 means that

(3.1)′ *$\pi^n(E)$ is the direct limit of $\{[E_i, S^{n+i}], \varepsilon_i^* \circ \Sigma\}$, where*

$$\varepsilon_i^* \circ \Sigma \colon [E_i, S^{n+i}] \xrightarrow{\Sigma} [SE_i, S^{n+i+1}] \xrightarrow{\varepsilon_i^*} [E_{i+1}, S^{n+i+1}].$$

For a CW cospectrum E of finite type, by the two bijections

$$[E, \Sigma^n S^0]_L \xrightarrow{\Sigma} [E \wedge S^1, \Sigma^n S^0 \wedge S^1]_L \xrightarrow{p_*} [E \wedge S^1, \Sigma^{n+1} S^0]_L,$$

where Σ is the one in Theorem 2.12 and $p \colon \Sigma^{n+1} S^0 \to \Sigma^n S^0 \wedge S^1$ is the projection shrinking the null subcollection $(\Sigma^{n+1} S^0)^{(-n)}$ in $\Sigma^{n+1} S^0$ to $*$, we obtain the bijection

$$\Sigma = p_*^{-1} \circ \Sigma \colon \pi^n(E) \longrightarrow \pi^{n+1}(E \wedge S^1). \tag{3.2}$$

Furthermore, $\pi^{n+2}(E \wedge S^2)$ has the structure of an abelian group. Thus, by using the bijection Σ^2 we can see that $\pi^n(E)$ has the structure of an abelian gruop.

DEFINITION 3.3 For any CW cospectrum E of finite type and any integer n, the *n-th cohomotopy group* $\pi^n(E)$ of E is the set $\pi^n(E)$ in (3.1) with the structure of an abelian group induced by the bijection Σ^2. Any map between CW cospectra of finite type induces the homomorphism between cohomotopy groups.

REMARK 3.4. The cohomotopy group $\pi^n(E)$ can be defined for any CW cospectrum E which is not necessarily of finite type. In fact, we can show that the suspension Σ of (3.2) is bijective for any CW cospectrum E (cf. Lemma 4.3.)

We shall deal with a proof and applications of the following theorem in this section.

THEOREM 3.5. *Assume that a CW cospectrum E of finite type satisfies the condition $\pi^*(E)=0$. Then, E is contractible in the homotopy category of CW cospectra and maps, i.e., the identity map $id \colon E \to E$ is homotopic to $*$.*

By using this theorem, we have the 'dual' of the J. H. C. Whitehead theorem.

THEOREM 3.6. *Let E and F be CW cospectra of finite type and $f: E \to F$ be a locally finite map. If $f^*: \pi^*(F) \to \pi^*(E)$ induced by f is an isomorphism, then $f: E \to F$ is a homotopy equivalence in the homotopy category of CW cospectra and maps.*

PROOF. In the first place, we show that

($*$) *for any CW cospectrum G of finite type, there are a CW cospectrum $\bar{G}$ of finite type and locally finite homotopy equivalence $q: G \to \bar{G} \wedge S^1$.*

For G in ($*$), consider the filtration

$$* = G^0 \subset G^1 \subset \cdots \subset G^n \subset \cdots \subset \bigcup_n G^n = G$$

