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Compact transformation groups on Z_2-cohomology spheres with orbit of codimension 1

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§1. Introduction

Let M be a connected closed smooth manifold and G be a compact connected Lie group which acts smoothly on M, and consider the following assumption:

(AI) *There is an orbit* $G\cdot x$ *of* $x\in M$ *such that* $\dim G\cdot x=\dim M-1$.

Then the following is well-known (cf., e.g., [4; IV, Th. 3.12, Th. 8.2]):

(1.1) *For a G-action on M with* (AI), *where M is simply connected, there is a triple* (K, K_1, K_2) *of subgroups of G with* $K\subset K_1\cap K_2$ *such that K is a principal isotropy subgroup with* $\dim G/K=n-1$ ($n=\dim M$), K_1 *and* K_2 *are non-principal ones with* $k_s=n-\dim G/K_s\geqq 2$ ($s=1, 2$), *and the G-manifold M can be decomposed into the union of two mapping cylinders of the projections* $G/K\to G/K_s$ ($s=1, 2$). (See (3.2–6).)

Based on (1.1), such actions are studied by several authors. For example, H. C. Wang [15] investigated such actions on the spheres S^n with even $n\neq 4$ or odd $n\geqq 33$, and W. C. Hsiang and W. Y. Hsiang [7] have given some examples which are not listed in [15].

The purpose of this paper is to classify such actions (G, M) with (AI) for the case that M is a *Z_2-cohomology sphere*, i.e.,

(AII) *M is simply connected and* $H^*(M; Z_2)\cong H^*(S^n; Z_2)$.

Typical examples of such (G, M) are seen among the linear actions (G, S^n, ψ) on S^n via representations $\psi: G\to SO(n+1)$. Moreover, we have the following example due to W. C. Hsiang and W. Y. Hsiang:

EXAMPLE 1.2 ([7; Example 5.3], cf. [4; Ch. I, §7 and Ch. V, §9]). For any odd integer $r\geqq 1$, consider the $(2m-1)$-manifold

$$W^{2m-1}(r) = \{(z_0, z)\in C\times C^m;\ |z_0|^2+|z|^2 = 2,\ z_0^r+z\cdot {}^tz = 0\}.$$

Then, this is a Z_2-cohomology sphere. Further, for any subgroup G of $SO(m)\times S^1$, the G-action on $W^{2m-1}(r)$ is defined by

$$(X, x)\cdot(z_0, z) = (x^2 z_0, x^r z\cdot {}^tX) \quad \text{for} \quad (X, x)\in SO(m)\times S^1,\ (z_0, z)\in W^{2m-1}(r).$$

This action $(G, W^{2m-1}(r))$ satisfies (AI) for the case

$$G = SO(m)\times S^1, \quad Spin(7)\times S^1\ (m=8) \quad \text{or} \quad G_2\times S^1\ (m=7),$$

since the principal isotropy subgroup K is isomorphic to $SO(m-2)\times Z_2$, $SU(3)\times Z_2$ or $S^3\times Z_2$, respectively; and then K_1 in (1.1) can be taken so that

$$Z(G)^\circ\cap K_1\cong Z_r \text{ (the cyclic group of order } r) \quad \text{and} \quad k_1 = 2,$$

($Z(G)^\circ$ denotes the identity component of the center $Z(G)$ of G). We notice that $W^{2m-1}(r)$ is the sphere S^{2m-1} if $r=1$, or m is odd and $r\equiv\pm 1 \bmod 8$. Moreover, $(G, W^{2m-1}(r))$ is linear if and only if $r=1$.

EXAMPLE 1.3 (see Proposition 9.4.2). Consider the subgroup

$$S^1(l, m) = \{(z^l, z^m)\in S^3\times S^3;\ z\in S^1(\subset C)\}(\cong S^1)$$

of $S^3\times S^3=Spin(4)$. Then, for any relatively prime integers l_s and m_s $(s=1, 2)$ with

$$l_s, m_s\equiv 1 \bmod 4, \quad 0<l_1-m_1\equiv 4 \bmod 8, \quad l_2-m_2\equiv 0 \bmod 8,$$

there is an action $(Spin(4), M)$ with $\dim M=7$, (AI) and (AII) such that $K_s^\circ = S^1(l_s, m_s)$, where $k_s=2$ and G/K_s is non-orientable $(s=1, 2)$. Further, this action induces an effective one $(SO(4), M)$.

For the condition that M is S^7 or the action is linear, we only know that the action is linear if $(l_1, m_1, l_2, m_2)=(1, -3, 1, 1)$.

Our main result is stated in Theorem 6.1, and is summarized as follows:

MAIN THEOREM. *Let an effective action* (G, M) *with* (AI) *and* (AII) *be given, and consider its non-principal orbits* G/K_s *with* $k_s=n-\dim G/K_s\geqq 2$ $(s=1, 2)$ *given in* (1.1). *Then we have the following five cases* (CI)–(CV):

(CI) k_1+k_2 *is odd, and* $n=k_1+k_2-1$ *or* $n=2k_1+2k_2-3$;

(CII) k_1 *and* k_2 *are even, and* $n=k_1=k_2$ *or* $n=k_1+k_2-1$;

(CIII) $k_s=2$, k_{3-s} *is even* ($s=1$ *or* 2), *and* $n=2k_1+2k_2-3$;

(CIV) (e) $k_1=k_2=2$ *and* $n=4$, *or* (o) $k_1=k_2=2$ *and* $n=7$;

(CV) k_1 *and* k_2 *are odd, and* $n=\chi(k_1+k_2-2)/2+1$ $(\chi=\chi(G/K_1)=\chi(G/K_2)$ $=1, 2, 3, 4$ *or* 6).

Furthermore, (G, M) *is the one given in Example* 1.3 *for the case* (CIV) (o), *and is isomorphic to the effective action induced from the action given in Example*

1.2 *for the cases* (CIII) *and* (CI) *with* k_1 *or* $k_2=2$ *and* $n=2k_1+2k_2-3$, *and is linear for the other cases.*

We prepare some known results on compact Lie groups in § 2. After studying (1.1) more precisely in § 3, we investigate the Poincaré polynomials of orbits of an action with (AI) and (AII) in § 5, and consider the five cases (CI)–(CV) in Proposition 5.10. In §§ 7–10, we prove the main result stated in Theorem 6.1 for these cases separately. The proof is done by showing some necessary conditions for G, K and K_s ($s=1, 2$) of an action (G, M) with (AI) and (AII) in the first half of each section, and by studying the existence and uniqueness of such actions with G, K and K_s satisfying the necessary conditions in the second half.

We notice that actions (G, M) with (AI) for cohomology real projective spaces M can be investigated by using the results in this paper. The classification of such (G, M) for cohomology complex projective spaces M have been done by F. Uchida [13].

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§ 2. Preliminaries

In this paper, groups are compact Lie groups and subgroups are closed subgroups, and U° denotes the identity component of a group U.

The following (2.1) is well-known (see [2], [9], [11]).

(2.1) *Suppose that a group U is connected, and acts effectively and transitively on the sphere S^{k-1} ($k\geqq 2$). Then the U-action on S^{k-1} is equivalent to the following linear action of U on S^{k-1} via the standard representation ι: $U\to SO(k)$ with an isotropy subgroup H.*

(i) *If k is odd, then U is simple and (U, k, ι, H) is*

$$(SO(k), k, \rho_k, SO(k-1)) \quad \text{or} \quad (G_2, 7, \varphi_2, SU(3)).$$

(ii) *If k is even, then U contains a simple normal subgroup U' such that the restricted U'-action on S^{k-1} is transitive and U/U' is of rank at most* 1, *and (U, k, ι, H) is*

$$(SO(k), k, \rho_k, SO(k-1))\,(k\neq 4), \qquad (SU(l), 2l, (\mu_l)_R, SU(l-1)),$$
$$(U(l), 2l, (\mu_l)_R, U(l-1)), \qquad (Sp(l), 4l, (\nu_l)_R, Sp(l-1)),$$
$$(Sp(l)\times S^i/Z_2, 4l, (\nu_l\otimes\mu_1^* \text{ (or } \nu_1^*))_R, Sp(l-1)\times S^i/Z_2)$$
$$(i = 1 \text{ or } 3;\ Z_2 \text{ is generated by } (-E, -1)),$$
$$(Spin(9), 16, \Delta_9, Spin(7)) \quad \text{or} \quad (Spin(7), 8, \Delta_7, G_2).$$

For a subgroup H of U, denote by $N(H, U)$ (or NH) and $Z(H, U)$ the normalizer and the centralizer of H in U, respectively. Then we see the following two lemmas by easy calculation.

LEMMA 2.2. *Let* (U, H) *be as in* (2.1). *Then* $N(H, U)/H$ *is isomorphic to*

S^3 *if* $U = Sp(l)\,(k=4l\geqq 8)$,

S^1 *if* $U = SO(2), SU(l), U(l)\,(k=2l\geqq 6), Sp(l)\times S^1/Z_2\ (k=4l)$,

Z_2 *otherwise.*

LEMMA 2.3. $Z(SU(l), O(2l)) \cong S^1\ (l\geqq 3)$, $\qquad Z(Sp(l), O(4l)) \cong S^3\ (l\geqq 1)$.

LEMMA 2.4. *Assume that* U/U° *is cyclic, and let* $\tau_1, \tau_2\colon U\to O(k)\ (k\geqq 2)$ *be representations of* U. *If the actions of* U *on* S^{k-1} *via* τ_1 *and* τ_2 *are both effective and transitive and their isotropy subgroups are conjugate to each other, then* τ_1 *is equivalent to* τ_2.

PROOF. U° also acts effectively and transitively on S^{k-1} via the restricted representations $\tau_1\,|\,U^\circ$ and $\tau_2\,|\,U^\circ$, which are equivalent by (2.1). Thus we may assume that $\tau_1\,|\,U^\circ=\tau_2\,|\,U^\circ=\tau$. Let $H_s(s=1, 2)$ be the isotropy subgroup of the U-action on S^{k-1} via τ_s at $p=(1, 0,\dots, 0)\in S^{k-1}$. Then H_2 is conjugate to H_1 by the assumption.

