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Bipartite decomposition of complete multipartite graphs

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1. Introduction

Graph theory is a subject of combinatorics in mathematics and it is one of the most flourishing branches of modern algebra with wide applications to various fields. The problem of decomposing a graph into a union of subgraphs each isomorphic to a given graph is an important subject of graph theory. There are many types of decomposition problems, such as, clique decomposition [7, 15], claw decomposition [18, 19, 20, 22, 24], path decomposition [9, 13, 14], cycle decomposition [4, 6, 16], bipartite decomposition [10, 11] and so on. Some of them are used, for example, for combinatorial file organization schemes in filing theory and some are used for construction schemes of designs of experiments in statistics.

We are concerned with a bipartite decomposition, which includes a claw decomposition as a special type. It will be used for a design of combinatorial file organization scheme.

Some results [5, 10, 11, 17, 24] are known about the decompositions of a complete graph K_m with m points. The problem of claw decomposition of a complete graph K_m has been raised and solved completely by Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [24]. The claw decomposition of a complete graph provides us a balanced file organization scheme of order two for binary-valued records. It is optimal in such a sense that it has the least redundancy among all possible balanced binary-valued file organization schemes of order two having the same parameters, provided the distribution of records has the property of invariance with respect to the permutation of attributes. Such a scheme is called HUBFS_2 [25]. Huang and Rosa [10] and Huang [11] have investigated a bipartite decomposition of a complete graph K_m by introducing the concept of the balance of points.

As for the decomposition of a complete multipartite graph, many authors [18, 19, 20, 21, 22, 24] have studied. The complete solution of the problem of claw decomposition of a complete bipartite graph has been given by Yamamoto et al. [24]. Ushio, Tazawa and Yamamoto [20] have given a theorem which states a necessary and sufficient condition for a complete m-partite graph $K_m(n,\dots, n)$ with m sets of n points each to have a claw decomposition. Moreover, Tazawa, Ushio and Yamamoto [18] have given a necessary and sufficient condition for a complete m-partite graph $K_m(n,\dots, n)$ to be decomposed into partite-claws, where a partite-claw is a particular type of claw. The former decomposition yields a generalized balanced multiple-valued file organization scheme of order two which is called $GHUBMFS_2$ [27]. The latter one yields an optimal balanced multiple-valued file organization scheme of order two, called $HUBMFS_2$ [26], in that it has the least redundancy among all possible balanced schemes with the same parameters for an equally likely distribution of multiple-valued records. The problem of balanced claw decomposition of a complete m-partite graph $K_m(n,\dots, n)$ has been solved completely by Ushio [22].

In this paper, we shall study the bipartite decomposition of complete multipartite graphs. In Section 3, a theorem which states a necessary and sufficient condition for a complete bipartite graph $K(n_1, n_2)$ to have a bipartite decomposition will be given (Theorem 3.2). Some corollaries will also be given. In Section 4, we shall investigate a bipartite decomposition of a complete m-partite graph $K_m(n_1,\dots, n_m)$ with $m\geq 3$. Especially when $n_1=\cdots=n_m=n$, it will be discussed that a bipartite decomposition yields a new type of balanced multiple-valued file organization scheme of order two by introducing the concept of the balance of points. Some theorems which deal with a balanced bipartite decomposition of a complete m-partite graph $K_m(n,\dots, n)$ will be given.

2. Preliminaries

This paper is concerned with graphs without loops or multiple lines. Any term not defined here can be found in [1, 8]. Let $G(V, X)$ be a graph, where V is the point set and X is the line set of the graph. A graph is called a *multipartite graph* if the point set V can be partitioned into m subsets $V_1,\dots, V_m$ such that no two points in the same subset are adjacent. Each subset V_i is called its *independent set.* A multipartite graph is said to be a *complete m-partite graph* if each point in V_i is adjacent to every point except those in V_i. The complete m-partite graph is denoted by $K_m(n_1,\dots, n_m)$, where n_i is the cardinality $|V_i|$ of V_i $(i=1,\dots, m)$. A *complete graph* K_m with m points may be regarded as a particular type of complete m-partite graph where $n_1=\cdots=n_m=1$. When $m=2$, a complete 2-*partite* graph $K_2(n_1, n_2)$ is usually called a complete *bipartite* graph and is denoted simply by $K(n_1, n_2)$. In particular, $K(1, c)$ with $c+1$ points and c lines is called a *claw* or *star* of degree c.

DEFINITION 1. Let G be a complete bipartite graph $K(k_1, k_2)$. A complete m-partite graph $K_m(n_1,\dots, n_m)$ with m independent sets of $n_1,\dots, n_m$ points each is said to have a $K(k_1, k_2)$*-decomposition* if it can be decomposed into a union of line-disjoint subgraphs each isomorphic to G. Each of those subgraphs is called a *block* of the original graph $K_m(n_1,\dots, n_m)$.

DEFINITION 2. A bipartite decomposition is said to be *balanced* if each point of $K_m(n_1,\dots, n_m)$ belongs to exactly the same number of blocks.

3. Bipartite decomposition of a complete bipartite graph

In this section, we shall discuss a bipartite decomposition of a complete bipartite graph.

3.1. Bipartite decomposition theorem of $K(n_1, n_2)$

Given two positive integers k_1 and k_2, suppose that for a positive integer n there exist two nonnegative integers x and y such that an equation $n=k_1x+k_2y$ holds. We call the ordered pair (x, y) a *solution vector* of the equation. Let $w(n; k_1, k_2)$ denote the number of distinct solution vectors, where $w(n; k_1, k_2)=0$ means that there does not exist any solution vector of the equation. We write $w(n)$, for short, instead of $w(n; k_1, k_2)$ throughout this paper. We assume $n_1\le n_2$ and $k_1\le k_2$ without loss of generality.

LEMMA 3.1. *Let n_1, n_2, k_1, k_2 be positive integers, where $n_1\le n_2$ and $k_1\le k_2$. A necessary condition for a complete bipartite graph $K(n_1, n_2)$ to have a $K(k_1, k_2)$-decomposition is that the following conditions* (i)–(iii) *hold*:

(i) *n_1n_2 is an integral multiple of k_1k_2.*

(ii) *$n_1 \ge k_1$ and $n_2 \ge k_2$.*

(iii) *$w(n_1)\ge 1$ and $w(n_2)\ge 1$.*

PROOF. Since $K(n_1, n_2)$ has n_1n_2 lines and every block in the $K(k_1, k_2)$-decomposition has k_1k_2 lines, the first condition is, obviously, necessary. If the second condition does not hold, then no $K(k_1, k_2)$ is a subgraph of $K(n_1, n_2)$, so that $K(n_1, n_2)$ does not have any $K(k_1, k_2)$-decomposition. Therefore, the condition (ii) is necessary. Let V_1, V_2 be the independent sets of $K(n_1, n_2)$. For each block B, let B_1 denote the independent set of B with cardinality k_1 and let B_2 denote that of B with cardinality k_2. For a point u in V_1, let $y(u)$ and $x(u)$, respectively, be the number of B_1's and that of B_2's such that u appears in B_1 and B_2. Then the point u is adjacent both to $k_2y(u)$ points of $y(u)$ B_2's and to $k_1x(u)$ points of $x(u)$ B_1's. In $K(n_1, n_2)$ the point u is adjacent to n_2 points of V_2. Therefore, we have

$$n_2 = k_1x(u) + k_2y(u). \tag{3.1}$$

If for a point v in V_2, we denote by $y(v)$ and $x(v)$ the respective numbers of B_1's and B_2's in which v appears, then by the similar discussion we have

$$n_1 = k_1x(v) + k_2y(v). \tag{3.2}$$

As seen in (3.1) and (3.2), the ordered pair $(x(v), y(v))$ is a solution vector of $n_1=k_1x+k_2y$ and the ordered pair $(x(u), y(u))$ is that of $n_2=k_1x+k_2y$. Thus we obtain $w(n_1)\geq 1$ and $w(n_2)\geq 1$, that is Condition (iii). This completes the proof.

We shall see in the following that the conditions stated in the above lemma are not sufficient.

THEOREM 3.2. *Let n_1, n_2, k_1, k_2 be positive integers with $n_1\leq n_2$ and $k_1\leq k_2$.*

(a) *When $w(n_1)=1$, i.e., when there exists only one solution vector (x_0, y_0) of $n_1=k_1x+k_2y$, a complete bipartite graph $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition if and only if there hold Conditions* (i)–(iii) *in Lemma* 3.1 *and the following Condition* (iv):

(iv) *There exists a nonnegative integer vector $(f_1,\dots,f_\beta)$ such that*

$$\sum_{q=1}^{\beta} f_q = n_1 \quad and \quad k_1x_0n_2 = \sum_{q=1}^{\beta} k_2y_qf_q, \tag{3.3}$$

where (x_q, y_q), $q=1,\dots,\beta$, are solution vectors of $n_2=k_1x+k_2y$.

(b) *When $w(n_1)\geq 2$, i.e., when the number of distinct solution vectors of $n_1=k_1x+k_2y$ is greater than or equal to 2, a complete bipartite graph $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition if and only if there hold Conditions* (i)–(iii) *in Lemma* 3.1.

The proof of this theorem will be given in the subsection 3.4. Under the restrictions imposed on a set of the original parameters, we have some corollaries.