given in Definition 2.8 (2). Then, we can constract inductively CW cospectra $\bar{G}^n$ of finite type and locally finite homotopy equivalences $q^n: G^n \to \bar{G}^n \wedge S^1$ for $n \geq 0$ with $\bar{G}^0 = *$, $\bar{G}^n \subset \bar{G}^{n+1}$ and $q^{n+1}|G^n = q^n$. In fact, assume that we have $\bar{G}^n$ and q^n as desired. Then, for the big attaching map $g: \vee_\alpha K_\alpha \wedge S^1 \to G^n (K_\alpha = \Sigma^{d_\alpha} S^0)$ with $G^{n+1} = C_g$ in Lemma 2.9, there are subcospectra L_α of $\bar{G}^n$ with $(q^n \circ g)(K_\alpha \wedge S^1) = L_\alpha \wedge S^1$ by Example 1.2 (3). Furthermore, by Theorem 2.12, there is a locally finite map $\bar{g}: \vee_\alpha K_\alpha \to \bar{G}^n$ such that $\bar{g} \wedge \mathrm{id}|K_\alpha \wedge S^1$ is homotopic to $q^n \circ g|K_\alpha \wedge S^1$ in $L_\alpha \wedge S^1$. Then, we can define $\bar{G}^{n+1} = C_{\bar{g}}$ and $q^{n+1}: G^{n+1} = C_g \to \bar{G}^{n+1} \wedge S^1 = C_{\bar{g} \wedge \mathrm{id}}$ naturally.

By the above construction, we see ($*$) by taking $\bar{G} = \bigcup_n \bar{G}^n$ and $q|G^n = q^n$.

Now, we prove the theorem. Let $\bar{E}$ and $\bar{F}$ be CW cospectra in ($*$) for $G = E$ and F in the theorem, respectively. Then, we have the bijections $[\bar{E}, \bar{F}]_L \overset{\Sigma}{\approx} [\bar{E} \wedge S^1, \bar{F} \wedge S^1]_L \approx [E, F]_L$ by Theorem 1.12 and ($*$). Thus, for a given locally finite map $f: E \to F$, we can find a locally finite map $\bar{f}: \bar{E} \to \bar{F}$ which is mapped to f by the composition of the above bijections; and $\bar{f}^*: \pi^*(\bar{F}) \to \pi^*(\bar{E})$ is isomorphic by (3.2), ($*$) and the assumption that f^* is isomorphic. Hence, $\pi^*(C_{\bar{f}}) = \pi^*(C_{\bar{f}} \wedge S^1) = 0$ by the exact sequence in Lemma 1.16, and these equalities imply that $[C_{\bar{f}}, H] = [C_{\bar{f}} \wedge S^1, H] = 0$ for any CW cospectrum H by Theorem 3.5. Therefore, $(\bar{f} \wedge \mathrm{id})^*: [\bar{F} \wedge S^1, H] \to [\bar{E} \wedge S^1, H]$ is isomorphic by the exact sequence in Lemma 1.16. These show that

(3.7) $\quad f^*: [F, H] \longrightarrow [E, H]$ *is isomorphic for any CW cospectrum H.*

By taking $H = E$ in (3.7), we have a map $g: F \to E$ with $[g \circ f] = [\mathrm{id}]$. Furthermore, by taking $H = F$ in (3.7), we see that $f^*[f \circ g] = [f \circ g \circ f] = [f] = f^*[\mathrm{id}]$ and so $[f \circ g] = [\mathrm{id}]$. Thus, f is a homotopy equivalence, and the theorem is proved assuming Theorem 3.5. q.e.d.

The following corollary ensures that the homotopy category of CW cospectra of finite type and maps is "stable".

COROLLARY 3.8. *Let E, F and G be CW cospectra of finite type.*

(1) *There is a natural homotopy equivalence* $E\wedge S^1\simeq\Sigma^1 E$.

(2) *The homotopy suspension*

$$\Sigma\colon [E, F]\longrightarrow [E\wedge S^1, F\wedge S^1],\quad \Sigma g = g\wedge \mathrm{id}.$$

is an isomorphism.