Now, take $a\in U-U^\circ$ such that aU° generates U/U° by the assumption on U, and set $x_s=\tau_s(a)$ $(s=1, 2)$. Then we see that

$$x_2\in N(=N(\tau(U^\circ), O(k)), \quad x_2^{-1}x_1\in Z(=Z(\tau(U^\circ), O(k))),$$

and $x_2\tau(g)p=\tau(g)p$ for some $g\in N(H_1^\circ, U^\circ)$. On the other hand,

$$(*) \qquad \tau(N(H_1^\circ, U^\circ))p\subset Zp \qquad \text{(by the above two lemmas)}.$$

Thus $\tau(g)p=yp$ for some $y\in Z$. Therefore $y^{-1}x_2yx_1^{-1}p=y^{-1}x_2yp=p$ and $y^{-1}x_2yx_1^{-1}\in Z$, which imply $y^{-1}x_2yx_1^{-1}=1$ and $x_2=yx_1y^{-1}$ since the U°-action on S^{k-1} via τ is effective and transitive. *q.e.d.*

LEMMA 2.5 (cf. [10; (5.4)]). *Let* $\tau_1, \tau_2\colon U\to GL(k; C)$ *be equivalent representations of* U. *Assume that* τ_1 *is irreducible or equivalent to a direct sum of an irreducible representation and a trivial representation of degree* 1. *If* $\tau_1(U)$ *and* $\tau_2(U)$ *are contained in* $U(k)$ *(resp.* $O(k)$*), then they are conjugate in* $U(k)$ *(resp.* $O(k)$*).*

The Poincaré polynomial $P(X; t)=\sum_i \dim H^i(X; Q)t^i$ of a space X will be denoted simply by $P(X)$. Now the following lemma can be proved by using [3], [14] and Hirsch's formula.

LEMMA 2.6. *Let U be a connected simple group and H be its connected subgroup with same rank. If* $\dim H^i(U/H; Q) \leqq 1$ *for* $i \geqq 0$, *then* $P(U/H)$ *is given as follows, where* A_l, B_l, C_l, D_l *are the classical groups of rank* l, G_2, F_4 *are the exceptional Lie groups, and* $U_1 \circ U_2$ *denotes an essentially direct product of groups* U_1 *and* U_2:

(1) $P(A_l/A_{l-1}\circ S^1) = (1-t^{2l+2})/(1-t^2)\,(l \geqq 1)$, (2) $P(B_l/D_l) = 1+t^{2l}\,(l \geqq 2)$,

(3) $P(B_l/B_1\circ D_{l-1}) = (1+t^{2l-2})(1-t^{4l})/(1-t^4)\,(l \geqq 4\colon \textit{even})$,

(4) $P(B_l/B_{l-1}\circ S^1) = (1-t^{4l})/(1-t^2)\,(l \geqq 2)$,

(5) $P(C_l/C_{l-1}\circ C_1) = (1-t^{4l})/(1-t^4)\,(l \geqq 3)$,

(6) $P(F_4/B_4) = 1+t^8+t^{16}$, (7) $P(G_2/A_1\circ A_1) = 1+t^4+t^8$,

(8) $P(G_2/A_2) = 1+t^6$, (9) $P(C_l/C_{l-1}\circ S^1) = (1-t^{4l})/(1-t^2)\,(l \geqq 3)$,

(10) $P(G_2/A_1\circ S^1) = (1-t^{12})/(1-t^2)$.

In the rest of this section, we prove the following

PROPOSITION 2.7. *Let H be a connected subgroup of a connected group U. Assume that*

(1) *H does not contain any positive dimensional normal subgroup of U, and*

(2) $r(U) = r(H)+1$ (“r” *denotes the rank*).

If $\dim U/H = 3-2(c(U)-c(H))$ (“c” *denotes the dimension of the center*), *then U is an essentially direct product of some copies of S^3 and a toral group.*

To prove this proposition, we set

$$\alpha(U, H) = \dim U - \dim H - 3(r(U) - r(H)).$$

LEMMA 2.8. *Let U be simple and H be its proper subgroup. Then* $\alpha(U, H) > 0$ *if* $r(U) \geqq 2$.

PROOF. Since U is simple, U acts almost effectively on U/H and we see

$$\dim U - \dim H - r(U) - r(H) \geqq 0 \quad \text{(by [4; IV, Cor. 5.4])}.$$

If $2r(H) > r(U)$, then this implies $\alpha(U, H) > 0$.

Suppose that $2r(H) \leqq r(U)$. By using the classification theorem of Lie groups, we see that

$$(*) \qquad r(V)^2 + 2r(V) \leqq \dim V < 4r(V)^2 \qquad \text{for any simple group } V.$$

By representing H as an essentially direct product of simple groups and a toral group, $(*)$ implies $\dim H \leqq 4r(H)^2$. This and $(*)$ for $V = U$ imply

$$\alpha(U, H) \geqq r(U)^2 - r(U) - 4r(H)^2 + 3r(H) > 0,$$

since $2r(H) \leqq r(U)$ and $r(U) \geqq 2$. *q.e.d.*

PROOF OF PROPOSITION 2.7. By [15; (9.1)] and the assumption (2), any connected simple normal subgroup with rank $\geqq 2$ of H is contained in a simple normal subgroup of U. Thus, by decomposing U and H into essentially direct products of simple groups and toral groups, we have

$$U = U_1 \circ \cdots \circ U_l \circ U', \quad H = H_1 \circ \cdots \circ H_l \circ H'$$

where U_i is simple with $r(U_i) \geqq 2$, $H_i \subset U_i$ $(1 \leqq i \leqq l)$, and U' (resp. H') is an essentially direct product of some copies of S^3 and a toral group of dimension $c(U)$ (resp. $c(H)$). Here $H_i \subsetneqq U_i$ by the assumption (1). Then we see easily that

$$\sum_{i=1}^{l} \alpha(U_i, H_i) = \dim U/H - 3 + 2(c(U) - c(H)).$$

Therefore, if the right hand side is zero, then $l=0$ by Lemma 2.8 as desired. *q.e.d.*

§ 3. Actions with orbit of codimension 1

Any action (G, M) induces the effective action $(G/N_0, M)$, where N_0 is the maximum subgroup of G acting trivially on M, ($N_0 = \bigcap_{x \in M} G_x$ and is normal in G). The action (G, M) is said to be *almost effective* if N_0 is finite. Two actions are said to be *essentially isomorphic* if their induced effective actions are isomorphic. Then we see easily the following

LEMMA 3.1. *Let (G, M) be a given action and K be its principal isotropy subgroup.*

(i) *If N is a normal subgroup of G with $N \subset K$, then N acts trivially on M.*

(ii) *The G-action on M is almost effective if and only if K does not contain and positive dimensional normal subgroup of G (i.e., the G-action on G/K is almost effective).*

(iii) *In the case* (i) *the isotropy subgroup $(G/N)_x$ $(x \in M)$ of the induced G/N-action on M is equal to G_x/N, and (G, M) is essentially isomorphic to $(G/N, M)$.*

(iv) *Especially, take $N = Z(G)^\circ \cap K$. Then $Z(G/N)^\circ = Z(G)^\circ/N$ and the restricted $Z(G/N)^\circ$-action on M of $(G/N, M)$ is effective.*

Now, we consider an action (G, M) with (AI). We notice that (1.1) can be restated more precisely as follows:

(3.2) *Let M be a G-manifold with* (AI) *and assume that $\pi_1(M)$ is finite.*

Then there are a principal orbit G/K *and two non-principal ones* G/K_1, G/K_2 *with* $\dim G/K=n-1$ $(n=\dim M)$ *and* $K\subset K_1\cap K_2$, *and* M *has an equivariant decomposition*

$$(3.3)\qquad M = M(\alpha) = X_1 \cup_\alpha X_2, \qquad X_s = G\times_{K_s} D^{k_s}, \qquad k_s = n - \dim G/K_s,$$

where the attaching map $\alpha\colon \partial X_1 = G/K \to G/K = \partial X_2$ *is given by* $\alpha(gK)=g\alpha^{-1}K\cdot$ $(g\in G)$ *for some* $\alpha\in NK(=N(K, G))$. *Here* K_s *acts on the unit disk* D^{k_s} *via a slice representation* $\sigma_s\colon K_s\to O(k_s)$ *so that* K_s *acts transitively on the boundary* ∂D^{k_s} *with the isotropy subgroup* $(K_s)_{p_s}=K$ *for some base point* $p_s\in\partial D^{k_s}$, *and the identification* $\partial X_s=G/K$ *is done by the equivariant diffeomorphism sending* $[g, p_s]\in G\times_{K_s}\partial D^{k_s}=\partial X_s$ *to* $gK\in G/K$.

(3.4) *In* (3.2), *the isotropy subgroups* K, K_1 *and* K_2 *can be chosen arbitrarily from their conjugate classes under the condition* $K\subset K_1\cap K_2$. *Especially, by choosing* $\alpha^{-1}K_2\alpha$ *instead of* K_2, *we have an equivariant decomposition*

$$(3.5)\qquad M = X_1 \cup X_2', \qquad X_1 \cap X_2' = G/K,$$

where X_1 *and* X_2' *are the mapping cylinders of the projections* $G/K\to G/K_1$ *and* $G/K\to G/\alpha^{-1}K_2\alpha$, *respectively.*

(3.6) *If* $H_1(M; Z_2)=0$ *in addition, then the non-principal orbits* G/K_1, G/K_2 *in* (3.2) *are singular, i.e.,* $\dim G/K_s>\dim G/K$ *and hence* $k_s\geqq 2$. (This is shown by [4; IV, Th. 3.12].)