COROLLARY 3.3. *For a set of parameters $n_1=n_2=n$, k_1, k_2 $(k_1\leq k_2)$, a complete bipartite graph $K(n, n)$ has a $K(k_1, k_2)$-decomposition if and only if they satisfy Conditions* (i) *and* (ii) *in Lemma* 3.1 *and the inequality $w(n)\geq 2$.*

PROOF. It is enough to show that when $w(n)=1$, the solution vector (x, y) of $n=k_1x+k_2y$ can not satisfy Condition (iv) of Statement (a) in Theorem 3.2. Assume that $w(n)=1$. Let (x, y) be the solution vector of $n=k_1x+k_2y$. From (3.3) we have $k_1xn=k_2yn$. Since $n=k_1x+k_2y$, we have $n=2k_1x=2k_2y$, which shows that $(0, 2y)$ and $(2x, 0)$ are also solution vectors of $n=k_1x+k_2y$. Consequently, the assumption that $w(n)=1$ implies $x=y=0$, which contradicts the fact that n is positive. This completes the proof.

COROLLARY 3.4. *When $k_1=k_2=k$, a complete bipartite graph $K(n_1, n_2)$ has a $K(k, k)$-decomposition if and only if*

$$n_1 \equiv 0 \quad and \quad n_2 \equiv 0 \pmod{k}.$$

When $k_1=1$, it can be shown that Theorem 3.2 is equivalent to the following corollary, which has been given by Yamamoto et al. [24].

COROLLARY 3.5. *A complete bipartite graph* $K(n_1, n_2)$ $(n_1 \leq n_2)$ *has a* $K(1, k_2)$*-decomposition if and only if*

(1) $n_2 \equiv 0 \pmod{k_2}$ *when* $n_1 < k_2$,

(2) $n_1 n_2 \equiv 0 \pmod{k_2}$ *when* $n_1 \geq k_2$.

3.2. Adjacency matrix and bipartite decomposition of $K(n_1, n_2)$

Let V_1, V_2 be the independent sets of $K(n_1, n_2)$, where $|V_1| = n_1$, $|V_2| = n_2$ and $V_1 \cap V_2 = \emptyset$. We label those points in V_1 and V_2 by $v_{11}, \ldots, v_{1n_1}$ and $v_{21}, \ldots, v_{2n_2}$, respectively. Consider a block $K(k_1, k_2)$ which is a subgraph of $K(n_1, n_2)$. Then the block is denoted by $\{B_1; B_2\}$, where B_i is a subset of V_i $(i = 1, 2)$. When $|B_1| = k_1$ and $|B_2| = k_2$, the block $\{B_1; B_2\}$ is said to be *A-type*. When $|B_1| = k_2$ and $|B_2| = k_1$, the block $\{B_1; B_2\}$ is said to be *B-type*. If $k_1 = k_2$, in particular, we refer to two types as *A*-type. In Fig. 1, a complete bipartite graph $K(5, 6)$ with two independent sets V_1, V_2 of 5, 6 points each is shown. For $k_1 = 2$ and $k_2 = 3$, an *A*-type block $\{B_1; B_2\}$ with $B_1 = \{v_{11}, v_{13}\}$ and $B_2 = \{v_{22}, v_{23}, v_{26}\}$ is also illustrated.

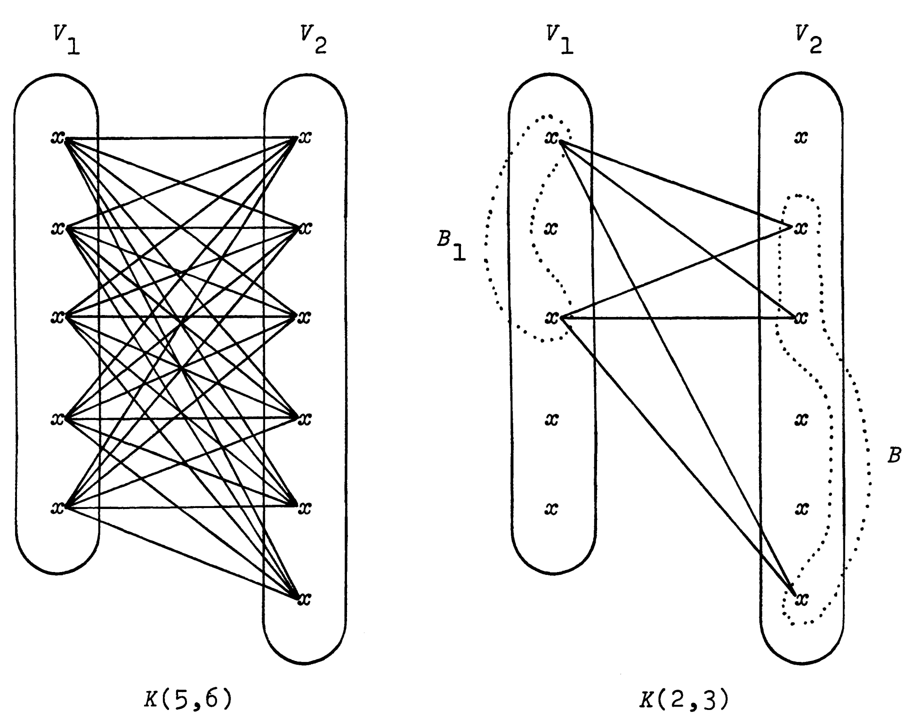

Fig. 1. A complete bipartite graph and an *A*-type block

To a block $\{B_1; B_2\}$ of $K(n_1, n_2)$, there corresponds a 0–1 matrix $M=\|m_{ij}\|$ of size $n_1 \times n_2$ which is defined by

$$(3.4) \qquad m_{ij} = \begin{cases} 1 & \text{if} \quad v_{1i} \in B_1 \quad \text{and} \quad v_{2j} \in B_2 \\ 0 & \text{otherwise.} \end{cases}$$

This matrix M is called an *adjacency matrix* of the block $\{B_1; B_2\}$. Note that the matrix M is reduced to a matrix of the form

$$(3.5) \qquad \begin{bmatrix} G_{|B_1|,|B_2|} & 0 \\ 0 & 0 \end{bmatrix}$$

by an appropriate permutation of rows and columns, where $G_{t,u}$ is a $t \times u$ matrix whose elements are all one. To a matrix M whose reduced matrix is of the form (3.5), there corresponds, obviously, a block $\{B_1; B_2\}$.

We call an adjacency matrix M of a block $\{B_1; B_2\}$ an *A-type matrix* or a *B-type matrix* according as the block $\{B_1; B_2\}$ is A-type or B-type. An A-type matrix is denoted by $M_A=\|m_{ij}^{(A)}\|$ and a B-type matrix is denoted by $M_B=\|m_{ij}^{(B)}\|$. It is easy to see that we have the following relations:

$$(3.6) \quad \sum_{i=1}^{n_1} m_{ij}^{(A)} = \begin{cases} k_1 & \text{if} \quad v_{2j} \in B_2 \\ 0 & \text{otherwise,} \end{cases} \qquad \sum_{j=1}^{n_2} m_{ij}^{(A)} = \begin{cases} k_2 & \text{if} \quad v_{1i} \in B_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.7) \quad \sum_{i=1}^{n_1} m_{ij}^{(B)} = \begin{cases} k_2 & \text{if} \quad v_{2j} \in B_2 \\ 0 & \text{otherwise,} \end{cases} \qquad \sum_{j=1}^{n_2} m_{ij}^{(B)} = \begin{cases} k_1 & \text{if} \quad v_{1i} \in B_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.8) \quad \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m_{ij}^{(A)} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m_{ij}^{(B)} = k_1 k_2.$$

Suppose that $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition. Let b_1 and b_2 be the number of A-type blocks and that of B-type blocks, respectively. If we let the p-th A-type block and the q-th B-type block correspond to a A-type matrix $M_A^{(p)}$ and a B-type matrix $M_B^{(q)}$, respectively, then it is easily seen that

$$(3.9) \qquad G_{n_1,n_2} = \sum_{p=1}^{b_1} M_A^{(p)} + \sum_{q=1}^{b_2} M_B^{(q)}.$$

Conversely, suppose that there exist b_1 A-type matrices $M_A^{(p)}$ and b_2 B-type matrices $M_B^{(q)}$ such that G_{n_1,n_2} can be expressed in the form (3.9). Consider a A-type block and a B-type block corresponding to $M_A^{(p)}$ and $M_B^{(q)}$, respectively. Then it is easily seen that a union of those A-type and B-type blocks is a complete bipartite graph $K(n_1, n_2)$. Thus we have the following theorem.

THEOREM 3.6. *A complete bipartite graph $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition if and only if there exist b_1 A-type matrices $M_A^{(p)}$ and b_2 B-type matrices $M_B^{(q)}$ such that G_{n_1,n_2} can be expressed in the form* (3.9).

3.3. Some lemmas

The following lemmas are useful for the proof of Theorem 3.2. With respect to the existence of a 0–1 matrix with given row sum and column sum vectors, we quote a result given by Yamamoto et al. [24, Corollary 1.3].