(3) *For any locally finite map* $f\colon F\to G$, *there holds the following exact sequence*:

$$[E, F]\xrightarrow{f_*}[E, G]\longrightarrow [E, C_f].$$

PROOF. (1) Let $H^{\pm 1}\colon S^1\wedge S^1\wedge I^+\to S^1\wedge S^1\wedge I^+$ be homeomorphisms with $H^{\pm 1}[x, y, 0]=[x, y, 0]$, $H^{+1}[x, y, 1]=[\nu(y), x, 1]$ and $H^{-1}[x, y, 1]=[y, \nu(x), 1]$ where $\nu\colon S^1\to S^1$ is given by $\nu(x)=1-x$ for $x\in S^1$. Then, we obtain a CW cospectrum $\tilde{E}=\{\tilde{E}_n, \tilde{\varepsilon}_n\}$ of finite type such that $\tilde{E}_n=E_n\wedge S^1\wedge I^+$ and $\tilde{\varepsilon}_n$ is the composition of

$$\begin{aligned}\tilde{E}_{n+1} = E_{n+1}\wedge S^1\wedge I^+ &\xrightarrow{\varepsilon_n\wedge\mathrm{id}} S^1\wedge E_n\wedge S^1\wedge I^+ \xrightarrow{T\wedge\mathrm{id}}\\ E_n\wedge S^1\wedge S^1\wedge I^+ &\xrightarrow{\mathrm{id}\wedge H(n)} E_n\wedge S^1\wedge S^1\wedge I^+ \xrightarrow{T\wedge\mathrm{id}}\\ &\qquad S^1\wedge E_n\wedge S^1\wedge I^+ = S\tilde{E}_n,\end{aligned}$$

where $H(n)=H^{(-1)^n}$ and T is the switching map. Furthermore, by using the inclusions $i_t\colon E_n\wedge S^1\wedge\{t\}^+\subset E_n\wedge S^1\wedge I^+ (t=0, 1)$, we have two locally finite functions

$$\tau\colon E\wedge S^1\longrightarrow \tilde{E}\quad\text{with}\quad \tau_n = i_0,\quad\text{and}$$

$$\kappa\colon \Sigma^1 E\longrightarrow \tilde{E}\quad\text{with}\quad \kappa_n = i_1\circ(\mathrm{id}\wedge\nu^n)\circ T\circ\varepsilon_n.$$

Now, we can show that the two induced homomorphisms

$$\tau^*\colon \pi^*(\tilde{E})\longrightarrow \pi^*(E\wedge S^1),\quad \kappa^*\colon \pi^*(\tilde{E})\longrightarrow \pi^*(\Sigma^1 E),$$

are both bijective; hence τ and κ are both homotopy equivalences, which implies (1). In fact, the bijectivity of κ^* is proved by (3.1)$'$ and by the commutative diagram

$$\begin{array}{ccccc}[\tilde{E}_{n-1}, S^{*+n-1}] & \xrightarrow{\Sigma} & [S\tilde{E}_{n-1}, S^{*+n}] & \xrightarrow{\tilde{\varepsilon}^*_{n-1}} & [\tilde{E}_n, S^{*+n}]\\ \downarrow{\scriptstyle \kappa^*_{n-1}} & & \swarrow{\scriptstyle (i_1\circ(\mathrm{id}\wedge\nu^n)\circ T)^*} & & \downarrow{\scriptstyle \kappa^*_n}\\ [E_n, S^{*+n-1}] & \xrightarrow[\Sigma]{} & [SE_n, S^{*+n}] & \xrightarrow[\varepsilon^*_n]{} & [E_{n+1}, S^{*+n}],\end{array}$$

where $(i_1\circ(\mathrm{id}\wedge\nu^n)\circ T)^*$ is bijective; and the one of τ^* is proved more easily.

(2) For E and F, consider the CW cospectra $\tilde{E}$ and $\tilde{F}$ given in the proof

of (1), respectively. Then, for any function $f: E \to F$, we obtain a function $\alpha(f)$: $\tilde{E} \to \tilde{F}$ such that

$$\alpha(f)_n = f_n \wedge \mathrm{id}: \tilde{E}_n = E_n \wedge S^1 \wedge I^+ \longrightarrow F_n \wedge S^1 \wedge I^+ = \tilde{F}_n,$$

