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Stationary pattern of some density-dependent diffusion system with competitive dynamics

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1. Introduction

Recently density-dependent diffusion equations have been extensively investigated. The first interesting equation is a model of gas flow through a homogeneous porous medium, which is of the form

$$u_t = \Delta(u^m) \qquad (m > 1), \tag{1}$$

where u describes the denisty of the gas. Because of the degeneracy of the diffusivity at $u=0$, there is an attractive phenomenon such as the finite speed of propagation of disturbances (see, for example, Aronson [2] and its bibliography). The second model is one species population model similar to (1),

$$u_t = \Delta(\phi(u)) + f(u), \tag{2}$$

where u means the population density, $\phi(u)$ is a monotone increasing function for $u>0$ with $\phi(0)=0$. This nonlinearity implies that dispersal is influenced by local population pressure and $f(u)$ is the population supply such as Fischer's type (Gurtin and MacCamy [4] and Newman [10]).

For systems of equations as an extension of (2), Shigesada et al. [14] proposed a model of two competing species with self- and cross-population pressures so as to discuss the problem of spatial segregation

$$\begin{cases} u_t = \Delta\{(d_{11}+d_{12}v)u\} + (R_1-a_1u-b_1v)u, \\ v_t = \Delta\{(d_{22}+d_{21}u)v\} + (R_2-a_2v-b_2u)v, \end{cases} \tag{3}$$

where d_{ij}, R_i, a_i and $b_i (i, j=1, 2)$ are positive constants or zero. When $d_{ij}>0 (i\neq j)$, the population pressure of each species is exerted on the other and raises its dipersive force. Aronson [2] has also proposed a similar population model of prey and predator interaction, which is represented by

$$\begin{cases} u_t = \rho u(1-u/K) - \alpha uv, \\ v_t = \Delta\{\psi(u)v\} - \mu v + \gamma uv, \end{cases} \tag{4}$$

where ρ, K, α, μ, γ are positive constants and

$$\psi(u) = \begin{cases} D(1-u/K)^p, & 0 < u \leqq K, \\ 0, & K < u, \end{cases}$$

for some $p>0$. The nonlinear diffusions in (3) and (4) are just derived from ecological situations where individuals are randomly walking and repulsively dispersing (Okubo [12]), though such a mechanism is unusual from a chemical or a physical viewpoint. When $d_{12}=d_{21}=0$ or $\psi(u)$ is constant in (3) or (4), that is, in the absence of cross-diffusions, these equations are reduced to usual reaction-diffusion equations so that a priori bounds on solutions can be easily obtained, which lead to the global existence of solutions (for example, Alikakos [1]). However, for systems containing cross-diffusions, there has not as yet been any general theory to obtain their a priori bounds. Therefore, at present, we must approach each problem one by one (Cosner [3]).

Masuda and Mimura [6] has recently proved the global existence of nonnegative solutions for a slightly simplified system of (3) in a finite interval $I\equiv(0, 1)$

$$\text{(5)} \qquad \begin{cases} u_t = \{(d_{11}+d_{12}v)u\}_{xx} + (R_1-a_1u-b_1v)u, \\ v_t = d_{22}v_{xx} + (R_2-a_2v-b_2u)v \end{cases}$$

subject to the initial conditions

$$\text{(6)} \qquad u(0, x) = u_0(x),\ v(0, x) = v_0(x), \quad x\in I,$$

and the boundary conditions

$$\text{(7)} \qquad u_x(t, 0) = u_x(t, 1) = v_x(t, 0) = v_x(t, 1) = 0, \quad t > 0.$$

In the sequel to the result above, the next study is to know asymptotic behavior of a solution $(u(t, x), v(t, x))$ of the problem (5)–(7). From an ecological viewpoint, it is interesting to investigate whether or not two species (u, v) form spatial segregation eventually. For this purpose, we discuss the stationary problem of (5)–(7) and study the existence of non constant solutions exhibiting segregation. Here we note that, in the absence of the cross-diffusion, i.e. $d_{12}=0$, (5) does not cause diffusion-induced instability, since the competitive interaction in (5) does not possess an activator-inhibitor mechanism, but, for suitable $d_{12}>0$, cross-diffustion-induced instability occurs. It was discussed by the use of the bifurcation technique that stationary small amplitude solutions of (5)–(7) exist (Mimura and Kawasaki [7]).