For $M(\alpha)$ in (3.3), we see immediately the following

LEMMA 3.7. *Let* $\alpha, \alpha'\colon \partial X_1=G/K\to G/K=\partial X_2$ $(\alpha, \alpha'\in NK)$ *be equivariant diffeomorphisms. Then* $M(\alpha)$ *is equivariantly diffeomorphic to* $M(\alpha')$ *if the following* (1) *or* (2) *is satisfied*:

(1) α *is G-diffeotopic to* α'.

(2) $\beta=\alpha^{-1}\alpha'$ *or* $\alpha'\alpha^{-1}$ *is extendable to an equivariant diffeomorphism on* X_s $(s=1$ *or* $2)$.

LEMMA 3.8. (2) *of Lemma* 3.7 *holds if the following* (1) *or* (2) *is satisfied*:

(1) β *is in the center of* G.

(2) β *is in* K_s, *and* $(K_s)_p=K$, $\sigma_s(\beta)p=-p$ *for some* $p\in\partial D^{k_s}$.

PROOF. (1) The equivariant diffeomorphism of $X_s=G\times_{K_s}D^{k_s}$ onto itself sending $[g, x]$ to $[\beta^{-1}g, x]$ is an extension of β.

(2) Suppose that $k_s\geqq 2$. Since $(K_s)_p=K$ and K_s° acts transitively on ∂D^{k_s} via $\sigma_s\,|\,K_s^\circ$, there exists γ in $N(K, K_s)\cap K_s^\circ$ satisfying $\sigma_s(\gamma)p_s=p$, where p_s is the base point in (3.3). Hence $\sigma_s(\gamma)$ is in $N(\sigma_s(K)^\circ, \sigma_s(K_s)^\circ)$. Therefore, by using

(*) in the proof of Lemma 2.4, we see that $Ap_s = p$ for some $A \in Z(\sigma_s(K_s)^\circ, O(k_s))$. Now we may assume that $\beta \in N(K, K_s) \cap K_s^\circ$, since $\beta \in N(K, K_s)$ and $\beta k \in K_s^\circ$ for some $k \in K$. Then we get

$$\sigma_s(\beta)p_s = \sigma_s(\beta)A^{-1}p = A^{-1}\sigma_s(\beta)p = -A^{-1}p = -p_s.$$

When $k_s = 1$, this equality $\sigma_s(\beta)p_s = -p_s$ is easily seen.

Therefore the equivariant diffeomorphism of $X_s = G \times_{K_s} D^{k_s}$ onto itself sending $[g, x]$ to $[g, -x]$ is an extension of β. *q.e.d.*

LEMMA 3.9 ([1; Prop. 3.9]). *Assume that $\sigma_s(K_s) \supset SO(k_s)$ $(s=1, 2)$ and σ_s is equivalent to $\sigma_s c_{\xi_s}$ for any $\xi_s \in NK \cap NK_s$, where $c_{\xi_s}(k) = \xi_s k \xi_s^{-1}$ $(k \in K_s)$. Then $M(\alpha)$ is equivariantly diffeomorphic to $M(\alpha')$ if and only if there exist $\gamma_s \in NK \cap NK_s$ $(s=1, 2)$ such that $\gamma_1 K$ and $\alpha^{-1}\gamma_2\alpha' K$ are contained in the same component of NK/K.*

§ 4. Extension of actions

In the first place, we prepare the following lemma due to F. Uchida.

LEMMA 4.1. *Let $\tilde{G}$ be a connected group and G be its connected subgroup. Suppose that the given $\tilde{G}$-action on M and the restricted G-action on M have principal orbits of same dimension. Then, for each $x \in M$, $\tilde{G} \cdot x = G \cdot x$ and $\tilde{G}_x \cap G = G_x$, and $\tilde{G} \cdot x$ is principal if and only if so is $G \cdot x$.*

PROOF. Since the union of all principal orbits is open and dense in M (cf. [4; IV, Th. 3.1]), we can choose $u \in M$ such that $\tilde{K} = \tilde{G}_u$ and $K = G_u$ are principal. Since $K = \tilde{K} \cap G$, the orbit $G \cdot u = G/K$ is a closed submanifold of a connected manifold $\tilde{G} \cdot u = \tilde{G}/\tilde{K}$, and these have the same dimension by the assumption. Hence

$$G/K = \tilde{G}/\tilde{K} \quad \text{and so} \quad \tilde{G} = G\tilde{K}. \tag{4.2}$$

Let $x \in M$. Then there exists $g \in \tilde{G}$ with $\tilde{K} \subset \tilde{G}_{g^{-1}x} = g^{-1}\tilde{G}_x g$, and so we see easily $\tilde{G} \cdot x = G \cdot x$ by using (4.2).

Now suppose that $G \cdot x$ is a principal orbit. Take $v \in \tilde{G} \cdot x = G \cdot x$ satisfying $\tilde{G}_v \supset \tilde{K}$. Then $G_v \supset \tilde{K} \cap G = K$, and hence $G_v = K$. Therefore we see $\tilde{G}_v = \tilde{K}$ by (4.2), which shows that $\tilde{G} \cdot x = \tilde{G} \cdot v$ is a principal orbit. The converse is clear. *q.e.d.*

In the rest of this section, let $\tilde{G} = G \times H$ for connected groups G and H, and assume that

(4.3) the given G-action on M in (3.2) can be extended to a $\tilde{G}$-action on M with orbit of codimension 1.

Then by Lemma 4.1 we see that a G-equivariant decomposition $M=X_1 \cup_\alpha X_2$ in (3.3) gives a $\tilde{G}$-equivariant decomposition

$$(4.4) \quad M = M(\tilde{\alpha}) = \tilde{X}_1 \cup_{\tilde{\alpha}} \tilde{X}_2, \quad \tilde{X}_s = \tilde{G} \times_{\tilde{K}_s} D^{k_s} \text{ with } G \cap \tilde{K} = K,$$
$$G \cap \tilde{K}_s = K_s, \quad X_s = \tilde{X}_s \ (s = 1, 2), \quad \alpha = \tilde{\alpha},$$

where $\tilde{G}/\tilde{K}$ is a principal orbit, $\tilde{G}/\tilde{K}_s$ $(s=1, 2)$ are non-principal ones with $\tilde{K} \subset \tilde{K}_1 \cap \tilde{K}_2$, and $\tilde{K}_s$ acts on D^{k_s} via $\tilde{\sigma}_s$ with $\tilde{\sigma}_s | K_s = \sigma_s$.

LEMMA 4.5. *Under the above situation, there is a homomorphism*

$$(4.6) \qquad \phi: H \longrightarrow NK \cap NK_1 \cap NK_2/K \quad (NL = N(L, G))$$

satisfying

$$\tilde{K} = \{(g, h) \in G \times H = \tilde{G};\ g\phi(h)^{-1} = K\}, \qquad \tilde{K}_s = \{(g, h) \in \tilde{G};\ g\phi(h)^{-1} \in K_s/K\}.$$

Furthermore the kernel of ϕ is finite if the restricted H $(\subset \tilde{G})$-action on M is almost effective.

PROOF. Fix a point $u \in \partial X_1$ with $\tilde{G}_u = \tilde{K}$. For any $h \in H$, there exists $g \in G$ with $h \cdot u = g^{-1} \cdot u$ by (4.2). Then $(g, h) \in \tilde{K}$ and

$$\tilde{L} \supset (g, h)L(g, h)^{-1} = gLg^{-1} \subset G \quad (L = K, K_1, K_2).$$

This implies $g \in NK \cap NK_1 \cap NK_2$. Set $\phi(h) = gK$. Then we see easily that ϕ is a homomorphism.

By considering the isotropy subgroups of the $\tilde{G}$-action at $u \in \partial X_1$ and $x_s = [1, 0] \in \tilde{G} \times_{\tilde{K}_s} D^{k_s}$, we have the lemma. *q.e.d.*

LEMMA 4.7. *Let there be given two extended $\tilde{G}$-actions on M in* (4.3), *and* $(\tilde{K}, \tilde{K}_1, \tilde{K}_2)$, $(\tilde{K}', \tilde{K}_1', \tilde{K}_2')$ *and ϕ, ϕ' be the corresponding isotropy subgroups in* (4.4) *and the homomorphisms of* (4.6); *and assume that*
(4.8) *there holds a commutative diagram*

$$\begin{array}{ccc} H & \xrightarrow{\phi} & NK \cap NK_1 \cap NK_2/K \\ \psi \downarrow & & \downarrow c_\beta \\ H & \xrightarrow{\phi'} & NK \cap NK_1 \cap NK_2/K \end{array}$$

for some automorphism ψ and $\beta \in NK \cap NK_1 \cap NK_2$, where $c_\beta(gK) = \beta g \beta^{-1} K$. Then there exists an automorphsim Ψ of $\tilde{G}$ with $\Psi(\tilde{K}) = \tilde{K}'$ and $\Psi(\tilde{K}_s) = \tilde{K}_s'$ $(s=1, 2)$.

PROOF. Set $\Psi(g, h) = (\beta g \beta^{-1}, \psi(h))$ $((g, h) \in \tilde{G})$. Then Ψ is the desired automorphism by Lemma 4.5. *q.e.d.*

LEMMA 4.9. *Let ϕ be the homomorphism of* (4.6). *Then $N(\tilde{K}, \tilde{G})/\tilde{K}$ is isomorphic to $Z(\mathrm{Im}\,\phi, NK/K)$.*

PROOF. For each $(g, h)\in N(\tilde{K}, \tilde{G})$, we see easily that $g\in N(K, G)$ and $g\phi(h^{-1})\in Z=Z(\mathrm{Im}\,\phi, NK/K)$. Consider the homomorphism

$$\xi\colon N(\tilde{K}, \tilde{G}) \longrightarrow Z, \quad \xi(g, h) = g\phi(h^{-1})\ ((g, h)\in N(\tilde{K}, \tilde{G})).$$

Since $(g, 1)$ is in $N(\tilde{K}, \tilde{G})$ for any $g\in G$ with $gK\in Z$, we see that ξ is an epimorphism. Clearly $\mathrm{Ker}\,\xi=\tilde{K}$. Thus $N(\tilde{K}, \tilde{G})/\tilde{K}\cong Z$. *q.e.d.*

§5. Orbits of an action with (AI) and (AII)

Now we assume that a G-manifold $M=M(\alpha)$ in (3.3) is a Z_2-cohomology sphere, i.e., M satisfies (AII). Throughout this section,

(5.1) we write K_2 instead of $\alpha^{-1}K_2\alpha$ for the sake of simplicity.