and this α induces

$$\alpha: [E, F] \longrightarrow [\tilde{E}, \tilde{F}].$$

Furthermore, we see easily that the diagram

$$\begin{array}{ccccc} [\Sigma^1 E, \Sigma^1 F] & \xleftarrow{\cong} & [E, F] & \xrightarrow{\Sigma} & [E \wedge S^1, F \wedge S^1] \\ \kappa_* \downarrow \cong & & \alpha \downarrow & & \cong \downarrow \iota_* \\ [\Sigma^1 E, \tilde{F}] & \underset{\cong}{\xleftarrow{\kappa^*}} & [\tilde{E}, \tilde{F}] & \underset{\cong}{\xrightarrow{\iota^*}} & [E \wedge S^1, \tilde{F}] \end{array}$$

is commutative by the definition of κ and τ in the proof of (1). Thus, Σ is an isomorphism.

(3) We can prove (3) by the same proof as that of Corollary 2.13 (2).

q.e.d.

The rest of this section is devoted to the proof of Theorem 3.5.

DEFINITION 3.9. Let E be a CW cospectrum such that each E_i is finite and $E_n = *$ for $n \leq 0$. Then, we define $E^{\{n\}}$ to be the union of cells e of E with $e_n = *$. We see easily that $E/E^{\{n\}}$ is finite and

$$E = E^{\{0\}} \supset E^{\{1\}} \supset \cdots \subset E^{\{n\}} \supset \cdots \supset * = \cap_{n \geq 0} E^{\{n\}}.$$

LEMMA 3.10. *Let E and $E^{\{n\}}$ be as in the above definition. If $F(n)$ is a null subcollection of each $E^{\{n\}}$, then $F = \cup_{n \geq 0} F(n)$ is null in E.*

PROOF. Let $e \neq *$ be any cell in E. Then, there is an integer $N \geq 0$ with $E^{\{n\}} \not\supset e$ for $n \geq N$ since $\cap_{n \geq 0} E^{\{n\}} = *$. Thus $E^{\{N\}} \supset \cup_{n \geq N} F(n) \not\supset e$. Also $\cup_{0 \leq n < N} F(n) \not\supset e$ by Lemmas 1.9 (1) and 2.3 (1). Therefore, $F = \cup_n F(n) \not\supset e$; hence F is null by Lemma 2.3 (1). q.e.d.

PROOF OF THEOREM 3.5. We may assume that $E_n = *$ for $n \leq 0$ by Lemma 1.13. By the assumption $\pi^*(E) = 0$, Lemma 2.9, Corollary 2.13 (2) and the five lemma, we see immediately that $[E, F]_L = 0$ for any finite CW cospectrum F. Hence, $[E, E^{\{n\}}/E^{\{n+1\}}]_L = 0$, because $E^{\{n\}}/E^{\{n+1\}}$ is finite. By using the exact sequence of Corollary 2.13 (2) for $E^{\{n+1\}} \subset E^{\{n\}}$, we can find inductively locally finite functions

$$h^n: E \wedge I^+ \longrightarrow E^{\{n\}}/F(n) \ (F(n): \text{null in } E^{\{n\}})$$

for $n \geq 0$ such that $h_0^0 = \mathrm{id}$, $h_1^n(E) \subset E^{\{n+1\}}/F(n) \cap E^{\{n+1\}}$ and $h_0^{n+1} = h_1^n$, ($h_t^n = h^n | E \wedge \{t\}^+$ for $t = 0, 1$). Now, we have locally finite functions

$$\bar{h}^n\colon E \wedge I^+ \xrightarrow{h^n} E^{\{n\}}/F(n) \xrightarrow{\text{pr}} E^{\{n\}}/F \cap E^{\{n\}} \subset E/F,$$

where $F=\bigcup_{n\geq 0} F(n)$ is null in E by the above lemma, and

$$H^n\colon E \wedge I^+ \longrightarrow (E/F)/(E^{\{n+1\}}/E^{\{n+1\}} \cap F)$$