In this paper, we deal with the following stationary problem:

$$\text{(8)} \qquad \begin{cases} 0 = \{(1+\alpha v)u\}_{xx} + \beta(R_1-au-bv)u, \\ 0 = \varepsilon^2 v_{xx} + (R_2-av-bu)v, \end{cases} \qquad x\in I,$$

subject to zero flux boundary conditions

(9) $$u_x(0) = u_x(1) = v_x(0) = v_x(1) = 0.$$

For the parameters R_1, R_2, a, b and α in (8), we make the following two conditions (A. 1) and (A. 2):

(A.1) $$a/b > R_1/R_2 > b/a,$$

which indicates that

$$(\bar{u}, \bar{v}) = ((R_1 a - R_2 b)/(a^2 - b^2), (R_2 a - R_1 b)/(a^2 - b^2)),$$

which is a positive constant solution of (8) and (9), is asymptotically stable in the unsteady kinetic system of (8), i.e.

(10) $$\begin{cases} u_t = \beta(R_1 - au - bv)u, \\ v_t = (R_2 - av - bu)v; \end{cases}$$

(A.2) $$\alpha > a/(b\bar{u} - a\bar{v}) > 0,$$

which states that the effect of the cross-diffusivity is not so weak. The necessity of these conditions will be mentioned in the next section. Under (A.1) and (A.2), we show the existence of large amplitude solutions exhibiting striking segragation when β and/or ε are sufficiently small. Especially, when ε is zero or sufficiently small, strong heterogeneity can be found in both arguments u and v, which shows a typical segregating phenomenon (see Figures 4 and 5). The results are obtained as the nice application of perturbation techniques developed by Mimura et al. [9] and Nishiura [11].

2. Preliminaries

In this section, we consider the bifurcation problem of (8) and (9) with respect to a trivial solution $(\bar{u}, \bar{v})$. By introducing pertubation variables $w_1 = u - \bar{u}$ and $w_2 = v - \bar{v}$, $W = (w_1, w_2)$ satisfies the equation

(11) $$0 = LW + N(W), \quad x \in I,$$

and

(12) $$W_x(0) = W_x(1) = 0,$$

where

$$L = \begin{bmatrix} 1 + \alpha\bar{v} & \alpha\bar{v} \\ 0 & \varepsilon^2 \end{bmatrix} \frac{d^2}{dx^2} + \begin{bmatrix} -\beta a\bar{u} & -\beta b\bar{u} \\ -b\bar{v} & -a\bar{v} \end{bmatrix}$$

and

$$N(W) = (-\beta(aw_1^2 + bw_1 w_2) + \alpha(w_1 w_2)_{xx}, -(bw_1 w_2 + aw_2^2)).$$

To study the bifurcation problem with respect to $W=0$, we consider the linear eigenvalue problem of the form

$$(13)\qquad \begin{cases} \lambda\Phi = L\Phi, \ x \in I, \\ \Phi_x(0) = \Phi_x(1) = 0. \end{cases}$$

It follows from the Fourier series expansion that the eigenvalues of (13) correspond to the ones of the matrix

$$\begin{bmatrix} -(1+\alpha\bar{v})\gamma^2 - \beta a\bar{u} & -\alpha\bar{u}\gamma^2 - \beta b\bar{u} \\ -b\bar{v} & -\varepsilon^2\gamma^2 - a\bar{v} \end{bmatrix},$$

where $\gamma = n\pi$. The resulting characteristic equation is

$$(14)\qquad \lambda^2 + P_n\lambda + Q_n = 0,$$

where

$$P_n = a(\beta\bar{u} + \bar{v}) + (1 + \alpha\bar{v} + \varepsilon^2)\gamma^2$$

and

$$Q_n = \varepsilon^2(1+\alpha\bar{v})\gamma^4 + \{\beta\varepsilon^2 a\bar{u} + av + \alpha\bar{v}(a\bar{v} - b\bar{u})\}\gamma^2 + \beta(a^2 - b^2)\bar{u}\bar{v}.$$

Since $P_n > 0$, it is obvious to see that no Hopf bifurcation occur from $W=0$, and that zero is an eigenvalue of (13) if and only if $Q_n = 0$. Thus, the bifurcation curves C_n are described by

$$C_n:\ \beta = \frac{-\varepsilon^2(1+\alpha\bar{v})\gamma^2 + \{\alpha(b\bar{u} - a\bar{v}) - a\}\bar{v}}{\varepsilon^2 a\bar{u} + (a^2 - b^2)\bar{u}\bar{v}\gamma^{-2}} \qquad (n = 1, 2, \ldots).$$

If β and ε are used as bifurcation parameters under (A.1) and (A.2), the bifurcation curves C_n between β and ε for fixed α are drawn in Figure 1.