Thus we consider a Z_2-cohomology sphere M with the decomposition

(5.2) $$M = X_1 \cup X_2, \qquad X_1 \cap X_2 = G/K,$$

where X_s is the mapping cylinder of the projection $f_s\colon G/K\to G/K_s$, $k_s=n-\dim G/K_s\geqq 2$ $(s=1, 2)$ and $\dim G/K=n-1$ $(n=\dim M)$, (cf. (3.5), (3.6)).

The following several results are due to H. C. Wang.

(5.3) ([15; (4.3) and (4.9)]) (i) *For the induced homomorphism $f_{s*}\colon \pi_1(G/K)\to\pi_1(G/K_s)$ of f_s,*

$$\pi_1(G/K) = \mathrm{Ker}f_{1*}\cdot\mathrm{Ker}f_{2*}, \qquad \pi_1(G/K_s) = f_{s*}(\mathrm{Ker}f_{3-s*})\ (s = 1, 2).$$

(ii) *Let $\Pi_s=(K_s^\circ\cap K)/K^\circ$. Then $K/K^\circ=\Pi_1\Pi_2$ and $(K/K^\circ)/\Pi_s\cong K_s/K_s^\circ$ is cyclic* $(s=1, 2)$.

LEMMA 5.4. (i) *If $k_1>2$ and $k_2>2$, then G/K and G/K_s $(s=1, 2)$ are simply connected, and hence K and K_s $(s=1, 2)$ are connected.*

(ii) *If $k_1=2$ and $k_2>2$, then G/K_1 is simply connected and*

$$K_1 = K_1^\circ,\ K = \cup_i b_1^i K^\circ,\ K_2 = \cup_i b_1^i K_2^\circ \quad \textit{for some } b_1\in K_1\cap K.$$

(iii) *If $k_1 = k_2 = 2$, then*

$$K = \cup_{i,j} b_1^i b_2^j K^\circ,\ K_1 = \cup_i b_2^i K_1^\circ,\ K_2 = \cup_i b_1^i K_2^\circ \ \textit{for some } b_s\in K_s^\circ\cap K\ (s = 1, 2).$$

PROOF. Suppose $k_s>2$. Then, from the homotopy exact sequence of the fibering $S^{k_s-1}\to G/K\xrightarrow{f_s} G/K_s$, it follows that $\mathrm{Ker}f_{s*}=1$ in (5.3). Thus we see (i) and the first half of (ii).

If $k_s=2$, then $\Pi_s=(K_s^\circ \cap K)/K^\circ$ is a proper subgroup of $K_s^\circ/K^\circ \cong S^1$ generated by $b_s K^\circ$ ($b_s \in K_s^\circ \cap K$). By (5.3) (ii), the homomorphism $\Pi_{3-s} \hookrightarrow \Pi_1 \Pi_2 = K/K^\circ \to K_s/K_s^\circ$ is epimorphic. Therefore the rest of the lemma follows immediately. *q.e.d.*

By using the Mayer-Vietoris exact sequence of the triad (M, X_1, X_2) in (5.2), we see

(5.5) ([15; (3.4)]) *For the cohomology with coefficient in Q or Z_2, f_s^*: $H^*(G/K_s) \to H^*(G/K)$ ($s=1, 2$) are monomorphic, and*

$$f_1^*(H^i(G/K_1)) \oplus f_2^*(H^i(G/K_2)) = H^i(G/K) \quad (0 < i < n-1),$$
$$P(G/K) = P(G/K_1) + P(G/K_2) - 1 + t^{n-1}.$$

Let θ: $G/K^\circ \to G/K$, θ_s: $G/K_s^\circ \to G/K_s$ and e_s: $G/K^\circ \to G/K_s^\circ$ be the natural projections, and consider the induced homomorphisms $H^*(G/K) \xrightarrow{\theta^*} H^*(G/K^\circ)$, $H^*(G/K_s) \xrightarrow{\theta_s^*} H^*(G/K_s^\circ) \xrightarrow{e_s^*} H^*(G/K^\circ)$ of cohomology with coefficient in Q.

(5.6) ([15; § 11]) *Suppose that k_1 or k_2 is equal to* 2. *Then*

(i) *θ^* is isomorphic, and hence $P(G/K)=P(G/K^\circ)$.*

(ii) *$H^*(G/K_s^\circ)=\theta_s^*(H^*(G/K_s)) \oplus \mathrm{Ker}\, e_s^*$ ($s=1, 2$).*

(iii) *If G/K_s is orientable, then θ_s^* is isomorphic, and hence $P(G/K_s)= P(G/K_s^\circ)$. If G/K_s is non-orientable, then $P(G/K)=(1+t^{2k_s-1})P(G/K_s)$ and $P(G/K_s^\circ)=(1+t^{k_s})P(G/K_s)$.*

(iv) *If k_s is odd, then G/K_s is orientable.*

In the followings, let $K \nsim 0$ in G mean that K *is non-homologous to zero in G.*

Lemma 5.7. (i) *If G/K_s is orientable and k_{3-s} is even, then $K_s^\circ \nsim 0$ in G.*

(ii) *If k_1 and k_2 are even, then $K^\circ \nsim 0$ in G.*

Proof. Let i: $G/K^\circ \to BK^\circ$ and i_s: $G/K_s^\circ \to BK_s^\circ$ be classifying maps, and r_s: $BK^\circ \to BK_s^\circ$ be the natural map induced from $K^\circ \hookrightarrow K_s^\circ$ ($s=1, 2$). Consider the commutative diagram

$$\begin{array}{ccccc}
H^*(BK_1^\circ) & \xrightarrow{r_1^*} & H^*(BK^\circ) & \xleftarrow{r_2^*} & H^*(BK_2^\circ) \\
\downarrow i_1^* & & \downarrow i^* & & \downarrow i_2^* \\
H^*(G/K_1^\circ) & \xrightarrow{e_1^*} & H^*(G/K^\circ) & \xleftarrow{e_2^*} & H^*(G/K_2^\circ) \\
\uparrow \theta_1^* & & \cong \uparrow \theta^* & & \uparrow \theta_2^* \\
H^*(G/K_1) & \xrightarrow{f_1^*} & H^*(G/K) & \xleftarrow{f_2^*} & H^*(G/K_2),
\end{array}$$

where $\mathrm{Im} f_1^* \cap \mathrm{Im} f_2^* \subset H^0(G/K)$ by (5.5).

(i) By the assumption and (5.6) (iii), θ_s^* is isomorphic. Further, r_{3-s}^* is epimorphic since K_{3-s}°/K° is an odd sphere. Then, in the above diagram, we have $\mathrm{Im} f_s^*(\theta_s^*)^{-1} i_s^* \subset \mathrm{Im} f_{3-s}^*$ by using (5.6) (ii). Thus $\mathrm{Im}\, i_s^* \subset H^0(G/K_s^\circ)$, and so $K_s^\circ \sim 0$ in G (cf. [5; § 10]).

(ii) $r_s^*(s=1, 2)$ are epimorphic since $K_s^\circ/K^\circ (s=1, 2)$ are odd spheres. Thus we have $\mathrm{Im}\, i^* \subset \mathrm{Im}\, \theta^* f_1^* \cap \mathrm{Im}\, \theta^* f_2^*$, and $\mathrm{Im}\, i^* \subset H^0(G/K^\circ)$. Then $K^\circ \sim 0$ in G. *q.e.d.*

LEMMA 5.8. (i) *If G/K_1 and G/K_2 are orientable, then*

$$(1-t^k)P(G/K_s^\circ) = (1+t^{k_{3-s}-1})(1-t^{n-1}) \quad (s = 1, 2 \text{ and } k = k_1+k_2-2).$$

(ii) *If G/K_1 is orientable and G/K_2 is not so, then $k_1=2$ and*

$$(1-t^{2k_2})P(G/K_1^\circ) = (1+t^{2k_2-1})(1-t^{n-1}), \quad (1-t^{2k_2})P(G/K_2) = (1+t)(1-t^{n-1}).$$

(iii) *If G/K_1 and G/K_2 are non-orientable, then $k_1=k_2=2$ and*

$$(1-t^3)P(G/K) = (1+t^3)(1-t^{n-1}), \quad (1-t^3)P(G/K_s) = 1-t^{n-1} \ (s = 1, 2).$$

PROOF. Suppose that G/K_s is orientable. Then, for (M, X_1, X_2) in (5.2), we have the isomorphisms $H^i(M, X_{3-s}) \cong H^i(X_s, \partial X_s) \cong H^{i-k_s}(G/K_s)$ by the excision and the Thom isomorphism. From the cohomology exact sequence of the pair (M, X_{3-s}), we get

$$(*) \qquad t^{k_s}P(G/K_s) - tP(G/K_{3-s}) = t^n - t.$$

By $(*)$ for $s=1, 2$ and (5.6) (iii), we have (i). If G/K_s is non-orientable, then we have $k_{3-s}=2$ by Lemma 5.4. Then (ii) and (iii) of the lemma follow from (5.5), (5.6) (iii) and $(*)$ by easy calculation. *q.e.d.*

For the polynomial in the above lemma, we see the following

LEMMA 5.9. *Let $P(t)$ be an integral polynomial on t satisfying*

$$(1-t^k)P(t) = (1+t^l)(1-t^{n-1}) \quad \textit{for some positive integers } k, l \textit{ and } n(\geqq 2).$$

(i) *Assume that l is odd and $P(t)=\prod_{i=1}^m (1+t^{u_i})$ for some integer $m \geqq 0$ and odd integers $u_i \geqq 1$ $(1 \leqq i \leqq m)$. Then*

$$2(n-1) = k, \quad l = n-1 \quad \text{and} \quad P(t) = 1 \qquad \text{if } n \text{ and } k \text{ are even},$$

$$n-1 = 2k \quad \text{and} \quad P(t) = (1+t^l)(1+t^k) \qquad \text{if } n \text{ and } k \text{ are odd},$$

$$n-1 = k \quad \text{and} \quad P(t) = 1+t^l \qquad \text{otherwise}.$$

(ii) *Assume that l, k are even, and the degree of $P(t)$ is less than $n-1$. Then*

$k=2l,\quad n-1=\chi l\quad$ *and* $\quad P(t)=(1-t^{n-1})/(1-t^l)\qquad$ *if* χ *is odd,*

$n-1=(\chi/2)k\quad$ *and* $\quad P(t)=(1+t^l)(1-t^{n-1})/(1-t^k)\qquad$ *if* χ *is even,*

where $\chi=P(1)=P(-1)$.