LEMMA 3.7. *Let $r_1,\dots, r_{n_1}$ and s be nonnegative integers. There exists a* 0–1 *matrix of size $n_1 \times n_2$ having the row sum vector $(r_1,\dots, r_{n_1})$ and the column sum vector $(s,\dots, s)$ if and only if*

$$\sum_{i=1}^{n_1} r_i = n_2 s \quad \text{and} \quad r_i \le n_2 \qquad \text{for all} \quad i. \tag{3.10}$$

Under the condition (3.10), such a matrix is straightforwardly constructed by the following

LEMMA 3.8. *(Algorithm) Form a sequence R in such a way that the first r_1 positions have* 1 *and the next r_2 positions have* 2,..., *and the last r_{n_1} positions have n_1, i.e.,*

$$R\colon \underbrace{1,\dots,1}_{r_1}, \underbrace{2,\dots,2}_{r_2},\dots, \underbrace{n_1,\dots,n_1}_{r_{n_1}}. \tag{3.11}$$

Form another sequence C in such a way that the subsequence 1,..., n_2 *is repeated s times, i.e.,*

$$C\colon 1,\dots, n_2, 1,\dots, n_2,\dots, 1,\dots, n_2. \tag{3.12}$$

Let $i_R(h)$ and $j_C(h)$ be the values in the h-th position of R and in the same position of C, respectively, and consider a set $E=\{(i_R(h), j_C(h)) \mid h=1,\dots, n_2 s\}$ of $n_2 s$ ordered pairs $(i_R(h), j_C(h))$. Define a 0–1 *matrix $M=\|m_{ij}\|$ of size $n_1 \times n_2$ by*

$$m_{ij}=\begin{cases} 1 & \text{if} \quad (i, j)\in E \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Then the matrix M is a 0–1 *matrix of size $n_1 \times n_2$ having the row sum vector $(r_1,\dots, r_{n_1})$ and the column sum vector $(s,\dots, s)$.*

PROOF. Since $r_i \le n_2$ for all i, it can be seen easily that $(i_R(h), j_C(h))= (i_R(h'), j_C(h'))$ if and only if $h=h'$. We observe from two sequences R and C that the row number i occurs r_i times in R for each $i=1,\dots, n_1$ and that the column number j occurs exactly s times in C for each $j=1,\dots, n_2$. Therefore, we have $\sum_{j=1}^{n_2} m_{ij}=r_i \ (i=1,\dots, n_1)$ and $\sum_{i=1}^{n_1} m_{ij}=s \ (j=1,\dots, n_2)$. This completes the proof.

For an ordered pair $(i_R(h), j_C(h))$, we call $i_R(h)$ the *row coordinate* and $j_C(h)$ the *column coordinate*.

We prove the following lemma related to Lemma 3.8.

LEMMA 3.9. *Let* $r_1,\dots, r_{n_1}$ *and* s *be nonnegative integers satisfying the condition* (3.10). *Suppose that* r_i, s *and* $n_2 s$ *are integral multiples of* k_2, k_1 *and* $k_1 k_2$, *respectively. Then the matrix* M *constructed by Lemma* 3.8 *can be written as the sum of A-type matrices* $M_A^{(p)}$ *of size* $n_1 \times n_2$, *i.e.*,

$$M = \textstyle\sum_{p=1}^{b_1} M_A^{(p)} \quad \textit{where} \quad b_1 = n_2 s/(k_1 k_2). \tag{3.14}$$

PROOF. Consider a sequence X composed of all elements in E, which is given in Lemma 3.8, i.e.,

$$X\colon e(1),\dots, e(T) \tag{3.15}$$

where $e(h)=(i_R(h), j_C(h))$ and $T=n_2 s$. Put $t=T/k_1$. Then $b_1=t/k_2$. In this sequence, if we select the first t elements as the first row, the next t elements as the second row,..., and the last t elements as the last row, then we have the following rectangular array of size $k_1 \times t$:

$$\begin{array}{cccc} e(1) & e(2) & \cdots & e(t) \\ e(t+1) & e(t+2) & \cdots & e(2t) \\ & & \cdots & \\ e(T-t+1) & e(T-t+2) & \cdots & e(T). \end{array} \tag{3.16}$$

Partition this array into b_1 subarrays, which are of size $k_1 \times k_2$, as follows:

$$A^{(1)} \quad A^{(2)} \quad \cdots \quad A^{(b_1)}. \tag{3.17}$$

Then each subarray $A^{(p)}$ has the following properties:

Property A. The values of the row coordinates of elements in each row of $A^{(p)}$ are all equal.

Property B. The values of the column coordinates of elements in each column of $A^{(p)}$ are all equal.

Since r_i are integral multiples of k_2 for all i, it can be easily checked that each $A^{(p)}$ has Property A. Since s is an integral multiple of k_1 and t is a common multiple of k_2 and n_2, it can be easily checked that each $A^{(p)}$ has Property B. Let $E^{(p)}$ be a set of all elements in $A^{(p)}$. If we define a 0–1 matrix $M^{(p)}=\|m_{ij}^{(p)}\|$ of size $n_1 \times n_2$ by

$$m_{ij}^{(p)} = \begin{cases} 1 & \text{if} \quad (i, j) \in E^{(p)} \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

then it can be seen from Properties A and B that the matrix $M^{(p)}$ is an A-type matrix.

Observing carefully the structures of those matrices $M^{(p)}$ and of the matrix

M, which is constructed by Lemma 3.8, and noting that $E=\bigcup_{p=1}^{b_1} E^{(p)}$ and $E^{(p)} \cap E^{(p')}=\emptyset$ for $p\neq p'$, we have

$$M = \sum_{p=1}^{b_1} M^{(p)} \quad \text{where} \quad b_1 = T/(k_1k_2). \tag{3.19}$$

This completes the proof.

From Lemma 3.9, we have

LEMMA 3.10. *Let $r_1,\dots, r_{n_1}$ and s be nonnegative integers which satisfy the condition* (3.10) *and all the conditions in Lemma* 3.9. *Suppose that n_2-r_i, n_1-s and $n_2(n_1-s)$ are integral multiples of k_1, k_2 and k_1k_2, respectively. Then a complete bipartite graph $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition.*

PROOF. Put $r'_i=n_2-r_i\ (i=1,\dots, n_1)$ and $s'=n_1-s$. Consider a sequence R' obtained from the replacement of r_i in (3.11) by r'_i and form another sequence C' in such a way that the subsequence $n_2,\dots, 1$ is repeated s' times, i.e.,

$$C'\colon n_2,\dots, 1, n_2,\dots, 1,\dots, n_2,\dots, 1. \tag{3.20}$$

Let $i_{R'}(h)$ and $j_{C'}(h)$ be the respective values in the h-th position of R' and in the same position of C'. We denote $\{(i_{R'}(h), j_{C'}(h))\mid h=1,\dots, n_2s'\}$ by E'. Define a 0–1 matrix $M'=\|m'_{ij}\|$ of size $n_1\times n_2$ by

$$m'_{ij} = \begin{cases} 1 & \text{if} \quad (i, j)\in E' \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Then M' has the row sum vector $(r'_1,\dots, r'_{n_1})$ and the column sum vector $(s',\dots, s')$. By the method similar to the proof of Lemma 3.9, the matrix M' can be written as the sum of B-type matrices $M_B^{(q)}$ of size $n_1\times n_2$, i.e.,

$$M' = \sum_{q=1}^{b_2} M_B^{(q)} \quad \text{where} \quad b_2 = n_2s'/(k_1k_2). \tag{3.22}$$

Let $S=\{(i, j)\mid i=1,\dots, n_1;\ j=1,\dots, n_2\}$. Since $r_i+r'_i=n_2$ for all i, we have the relations

$$E \cup E' = S \quad \text{and} \quad E \cap E' = \emptyset, \tag{3.23}$$

where E is given in Lemma 3.8. Therefore, since S, E and E' are able to be identified with G_{n_1,n_2}, M and M', respectively, where M is given in (3.13), we have $G_{n_1,n_2}=M+M'$. Thus by (3.14) and (3.22), G_{n_1,n_2} is in the form (3.9). Hence, we have the desired result. This completes the proof.

Finally, we shall give a lemma, which may be called an extension lemma.

LEMMA 3.11. *If $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition, then $K(dn_1, dn_2)$*

has a $K(dk_1, dk_2)$-decomposition for a positive integer d.

PROOF. Let V_1, V_2 be the independent sets of the $K(dn_1, dn_2)$, where $|V_i| = dn_i$ $(i=1, 2)$. Divide V_i into n_i subsets of d points each. Construct a new graph G with a point set, where the point set consists of just constructed subsets and two points are adjacent if and only if the subsets come from distinct independent sets of $K(dn_1, dn_2)$. Then G is a complete bipartite graph $K(n_1, n_2)$. If we note that the cardinality of each subset identified with a point of G is d and that $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition, we can see that the desired result is obtained. This completes the proof.

3.4. Proof of Theorem 3.2

3.4.1. Proof of Statement (a)

(*Necessity*) Suppose that $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition. Let V_1, V_2 be the independent sets of $K(n_1, n_2)$. Let b_1 be the number of A-type blocks of the $K(k_1, k_2)$-decomposition of $K(n_1, n_2)$. Consider $x(u)$, $y(u)$, $x(v)$ and $y(v)$ appeared in the proof of Lemma 3.1. Then in those A-type blocks, there exist $k_2y(u)$ lines incident to u for each point u in V_1 and there exist $k_1x(v)$ lines incident to v for each point v in V_2. Since the sum of $k_2y(u)$ over all u in V_1 is the number of all lines in those A-type blocks and the same thing also holds for the sum of $k_1x(v)$ over all v in V_2, the equality

$$(3.24) \qquad \sum_{v\in V_2} k_1x(v) = \sum_{u\in V_1} k_2y(u)$$

holds. Let (x_0, y_0) denote the solution vector of $n_1 = k_1x + k_2y$. Then since $w(n_1) = 1$, it is observed that $x(v) = x_0$ and $y(v) = y_0$ for all v in V_2. Thus by (3.24) we have

$$(3.25) \qquad k_1x_0n_2 = \sum_{u\in V_1} k_2y(u).$$

For each solution vector (x_q, y_q) of $n_2 = k_1x + k_2y$ $(q=1,\dots, \beta)$, let f_q be the number of u's in V_1 such that $(x(u), y(u)) = (x_q, y_q)$. Then we have

$$(3.26) \qquad \sum_{q=1}^{\beta} f_q = n_1 \quad \text{and} \quad \sum_{u\in V_1} y(u) = \sum_{q=1}^{\beta} y_qf_q \quad \text{where} \quad \beta = w(n_2).$$

Applying (3.26) to (3.25), we obtain the second expression in (3.3). Hence, Condition (iv) is necessary.