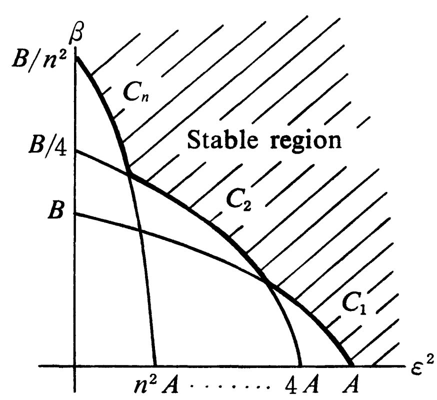

Figure 1. Schematic bifurcation curve in the (β, ε) space;
$A = \dfrac{\{\alpha(b\bar{u} - a\bar{v}) - a\}\bar{v}}{(1+\alpha\bar{v})\pi^2}$ and $B = \dfrac{\{\alpha(b\bar{u} - a\bar{v}) - a\}\bar{v}}{(a^2 - b^2)\bar{u}\bar{v}\pi^2}$.

Using the Lyapunov-Schmidt method, one can know the existence of small amplitude solutions of the problem (11), (12) for any (β, ε) in some neighborhood of each C_n (Mimura and Kawasaki [7]).

We are concerned with non-constant solutions for (β, ε) far away from the primary bifurcation curves, where we may expect that large amplitude solutions exist. In this paper, we study the problem (8) and (9) in the case when $\beta \ll 1$ and/or $\varepsilon \ll 1$.

3. Singular perturbation problem $(0 \leqq \varepsilon \ll 1)$

In this section, we look for nonconstant solutions of the problem (8), (9) when ε is a sufficiently small parameter for fixed $\beta > 0$. We transform u into w through $(1+\alpha v)u = w$. The resulting problem is

$$(15) \qquad \begin{aligned} 0 &= w_{xx} + \beta\left(R_1 - \frac{aw}{1+\alpha v} - bv\right)\frac{w}{1+\alpha v}, \\ 0 &= \varepsilon^2 v_{xx} + \left(R_2 - av - \frac{bw}{1+\alpha v}\right)v, \end{aligned} \qquad x \in I,$$

subject to

$$(16) \qquad w_x(0) = w_x(1) = v_x(0) = v_x(1) = 0.$$

Thus, this problem is reduced to a standard two-point boundary value problem of semilinear type. We deal with (15) and (16) instead of the original problem (8), (9). First, let us see what are the nonlinearities in (15): Put, for simplicity,

$$f(w, v) = \left(R_1 - \frac{aw}{1+\alpha v} - bv\right)\frac{w}{1+\alpha v}$$

and

$$g(w, v) = \left(R_2 - av - \frac{bw}{1+\alpha v}\right)v.$$

The curves of $f = g = 0$ under (A.1) and (A.2) are drawn in Figure 2. Here we note that $f = 0$ has a *humped* effect from (A.2).

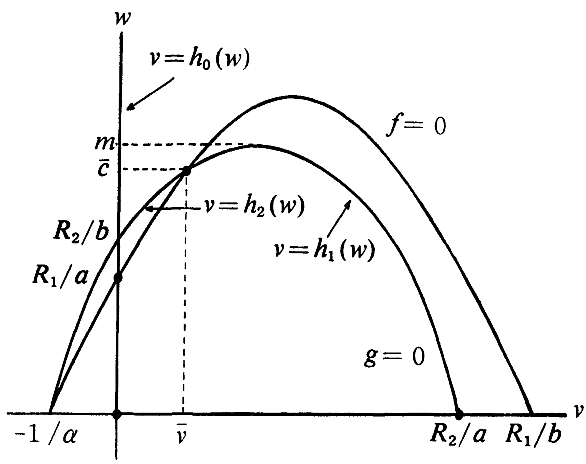

Figure 2. The zero curves of f and g; $\bar{c}=(1+\alpha\bar{v})\bar{u}$.