PROOF. Put $\chi=P(1)$. Then the given equality divided by $1-t$ shows $k\chi=2(n-1)$.

(i) $\chi=2^m$ by the assumption on $P(t)$. Thus $2^m k=2(n-1)$. If $m=0$, then we have the first case. If $m\geqq 1$, then $n-1=2^{m-1}k$ and $P(t)=(1+t^l)(1-t^{n-1})/(1-t^k)=(1+t^l)\prod_{j=0}^{m-2}(1+t^{2^j k})$. Thus we have the other cases by the assumption on $P(t)$, because $1+t$ is a factor of $1+t^h$ if and only if h is odd, and because $(1+t)^2$ is not a factor of $1+t^h$.

(ii) Since $k\chi=2(n-1)$, the second case is trivial. Assume that χ is odd. By multiplying the given equality by $(1+t^{n-1})/(1-t^k)$, we obtain

$$(*)\qquad P(t)+t^{n-1}P(t)=(1+t^l)\sum_{i=0}^{\chi-1}t^{ki},\ \text{where}\ \deg P(t)<n-1.$$

Since $n-1=\chi k/2$ and χ is odd, $(*)$ implies that $n-1=ik+l$ for some i, and hence l is an odd multiple of $k/2$. Thus, $(*)$ implies that $\deg P(t)=k(\chi-1)/2$ and $n-1+\deg P(t)=l+(\chi-1)k$. Therefore we have $l=k/2$ and the first cases. $P(-1)=P(1)$ is now trivial. *q.e.d.*

Now we are ready to prove the following proposition, where *each* (e) *holds if n is even,* and *each* (o) *holds if n is odd.*

PROPOSITION 5.10 (cf. [15; (5.2), (8.3), (11.7), (11.9)]).

(CI) *Assume that* k_1 *is odd and* k_2 *is even. Then* G/K_2 *is simply connected,* G/K_1 *is orientable,* $K_1^\circ \sim 0$ *in* G, *and*

(e) $n=k_1+k_2-1,\quad P(G/K_{3-s}^\circ)=1+t^{k_s-1}\ (s=1,2)$,

(o) $n=2k+1\ (k=k_1+k_2-2),\quad P(G/K_{3-s}^\circ)=(1+t^{k_s-1})(1+t^k)\ (s=1,2)$.

(CII) *Assume that* k_1, k_2 *are even, and* G/K_1, G/K_2 *are orientable. Then* K°, K_1° *and* $K_2^\circ \sim 0$ *in* G, *and*

(e) $k_1=k_2=n,\quad K_1=K_2=G$,

(o) $n=k_1+k_2-1,\quad P(G/K_{3-s}^\circ)=1+t^{k_s-1}\ (s=1,2)$.

(CIII) *Assume that* k_1, k_2 *are even,* G/K_1 *is orientable and* G/K_2 *is non-orientable. Then* K° *and* $K_1^\circ \sim 0$ *in* G, $k_1=2$, *n is odd, and*

(o) $n=2k_2+1,\quad P(G/K_1^\circ)=1+t^{2k_2-1},\quad P(G/K_2^\circ)=(1+t)(1+t^{k_2})$,

$P(G/K_2)=1+t,\quad P(G/K^\circ)=(1+t)(1+t^{2k_2-1})$.

(CIV) *Assume that* k_1, k_2 *are even, and* G/K_1, G/K_2 *are non-orientable. Then* $k_1 = k_2 = 2$, $K^\circ \nsim 0$ *in* G, *and*

(e) $n = 4$, $P(G/K_s^\circ) = 1 + t^2$, $P(G/K_s) = 1$ $(s = 1, 2)$, $P(G/K^\circ) = 1 + t^3$,

(o) $n = 7$, $P(G/K_s^\circ) = (1+t^2)(1+t^3)$, $P(G/K_s) = 1 + t^3$ $(s = 1, 2)$,

$$P(G/K^\circ) = (1+t^3)^2.$$

(CV) *Assume that* k_1, k_2 *are odd. Then* K, K_1 *and* K_2 *are connected, the Euler characteristic* $\chi = P(G/K_1; -1)$ *of* G/K_1 *is equal to that of* G/K_2, $n - 1 = \chi k/2$ $(k = k_1 + k_2 - 2)$, *and*

$$k_1 = k_2, \quad P(G/K_s) = (1 - t^{n-1})/(1 - t^{k/2}) \ (s = 1, 2) \qquad \textit{if } \chi \textit{ is odd},$$

$$P(G/K_{3-s}) = (1 + t^{k_s - 1})(1 - t^{n-1})/(1 - t^{k/2}) \ (s = 1, 2) \qquad \textit{if } \chi \textit{ is even}.$$

Proof. For a connected subgroup H of G with $H \nsim 0$ in G, we have $P(G/H) = \prod_{i=1}^m (1 + t^{u_i})$ for some odd integers u_i (cf. [12; Satz VI]). Thus the proposition follows immediately from (5.6) and Lemmas 5.7–5.9. *q.e.d.*

§6. The statement of the main result

Now we state our main result by the following classification theorem, where the cases (CI)–(CV) are the ones in Proposition 5.10, φ_1: $Spin(7) \to SO(7)$, φ_4: $SU(4) \to SO(6)$, φ_4: $F_4 \to SO(26)$ are the irreducible representations, and "$\sim_\ell$" denotes "locally isomorphic".

Theorem 6.1. *Let* (G, M) *be an effective action with* (AI) *and* (AII), *and consider* K_s *and* k_s *in* (3.2).

(CI) *The case that* k_1 *is odd* $\geqq 3$ *and* k_2 *is even* $\geqq 2$:

(e) *If* n *is even, then* $n = k_1 + k_2 - 1$, $M = S^n$ *and* (G, M) *is essentially isomorphic to one of the linear actions*

$$(Spin(7), S^{14}, \varphi_1 \oplus \Delta_7) \quad (k_1 = 7, k_2 = 8),$$

$$(Sp(l) \times S^3, S^n, (\nu_l \otimes \nu_1^*) \oplus S^2\nu_1) \quad (k_1 = 3, k_2 = 4l \geqq 4),$$

$$(U_1 \times U_2, S^n, \iota_1 \oplus \iota_2) \ ((U_s, k_s, \iota_s) \ (s = 1, 2) \textit{ are the ones in } (2.1)),$$

where the G*-action on* G/K_2 *is almost effective for the first two actions and is not for the last one.*

(o) *If* n *is odd, then* $n = 2k_1 + 2k_2 - 3$ *and* (G, M) *is so to one of the actions*

$(Spin(7) \times S^1, W^{15}(r))(k_1 = 7, k_2 = 2)$, $(SO(l+1) \times S^1, W^{2l+1}(r))(k_1 = l \geqq 3, k_2 = 2)$

given in Example 1.2, *where* $Z(G)^\circ \cap K_2 = Z_r$ $(r$: *odd* $\geqq 1)$, *and the linear actions*

$$(SU(5)\ (or\ U(5)),\ S^{19},\ (\Lambda^2\mu_5)_R)\ (k_1 = 5,\ k_2 = 6),$$
$$(Spin(10)\ (or\ Spin(10)\times S^1),\ S^{31},\ (\Delta_{10}^+)_R\ (or\ (\Delta_{10}^+ \otimes \mu_1)_R))\ (k_1 = 7,\ k_2 = 10),$$
$$(SU(l+1)\times S^3\ (or\ U(l+1)\times S^3),\ S^{4l+3},\ (\mu_{l+1}\otimes\mu_2)_R)\ (k_1 = 3,\ k_2 = 2l \geqq 4),$$
$$(Sp(l+1)\times Sp(2),\ S^{8l+7},\ (\nu_{l+1}\otimes\nu_2^*)_R)\ (k_1 = 5,\ k_2 = 4l \geqq 4).$$

(CII) *The case that* k_1, k_2 *are even* $\geqq 2$ *and* G/K_1, G/K_2 *are orientable*:

(e) *If* n *is even, then* $n=k_1=k_2$ *and* (G, M) *is essentially isomorphic to one of the linear actions* $(U, S^n, \iota\oplus\theta)$, *where* (U, n, ι) *is the one in* (2.1).