(*Sufficiency*) We assume that a set of parameters n_1, n_2, k_1, k_2 satisfies Conditions (i)–(iii) in Lemma 3.1. Since by Condition (iii) each of $n_1 = k_1x + k_2y$ and $n_2 = k_1x + k_2y$ has at least one solution vector, a common divisor of k_1 and k_2 is a divisor of n_1 and is also that of n_2. Therefore, it follows from Lemma 3.11 that it is enough to show the sufficiency of Condition (iv) only when k_1 and k_2 are relatively prime. The sufficiency will be shown by Lemma 3.10. Consider a vector $(r_1,\dots, r_{n_1})$ and an integer s such that

$$(3.27)\qquad (r_1,\dots,r_{n_1}) = (\underbrace{k_2y_1,\dots,k_2y_1}_{f_1},\underbrace{k_2y_2,\dots,k_2y_2}_{f_2},\dots,\underbrace{k_2y_\beta,\dots,k_2y_\beta}_{f_\beta}),$$

$$(3.28)\qquad s = k_1x_0.$$

Then the second condition in (3.10) is satisfied. Clearly, r_i is an integral multiple of k_2 for every i and s is an integral multiple of k_1. From (3.3) in Condition (iv) and (3.28) we have

$$(3.29)\qquad n_2s = n_2k_1x_0 = \textstyle\sum_{q=1}^{\beta} k_2y_qf_q,$$

which implies that the first condition in (3.10) holds. Therefore, n_2s is an integral multiple of k_1 and is also that of k_2. Since k_1 and k_2 are relatively prime, n_2s is an integral multiple of k_1k_2. Noting that r_i has the form k_2y_q from (3.27) and that $n_2=k_1x_q+k_2y_q$, it follows that n_2-r_i is an integral multiple of k_1 for each $i=1,\dots,n_1$. Similarly, n_1-s is an integral multiple of k_2, since $n_1=k_1x_0+k_2y_0$. As seen in Condition (i) and in the above, $n_2(n_1-s)$ is an integral multiple of k_1k_2. Hence, from Lemma 3.10 $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition. This completes the proof of Statement (a) in Theorem 3.2.

3.4.2. Proof of Statement (b)

As stated in the previous subsection, it is enough to show that Statement (b) holds only when k_1 and k_2 are relatively prime. There are two cases: $w(n_2)=1$ and $w(n_2)\geq 2$.

Case (1). $w(n_2)=1$: In this case, it is easy to see that $n_2<2k_1k_2$. Since k_1 and k_2 are relatively prime, each of solution vectors of $n_1=k_1x+k_2y$ is of the form $(z_1+\mu k_2, z_2+\nu k_1)$ for some nonnegative integers μ and ν, where $z_1<k_2$ and $z_2<k_1$. Therefore, noting $n_1\leq n_2<2k_1k_2$, we have $w(n_1)=2$, since $w(n_1)\geq 2$. Two solution vectors (x_1, y_1) and (x_2, y_2) of $n_1=k_1x+k_2y$ have the following relations:

$$(3.30)\qquad x_1<k_2,\quad x_2=x_1+k_2,\quad y_1=y_2+k_1,\quad y_2<k_1.$$

Let (x_0, y_0) be the solution vector of $n_2=k_1x+k_2y$, so that $x_0<k_2$ and $y_0<k_1$. Put $f_1=(k_1x_0n_1-k_2y_2n_2)/(k_1k_2)$ and $f_2=n_2-f_1$. Since k_1 and k_2 are relatively prime, from Condition (i) it can be seen that $k_1x_0n_1$ and $k_2y_2n_2$ are integral multiples of k_1k_2. Therefore, f_1 is an integer. Using two inequalities $x_0<k_2$ and $n_1\leq n_2$, we lead that $0\leq f_1\leq n_1$, so that $0\leq f_2\leq n_2$. Put

$$(3.31)\qquad r_i=\begin{cases} k_1x_1 & (i=1,\dots,f_1)\\ k_1x_2 & (i=f_1+1,\dots,n_2),\end{cases}$$

$$(3.32)\qquad s=k_2y_0.$$

Here, note that $f_1+f_2=n_2$ and $k_2y_0n_1=k_1x_1f_1+k_1x_2f_2$. The latter fact can be seen after some calculations. From these facts it follows that all the assumptions in Lemma 3.10 are satisfied. Hence, $K(n_1, n_2)$ has a $K(k_1, k_2)$-decomposition.

Case (2). $w(n_2)\geq 2$: In this case, put $n'_i=n_i-(w(n_i)-2)k_1k_2$ $(i=1, 2)$. Then we show the following

LEMMA 3.12. *The equality* $w(n'_1)=w(n'_2)=2$ *holds.*

PROOF. Let (x_{1p}, y_{1p}), $p=1,\dots,\alpha$, be solution vectors of $n_1=k_1x+k_2y$, where $\alpha=w(n_1)$, $x_{11}<\cdots<x_{1\alpha}$ and $y_{11}>\cdots>y_{1\alpha}$. Since k_1 and k_2 are relatively prime, we have $x_{11}<k_2$ and $y_{1\alpha}<k_1$. Furthermore, we have

$$x_{1p} = x_{11} + (p-1)k_2 \quad \text{and} \quad y_{1p} = y_{1\alpha} + (\alpha-p)k_1 . \tag{3.33}$$

Therefore, substituting (3.33) into $n_1=k_1x_{1p}+k_2y_{1p}$, we obtain $n'_1=k_1x_{11}+k_2y_{1\alpha}+k_1k_2$, which has two solution vectors $(x_{11}, y_{1\alpha}+k_1)$ and $(x_{11}+k_2, y_{1\alpha})$. Hence, $w(n'_1)=2$. Similarly, $w(n'_2)=2$. This completes the proof.

We use the following reduction: $K(n_1, n_2)$ can be decomposed into four subgraphs $K(n'_1, n'_2)$, $K(n'_1, t_2k_1k_2)$, $K(n'_2, t_1k_1k_2)$ and $K(t_1k_1k_2, t_2k_1k_2)$, where $t_i=w(n_i)-2$ $(i=1, 2)$. Clearly, the last subgraph has a $K(k_1, k_2)$-decomposition. Since $w(n'_1)=w(n'_2)=2$, n'_1 and n'_2 can be represented as $n'_1=k_1x+k_2y$ and $n'_2=k_1x'+k_2y'$, respectively. From these representations, it follows that each of the middle two subgraphs has a $K(k_1, k_2)$-decomposition. Thus it remains only to prove that the first subgraph $K(n'_1, n'_2)$ has a $K(k_1, k_2)$-decomposition. Obviously, n'_1 and n'_2 satisfy Conditions (i)–(iii) of Lemma 3.1.

We assume first that $n'_1\geq n'_2$. From $w(n'_1)=w(n'_2)=2$, as seen in the proof of Lemma 3.12, n'_i can be written as

$$n'_i = k_1x_{i1} + k_2y_{i1} = k_1x_{i2} + k_2y_{i2} \qquad \text{for} \quad i = 1, 2, \tag{3.34}$$

where

$$x_{i1} < k_2, \quad x_{i2} = x_{i1} + k_2, \quad y_{i1} = y_{i2} + k_1, \quad y_{i2} < k_1 . \tag{3.35}$$

There are two subcases to consider.

Case (2.1). $k_1x_{11}n'_2\geq k_2y_{22}n'_1$: Put $f_{21}=(k_1x_{11}n'_2-k_2y_{22}n'_1)/(k_1k_2)$ and $f_{22}=n'_1-f_{21}$. Then f_{21} is nonnegative. Since $x_{11}<k_2$ and $n'_1\geq n'_2, f_{22}$ is also nonnegative. Since k_1 and k_2 are relatively prime, from Condition (i) it can be seen that $k_1x_{11}n'_2$ and $k_2y_{22}n'_1$ are both integral multiples of k_1k_2. Therefore, we conclude that f_{21} and f_{22} are nonnegative integers satisfying $f_{21}+f_{22}=n'_1$. Put

$$r_i = \begin{cases} k_2y_{21} & (i = 1,\dots,f_{21}) \\ k_2y_{22} & (i = f_{21} + 1,\dots, n'_1), \end{cases} \tag{3.36}$$

(3.37) $$s = k_1 x_{11}.$$

Note that $k_1x_{11}n'_2 = k_2y_{21}f_{21} + k_2y_{22}f_{22}$ and $f_{21}+f_{22}=n'_1$. Thus by the discussion similar to that in Case (1), it follows from Lemma 3.10 that $K(n'_1, n'_2)$ has a $K(k_1, k_2)$-decomposition.

