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On codimension 1 submanifolds in a manifold with abelian fundamental group

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Introduction. The results of this paper were obtained several years ago and later H. Imanishi used Proposition A below in his study [1] on the codimension 1 foliations on manifolds with abelian fundamental group.

All statements will be given in the C^1 category. (They are also true in the C^0 category with local flatness of submanifolds.) A submanifold N of a manifold M is called proper if it is a closed subset of M and $N \cap \partial M = \partial N$.

PROPOSITION A. *Let M be a manifold with abelian fundamental group. Let N be a proper submanifold of codimension* 1 *in M with connected complement. Then N is connected.*

PROPOSITION B. *Let M be a connected manifold with abelian fundamental group. Let N be a proper submanifold of codimension* 1 *in M, and $i_*: \pi_1(N) \to \pi_1(M)$ the homomorphism induced by the inclusion $i: N \to M$. Then the following two statements hold*:

(1) *If $M-N$ is connected, then* Coker i_* $(=\pi_1(M)/i_*(\pi_1(N)))$ *is a cyclic group.*

(2) *If N is* 1*-sided in M, then i_* is surjective. Here N is called* 1*-sided in M if $T(N)-N$ is connected for a tubular neighborhood $T(N)$ of N in M.*

We shall apply (2) in Proposition B to the case of $M=T^n$, the n-dimensional torus.

COROLLARY. *Let N be a non-orientable closed submanifold of codimension* 1 *in T^n. Then* rank $H_1(N) \geq n$.

PROOF. Since N is 1-sided in T^n, $i_*: \pi_1(N) \to \pi_1(T^n)$ is surjective. Hence, by the Hurewicz isomorphism theorem, $i_*: H_1(N) \to H_1(T^n)$ is surjective also. Thus we obtain rank $H_1(N) \geq n$.

1. Proof of Proposition A. Suppose to the contrary that N is not connected. Then, N is a union of two non-void disjoint manifolds N_1, N_2. Let p be the base point of $\pi_1(M)$. (In the following we shall denote, for simplicity, by I the interval [0, 1].) Since $M-N$ is connected, there are C^1 loops $\alpha, \beta: I \to M$

with base point p satisfying the following condition: α (resp. β) intersects N_1 (resp. N_2) transversally at $t=1/2$, and never meets $N=N_1 \cup N_2$ except at $t=1/2$.

Since $\pi_1(M)$ is abelian, we have $\alpha\beta \sim \beta\alpha$. Hence there is a continuous map $F: I^2 \to M$ such that $F(t, s)=\alpha\beta(t)$ for $(t, s) \in I \times [0, 1/9]$, $F(t, s)=\beta\alpha(t)$ for $(t, s) \in I \times [8/9, 1]$ and $F(0, s)=F(1, s)=p$ for $s \in I$. Without loss of generality, we may assume that F is transversal to N on I^2. In particular, $F^{-1}(N)$ is a properly embedded 1-dimensional submanifold of I^2. (As to the transversality theorems, refer to, for example, M. W. Hirsch [2] for the C^1 category and R. Kirby and L. C. Siebenmann [3] for the C^0 category.)

By the definition of α and β, $\partial I^2 \cap F^{-1}(N)$ consists of $(1/4, 0)$, $(3/4, 1) \in F^{-1}(N_1)$ and $(3/4, 0)$, $(1/4, 1) \in F^{-1}(N_2)$. Let J_1 be the connected component of $F^{-1}(N_1)$ which contains the point $(1/4, 0)$ (as endpoint). Any properly embedded connected submanifold of dimension 1 in I^2 is homeomorphic to S^1 or I. Since J_1 has an endpoint, it must be homeomorphic to I. The other endpoint of J_1 is necessarily $(3/4, 1)$, because $F^{-1}(N_1)$ has no endpoint in the interior of I^2. Similarly, there is a connected component J_2 of $F^{-1}(N_2)$ whose endpoints are $(3/4, 0)$ and $(1/4, 1)$. Now the Jordan-Schönflies theorem (refer to [4], for example) is applicable. Thus J_1 and J_2 cannot be disjoint, which contradicts the fact that N_1 and N_2 are disjoint. Hence Proposition A is proved.

We notice that Proposition A can be proved more easily in the case where the tubular neighborhood $T(N)$ is homeomorphic to $N \times [-1, 1]$.