(o) *Let* V_s *be the maximum connected normal subgroup of* G *acting trivially on* G/K_s° $(s=1, 2)$, *and set* $V=V_1\times V_2$. *If* n *is odd, then* $n=k_1+k_2-1$ *and* (G, M) *is essentially isomorphic to one of the linear actions*

$$(Spin(8),\ S^{15},\ \Delta_8^+\oplus \Delta_8^-)\quad (k_1 = k_2 = 8;\ V = 1),$$
$$(SU(4),\ S^{13},\ \varphi_4 \oplus (\mu_4)_R)\quad (k_1 = 6,\ k_2 = 8;\ V = 1),$$
$$(SU(4) \times S^1,\ S^{13},\ \varphi_4 \oplus (\mu_4 \otimes \mu_1^*)_R)\quad (k_1 = 6,\ k_2 = 8;\ V = S^1),$$
$$(U_1 \times U_2,\ S^n,\ \iota_1 \oplus \iota_2)\ ((U_s,\ k_s,\ \iota_s)\ (s = 1, 2)\ \textit{are the ones in}\ (2.1);\ G \sim_{\ell} V),$$
$$(Sp(l_1) \times Sp(l_2) \times S^3,\ S^n,\ (\nu_{l_1} \otimes \nu_1^*) \oplus (\nu_{l_2} \otimes \nu_1^*))$$
$$(k_1 = 4l_1,\ k_2 = 4l_2;\ G \sim_{\ell} V \times S^3),$$
$$(Sp(l) \times S^3\ (or\ Sp(l) \times S^3 \times S^1),\ S^{4l+3},\ \nu_1(or\ \nu_1 \otimes \mu_1^*) \oplus (\nu_l \otimes \nu_1^*))$$
$$(k_1 = 4l \geqq 4,\ k_2 = 4;\ G \sim_{\ell} V \times S^3),$$
$$(Q_1 \times Q_2 \times S^1,\ S^n,\ (\iota_1 \otimes \mu_1^{*r_2}) \oplus (\iota_2 \otimes \mu_1^{*r_1}))$$
$$((Q_s,\ k_s,\ \iota_s) = (Sp(l_s),\ 4l_s,\ \nu_{l_s})\ or\ (SU(l_s),\ 2l_s,\ \mu_{l_s})\ (s = 1, 2);\ G \sim_{\ell} V \times S^1),$$
$$(Sp(l)\ (or\ SU(l)) \times S^1,\ S^n,\ \mu_1^{r_1} \oplus (\nu_l(or\ \mu_l) \otimes \mu_1^{*r_2}))$$
$$(k_1 = 2,\ k_2 = 4l\ (or\ 2l);\ G \sim_{\ell} V \times S^1),$$

where $Z(G)^\circ \cap K_s=Z_{r_s}$ $(s=1, 2)$ *for relatively prime integers* r_1 *and* r_2 *(with* $r_1\geqq r_2$ *if* $Q_1=Q_2$*).*

(CIII) *The case that* k_1, k_2 *are even,* G/K_1 *is orientable and* G/K_2 *is not so*: *Then,* $k_1=2$, $n=2k_2+1$, *and* (G, M) *is essentially isomorphic to one of the actions*

$$(SO(2l+1) \times S^1,\ W^{4l+1}(r))\quad (k_2 = 2l),\quad (G_2 \times S^1,\ W^{13}(r))\quad (k_2 = 6)$$

given in Example 1.2, *where* $Z(G)^\circ \cap K_1=Z_r$ $(r$: *odd*).

(CIV) *The case that* k_1, k_2 *are even, and* G/K_1, G/K_2 *are non-orientable*:

(e) *If* n *is even, then* $n=4$, $k_1=k_2=2$ *and* (G, M) *is so to the linear action*

$$(SO(3),\ S^4,\ S^2\rho_3 - \theta).$$

(o) *If n is odd, then $n=7$, $k_1=k_2=2$, $G=SO(4)$, and (G, M) is the action given in Example* 1.3.

(CV) *The case that k_1, k_2 are odd*:

Then, $\chi(G/K_1)=\chi(G/K_2)$ $(=\chi=1, 2, 3, 4$ *or* $6)$, $n-1=\chi(k_1+k_2-2)/2$, *and* (G, M) *is essentially isomorphic to one of the linear actions*

$$(U, S^n, \iota\oplus\theta) \quad (\chi=1, k_1=k_2=n, ((U, n, \iota) \text{ is the one in } (2.1))),$$

$$(SU(3), S^7, \mathrm{Ad}) \quad (\chi=3, k_1=k_2=3),$$

$$(Sp(3), S^{13}, \Lambda^2\nu_3-\theta) \quad (\chi=3, k_1=k_2=5),$$

$$(F_4, S^{25}, \varphi_4) \quad (\chi=3, k_1=k_2=9),$$

where G/K_s $(s=1, 2)$ *is a point*, $P_2(C)$, $P_2(H)$, $P_2(Cay)$, *respectively, and*

$$(SO(5), S^9, \mathrm{Ad})\ (\chi=4, k_1=k_2=3), \quad (G_2, S^{13}, \mathrm{Ad})\ (\chi=6, k_1=k_2=3),$$

$$(U_1\times U_2, S^{k_1+k_2-1}, \iota_1\oplus\iota_2)\ (\chi=2, ((U_s, k_s, \iota_s)\ (s=1, 2) \text{ are the ones in } (2.1))),$$

where $(G/K_1, G/K_2)=(P_3(C), SO(5)/SO(2)\times SO(3))$, $(G_2/U(2), G_2/U(2)')$ $(U(2)'$ *is the subgroup of* G_2 *which is isomorphic but not conjugate to* $U(2))$, (S^{k_2-1}, S^{k_1-1}), *respectively.*

We shall prove this theorem for the cases (CI)–(CV) separately in the following §§7–10.

§7. The case (CI)

In the rest of this paper, we shall classify almost effective actions with (AI) and (AII) up to essentially isomorphisms for convenience sake. Thus we assume that an action (G, M) satisfies (AI), (AII) and the following three conditions:

(BI) *The G-action on M is almost effective, i.e., K does not contain any positive dimensional normal subgroup of G* (cf. Lemma 3.1).

(BII) *The restricted $Z(G)^\circ$-action on M is effective* (cf. Lemma 3.1).

(BIII) *G is the direct product of some copies of simply connected simple groups and a toral group*, (since there is a finite covering $G^*\to G$ such that G^* satisfies (BIII)).

7.1. In the first half of this section, we prove the following (7.1.1–2) which gives necessary conditions for the case (CI).

(7.1.1) *For the case* (CI) (e), *we have the following table*:

n	k_1	k_2	G	K_1°	K_2	K°
(1) 14	7	8	$Spin(7)$	G_2	$Spin(6)$	$SU(3)$
(2) $4l+2\geqq 6$	3	$4l$	$Sp(l)\times S^3$	$Sp(l-1)\circ S^3$	$Sp(l)\times S^1$	$Sp(l-1)\circ S^1$
(3) k_1+k_2-1	k_1	k_2	$U_1\times U_2$	$U_1\times U_2'$	$U_1'\times U_2$	$U_1'\times U_2'$

Here, the G-action on G/K_1° is almost effective in (2), *and* $U_s/U_s'\approx S^{k_s-1}$ $(U_s'\subset U_s)$ *in* (3) $(s=1, 2)$.

(7.1.2) *For the case* (CI) (o), *let G' be a minimal connected normal subgroup of G such that the restricted G'-action on G/K° is transitive, i.e., the restricted G'-action on M satisfies* (AI). *Then we have $G=G'\circ H$ for an essentially direct product H of some copies of S^3 and a toral group, and the following table*:

n	k_1	k_2	G'	$(G'\cap K_1)^\circ\sim_{\mathfrak{L}}$	$G'\cap K_2\sim_{\mathfrak{L}}$	$(G'\cap K)^\circ\sim_{\mathfrak{L}}$
(1) 19	5	6	$SU(5)$	$Sp(2)$	$SU(3)\times S^3$	$S^3\times S^3$
(2) 23	5	8	$Spin(8)$	$Sp(2)$	$Sp(2)\times S^3$	$S^3\times S^3$
(3) 31	7	10	$Spin(10)$	$Spin(7)$	$SU(5)$	$SU(4)$
(4) 15	7	2	$S^1\times Spin(7)$	G_2	$S^1\times SU(3)$	$SU(3)$
(5) 11	3	4	$S^3\times SU(3)$	S^3	$S^3\times S^1$	S^1
(6) $2l+1\geqq 7$	l	2	$S^1\times Spin(l+1)$	$Spin(l)$	$S^1\times Spin(l-1)$	$Spin(l-1)$
(7) $4l+3\geqq 11$	3	$2l$	$SU(l+1)\times S^3$	$SU(l-1)\times S^3$	$SU(l)\times S^1$	$SU(l-1)\times S^1$
(8) 23	5	8	$Sp(2)\times Sp(3)$	$S^3\times Sp(2)$	$S^3\times Sp(2)\times S^3$	$S^3\times S^3\times S^3$
(9) $8l+7\geqq 15$	5	$4l$	$Sp(l+1)\times Sp(2)$	$Sp(l-1)\times Sp(2)$	$Sp(l)\times S^3\times S^3$	$Sp(l-1)\times S^3\times S^3$

Here, the normal subgroup of $(G'\cap K_1)^\circ$ locally isomorphic to S^3 (resp. $Sp(2)$) is contained in the normal subgroup $SU(3)$ (resp. $Sp(3)$) of G' in (5) (*resp. in* (8)), *but is not so in any simple normal subgroup of G' in* (7) (*resp. in* (9)). *Further, $G=G'$ in* (4), (6) *for $k_1\neq 3$, and* (9).

We prove (7.1.1–2) in the following subsections 7.2–3.

7.2 (Proof of (7.1.1)). It is known that a homogeneous space is a sphere if it is a Q-cohomology even sphere. Hence $G/K_2\approx S^{k_1-1}$ by Proposition 5.10 (CI) (e). Furthermore $K_1^\circ/K^\circ\approx S^{k_1-1}$ and $K_2/K^\circ\approx S^{k_2-1}$. Then by using (2.1) we see easily that (1) holds if G is simple and simply connected.