Case (2.2). $k_1x_{11}n'_2 < k_2y_{22}n'_1$: Put $f_{21} = (k_1k_2n'_2 + k_1x_{11}n'_2 - k_2y_{22}n'_1)/(k_1k_2)$ and $f_{22} = n'_1 - f_{21}$. Though we need the tedious calculations, by the discussion similar to that in Case (2.1) we can show that f_{21} and f_{22} are nonnegative integers satisfying $f_{21}+f_{22}=n'_1$. Consider r_i given in (3.36) and put

(3.38) $$s = k_1 x_{12}.$$

Then from the method similar to Case (2.1), $K(n'_1, n'_2)$ has a $K(k_1, k_2)$-decomposition.

In the case when $n'_1 < n'_2$, if we exchange n'_1 and n'_2, it can be shown from the method in the case $n'_1 \geq n'_2$ that $K(n'_1, n'_2)$ has a $K(k_1, k_2)$-decomposition. This completes the proof of Statement (b) in Theorem 3.2.

4. Bipartite decomposition of a complete multipartite graph

In this section, we shall discuss a bipartite decomposition of a complete m-partite graph with $m \geq 3$.

4.1. Bipartite decomposition theorem of $K_m(n_1,\dots, n_m)$

4.1.1. Necessary conditions and claw decomposition theorem

Let V_i $(i=1,\dots, m)$ be m independent sets of $K_m(n_1,\dots, n_m)$, where n_i is the cardinality of V_i. Let $N = \sum_{i=1}^{m} n_i$. With respect to a $K(k_1, k_2)$-decomposition of $K_m(n_1,\dots, n_m)$, we have the following theorem, where we assume $k_1 \leq k_2$ and $n_1 \leq \cdots \leq n_m$ without loss of generality.

THEOREM 4.1. *If a complete m-partite graph $K_m(n_1,\dots, n_m)$ has a $K(k_1, k_2)$-decomposition, where $k_1 \leq k_2$ and $n_1 \leq \cdots \leq n_m$, then the following conditions hold:*

(i) $\sum_{i<j} n_in_j$ *is an integral multiple of* k_1k_2.

(ii) $(\sum_{i<j} n_in_j)/k_2 \geq N - n_m$.

(iii) $w(N-n_i) \geq 1$ *for* $i = 1,\dots, m$.

PROOF. Since $K_m(n_1,\dots, n_m)$ has $\sum_{i<j} n_in_j$ lines and every block in the $K(k_1, k_2)$-decomposition has k_1k_2 lines, Condition (i) is, obviously, necessary. Suppose that $K_m(n_1,\dots, n_m)$ can be decomposed into a union of line-disjoint b blocks. We write those blocks as $B^{(p)} = \{B_1^{(p)}; B_2^{(p)}\}$ $(p=1,\dots, b)$, where $b = (\sum_{i<j} n_in_j)/(k_1k_2)$, $|B_1^{(p)}| = k_1$ and $|B_2^{(p)}| = k_2$. Let $V^{(1)} = \bigcup_{p=1}^{b} B_1^{(p)}$ and $V^{(2)} = \bigcup_{p=1}^{b} B_2^{(p)}$. Then it can be shown that at most n_m points of $K_m(n_1,\dots, n_m)$ do not belong to $V^{(1)}$. If not, i.e., if there exist at least n_m+1 points which do not belong

to $V^{(1)}$, then those points belong only to $V^{(2)}$ and, moreover, they are not adjacent with each other. Because all lines in $K_m(n_1,..., n_m)$ are covered by all lines joining points in $V^{(1)}$ and points in $V^{(2)}$. This contradicts the fact that among those points there exist at least two points being adjacent, since the cardinality of each independent set of $K_m(n_1,..., n_m)$ is less than or equal to n_m. Therefore, at least $N-n_m$ points belong to $V^{(1)}$. Since k_1 points of $B^{(p)}$ are all distinct for each block $B^{(p)}$, the number of blocks is at least $(N-n_m)/k_1$ which implies $b \geq (N-n_m)/k_1$. Thus we have $(\sum_{i<j} n_i n_j)/k_2 \geq N-n_m$. Condition (ii) is, therefore, necessary. For a point v of V_i, let $y_i(v)$ and $x_i(v)$ be the number of $B_1^{(p)}$'s and that of $B_2^{(p)}$'s in which v appears, respectively. As there exist $N-n_i$ lines incident to v, we have $N-n_i=k_1x_i(v)+k_2y_i(v)$. The vector $(x_i(v), y_i(v))$ is a solution vector of $N-n_i=k_1x+k_2y$. Therefore, we have $w(N-n_i) \geq 1$. Condition (iii) is, therefore, neccessary. This completes the proof.

When $k_1=1$ and $n_1=\cdots=n_m$, we have the following claw decomposition theorem, which has been proved by Ushio, Tazawa and Yamamoto [20].

THEOREM 4.2. *A complete m-partite graph $K_m(n,..., n)$ has a $K(1, k_2)$-decomposition if and only if the following conditions hold:*

(i) $\binom{m}{2}n^2$ *is an integral multiple of* k_2.

(ii) $mn \geq 2k_2$.

Note that Condition (iii) of Theorem 4.1 always holds when $k_1=1$. In fact, for any positive integer n, the vector $(x, y)=(n-[n/k_2]k_2, [n/k_2])$ ($[a]$ denote the greatest integer not exceeding a) is a solution vector of $n=k_1x+k_2y$ with $k_1=1$, so that we always have $w(n) \geq 1$.

4.1.2. Example of a bipartite decomposition constructed cyclically

In the following, we shall give an illustrative example of bipartite decomposition of a complete m-partite graph, which is constructed cyclically. It is an example suggestive of an application to a combinatorial balanced multiple-valued file organization scheme of order two.

EXAMPLE 1. Consider a complete 5-partite graph $K_5(3, 3, 3, 3, 3)$ with 5 independent sets, each of them having 3 points. We label 15 points of $K_5(3, 3, 3, 3, 3)$ sequentially as $v_1,..., v_{15}$ and we denote its independent sets by $V_i=\{v_i, v_{i+5}, v_{i+10}\}$ $(i=1,..., 5)$. When $k_1=2$ and $k_2=3$, 15 blocks are given as follows:

$B^{(1)} = \{v_1, v_2; v_3, v_8, v_{13}\}$ $\quad B^{(9)} = \{v_9, v_{10}; v_{11}, v_1, v_6\}$

$B^{(2)} = \{v_2, v_3; v_4, v_9, v_{14}\}$ $\quad B^{(10)} = \{v_{10}, v_{11}; v_{12}, v_2, v_7\}$

$B^{(3)} = \{v_3, v_4; v_5, v_{10}, v_{15}\}$ $\quad B^{(11)} = \{v_{11}, v_{12}; v_{13}, v_3, v_8\}$

$B^{(4)} = \{v_4, v_5; v_6, v_{11}, v_1\}$ $\quad B^{(12)} = \{v_{12}, v_{13}; v_{14}, v_4, v_9\}$

$B^{(5)} = \{v_5, v_6; v_7, v_{12}, v_2\}$ $B^{(13)} = \{v_{13}, v_{14}; v_{15}, v_5, v_{10}\}$
$B^{(6)} = \{v_6, v_7; v_8, v_{13}, v_3\}$ $B^{(14)} = \{v_{14}, v_{15}; v_1, v_6, v_{11}\}$
$B^{(7)} = \{v_7, v_8; v_9, v_{14}, v_4\}$ $B^{(15)} = \{v_{15}, v_1; v_2, v_7, v_{12}\}$.
$B^{(8)} = \{v_8, v_9; v_{10}, v_{15}, v_5\}$

It can be easily checked that these 15 blocks give a $K(2, 3)$-decomposition of $K_5(3, 3, 3, 3, 3)$. Let $B_1^{(p)}=\{v_p, v_{p+1}\}$ and $B_2^{(p)}=\{v_{p+2}, v_{p+7}, v_{p+12}\}$, where the indices of points are reduced modulo 15 to the set of residues $\{1,\dots, 15\}$. Then $B^{(p)}$ can be expressed with $B_1^{(p)}$ and $B_2^{(p)}$, i.e., $B^{(p)}=\{B_1^{(p)}; B_2^{(p)}\}$, $p=1,\dots, 15$. From this observation we see that these blocks are constructed cyclically. In this bipartite decomposition, the following properties can be seen:

(1) Each block contains exactly 5 points and exactly 6 lines (*property of uniformity*).

(2) Each line appears in exactly one block (*property of uniqueness*).

(3) Each point appears in exactly 5 blocks (*property of balanceability*).

(4) Given any line, the block number of the block containing the line can be computed algebraically (*property of identifiability*). This example is also that of balanced bipartite decomposition (*to be continued*).

Properties (1)–(4) are essential for a balanced multiple-valued file organization scheme of order two, namely, $BMFS_2$. Therefore, we can see that a balanced bipartite decomposition will be applied to a new type of $BMFS_2$. Such a scheme will be called a bipartite-type $BMFS_2$. With respect to a $BMFS_2$, the reader is referred to [26].

In the next section, we shall investigate a balanced bipartite decomposition of a complete m-partite graph.

4.2. Balanced bipartite decomposition of $K_m(n,\dots, n)$

In this section, we shall restrict our discussion to the case that $n_1=\cdots=n_m=n$ and investigate a balanced bipartite decomposition of $K_m(n,\dots, n)$.