such that $H^n|E\wedge[1-(r+1)^{-1}, 1-(r+2)^{-1}]^+$ for $r\geq 0$ is equal to

$$\text{pr}\circ\bar{h}^r\circ j_r \ (0\leq r\leq n), \quad * \ (r>n)$$

where $j_r\colon [1-(r+1)^{-1}, 1-(r+2)^{-1}]\to[0, 1]$ is a homeomorphism. Because each E_n is finite, there is $\beta(n)$ for each n such that $(E^{\{\beta(n)\}}/E^{\{\beta(n)\}}\cap F)_n=*$. Moreover, we can take $\beta(n)$ so that $\beta(n')\geq\beta(n)$ if $n'\geq n$. Thus, we can find a homotopy

$$H\colon E \wedge I^+ \longrightarrow E/F,\ H_n=(H^{\beta(n)})_n,$$

with $H_0=\text{id}$ and $H_1=*$. q.e.d.

§4. The representation of homology theories by *CW* cospectra

In the first place, we notice that any CW cospectrum E defines a homology theory $E_*(-)$ on the homotopy category NCW' of nice complexes and nice maps which is equivalent to the homotopy category CW' of CW complexes by Proposition 1.3.

For any CW cospectrum E, any nice complex X and any integer n, we consider the CW cospectra $\Sigma^n E$ and $E(X)$ in Example 1.6, and the homotopy set

(4.1) $$E_n(X)=[\Sigma^n E, X] \quad \text{(where } X \text{ stands for } E(X))$$

of maps between CW cospectra. Then, we have the following

Proposition 4.2. *For any CW cospectrum E, $E_*(-)$ in (4.1) forms a reduced homology theory on NCW'.*

This proposition is an immediate consequence of the following

Lemma 4.3. (1) *The homotopy suspension*

$$\Sigma\colon [E, X] \longrightarrow [E\wedge S^1, X\wedge S^1], \quad \Sigma f=f\wedge \text{id}.$$

is a bijection.

(2) *If $f\colon X\to Y$ is a nice map, then the sequence*

$$[E, X] \xrightarrow{f_*} [E, Y] \xrightarrow{i_*} [E, C_f]$$

is exact, where C_f is the mapping cone of f and $i\colon Y\subset C_f$ is the inclusion.

PROOF. (1) Consider the commutative diagram

$$\begin{array}{ccccc}
[E_n, S^nX] & \xrightarrow{\Sigma} & [SE_n, S^{n+1}X] & \xrightarrow{\varepsilon_n^*} & [E_{n+1}, S^{n+1}X] \\
\Sigma' \downarrow & \nearrow^{T'_* \circ T^*} & \Sigma' \downarrow & & \Sigma' \downarrow \\
[E_n \wedge S^1, S^nX \wedge S^1] & \xrightarrow[\Sigma]{} & [SE_n \wedge S^1, S^{n+1}X \wedge S^1] & \xrightarrow[(\varepsilon_n \wedge \mathrm{id})^*]{} & [E_{n+1} \wedge S^1, S^{n+1}X \wedge S^1]
\end{array}$$

where Σ' are also the homotopy suspensions and T and T' are the switching maps. (We see that $\Sigma' \circ (T'_* \circ T^*) = \Sigma$ because $\mathrm{id} \wedge g = (T' \wedge \mathrm{id}) \circ (\mathrm{id} \wedge \rho^{-1}) \circ (g \wedge \mathrm{id}) \circ (\mathrm{id} \wedge \rho) \circ (T \wedge \mathrm{id})$ where $\rho \colon S^1 \wedge S^1 \to S^1 \wedge S^1$, $\rho(t, s) = (s, 1-t)$, is homotopic to the identity.) In this diagram, Proposition 1.15 means that

$$[E, X] = \operatorname{dir\,lim} \{[E_n, S^nX], \varepsilon_n^* \circ \Sigma\}; \tag{4.4}$$

and therefore

$$[E \wedge S^1, X \wedge S^1] = \operatorname{dir\,lim} \{[E_n \wedge S^1, S^nX \wedge S^1], \quad (\varepsilon_n \wedge \mathrm{id})^* \circ \Sigma\}.$$

Thus, the assertion (1) follows immediately from the above diagram.