Let N be the maximum connected normal subgroup of G acting trivially on G/K_1°. Then, by (BIII) and Proposition 5.10 (CI) (e), we have

(7.2.1) $\quad G = U\times W\times N$ and $K_1^\circ = (U'\circ V)\times N$,

where U is simple $(k_2\geqq 4)$ or $S^1 (k_2=2)$ acting transitively on G/K_1°, $U'=(U\cap K_1^\circ)^\circ$

and $W \cong V$ with $r(W) \leqq 1$. Also U contains a subgroup locally isomorphic to $U' \times V$. (Cf. [8; Proof of Th.I].)

Since K_1°/K° is an even sphere, we see that there exists only one simple normal factor M_1 of K_1° acting non-trivially, hence transitively, on K_1°/K° by (2.1). Now we divide our proof into three cases;

$$\textbf{(a)}\quad M_1 \subset U', \quad \textbf{(b)}\quad M_1 = V \quad \text{and} \quad \textbf{(c)}\quad M_1 \subset N,$$

where we have $N=1$, 1 and M_1, respectively, by (BI).

Case (**a**). In this case, the simple group U acts transitively on G/K°, and hence $(U, U')=(Spin(7), G_2)$ by the first observation. We see that U does not contain any subgroup locally isomorphic to $U' \times S^1$ (cf. [3; p. 219] and [14; Th.II]). Then $V=1$ and G is simple. Thus we obtain (1).

Case (**b**). By (7.2.1) and Proposition 5.10 (CI) (e), we get

$$(*) \qquad P(U) = (1 + t^{k_2-1})P(U').$$

From $V=M_1$ and $r(V) \leqq 1$, we see that $V \cong W \cong S^3$, $k_1=3$ and $K^\circ = U' \circ V'$ $(S^1 \cong V' \subset V)$. Since U contains a subgroup locally isomorphic to $U' \times V$ (by (7.2.1)), we have $k_2 \equiv 0 \bmod 4$ by using $(*)$ and Hirsch's formula.

If $k_2=4$, then $(U, U')=(S^3, 1)$ by $(*)$, and we obtain (2) for $k_2=4$.

Suppose that $k_2 \geqq 8$. Then $r(U) \geqq 2$ by $(*)$. In $G=U \times W$, $W(\cong S^3)$ acts transitively on $G/K_2 \approx S^2$, and $K_2 = U \times W'$ $(S^1 \cong W' \subset W)$. Since $K_2/K^\circ \approx S^{k_2-1}$ $(k_2 \geqq 8)$, we see easily that $U(\subset K_2)$ acts transitively on K_2/K° with isotropy subgroup U'. Therefore $U/U' \approx S^{k_2-1}$, and U contains a subgroup locally isomorphic to $U' \times S^3$. By (2.1) we have $(U, U')=(Sp(l), Sp(l-1))$ and (2) for $k_2 \geqq 8$.

Case (**c**). By (7.2.1) and $M_1=N$, we get

$$G = U \times W \times N \supset K_1^\circ = (U' \circ V) \times N \supset K^\circ = (U' \circ V) \times N',$$

where $N/N' \approx S^{k_1-1} (N' \subset N)$.

If the $N(\subset G)$-action on G/K_2 is trivial, then $K_2 = Q \times N$ $(Q \subset U \times W)$, and hence $K_2/K^\circ \approx Q/(U' \circ V) \times S^{k_1-1}$. This is contrary to $K_2/K^\circ \approx S^{k_2-1}$. Therefore the $U \times W(\subset G)$-action on $G/K_2 \approx S^{k_1-1}$ is trivial, and $K_2 = U \times W \times N''$ for $N'' \subset N$. From $S^{k_2-1} \approx K_2/K^\circ = (U \times W)/(U' \circ V) \times (N''/N')$, it follows that $N''=N'$ and $(U \times W)/(U' \circ V) \approx S^{k_2-1}$. By setting $U_1=N$, $U_2=U \times W$, $U_1'=N'$ and $U_2' = U' \circ V$, we obtain (3).

This completes the proof of (7.1.1).

7.3 (Proof of (7.1.2)). To begin with we show the following

LEMMA 7.3.1. *If* $(k_1, k_2)=(3, 2)$, *then we obtain* (6) *of* (7.1.2) *for* $k_1=3$.

PROOF. By Proposition 5.10 (CI) (o) and the assumption, we get

$$(*) \qquad n = 7,\ K_1^\circ \nsim 0 \text{ in } G \quad \text{and} \quad P(G) = (1+t)(1+t^3)P(K_1^\circ).$$

This and $K_1^\circ/K^\circ \approx S^2$, $K_2/K^\circ \approx S^1$ imply $r(G)=r(K_2)+1$, $c(G)=c(K_2)-1$ and $\dim G/K_2=5$.

Let N be the maximum connected normal subgroup of G acting trivially on G/K_2. Then $G=U\times N$ and $K_2=H\times N$ $(H\subset U)$, where $N=1$ or S^1 by $K_2/K^\circ\approx S^1$ and (BI). By Proposition 2.7, the first observation implies that U is the direct product of some copies of S^3 and a toral group, and so is G.

Now, put

$$G = U_1 \times\cdots\times U_m \times T^l, \quad U_i \cong S^3 \ (1 \leqq i \leqq m).$$

Since $G\sim_{\mathfrak{g}} K_1^\circ\times S^3\times S^1$ and $K_1^\circ \nsim 0$ in G by $(*)$, we get
(a) $K_1^\circ=\{(u_1,\dots, u_m, 1, v)\in U_1\times\cdots\times U_m\times S^1\times T^{l-1}=G;\ u_1=u_2\}$, or
(b) $K_1^\circ=\{(g(v), u_2,\dots, u_m, 1, v)\in G;\ g(v)\in U_1\}$ for a homomorphism $g\colon T^{l-1}\to U_1$.
Then the $U_1\times S^1$-action on G/K_1° is transitive. Further, from $K_1^\circ/K^\circ\approx S^2$ and (BI), it follows that $m=2$ or 3 in (a), $m=2$ in (b), and the restricted $G'=U_1\times U_m\times S^1(\cong Spin(4)\times S^1)$-action on G/K° is transitive, as desired. *q.e.d.*

Let $N_s (s=1, 2)$ be the maximum connected normal subgroup of K_s° acting trivially on $K_s^\circ/K^\circ\approx S^{k_s-1}$. Then by (2.1) we have

$$\text{(7.3.2)} \quad K_1^\circ = N_1\circ M_1, \quad K_2 = N_2\circ M_2\circ J \quad \textit{and}$$
$$K^\circ = N_1\circ M_1' = N_2\circ M_2'\circ J' \quad (M_s' = (M_s \cap K^\circ)^\circ\ (s = 1, 2)),$$

where $J\cong J'$, $r(J)\leqq 1$ *and* M_s *is simple* $(k_s\geqq 3)$ *or* S^1 $(k_s=2)$ *acting transitively on* $K_s^\circ/K^\circ\approx S^{k_s-1}$. *Also here*, M_2' *is simple* $(k_2\geqq 6)$ *or trivial* $(k_2=2, 4)$.

In the rest of this subsection, we use the notations M_s, N_s, J and J' in the above sense.

One of the following three cases occurs in (7.3.2).

(7.3.3) (α) $k_2\geqq 6$ *and* $M_2'\subset N_1$ (*hence* $M_1'\subset N_2\circ J'$),
(β) $k_2\geqq 6$ *and* $M_2'\subset M_1'$ (*hence* $N_1\subset N_2\circ J'$),
(γ) $k_2=2$ *or* 4 (*hence* $M_2'=1$ *and* $N_1\circ M_1'=N_2\circ J'$).

In the case (β), M_1' contains the simple normal subgroup M_2'. Then (2.1) shows the following table:

(7.3.4) *The case* (β):

k_1	k_2	$M_1 \sim_\ell$	$M_2 \sim_\ell$	$M_1' \sim_\ell$	$M_2' \sim_\ell$
(i) 5	6	$Sp(2)$	$SU(3)$	$S^3 \times S^3$	S^3
(ii) 5	8	$Sp(2)$	$Sp(2)$	$S^3 \times S^3$	S^3
(iii) 7	8	G_2	$SU(4)$	$SU(3)$	$SU(3)$
(iv) 7	10	$Spin(7)$	$SU(5)$	$SU(4)$	$SU(4)$

LEMMA 7.3.5. *If G is simple, then we obtain* (1), (2) *and* (3) *of* (7.1.2).

PROOF. By Proposition 5.10 (CI) (o), we have

$$(*) \qquad P(G) = (1+t^{k_2-1})(1+t^{k_1+k_2-2})P(K_1^\circ).$$

This and the assumption show that $k_2 \geqq 6$ and K_1° is simple. Hence, $K_1^\circ = M_1$ and the case (β) of (7.3.3) occurs. By using (*) and (7.3.4), we have the lemma immediately. Here we note that any simple groups do not satisfy $P(G)=(1+t^3)\cdot(1+t^7)(1+t^{11})(1+t^{13})$ (cf., e.g., [8; Ch. V], [12; Kap. III]). *q.e.d.*

From now on, we assume that $(k_1, k_2) \neq (3, 2)$ and G is not simple, and prepare several lemmas.

The following result is due to H. C. Wang [15; (8.5)].

(7.3.6) *G, K_1°, K_2 and K° do not satisfy*

$$G = U_1 \circ U_2 \ (U_s \neq 1), \quad K_1^\circ = Q_1 \circ Q_2, \quad K_2 = R_1 \circ R_2 \quad and \quad K^\circ = P_1 \circ P_2$$

for $Q_s \cup R_s \subset U_s$, $P_s \subset Q_s \cap R_s$ $(s = 1, 2)$.

LEMMA 7.3.7 (cf. [15; (8.6)]). *The G-action on G/K_1° is almost effective if* $(k_1, k_2) \neq (3, 2)$.