4.2.1. Line length and turning in $K_m(n,\dots, n)$

The concepts of line length and turning are used for a construction of a balanced bipartite decomposition of $K_m(n,\dots, n)$. We use the following labeling scheme for $K_m(n,\dots, n)$. Let the points of $K_m(n,\dots, n)$ be labeled by $v_1,\dots, v_{mn}$. Consider the *length* of v_i, v_j defined by

$$l(v_i, v_j) = \min\{|i-j|, mn - |i-j|\}. \tag{4.1}$$

Let v_i, v_j be adjacent if and only if the length of v_i, v_j is not divisible by m. The m disjoint independent sets of $K_m(n,\dots, n)$ with this labeling are

$$V_i = \{v_i, v_{i+m},\dots, v_{i+(n-1)m}\}, \qquad i = 1,\dots, m. \tag{4.2}$$

The lengths of the lines of $K_m(n,\dots, n)$ are integers in the set $\{1, 2,\dots, [mn/2]\}$. From the definition of adjacency of points, those integers are not divisible by m. We denote the set of line lengths of $K_m(n,\dots, n)$ by L, i.e.,

$$L = \{1,\dots, [mn/2]\} - \{m,\dots, [n/2]m\}. \tag{4.3}$$

If l is such a line length and $l \neq mn/2$, there are exactly mn lines in $K_m(n,\dots, n)$ having length l. If $l = mn/2$, there are $mn/2$ lines of length l.

By the *turning of a line* (v_i, v_j) of $K_m(n,\dots, n)$ we mean the increasing of both indices by one, whereby we obtain a line (v_{i+1}, v_{j+1}) of $K_m(n,\dots, n)$ from the line (v_i, v_j). The indices are reduced modulo mn to the set of residues $\{1,\dots, mn\}$. By the *turning of a block* we mean the simultaneous turnings of all lines of the block. Obviously, the turning operation is a cyclic permutation of length mn on the point set of $K_m(n,\dots, n)$.

Sometimes we may write, for simplicity, the mn points of $K_m(n,\dots, n)$ as $1,\dots, mn$ instead of $v_1,\dots, v_{mn}$. When two independent sets of a block B are $B_1 = \{i_1,\dots, i_{k_1}\}$ and $B_2 = \{j_1,\dots, j_{k_2}\}$, we denote the block by

$$B = \{B_1; B_2\} = \{i_1,\dots, i_{k_1}; j_1,\dots, j_{k_2}\}. \tag{4.4}$$

As seen in Section 2, note that the block B is a complete bipartite subgraph with

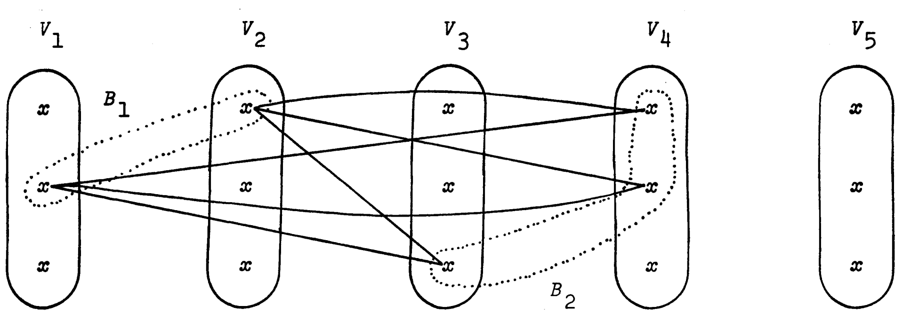

Fig. 2. A block B of $K_5(3, 3, 3, 3, 3)$

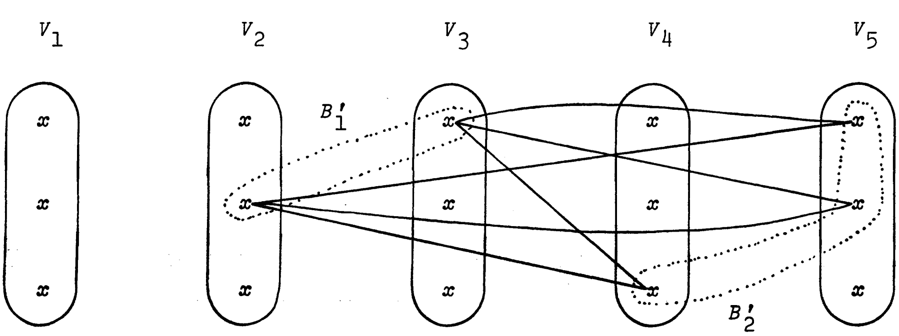

Fig. 3. The block B' obtained by a turning of B in Fig. 2

the independent sets B_1 and B_2 in $K_m(n,\dots, n)$. In Fig. 2 and 3, we illustrate two blocks of a complete 5-partite graph $K_5(3, 3, 3, 3, 3)$ with 5 independent sets V_i $(i=1,\dots, 5)$, each of them having 3 points. For $k_1=2$ and $k_2=3$, a block $B=\{B_1; B_2\}$ with $B_1=\{v_2, v_6\}$ and $B_2=\{v_4, v_9, v_{13}\}$ is given in Fig. 2. Another block $B'=\{B'_1; B'_2\}$ with $B'_1=\{v_3, v_7\}$ and $B'_2=\{v_5, v_{10}, v_{14}\}$, which is obtained by a turning of B, is also given in Fig. 3.

In addition to these considerations, we shall provide the following lemma which is useful for the balanced bipartite decomposition constructed cyclically.

LEMMA 4.3. *Let $K_m(n,\dots, n)$ contain a block B whose line lengths are all distinct and are not equal to $mn/2$. Suppose that B is turned $mn-1$ times. Then all of the original block B and the produced $mn-1$ blocks are line-disjoint. Moreover, for each line in B, all lines of $K_m(n,\dots, n)$ having the same length as the line appear in these mn blocks.*

PROOF. Let $B_1=\{i_1,\dots, i_{k_1}\}$ and $B_2=\{j_1,\dots, j_{k_2}\}$ be the two independent sets of the block B. Put the lengths

$$l_{pq} = l(i_p, j_q) \quad (p = 1,\dots, k_1;\ q = 1,\dots, k_2). \tag{4.5}$$

We first show that in turning B $mn-1$ times, no line duplication occurs. Since line length is preserved under the turning operation, if the same line of length l_{pq} appears in B turned through m_1 positions and in B turned through m_2 positions where $0\le m_1<m_2\le mn-1$, then we have the unordered pair equality

$$\{i_p + m_1, j_q + m_1\} = \{i_p + m_2, j_q + m_2\}. \tag{4.6}$$

There are two cases to consider.

Case (1). $i_p+m_1\equiv i_p+m_2$ and $j_q+m_1\equiv j_q+m_2 \pmod{mn}$: In this case, we have $m_1\equiv m_2 \pmod{mn}$.

Case (2). $i_p+m_1\equiv j_q+m_2$ and $j_q+m_1\equiv i_p+m_2 \pmod{mn}$: In this case, since $0\le m_1<m_2\le mn-1$, we have $i_p=j_q$, which implies that $m_1\equiv m_2 \pmod{mn}$. In two cases above, we conclude that $m_1\equiv m_2 \pmod{mn}$, which contradicts the fact that $0\le m_1<m_2\le mn-1$. Therefore, no line duplication occurs in the turnings. Using this result and the assumption that l_{pq} is not equal to $mn/2$ for each p and q, mn blocks produced by the turnings contain mn lines of length l_{pq}. This completes the proof.

EXAMPLE 1 (*continued*). The set of line lengths of $K_5(3, 3, 3, 3, 3)$ with $m=5$ and $n=3$ is $\{1, 2, 3, 4, 6, 7\}$. Line lengths of $B^{(1)}$ are 1, 2, 3, 4, 6, 7 which are all distinct and are not equal to $mn/2$. As those blocks $B^{(p)}$ $(p=2,\dots, 15)$ are produced by turnings of $B^{(1)}$, all of the original block $B^{(1)}$ and the produced 14 blocks $B^{(p)}$ are line-disjoint. Moreover, for each of line lengths 1, 2, 3, 4, 6, 7 those 15 blocks contain all lines of $K_5(3, 3, 3, 3, 3)$ having the same length as that. Since the set of line lengths of $B^{(1)}$, i.e., $\{1, 2, 3, 4, 6, 7\}$ is equal to the set of line lengths of $K_5(3, 3, 3, 3, 3)$, those 15 blocks give a $K(2, 3)$-decomposition of $K_5(3, 3, 3, 3, 3)$. Since those blocks are constructed cyclically, we see that they give a balanced $K(2, 3)$-decomposition of $K_5(3, 3, 3, 3, 3)$ (*to be continued*).

Note that a bipartite decomposition constructed cyclically is always balanced.
Consider the *size* of v_i, v_j defined by

(4.7) $$s(v_i, v_j) = |i-j|.$$

It can be seen that the lengths of lines with the same size are all equal. Let S be the set of sizes of all lines of $K_m(n,\dots, n)$. Then we have

(4.8) $$S = \{1,\dots, mn-1\} - \{m,\dots, (n-1)m\}.$$

We denote by $l(s)$ the length of lines whose size is s and denote by $L(S')$ the set of lengths of lines having sizes in a subset S' of S. Then we have the following lemma.

LEMMA 4.4. *When both m and n are odd, consider a set of sizes of lines*
(4.9) $$S' = \{\mu m + \nu \mid \mu = 0, 1,\dots, n-1;\ \nu = 1,\dots, (m-1)/2\}.$$
Then we have

(4.10) $$L(S') = L,$$

where L is given in (4.3).