(2) We can prove (2) by the same proof as that of Corollary 2.13 (2).

q.e.d.

We are in a position to prove the following representation theorem.

THEOREM 4.5. *Let h_* be a given reduced homology theory satisfying*
(4.6) *$h_m(S^n)$ is finitely generated for any $n \geq 0$ and m, and there is an integer N with $h_m(S^0) = 0$ for $m \leq N$.*

Then, there exist a CW cospectrum E of finite type and a natural equivalence

$$T_* \colon E_*(X) \cong h_*(X)$$

for any finite nice complex X, where E_* is the homology theory given in Proposition 4.2.

As an application of this theorem, we have a CW cospectrum which is "dual" to a given CW spectrum with certain conditions by the following corollary. (For the notion of CW spectra and the notations, see [1] or [7, Ch. 8].)

COROLLARY 4.7. *Let $\bar{E}$ be a CW spectrum whose homotopy group $\pi_*(\bar{E})$ satisfies the following condition:*
(4.8) *$\pi_*(\bar{E})$ is finitely generated and there is an integer N with $\pi_n(\bar{E}) = 0$ for $n \leq N$.*
Then, there exist a CW cospectrum E of finite type and a natural equivalence

$$[\Sigma^* E, X] \cong [\Sigma^* S^0, \bar{E} \wedge X] \qquad \textit{for any finite nice complex} \quad X.$$

PROOF. $\bar{E}_*(X) = [\Sigma^* S^0, \bar{E} \wedge X]$ is a reduced homology theory (cf., e.g..

[7, 8.33]) and it satisfies (4.6) by the assumption (4.8). Thus, the corollary follows immediately from the above theorem. q. e. d.

The rest of this section is devoted to the proof of Theorem 4.5.

For a given reduced homology theory h_*, nice complexes X, Y and an element $u \in h_n(X)$, we consider the map

$$(4.9) \qquad T_u\colon [X, Y] \longrightarrow h_n(Y),\ T_u(f) = f_*(u).$$

If $Y=S^k$, $\dim X \leq 2(k-1)$ and $k \geq 2$, then the cohomotopy set $[X, S^k]$ is an abelian group and T_u is a homomorphism (cf., e.g., [5, p. 421, p. 458]).

LEMMA 4.10. *Let h_* be a reduced homology theory satisfying* (4.6) *with $N=0$, X be a finite nice complex with $\dim X \leq 2r$ and $u \in h_{2r}(X)$ where $r \geq 1$. Assume that T_u of* (4.9) *is isomorphic for $k > r+1$ and epimorphic for $k=r+1$. Then, there exist a nice map $f\colon X \to W$ ($W = \vee_{j=1}^{t} S_j^{r+1}$ is the wedge sum of t copies S_j^{r+1} of S^{r+1}) and an element $\bar{u} \in h_{2r+1}(\bar{X})$ ($\bar{X} = C_f$ is the mapping cone of f) satisfying the following conditions*:

1) $\mathrm{pr}_*(\bar{u}) = \sigma(u)$, *where* $\mathrm{pr}\colon \bar{X} \to SX$ *is the projection and* $\sigma\colon h_{2r}(X) \to h_{2r+1}(SX)$ *is the suspension isomorphism.*

2) $T_{\bar{u}}\colon [\bar{X}, S^k] \to h_{2r+1}(S^k)$ *is isomorphic for* $k \geq r+2$.