We first consider the reduced problem ($\varepsilon=0$) of (15) and (16),

$$0 = w_{xx} + \beta f(w, v), \qquad x \in I, \tag{17}$$
$$0 = g(w, v),$$

and

$$w_x(0) = w_x(1) = 0. \tag{18}$$

Once a function $v=h(w)$ can be obtained from the second equation of (17), we have a scalar equation with respect to w,

$$0 = w_{xx} + \beta f(w, h(w)). \tag{19}$$

From Figure 2, we have three different functions, say $v=h_0(w)(\equiv 0)$, $v=h_1(w)$ and $v=h_2(w)$ ($h_0<h_2<h_1$). Here we define $h(w)$ by

$$h(w; s) = \begin{cases} h_0(w), & w \in R^+\backslash(s, m), \\ h_1(w), & w \in (s, m), \end{cases}$$

for any fixed $s \in (R_2/b, m)$, and then seek nonconstant solutions of the problem (19), (18). Since $F(w; s)=f(w, h(w; s))$ is a discontinuous function (see Figure 3), we define a weak solution of the problem (19), (18) by

$$w \in H^1(I)$$

and

$$(w_x, \phi_x) = (\beta F(w; s), \phi) \qquad \text{for all} \quad \phi \in H^1(I).$$

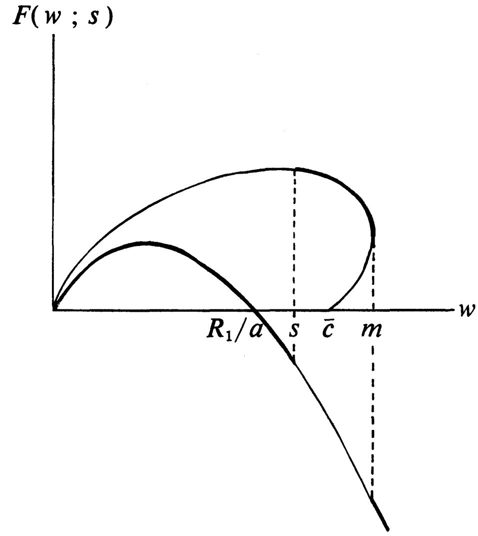

Figure 3. The nonlinearity of $F(w; s)$ with a point of discontinuity $w=s$.

PROPOSITION 1. *Let s be fixed arbitrarily such that $F(w; s)$ is monotone decreasing on (s, m). Then the problem* (19), (18) *has a family of periodic solutions*

$$\{w_n^s(x)\}_{n \geqq n_0} \subset C^1(I)$$

satisfying $R_1/a < w_n^s(x) < m$, where n is the mode number and n_0 is some integer depending on $\beta F(w; s)$.

PROOF. The proof is almost the same as that in Mimura and et al. [9; Theorem 1], so we omit it.

Using this proposition, we have the following existence theorem of the reduced problem:

THEOREM 2. *Consider the problem*

$$(20) \qquad \begin{cases} 0 = \{(1+\alpha v)u\}_{xx} + \beta(R_1 - au - bv)u, \\ 0 = \qquad\qquad\qquad (R_2 - av - bu)v, \end{cases} \qquad x \in I,$$

subject to zero flux boundary conditions (9). *Under the assumption of Proposition* 1, *there exists n_0 depending on βf and g such that a family of periodic solutions*

$$\{(u_n^s(x),\ v_n^s(x))\}_{n \geqq n_0} \subset L^2(I) \times L^2(I)$$

(*see Figure* 4).

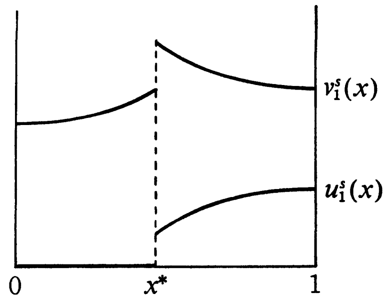

Figure 4. Spatial patterns of $(u_n^s(x), v_n^s(x))$ with $n=1$; x^* is a point satisfying $w_1^s(x^*)=s$.

PROOF. From the solution $w_n^s(x)$ in Proposition 1, we construct $v_n^s(x)$ by

$$h(w_n^s(x); s) = \begin{cases} h_0(w_n^s(x)) & \text{for} \quad w_n^s(x) < \text{s}, \\ h_1(w_n^s(x)) & \text{for} \quad w_n^s(x) > \text{s}. \end{cases}$$

This function is discontinuous because of the discontinuity of h. It is seen that $(w_n^s(x), v_n^s(x))$ is a solution of the problem (17), (18). Hence, by defining $u_n^s(x)$ by $u_n^s(x)=w_n^s(x)/(1+\alpha v_n^s(x))$, $(u_n^s(x), v_n^s(x))$ becomes a solution of the reduced problem of (8) and (9). Thus, the proof is completed.