PROOF. Put $n=2q+1$. Divide the set L into two subsets as follows:

(4.11) $$L = L_1 \cup L_2,$$

where

(4.12) $$L_1 = \{\mu m + \nu \mid \mu = 0, 1,\dots, q;\ \nu = 1,\dots, (m-1)/2\},$$

(4.13) $$L_2 = \{((2\mu+1)m-1)/2 + \nu \mid \mu = 0, 1,\dots, q-1;\ \nu = 1,\dots, (m-1)/2\}.$$

Divide the set S' into two subsets as follows:

(4.14) $$S' = S'_1 \cup S'_2,$$

where

(4.15) $$S'_1 = \{\mu m + \nu \mid \mu = 0, 1,\dots, q;\ \nu = 1,\dots, (m-1)/2\},$$

(4.16) $$S'_2 = \{(q+\mu+1)m + \nu \mid \mu = 0, 1,\dots, q-1;\ \nu = 1,\dots, (m-1)/2\}.$$

We shall show that $L(S_1')=L_1$ and $L(S_2')=L_2$.

Case (1). For any s in S_1', since $s \leq qm+(m-1)/2$, it follows that $mn-s \geq qm+(m+1)/2>s$. Therefore, $l(s)=\min\{s, mn-s\}=s$. Thus $L(S_1')=L_1$.

Case (2). For any s in S_2', since $s \geq (q+1)m+1$, it follows that $mn-s \leq qm-1<s$. Therefore, $l(s)=\min\{s, mn-s\}=mn-s$. Divide the set L_2 into q subsets as follows:

(4.17) $$L_2 = L_2^{(1)} \cup \cdots \cup L_2^{(q)},$$

where

(4.18) $$L_2^{(i)} = \{((2q-2i+1)m-1)/2 + v \mid v = 1,\dots, (m-1)/2\} \quad \text{for } i = 1,\dots, q.$$

Divide the set S_2' into q subsets as follows:

(4.19) $$S_2' = S_2'^{(1)} \cup \cdots \cup S_2'^{(q)},$$

where

(4.20) $$S_2'^{(i)} = \{(q+i)m + v \mid v = 1,\dots, (m-1)/2\} \qquad \text{for} \quad i = 1,\dots, q.$$

We show that $L(S_2'^{(i)})=L_2^{(i)}$ for each i. For any s in $S_2'^{(i)}$, let $s=(q+i)m+v$. Since $l(s)=mn-s$, it follows that $l(s)=((2q-2i+1)m-1)/2+t_v$, where $t_v=(m+1)/2-v$. Therefore, since $\{t_v \mid v=1,\dots, (m-1)/2\}=\{v \mid v=1,\dots, (m-1)/2\}$, by (4.18) we have $L(S_2'^{(i)})=L_2^{(i)}$ for each $i=1,\dots, q$. Thus $L(S_2')=L_2$. This completes the proof.

EXAMPLE 1 (*continued*). Both $m(=5)$ and $n(=3)$ are odd. The set L of line lengths of $K_5(3, 3, 3, 3, 3)$ is $\{1, 2, 3, 4, 6, 7\}$. Consider a set S' of sizes of lines in (4.9). Then we have $S'=\{1, 2, 6, 7, 11, 12\}$. As seen in (4.11), $L_1=\{1, 2, 6, 7\}$ and $L_2=\{3, 4\}$. As seen in (4.14), $S_1'=\{1, 2, 6, 7\}$ and $S_2'=\{11, 12\}$. In this example, we can concretely observe that $L(S_1')=L_1$, $L(S_2')=L_2$, and thus $L(S')=L$ (*to be continued*).

4.2.2. Balanced bipartite decomposition constructed cyclically

With respect to a balanced bipartite decomposition of $K_m(n,\dots, n)$ which is constructed cyclically, we have the following theorem.

THEOREM 4.5. *If*

(4.21) $$(m-1)n \equiv 0 \pmod{2k_1k_2},$$

then a complete m-partite graph $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition which is constructed cyclically.

PROOF. The proof is shown by a construction algorithm in which we use line length and turning. For a set of parameters m, n, k_1, k_2 satisfying (4.21), we write as $(m-1)n=2pk_1k_2$. There are two cases to consider.

Case (1). *n is even*: Put $n=2q$. Then we have $(m-1)q=pk_1k_2$. Let t be the greatest common divisor of p and q. Then we can write as $p=tp'$ and $q=tq'$, where p' and q' are relatively prime. Since $(m-1)q=pk_1k_2$, we have $(m-1)q'=p'k_1k_2$. Therefore, k_1k_2 is an integral multiple of q'. For two positive integers c and d satisfying $q'=cd$ such that k_1 and k_2 are integral multiples of c and d, respectively, put $k_1=ck'_1$ and $k_2=dk'_2$. Then we have $m-1=p'k'_1k'_2$. The set L given in (4.3) can be written as

$$L = \{\mu m + \nu \mid \mu = 0, 1,\dots, q-1;\ \nu = 1,\dots, m-1\}. \tag{4.22}$$

It is checked that

$$|L| = (m-1)q = p'k'_1k'_2tq' = tp'k_1k_2 \tag{4.23}$$

and that for any l in L we have $l\neq mn/2$. Divide the set L into t subsets as follows:

$$L = L_1 \cup \cdots \cup L_t, \tag{4.24}$$

where

$$L_i = \{((i-1)q'+\mu)m + \nu \mid \mu = 0, 1,\dots, q'-1;\ \nu = 1,\dots, p'k'_1k'_2\} \quad \text{for} \quad i = 1,\dots, t. \tag{4.25}$$

For each $i=1,\dots, t$, subdivide the set L_i into p' subsets as follows:

$$L_i = L_i^{(1)} \cup \cdots \cup L_i^{(p')}, \tag{4.26}$$

where

$$L_i^{(j)} = \{h_{ij} + \mu m + \nu \mid \mu = 0, 1,\dots, q'-1;\ \nu = 1,\dots, k'_1k'_2\} \tag{4.27}$$

for $j=1,\dots, p'$, where $h_{ij}=(i-1)q'm+(j-1)k'_1k'_2$. Obviously, $|L_i^{(j)}|=k_1k_2$ for each i and j. For each $i=1,\dots, t$ and $j=1,\dots, p'$, form a block $B_i^{(j)}$ in such a way that the set of lengths of lines of the block $B_i^{(j)}$ is $L_i^{(j)}$. It is as follows:

$$B_i^{(j)} = \{B_{i1}^{(j)};\ B_{i2}^{(j)}\}, \tag{4.28}$$

where

$$B_{i1}^{(j)} = \{\mu m + \nu \mid \mu = 0, 1,\dots, c-1;\ \nu = 1,\dots, k'_1\}, \tag{4.29}$$

$$B_{i2}^{(j)} = \{h_{ij} + (\mu c-1)m + \nu k'_1 + 1 \mid \mu = 1,\dots, d;\ \nu = 1,\dots, k'_2\}. \tag{4.30}$$

Let L' be the set of lengths of lines of $B_i^{(j)}$. We shall show that $L'=L_i^{(j)}$. Consider $t_1=\mu'm+\nu'$ be a point of $B_{i1}^{(j)}$ and consider $t_2=h_{ij}+(\mu''c-1)m+\nu''k_1'+1$ be a point of $B_{i2}^{(j)}$. Then

$$(4.31)\qquad t_2-t_1 = h_{ij}+(\mu''c-\mu'-1)m+\nu''k_1'-\nu'+1.$$

It can easily be observed that $1\leq t_2-t_1<mn/2$, which shows from the definition that $t_2-t_1\in L'$. While since $0\leq\mu''c-\mu'-1\leq q'-1$ and $1\leq\nu''k_1'-\nu'+1\leq k_1'k_2'$, it follows from (4.27) that $t_2-t_1\in L_i^{(j)}$. Evaluating the cardinalities of L' and $L_i^{(j)}$, we have $L'=L_i^{(j)}$. It can be seen that tp' blocks $B_i^{(j)}$ ($i=1,\ldots,t$; $j=1,\ldots,p'$) are line-disjoint, because all the line lengths of $B_i^{(j)}$'s are distinct. The turnings of $B_i^{(j)}$ $mn-1$ times yield $mntp'$ line-disjoint blocks of $K_m(n,\ldots,n)$ by Lemma 4.3. Since $mntp'=mnp=\binom{m}{2}n^2/(k_1k_2)$, we have a $K(k_1,k_2)$-decomposition. As the turning is a cyclic permutation of length mn on the point set of $K_m(n,\ldots,n)$, the $K(k_1,k_2)$-decomposition is constructed cyclically. Thus we have a balanced $K(k_1,k_2)$-decomposition of $K_m(n,\ldots,n)$.

