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On strong oscillation of even order differential equations with advanced arguments

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This paper is concerned with the oscillatory behavior of solutions of linear functional differential equations of the form

$$x^{(n)}(t) + p(t)x(g(t)) = 0, \tag{1}$$

where n is even, p: $[a, \infty)\to(0, \infty)$ is continuous, g: $[a, \infty)\to R$ is continuously differentiable, $g'(t)>0$ and $\lim_{t\to\infty} g(t)=\infty$. By a proper solution of equation (1) is meant a function x: $[T_x, \infty)\to R$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)|; t\geq T\}>0$ for any $T\geq T_x$. A proper solution of (1) is called oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise. Equation (1) is said to be oscillatory if all of its solutions are oscillatory; otherwise equation (1) is said to be nonoscillatory. Equation (1) is said to be *strongly oscillatory* or *strongly nonoscillatory* according as the equation

$$x^{(n)}(t) + kp(t)x(g(t)) = 0 \tag{2}$$

is oscillatory or nonoscillatory for every $k>0$.

Recently Naito [2] has proved the following theorem for the strong oscillation and nonoscillation of retarded equations of the form (1).

THEOREM 1. *Suppose that* $g(t)\leq t$ *for* $t\geq$ a *and*

$$\liminf_{t\to\infty} g(t)/t > 0. \tag{3}$$

Equation (1) *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds = \infty, \tag{4}$$

and equation (1) *is strongly nonoscillatory if and only if*

$$\lim_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds = 0. \tag{5}$$

A question naturally arises as to what will happen for the advanced case of (1). The purpose of this paper is to give an answer to this question by showing that a similar conclusion holds in this case.

THEOREM 2. *Suppose that* $g(t) \geq t$ *for* $t \geq a$ *and*

$$\limsup_{t\to\infty} g(t)/t < \infty. \tag{6}$$

Equation (1) *is strongly oscillatory if and only if* (4) *holds, and equation* (1) *is strongly nonoscillatory if and only if* (5) *holds.*

Since conditions (4) and (5) do not involve $g(t)$, it follows from Theorem 2 that the strong oscillation (or strong nonoscillation) of equation (1) with $g(t)$ satisfying (6) is equivalent to that of the corresponding ordinary differential equation

$$x^{(n)} + p(t)x = 0. \tag{7}$$

We first give sufficient conditions for (1) to be oscillatory or nonoscillatory.

THEOREM 3. *Suppose that* $g(t) \geq t$ *for* $t \geq a$. *Equation* (1) *is oscillatory if*

$$\int_a^\infty t^{n-2} p(t)dt = \infty \tag{8}$$

or if (8) *fails but one of the following holds*:

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} p(s)ds > (n-1)!, \tag{9}$$

$$\liminf_{t\to\infty} t \int_t^\infty s^{n-2} p(s)ds > (n-1)!/4. \tag{10}$$

PROOF. From the proof of Theorem 1 of Naito [2] we easily see that (1) is oscillatory if the second order equation

$$u''(t) + \frac{1}{(n-1)!}(t-T)^{n-2} p(t)u(g(t)) = 0 \tag{11}$$

is oscillatory for any $T \geq a$. By a comparison theorem of Kusano and Naito [1, Theorem 1] (11) is oscillatory if so is the ordinary differential equation

$$v'' + \frac{1}{(n-1)!}(t-T)^{n-2} p(t)v = 0. \tag{12}$$

From the classical results of Fite and Hille (see Swanson [3]) it follows that one of the conditions (8)–(10) guarantees the oscillation of (12). This completes the proof.

THEOREM 4. *Suppose that* $g(t) \geq t$ *for* $t \geq a$. *Equation* (1) *is nonoscillatory if*

$$\limsup_{t\to\infty} g(t) \int_t^\infty s^{n-2} p(s)ds < (n-2)!/4. \tag{13}$$

PROOF. Equation (1) is nonoscillatory if the second order equation

$$u''(t) + \frac{1}{(n-2)!}(t-T)^{n-2}p(t)u(g(t)) = 0 \tag{14}$$

is nonoscillatory for some $T \geq a$. This follows from Theorem 3 of Naito [2]. Applying a result of Kusano and Naito [1, Theorem 4] it can be shown that (14) is nonoscillatory if the following ordinary differential equation is nonoscillatory:

$$v'' + \frac{(g^{-1}(t)-T)^{n-2}p(g^{-1}(t))}{(n-2)!g'(g^{-1}(t))}v = 0. \tag{15}$$

The nonoscillation of (15), in turn, is guaranteed by Hille's criterion (see [3]):

$$\limsup_{t\to\infty} t\int_t^\infty \frac{(g^{-1}(s)-T)^{n-2}p(g^{-1}(s))}{(n-2)!g'(g^{-1}(s))}ds < \frac{1}{4}$$

which is equivalent to (13). This completes the proof.

We are now in a position to prove Theorem 2 regarding the strong oscillation and nonoscillation of (1).

PROOF OF THEOREM 2. (Strong oscillation) If (4) holds, then clearly

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}(kp(s))ds = \infty,$$

for any $k>0$, and so, by Theorem 3, (2) is oscillatory for any $k>0$. Conversely, suppose that (1) is strongly oscillatory. If (8) holds, then (4) is trivial. If (8) does not hold, then since (2) is oscillatory for any $k>0$, it follows from Theorem 4 that

$$\limsup_{t\to\infty} g(t)\int_t^\infty s^{n-2}(kp(s))ds \geq (n-2)!/4$$

for any $k>0$. In view of (6) this is possible only if (4) holds.

(Strong nonoscillation) If (5) holds, then

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}(kp(s))ds = 0 < (n-2)!/4$$

for every $k>0$, so that (1) is strongly nonoscillatory by Theorem 4. Conversely, if (1) is strongly nonoscillatory, then from Theorem 3 we see that

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}(kp(s))ds \leq (n-1)!$$

for every $k>0$. But this is true only if (5) holds. The proof is thus complete.

We say that equation (1) is *conditionally oscillatory* if it is neither stronly oscillatory nor strongly nonoscillatory. Thus for a conditionally oscillatory equation (1) there is a positive constant k_0 such that (2) is oscillatory for all $k>k_0$ and (2) is nonoscillatory for all positive $k<k_0$. The following result is an immediate consequence of Theorem 2.

COROLLARY. *Suppose that* $g(t)\geq t$ *for* $t\geq a$ *and* (6) *holds. Then equation* (1) *is conditionally oscillatory if and only if equation* (7) *is conditionally oscillatory.*

EXAMPLE. Consider the equations

$$x^{(n)}(t)+t^{\alpha}x(t+r)=0, \qquad t\geq 1, \tag{16}$$

$$x^{(n)}(t)+t^{\alpha}x(ct)=0, \qquad t\geq 1, \tag{17}$$

where α, $r>0$ and $c>1$ are constants. In view of the example in [2], Theorem 2 and Corollary above, equations (16) and (17) are strongly oscillatory, conditionally oscillatory or strongly nonoscillatory according as $\alpha>-n$, $\alpha=-n$ or $\alpha<-n$.

References

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