2. Proof of Proposition B. Let $p \in M-N$ be the base point of $\pi_1(M)$, $q \in N$ be that of $\pi_1(N)$. Suppose that

(*) We can choose a C^1 loop $\sigma: I \to M$ with base point p such that $\sigma(t) \notin N$ for all $t \neq 1/2$, $\sigma(1/2)=q$, and σ is transversal to N at q.

We denote the restrictions of σ to $[0, 1/2]$ and $[1/2, 1]$ by σ_0 and σ_1 respectively. Then, the homomorphism $i_*: \pi_1(N) \to \pi_1(M)$ is given by $i_*(\gamma)=\sigma_0 \cdot \gamma \cdot \sigma_0^{-1}$ for $\gamma \in \pi_1(N)$. Now, we have Proposition B by proving that

(**) $G=\pi_1(M)$, where G is the subgroup of $\pi_1(M)$ generated by the loop σ and the subgroup $i_*(\pi_1(N))$.

In fact, if $M-N$ is connected, then it is easy to choose σ in (*). Hence, (**) implies (1). If N is 1-sided in M, then σ in (*) can be chosen in a tubular neighborhood of N, hence $\sigma \in i_*(\pi_1(N))$. Thus (**) implies (2).

To prove (**), we shall show that

(***) If a C^1 loop $\gamma: I \to M$ with base point p is transversal to N, then γ belongs to G.

We prove this by induction on the number $n(\gamma)$ of points of $\gamma^{-1}(N)$.

Step 1. The case of $n(\gamma)=0$. This means that γ is a loop in $M-N$. Put $\eta=\sigma^{-1}\gamma\sigma\gamma^{-1}$. Since $\pi_1(M)$ is abelian, η is null homotopic. Hence there is a continuous map $F: I^2 \to M$ such that $F(t, 0)=\eta(t)$ for $t \in I$, and $F(t, 1)=F(0, s)=$

$F(1, s)=p$ for $t, s \in I$. As before, we may assume that F is transversal to N on I^2. In particular, $F^{-1}(N)$ is a properly embedded 1-dimensional submanifold in I^2. Let J be the connected component of $F^{-1}(N)$ which is homeomorphic to I. Here we need to remind that the loop σ meets N just once and $n(\gamma)=0$. So, J is determined uniquely. By the Jordan-Schönflies theorem, J and a part of the boundary of I^2 bounds a disc. Now it is easy to see that $\gamma \sim \sigma_0 \cdot F(J) \cdot \sigma_0^{-1}$. $F(J)$ is regarded as an element of $\pi_1(N)$, because $F(J)$ is contained in N and both end-points of J are mapped by F onto q, the base point of $\pi_1(N)$. Thus we obtain that $\gamma = i_*(F(J))$ i.e. $\gamma \in i_*(\pi_1(N)) \subset G$.

Step 2. The case of $n(\gamma)>0$. Let m be a positive integer, and let us assume that if $n(\gamma)<m$, then γ belongs to G. Let $\gamma \in \pi_1(M)$ be a loop with $n(\gamma)=m$. Let t_0 be the first point $t \in I$ that $\gamma(t)$ is contained in N. We denote by γ_0 and γ_1 the restrictions of γ to $[0, t_0]$ and $[t_0, 1]$, respectively. We take a curve ξ in N from $\gamma(t_0)$ to q $(=\sigma(1/2))$. (Note that N is connected. This is a result of Proposition A.) Since the tubular neighborhood $T(N)$ is trivial over ξ, either $\gamma_0 \xi \sigma_0^{-1}$ or $\gamma_0 \xi \sigma_1$ is homotopic to a loop α in $M-N$ (with base point p). Let us assume that the first is so. (The other case is analogous.) Then, it is easy to see that $\sigma_1^{-1} \xi^{-1} \gamma_1$ is homotopic to a loop β with $n(\beta)<m$. We have

$$\gamma \sim (\gamma_0 \xi \sigma_0^{-1}) \sigma (\sigma_1^{-1} \xi^{-1} \gamma_1) \sim \alpha \sigma \beta.$$

By step 1, α belongs to G, while by induction hypothesis, β belongs to G. Thus we obtain that $\gamma \sim \alpha \sigma \beta$ belongs to G.

This completes the proof of (***), hence that of Proposition B.

References

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