Case (2). *n is odd*: Put $n=2q+1$. Then we have $(m-1)(2q+1)=2pk_1k_2$, which implies that m is odd and that pk_1k_2 is an integral multiple of $2q+1$. Let t be the greatest common divisor of p and $2q+1$. Then we can write as $p=tp'$ and $2q+1=tq'$, where p' and q' are relatively prime and where q' is odd. Since $(m-1)(2q+1)=2pk_1k_2$, we have $(m-1)q'=2p'k_1k_2$. Therefore, k_1k_2 is an integral multiple of q'. For two positive integers c and d satisfying $q'=cd$ such that k_1 and k_2 are integral multiples of c and d, respectively, put $k_1=ck_1'$ and $k_2=dk_2'$. Then we have $(m-1)/2=p'k_1'k_2'$. Consider a set L given in (4.3). It is checked that

$$(4.32)\qquad |L| = (m-1)(2q+1)/2 = p'k_1'k_2'tq' = tp'k_1k_2$$

and that for any l in L we have $l\neq mn/2$. Consider a set S' given in (4.9). Since both m and n are odd, from Lemma 4.4 we have $L(S')=L$. Divide the set S' into t subsets as follows:

$$(4.33)\qquad S' = S_1\cup\cdots\cup S_t,$$

where S_i is the same form as in (4.25) for $i=1,\ldots,t$. For each $i=1,\ldots,t$, subdivide the set S_i into p' subsets as follows:

$$(4.34)\qquad S_i = S_i^{(1)}\cup\cdots\cup S_i^{(p')},$$

where $S_i^{(j)}$ is the same form as in (4.27) for $j=1,\ldots,p'$. By the discussion similar to that in Case (1), for each $i=1,\ldots,t$ and $j=1,\ldots,p'$, we can form the block $B_i^{(j)}$ given in (4.28) in such a way that the set of sizes of lines of the block $B_i^{(j)}$ is $S_i^{(j)}$. Since $L(S')=L$, it follows that all the line lengths of $B_i^{(j)}$ ($i=1,\ldots,t$; $j=1,\ldots,p'$) are distinct. Therefore, it can be seen that tp' blocks $B_i^{(j)}$ are line-disjoint. The turnings of $B_i^{(j)}$ $mn-1$ times yield $mntp'$ line-disjoint blocks of $K_m(n,\dots, n)$ by Lemma 4.3. Since $mntp'=mnp=\binom{m}{2}n^2/(k_1k_2)$, similarly as in Case (1), we have a balanced $K(k_1, k_2)$-decomposition of $K_m(n,\dots, n)$, which is constructed cyclically. This completes the proof.

Example 1 (*continued*). A set of parameters $m(=5)$, $n(=3)$, $k_1(=2)$, $k_2(=3)$ satisfies $(m-1)n\equiv 0 \pmod{2k_1k_2}$. Both m and n are odd. In Case (2) of the proof of Theorem 4.5, we have $p=1$, $p'=1$, $t=1$, $q'=3$, $c=1$, $d=3$, $k_1'=2$, $k_2'=1$. Two sets are given as $L=\{1, 2, 3, 4, 6, 7\}$ and $S'=\{1, 2, 6, 7, 11, 12\}$. Since $t=1$ and $p'=1$, we have a block $B=\{B_1; B_2\}$, where $B_1=\{1, 2\}$ and $B_2=\{3, 8, 13\}$. The turnings of B $14(=mn-1)$ times yield $15(=\binom{m}{2}n^2/(k_1k_2))$ line-disjoint blocks of $K_5(3, 3, 3, 3, 3)$. They give a balanced $K(2, 3)$-decomposition of $K_5(3, 3, 3, 3, 3)$, which is constructed cyclically.

4.2.3. Balanced bipartite decomposition theorem of $K_m(n,\dots, n)$

In this section, when $k_1\neq k_2$, we shall give a balanced $K(k_1, k_2)$-decomposition theorem of $K_m(n,\dots, n)$. The following lemma is useful for a balanced bipartite decomposition.

Lemma 4.6. *If a complete m-partite graph $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition, then a complete m-partite graph $K_m(dn,\dots, dn)$ has a balanced $K(dk_1, dk_2)$-decomposition for a positive integer d.*

Proof. This lemma can be verified similarly as Lemma 3.11.

Theorem 4.7. *When $k_1\neq k_2$, a complete m-partite graph $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition if and only if the following conditions hold:*

(i) $\binom{m}{2}n^2$ *is an integral multiple of* k_1k_2.

(ii) $(m-1)n$ *is a common multiple of* $2k_1$ *and* $2k_2$.

Proof. (*Necessity*) Suppose that $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition. Let b be the number of the total blocks and let r be the number of blocks such that each point of $K_m(n,\dots, n)$ belongs to exactly r blocks. A block B has k_1+k_2 points and k_1k_2 lines and is denoted by $B=\{B_1; B_2\}$, where $|B_1|=k_1$ and $|B_2|=k_2$. We have obviously

$$\binom{m}{2}n^2 = bk_1k_2, \tag{4.35}$$

$$mnr = b(k_1+k_2). \tag{4.36}$$

From (4.35) and (4.36) we have

$$b = m(m-1)n^2/(2k_1k_2), \tag{4.37}$$

(4.38) $$r = (k_1+k_2)(m-1)n/(2k_1k_2).$$

For a point v, let $r_1(v)$ and $r_2(v)$ be the number of B_1's and that of B_2's in which v appears, respectively. Counting in two ways the total number of lines to which v is incident, we obtain

(4.39) $$r_1(v)k_2 + r_2(v)k_1 = (m-1)n.$$

Obviously,

(4.40) $$r_1(v) + r_2(v) = r.$$

Since $k_1 \neq k_2$, we have from (4.37)–(4.40)

(4.41) $$r_1(v) = (m-1)n/(2k_2),$$

(4.42) $$r_2(v) = (m-1)n/(2k_1).$$

Therefore, r_1 and r_2 do not depend on the particular point v. Thus Conditions (i) and (ii) are necessary. Note that (4.41) and (4.42) imply (4.38).

(Sufficiency) There are two cases to consider.

Case (1). $(m-1)n \equiv 0 \pmod{2k_1k_2}$: In this case, from Theorem 4.5 it follows that $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition, which is constructed cyclically.

Case (2). $(m-1)n \not\equiv 0 \pmod{2k_1k_2}$: Let d be the greatest common divisor of k_1 and k_2. In this case, $d \neq 1$. If $d=1$, then from Condition (ii) we have $(m-1)n \equiv 0 \pmod{2k_1k_2}$, which is a contradiction. Therefore, $d \neq 1$. Put $k_1 = dk'_1$ and $k_2 = dk'_2$, where k'_1 and k'_2 are relatively prime. Then from Condition (ii) we have $(m-1)n \equiv 0 \pmod{2dk'_1k'_2}$. Therefore, we can write from Condition (i) as

(4.43) $$b = (mn/d)\{(m-1)n/(2dk'_1k'_2)\}.$$

There are two subcases with respect to mn/d.

Case (2.1). $mn \equiv 0 \pmod{d}$: Since $(m-1)n \equiv 0 \pmod{2dk'_1k'_2}$ and $mn \equiv 0 \pmod{d}$, put $(m-1)n = 2dk'_1k'_2t$ and $mn = du$. Then we have $n = mn-(m-1)n = d(u-2k'_1k'_2t)$. Therefore, we have $n \equiv 0 \pmod{d}$. Putting $n = dn'$, we have $(m-1)n' \equiv 0 \pmod{2k'_1k'_2}$. From Theorem 4.5 it follows that $k_m(n',\dots, n')$ has a balanced $K(k'_1, k'_2)$-decomposition. From Lemma 4.6 it follows that $K_m(dn',\dots, dn')$ has a balanced $K(dk'_1, dk'_2)$-decomposition. Since $dn' = n$, $dk'_1 = k_1$ and $dk'_2 = k_2$, it follows that $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition.

Case (2.2). $mn \not\equiv 0 \pmod{d}$: Let e be the greatest common divisor of n and d. Then $e \neq 1$. Suppose that $e=1$. Then since $(m-1)n \equiv 0 \pmod{2dk'_1k'_2}$, we have $m-1 \equiv 0 \pmod{d}$ which implies that m and d are relatively prime. Therefore, mn and d are relatively prime. In (4.43), since mn and d are relatively prime, we have $(m-1)n/(2dk_1'k_2')\equiv 0 \pmod{d}$. This implies that $(m-1)n\equiv 0$ $(\mathrm{mod}\ 2d^2k_1'k_2')$. Since $k_1=dk_1'$ and $k_2=dk_2'$, we have $(m-1)n\equiv 0 \pmod{2k_1k_2}$, which is a contradiction. Therefore, $e\neq 1$. We can write as $n=en'$ and $d=ed'$, where n' and d' are relatively prime. Since $(m-1)n'\equiv 0 \pmod{2d'k_1'k_2'}$, we have $m-1\equiv 0 \pmod{d'}$ which implies that m and d' are relatively prime. Therefore, mn' and d' are relatively prime. We can write (4.43) as

$$b = (mn'/d')\{(m-1)n'/(2d'k_1'k_2')\}. \tag{4.44}$$

In (4.44), since mn' and d' are relatively prime, we have $(m-1)n'/(2d'k_1'k_2')\equiv 0$ $(\mathrm{mod}\ d')$. This implies that $(m-1)n'\equiv 0 \pmod{2d'^2k_1'k_2'}$. Therefore, from Theorem 4.5 it follows that $K_m(n',\dots, n')$ has a balanced $K(d'k_1', d'k_2')$-decomposition. From Lemma 4.6 it follows that $K_m(en',\dots, en')$ has a balanced $K(ed'k_1', ed'k_2')$-decomposition. Since $en'=n$, $ed'k_1'=dk_1'=k_1$ and $ed'k_2'=dk_2'=k_2$, it follows that $K_m(n,\dots, n)$ has a balanced $K(k_1, k_2)$-decomposition. This completes the proof.

When $k_1=1$, Conditions (i) and (ii) of Theorem 4.7 are simplified to the following corollary, which has been given by Ushio [22].

Corollary 4.8. *A complete m-partite graph $K_m(n,\dots, n)$ has a balanced $K(1, k_2)$-decomposition if and only if*

$$(m-1)n \equiv 0 \pmod{2k_2}.$$

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