Let us mention the spatial structure of the solution obtained in Theorem 2 which shows segregating pattern. For brevity, we restrict the case when $n=1$ and that $w_1^s(x)$ is monotone increasing. Then it turns out that the solution of the reduced problem, say $(u(x), v(x))$ has only one point of discontinuity $x=x^*$. $(0<x^*<1)$ such that

$$u(x) = w_1^s(x), \quad v(x) = 0 \quad \text{on} \quad (0, x^*)$$

and

$$u(x) = w_1^s(x)/(1+\alpha v_1^s(x)), \quad v(x) = v_1^s(x) \quad \text{on} \quad (x^*, 1).$$

Thus, we find that in the subregion $(0, x^*)$ one species (u) is non zero and the other (v) is zero and that on the other hand, in $(x^*, 1)$, $u(x)$ is monotone increasing but $v(x)$ is monotone decreasing, because $h_1(w_1^s(x))$ is monotone decreasing in this subregion (see Figure 4). This structure shows spatial segregation between two species.

We next consider the case when ε is not zero but sufficiently small. The discontinuities of $(u(x), v(x))$ at $x=x^*$ suggest us that internal transition layers appear in both arguments. Thus, this case is a singular perturbation problem for two-point boundary value problems of Neumann type.

THEOREM 3. *Suppose that* s^* *defined by*

$$\int_{h_0(s^*)}^{h_1(s^*)} g(s^*, v)dv = 0$$

satisfies the assumption of Proposition 1. *Let* $(u(x), v(x))$ *be any solution of the problem* (20), (18) *when* $s=s^*$. *Then there exist some positive constants* ε_0 *and* β_0 *such that for each fixed* $\beta \in (0, \beta_0)$, *a family of nonnegative solutions* $(u(x; \varepsilon), v(x; \varepsilon))$ *of the problem* (8), (9) *esists for any* $\varepsilon \in (0, \varepsilon_0)$ *which satisfies*

$$\begin{cases} \lim_{\varepsilon \downarrow 0} u(x; \varepsilon) = u(x) \\ \lim_{\varepsilon \downarrow 0} v(x; \varepsilon) = v(x) \end{cases} \quad \textit{almost everywhere in } I$$

(*see Figure* 5).

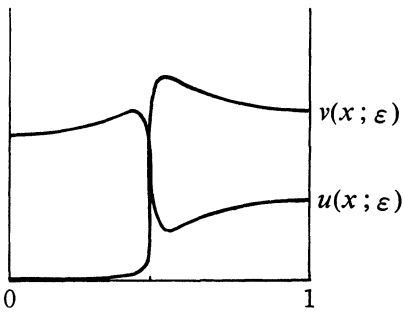

Figure 5. Spatial patterns of $(u(x; \varepsilon), v(x; \varepsilon))$ where ε is sufficiently small.

PROOF. We first discuss the problem (15), (16). Applying Theorem 2 in Mimura et al. [9] to this problem, we can know the existence of a family of solutions $(w(x; \varepsilon), v(x; \varepsilon))$ which satisfies

$$\begin{cases} \lim_{\varepsilon \downarrow 0} w(x; s) = w(x) & \textit{uniformly on } I, \\ \lim_{\varepsilon \downarrow 0} v(x; s) = v(x) & \textit{almost everywhere in } I, \end{cases}$$

where $(w(x), v(x))$ is a solution of (17) and (18) obtained in Proposition 1 when $s=s^*$. Thus, using the relation $(1+\alpha v)u=w$, we find a pair of functions $(u(x; \varepsilon), v(x; \varepsilon))$ which is a solution of the problem (15), (16). Moreover it turns out that

$$\lim_{\varepsilon \downarrow 0} u(x; \varepsilon) = \lim_{\varepsilon \downarrow 0} w(x; \varepsilon)/\{1+\alpha v(x; \varepsilon)\}$$
$$= w(x)/\{1+\alpha v(x)\} = u(x) \quad \text{almost everywhere in } I.$$