PROOF. We notice that $[X, S^k]$ is finitely generated for $k \geq r+1$, because X is finite and the homotopy groups of S^k is finitely generated. Suppose that Ker T_u of (4.9) for $k=r+1$ is generated by $\{g_1, \ldots, g_t\}$ and set $W = \vee_j S_j^{r+1}$ ($S_j^{r+1} = S^{r+1}$) and $f = i_1 \circ g_1 + \cdots + i_t \circ g_t\colon X \to W$ ($i_j\colon S^{r+1} = S_j^{r+1} \subset W$). Then, in the exact sequence

$$\cdots \longrightarrow h_{2r+1}(C_f) \xrightarrow{\mathrm{pr}_*} h_{2r+1}(SX) \xrightarrow{(Sf)_*} h_{2r+1}(SW) \longrightarrow \cdots,$$

we see that $(Sf)_*(\sigma(u)) = \sigma(f_*(u)) = \sigma(\sum_{j=1}^{t}(i_j \circ g_j)_*(u)) = \sigma(\sum_{j=1}^{t} i_{j*}(T_u(g_j))) = 0$. Hence, there is an element $\bar{u} \in h_{2r+1}(C_f)$ with $\mathrm{pr}_*(\bar{u}) = \sigma(u)$. Now, consider the the commutative diagrams

$$\begin{array}{ccccccc}
[SW, S^k] & \xrightarrow{(Sf)^*} & [SX, S^k] & \xrightarrow{\mathrm{pr}^*} & [C_f, S^k] & \longrightarrow & [W, S^k]\ (=0) \\
 & & \downarrow{\scriptstyle T_{\sigma(u)}} & & \downarrow{\scriptstyle T_{\bar{u}}} & & \\
 & & h_{2r+1}(S^k) & = & h_{2r+1}(S^k), & &
\end{array}$$

$$\begin{array}{ccccc}
[SW, S^k] & \xrightarrow{(Sf)^*} & [SX, S^k] & \xrightarrow{T_{\sigma(u)}} & h_{2r+1}(S^k) \\
\Sigma \uparrow \cong & & \Sigma \uparrow \cong & & \sigma \uparrow \cong \\
[W, S^{k-1}] & \xrightarrow{f^*} & [X, S^{k-1}] & \xrightarrow{T_u} & h_{2r}(S^{k-1})
\end{array}$$

for $k \geq r+2$, where the first line is exact. If $k \geq r+3$, then T_u is isomorphic by assumption and so is pr* because $[SW, S^k]=0$; hence so is $T_{\bar{u}}$. For the case $k=r+2$, $\operatorname{Im} f_*=\operatorname{Ker} T_u$ by the definition of f and so $\operatorname{Ker} T_{\sigma(u)}=\operatorname{Im}(Sf)^*=\operatorname{Ker}\mathrm{pr}^*$. Thus, $T_{\bar{u}}$ is isomorphic and (2) is proved. q.e.d.

PROOF OF THEOREM 4.5. We prove the theorem for the case $N=0$ in (4.6). Then, the one for the general case follows easily.

We shall construct a CW cospectrum $E=\{E_n, \varepsilon_n\}$ of finite type and elements $u_n \in h_n(E_n)$ satisfying the following conditions:

(4.11) *$E_n = *$ for $n \leq 2$ and $\dim E_n \leq n$,*

(4.12) *$T_{u_n}: [E_n, S^k] \to h_n(S^k)$ is isomorphic for $k>[n/2]+1$ and epimorphic for $k=[n/2]+1$, and*

(4.13) *$\sigma(u_n)=\varepsilon_{n*}(u_{n+1})$ where $\sigma: h_n(E_n)\to h_{n+1}(SE_n)$ is the suspension isomorphism.*

Then, after getting E and u_n satisfying above conditions, we have the map

$$T_k=\operatorname{dir\,lim}_n T_{u_{n+k}}: [\Sigma^k E, X]=\operatorname{dir\,lim}_n [E_{n+k}, S^n X] \longrightarrow \operatorname{dir\,lim}_n h_{n+k}(S^n X)=h_k(X)$$

for any nice complex X, and this is a natural transformation between the homology theories on the category of nice complexes and nice maps. In the case $X=S^i$, T_k is an isomorphism because $T_{u_{n+k}}: [E_{n+k}, S^{n+i}]\to h_{n+k}(S^{n+i})$ is so for $n>k-2i+2$ by (4.12). Thus, $T_*: E_*(X)\to h_*(X)$ is a natural equivalence for any finite nice complex X by the five lemma; and the proof of the theorem is reduced to the construction of E_n and u_n.