Thus, we may only show the nonnegativity of the solution $(u(x; \varepsilon), v(x; \varepsilon))$. It is obvious from $w(x; \varepsilon)>0$ that $u(x; \varepsilon)=w(x; \varepsilon)/\{1+\alpha v(x; \varepsilon)\}$ is positive. Then it suffices to show the nonnegativity of $v(x; \varepsilon)$. We first note that there exists $\varepsilon_1>0$ such that for $0<\varepsilon<\varepsilon_1$ and some $\delta>0$

(21) $$R_1/a < R_2/b - \delta < w(x; \varepsilon) < m + \delta,$$

since $R_2/b < w(x) < m$. Presuppose that there is a point $\bar{x} \in I$ such that $v(\bar{x}; \varepsilon) < 0$. Then the inequalities (21) lead to $g(v(\bar{x}; \varepsilon), w(\bar{x}; \varepsilon)) < 0$ for $0 < \varepsilon < \varepsilon_1$. This shows $v_{xx}(\bar{x}; \varepsilon) < 0$ which contradicts to $v(\bar{x}; \varepsilon) < 0$. Thus the proof is completed.

4. Regular perturbation problem ($0 \leqq \beta \ll 1$)

In this section, we seek large amplitude solutions of the problem (8), (9) in the case when β is sufficiently small. In the same manner as that in the previous section, the simpler system (15) subject to (16) is considered instead of the original one. This type of problems were discussed by Keener [5] and Nishiura [11].

When $\beta \downarrow 0$, (15) and (16) are reduced to the following problem for new variables (c, v) by using the zero flux boundary conditions:

(22) $$0 = \varepsilon^2 v_{xx} + g(c, v), \quad x \in I,$$

(23) $$\int_I f(c, v) dx = 0$$

and

(24) $$v_x(0) = v_x(1) = 0,$$

where $w = c$ is a constant function. The nonlinearity of g with respect to v shows that there are two critical values c_- and c_+ such that $g = 0$ has only one non-negative solution $v_-(c)(\equiv 0)$ for fixed c satisfying $c_+ < c$, three $v_-(c)(\equiv 0)$, $v_0(c)$ and $v_+(c)(v_- < v_0 < v_+)$ for $c_- < c < c_+$ and two $v_-(c)(\equiv 0)$ and $v_+(c)$ for $0 < c < c_-$ (see Figure 6).

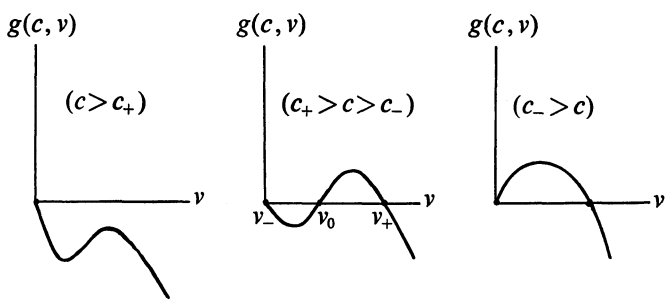

Figure 6. Dependency of c on the nonlinearity of $g(c, v)$.

Our aim in this section is to look for nonconstant solutions (c, v) of the reduced problem (22)–(24) for $c_- < c < c_+$. First suppose that one of the variables c is arbitrarily fixed and then consider the problem (22), (24) with respect to v only. This simplification makes the problem considerably treatable, because a phase-plane analysis can be applied. Energy form of (22) subject to (24) is

$$(\varepsilon^2/2)(v_x)^2 + G(c, v) = E, \tag{25}$$

where potential energy G is given by

$$G(c, v) = \int_{v_0(c)}^{v} g(c, s)ds \tag{26}$$

and energy E is a nonnegative constant which parametrizes all nonconstant solutions $v(x; c)$. We note that, if strictly monotone increasing solutions are obtained, other solutions can be constructed from them. Hence, it is sufficient to consider monotone increasing solutions obtined from the equation

$$\frac{dv}{dx} = \varepsilon^{-1}[2\{E-G(c, v)\}]^{1/2} \tag{27}$$

instead of (25). Putting

$$E_+(c) = \int_{v_0(c)}^{v_+(c)} g(c, s)ds$$

and

$$E_-(c) = \int_{v_0(c)}^{v_-(c)} g(c, s)ds,$$

we define $E^*(c)$ by

$$E^*(c) = \min(E_-(c), E_+(c)).$$

Then it is found that (27) has a strictly monotone increasing solution $v(x; E, c)$ for $0<E<E^*(c)(c_-<c<c_+)$, by solving the inverse function of

$$\varepsilon\int_{\xi_-}^{v} [2\{E-G(c, s)\}]^{-1/2}ds = x, \tag{28}$$

where $\varepsilon=\varepsilon(E, c)$ is given by

$$\varepsilon^{-1} = \int_{\xi_-}^{\xi_+} [2\{E-G(c, s)\}]^{-1/2}ds \tag{29}$$

and $\xi_\pm=\xi_\pm(E, c)$ are solutions of $E-G(c, v)=0$ satisfying $v_-(c)<\xi_-(E, c)<\xi_+(E, c)<v_+(c)$. More precisely, the following Lemma is known (see [11, 13], for instance):

LEMMA 4. *Consider the problem* (22), (24) *for each fixed* $c\in(c_-, c_+)$. *Then strictly monotone increasing solutions* $v(x; E, c)$ *are given by an* E-*parameter family of solutions* $(\varepsilon(E, c), v(x; E, c))$ *for* $0<E<E^*(c)$. *Moreover, it holds for* $c\in(c_-, c_+)$,

(i) $\lim_{E\downarrow 0} v(x; E, c) = v_0(c)$ *uniformly in* I,

(ii) $\lim_{E\downarrow 0} \varepsilon(E, c) = \{g_v(c, v_0(c))\}^{1/2}/\pi = A^{1/2}$

(for the definition of A, *see Figure* 1),

(iii) $\lim_{E\downarrow E^*(c)} v(x; E, c) =$

$$\begin{cases} v_-(c) \text{ compact uniformly on } [0, 1) \text{ for } G(c, v_-(c)) < G(c, v_+(c)), \\ v_+(c) \text{ compact uniformly on } (0, 1] \text{ for } G(c, v_-(c)) > G(c, v_+(c)), \\ v_c(x) = \begin{cases} v_-(c), & 0 \le x < M, \\ v_+(c), & M < x \le 1, \end{cases} \quad \text{for } G(c, v_-(c)) = G(c, v_+(c)), \end{cases}$$

where $m_- = m_-(c) = |G_{vv}(c, v_-(c))|$, $m_+ = m_+(c) = |G_{vw}(c, v_+(c))|$ *and* $M = \sqrt{m_+}/(\sqrt{m_+} + \sqrt{m_-})$,

(iv) $\lim_{E\downarrow E^*(c)} \varepsilon(E, c) = 0.$

The limit processes (i) and (ii) indicate that $(\varepsilon, v) = (A^{1/2}, v_0(c))$ is a primary bifurcation point of the trivial solution $v_0(c)$. On the other hand, (iii) and (iv) result from singular perturbation analysis as the diffusion coefficient ε^2 tends to zero.

The next problem is to consider

$$F(E, c) \equiv \int_I f(c, v(x; E, c))dx = 0 \tag{30}$$

by substituting the solution $v(x; E, c)$ into (23). If (30) has a solution $c = c(E)$, then it turns out that $(c(E), v(x; E, c(E)))$ is a solution of (22)–(24).

Define T in the (E, c) space by

$$T = \cup_{c_- < c < c_+} (0, E^*(c)) \times \{c\},$$

which is the domain of $\varepsilon = \varepsilon(E, c)$, and denote by B_0, B_- and B_+ each boundary of T, $E = 0$, $E = E^*_-(c)$ for $c_- < c < c^*$ and $E = E^*_+(c)$ for $c^* < c < c_+$, respectively, where c^* is some number defined from $E^*_-(c^*) = E^*_+(c^*)$. The problem is to consider whether or not the component of solutions $(c(E), v(x; E, c(E)))$ of (22)–(24) in T, say S_0 exists globally with respect to E. This problem has investigated by Mimura and Nishiura [8] and more precisely by Nishiura [11] who argued the problems for diffusive activator-inhibitor systems without cross-diffusions. Applying their results to (30), we have

THEOREM 6. *The component* S_0 *in* T, *containg* $(0, \bar{c})$, *exists globally with respect to* $E \in (0, E^*(c^*))$ *or* $\varepsilon \in (0, A^{1/2})$, *in other words*,

$$\bar{S}_0 \cap (B_- \cup B_+) = \{(E^*(c^*), c^*)\},$$

where $\bar{c}=(1+\alpha\bar{v})\bar{u}$.