Now, assume inductively that we have finite nice complexes E_n and elements $u_n \in h_n(E_n)$ with (4.11–12) for $n \leq 2r (r\geq 1)$. (Since $h_n(S^k)=0$ for $k\geq n$ by (4.6) with $N=0$, (4.12) holds for $E_n=*(n\leq 2)$.) Then, the assumptions in Lemma 4.10 hold for $X=E_{2r}$ and $u=u_{2r}$. Thus, by Lemma 4.10, we can find a finite nice complex $\bar{E}_{2r}$ (which is the mapping cone C_f of a nice map $f: E_{2r}\to \vee_j S_j^{r+1}$) and $\bar{u}_{2r}=\bar{u}\in h_{2r+1}(\bar{E}_{2r})$ such that $\mathrm{pr}_*(\bar{u}_{2r})=\sigma(u_{2r})$ for $\mathrm{pr}: \bar{E}_{2r}\to SE_{2r}$ and $T_{\bar{u}}$: $[\bar{E}_{2r}, S^k]\to h_{2r+1}(S^k)$ is isomorphic for $k\geq r+2$. Now, let $h_{2r+1}(S^{r+1})$ be generated by $\{a_1,\dots, a_s\}$ by (4.6), and set

$$E_{2r+1} = \bar{E}_{2r} \vee (\vee_{l=1}^{s} S_l^{r+1})\ (S_l^{r+1}=S^{r+1}),$$
$$u_{2r+1} = i_*(\bar{u}) + \textstyle\sum_{l=1}^{s} i_{l*}(a_l)\in h_{2r+1}(E_{2r+1})$$

($\bar{u}=\bar{u}_{2r}$, $i: \bar{E}_{2r}\subset E_{2r+1}$, $i_l: S^{r+1}=S_l^{r+1}\subset E_{2r+1}$). Then, for $k\geq r+2$,

$$T_{u_{2r+1}} = T_{\bar{u}}\circ i^*: [E_{2r+1}, S^k] \xrightarrow{i^*} [\bar{E}_{2r}, S^k] \underset{\cong}{\xrightarrow{T_{\bar{u}}}} h_{2r+1}(S^k),$$

is isomorphic, because so is i_* by the definition. For $k=r+1$, $T_{u_{2r+1}}$ is clearly epimorphic by the definition. Thus, (4.12) holds for $n=2r+1$.

For $n=2r+2$, set

$$E_{2r+2}=SE_{2r+1} \quad \text{and} \quad u_{2r+1}=\sigma(u_{2r+1})\in h_{2r+2}(E_{2r+2}).$$

Then, (4.12) holds for $n=2r+2$ by the right square of the lower diagram in the proof of Lemma 4.10.

Thus, we obtain finite nice complexes E_n and elements $u_n\in h_n(E_n)$ satisfying (4.11–12) by induction. Furthermore, in the above construction, let ε_{2r}: $E_{2r+1}\to \bar{E}_{2r}\to SE_{2r}$ be the composition of the projections shrinkig $\vee_l S_l^{r+1}$ and $\vee_j S_j^{r+1}$ to $*$ respectively, and $\varepsilon_{2r+1}=\mathrm{id}$. Then, (4.13) holds for $n=2r$, $2r+1$ by the definition. Moreover, we have a CW cospectrum $E=\{E_n, \varepsilon_n\}$ which is given in Example 1.6 (1). Therefore, by Example 2.6 and (4.4), we may take E of finite type.

These complete the proof of Theorme 4.5. q.e.d.

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