Thus we can find the existence of solutions $(c(E), v(x; E, c(E)))$ of (22)–(24) and then obtain a solution $(u(x; E, c(E)), v(x; E, c(E)))$ of the original problem (8), (9) in the limit $\beta \downarrow 0$, by using the relation $(1+\alpha v)u=c$. Consequently, we clearly found that two competing species exhibit spatial segregation, because u (resp. v) is strictly monotone decreasing (resp. increasing) (Figure 7). Moreover, we can see a striking segregating phenomenon in (u, v) in the limit $E \to E^*(c^*)$ or $\varepsilon \to 0$ (Figure 8).

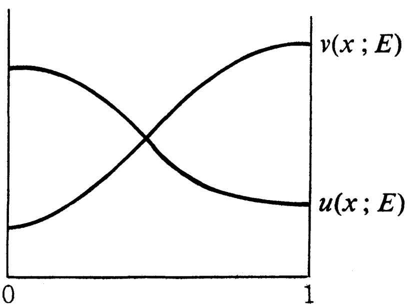

Figure 7. Spatial patterns of $(u(x; E), v(x; E))$.

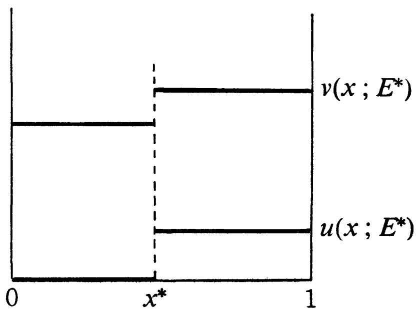

Figure 8. Spatial patterns of $(u(x; E), v(x; E))$ in the limit $\varepsilon \downarrow 0$.

For the case when $0<\beta \ll 1$, we prefer to avoid the discussion and invoke the paper by Nishiura [11].

5. Concluding remarks

We have found large amplitude solutions of stationary problems of competitive interaction and self- and cross-diffusions. The appearance of strong heterogeneity in *both* arguments is of great interest from mathematical and ecological viewpoints. This is caused by the quasilinearity involved in the system. In the case when $\beta \downarrow 0$, the phenomenon is quite different from ones appearing in models of semilinear type where one of the components is nearly flat (Keener [5], Nishiura [11]).

We have not discussed the stability problem of the solutions obtained here. This problem has been still open. Therefore we only numerically show that a solution of the unsteady problem of (8) and (9) tends to a non constant steady state constructed in Section 4 (Figure 9).

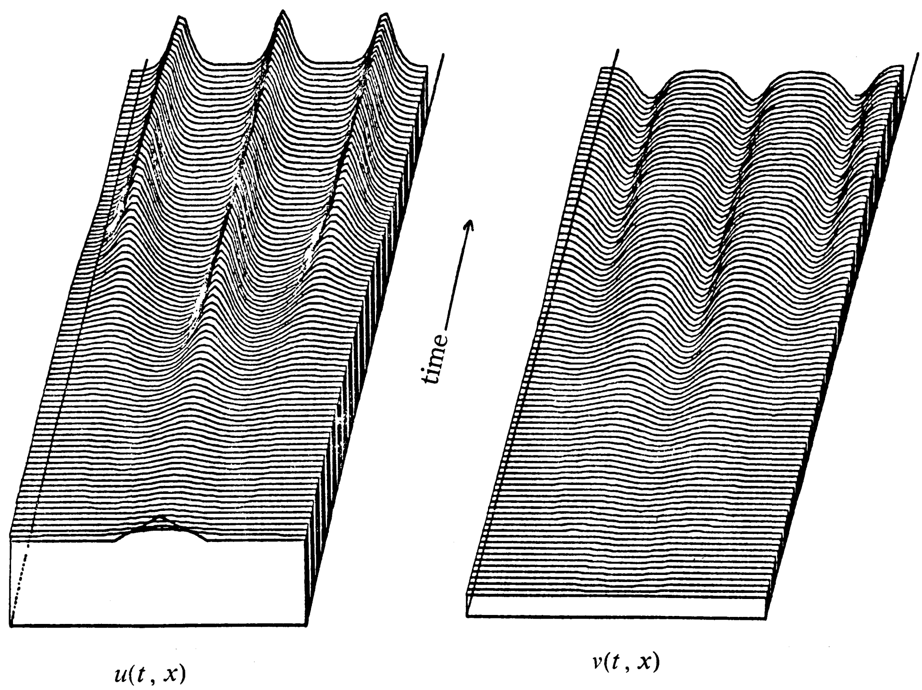

Figure 9. Segregation in the evolutional problem of (8) and (9) where β is sufficiently small.

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