PROOF. Suppose that the G-action on G/K_1° is not almost effective. Let $U_2(\neq 1)$ be the maximum connected normal subgroup of G acting trivially on G/K_1°. By (BI) and $K_1^\circ/K^\circ \approx S^{k_1-1}(k_1-1\colon \text{even})$, we get

$$(*) \quad G = U_1 \circ U_2, \quad K_1^\circ = U_1' \circ U_2 \quad \text{and} \quad K^\circ = U_1' \circ U_2' \quad \text{for} \quad U_s' \subset U_s \ (s = 1, 2),$$

where $M_1 = U_2$, $N_1 = U_1'$ and $M_1' = U_2'$ in (7.3.2). Then, by (7.3.6), the normal subgroup M_2 of K_2 in (7.3.2) satisfies

$$(**) \qquad M_2 \not\subset U_s \quad (s = 1, 2).$$

Now we derive a contradiction for each case of (α), (β), (γ) in (7.3.3).

In the case (α) or (β), $M_2'(\neq 1)$ is contained in $M_2 \cap U_s$ ($s=1$ or 2), which is a normal subgroup of the simple group M_2, and hence $M_2 \cap U_s = M_2$ ($s=1$ or 2). This is contrary to (**).

Consider the case (γ). In this case, $M_2 \cong S^3$ ($k_2=4$) or S^1 ($k_2=2$), and $J \cong J'=1$ if $M_2 \cong S^1$. Let g_s be the projection of G onto $\overline{U}_s = U_s/U_1 \cap U_2 \sim_{\mathfrak{l}} U_s$ ($s=1$ or 2). Thus (**) shows $g_s(M_2) \neq 1$ ($s=1, 2$).

Now, suppose that M_1' or M_2 is semi-simple and $N_2 \supset M_1'(=U_2')$. Then $M_1' \circ M_2 \subset K_2$ and $\operatorname{Ker}(g_2 | M_1' \circ M_2)$ is finite since $g_2(M_1') \neq 1$ and $g_2(M_2) \neq 1$. Thus $\overline{U}_2(\sim_{\mathfrak{l}} U_2 = M_1)$ contains a subgroup locally isomorphic to $M_1' \times M_2$, and this contradicts $r(M_1) = r(M_1')$. Therefore we have

(***) If M_1' or M_2 is semi-simple, then $M_1' \not\subset N_2$.

Next, suppose that $1 \neq J' \subset U_{3-s}'$($s=1$ or 2). Then $M_2 \cong S^3$ and $1 \neq J' \subset \operatorname{Ker} g_s \cap (M_2 \circ J)$, which is a normal subgroup of $M_2 \circ J$. From $J' \not\subset M_2$ and $J' \not\subset J$, it follows that $\operatorname{Ker} g_s \cap (M_2 \circ J) = M_2 \circ J$, and this contradicts $g_s(M_2) \neq 1$. Thus we have

(****) If $J' \neq 1$, then $J' \not\subset N_1$ $(= U_1')$ and $J' \not\subset M_1'$ $(= U_2')$.

By (2.1), M_1' is simple with $r(M_1') \geqq 2$ ($k_1 \geqq 7$) or locally isomorphic to $S^3 \times S^3$ ($k_1=5$). Then we have $M_1' \subset N_2$ or $S^3 \sim_{\mathfrak{l}} J' \subset M_1'$ in $K^\circ = N_1 \circ M_1' = N_2 \circ J'$ when $k_1 \geqq 5$. This contradicts (***) and (****).

Finally, suppose that $(k_1, k_2) = (3, 4)$. Then we get

$$U_2 = M_1 \sim_{\mathfrak{l}} S^3, \quad U_2' = M_1' \cong S^1 \quad \text{and} \quad M_2 \cong S^3.$$

Consider the normal subgroup $V = (U_2 \cap K_2)^\circ$ of $K_2 = N_2 \circ M_2 \circ J$. Here $J \cong J' \cong S^1$ since $S^1 \cong M_1' \not\subset N_2$ (by (***)) in $K^\circ = N_1 \circ M_1' = N_2 \circ J'$. Clearly we have $S^3 \sim_{\mathfrak{l}} U_2 \supset V \supset U_2' \cong S^1$. Hence $V = U_2$ or U_2'. If $V = U_2$, then $U_2 = M_2$ or $U_2 \subset N_2$, and this contradicts (**) and (***). If $V = U_2'$, then $U_2' \subset N_2 \circ J$, and $M_2 \subset Z(U_2', G) = U_1 \circ U_2'$. Thus $M_2 \subset U_1$, and this contradicts (**). Therefore the proof of the lemma is completed. *q.e.d.*

Let us set $G = U_1 \times \cdots \times U_m$ ($m \geqq 2$), where U_i ($1 \leqq i \leqq m$) is simple, and some U_i is a toral group if G is not semi-simple. Let ξ_i: $G \to U_i$ be the natural projection, and set

(7.3.8) $\Gamma_i = \xi_i(K_1^\circ)$, $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$, $L_i = (U_i \cap K_1)^\circ$ and $L = L_1 \times \cdots \times L_m$,

where $L \subset K_1^\circ \subset \Gamma \subset G$, $K_1^\circ \sim 0$ in Γ, and L_i is a normal subgroup of K_1°.

Then, by Lemma 7.3.7, we have

(7.3.9) $L_i \sim 0$ *in* U_i, *and* L_i *is simple or trivial* ($1 \leqq i \leqq m$).

Since L is a semi-simple normal subgroup of K_1°, there exists uniquely a connected normal subgroup V of K_1° such that $K_1^\circ = V \circ L$. Let us set $V = V_0 \circ V_1 \circ \cdots \circ V_l$, where V_0 is a toral group and V_j $(1 \leqq j \leqq l)$ is simple. Then we get

(7.3.10) $\Gamma_i = \xi_i(V) \circ \xi_i(L) = \xi_i(V_0) \circ \xi_i(V_1) \circ \cdots \circ \xi_i(V_l) \circ L_i$, *where* $\xi_i(V_j) = 1$ *or* $\sim_{\ell} V_j$ *and at least two of* $\xi_i(V_j)$ *are non-trivial for each* $1 \leqq j \leqq l$.

LEMMA 7.3.11. *Γ contains a normal subgroup locally isomorphic to* $V \times V \times L$.

PROOF. From (7.3.10), it follows immediately that Γ contains a normal subgroup locally isomorphic to $(V/V_0) \times (V/V_0) \times L$. Hence the lemma holds if V is semi-simple.

Suppose that V is not semi-simple $(V_0 \neq 1)$. Since $K_1^\circ \not\sim 0$ in G, we may assume that U_1 is a toral group, and $\mathrm{Ker}(\xi_1 \mid V_0)$ is finite. Thus $\dim \Gamma_1 = \dim V_0 = r$. Since $L_1 = 1$ by (7.3.9), we see easily that $\mathrm{Ker}(\xi_2 \times \cdots \times \xi_m \mid V_0)$ is also finite. Then the center of $\Gamma' = \Gamma/\Gamma_1$ is of dimension $c(\Gamma') \geqq r$. Therefore $c(\Gamma) \geqq 2r$, and hence we have the lemma if V is not semi-simple. *q.e.d.*

Since $r(G) = r(K_1^\circ) + 2$ by Proposition 5.10 (CI) (o), we shall divide our proof into three cases;

(**a**) $r(\Gamma) = r(K_1^\circ)$, (**b**) $r(\Gamma) = r(K_1^\circ) + 1$ and (**c**) $r(\Gamma) = r(K_1^\circ) + 2 (= r(G))$.

Case (**a**). By the assumption and Lemma 7.3.11, we get $V = 1$ and $K_1^\circ = L$. Then Lemma 7.3.7 and Proposition 5.10 (CI) (o) imply $m = 2$ and $L_i \subsetneqq U_i$ $(i = 1, 2)$. By (7.3.9), we may assume that $L_1 = M_1$ and $L_2 = N_1$ in (7.3.2). Thus we get

(7.3.12) $G = U_1 \times U_2$, $K_1^\circ = M_1 \times N_1$, $K^\circ = M_1' \times N_1$

$(M_1 \subsetneqq U_1,\ N_1 \subsetneqq U_2)$ *and* $P(G) = (1 + t^{k_2-1})(1 + t^k)P(K_1^\circ)$ $(k = k_1 + k_2 - 2)$.

LEMMA 7.3.13. *In the case* (**a**), *we obtain* (4), (5) *and* (6) *of* (7.1.2) *with* $H = 1$.

PROOF. By (7.3.6), we have $M_2 \ntriangleleft U_s (s = 1, 2)$. Thus, by [15; (9.1)] and $r(G) = r(K_2) + 1$, we see easily that $r(M_2) = 1$, and so $k_2 = 4$ $(M_2 \cong S^3)$ or 2 $(M_2 \cong S^1)$. Then, by (7.3.9) and (7.3.12), we have

$N_1 = 1$, $K_1^\circ = M_1 \subset U_1$, $K^\circ = M_1'$ and $P(U_1) = (1 + t^k)P(M_1)$, $P(U_2) = 1 + t^{k_2-1}$,

where $M_1 \sim_{\ell} SO(k_1)$ or G_2 $(k_1 = 7)$.

Suppose that $M_1 \sim_{\ell} G_2$ $(k_1 = 7)$. Then the above result implies $k_2 = 2$, $U_1 = Spin(7)$, $U_2 = S^1$, and we obtain (4).

Next, suppose that $M_1 \sim_{\ell} SO(k_1)$. Then $P(U_1) = (1 + t^3)(1 + t^7) \cdots (1 + t^{2k_1-3})(1 + t^k)$, and this shows that $k_2 = 4$, $U_1 = SU(3)$ if $k_1 = 3$, and $k_2 = 2$,

$U_1=Spin(k_1+1)$ if $k_1\geqq 5$. Therefore we obtain easily (5) and (6). *q.e.d.*

Case (**b**). By (7.3.10), Lemma 7.3.11 and the assumption, we get, for $K_1^\circ=V\circ L$,

$r(V)=1,\ \xi_j(V)\neq 1,\ \Gamma_j\sim_\ell V\times L_j\ (j=1,2)$ and $\xi_i(V)=1,\ \Gamma_i=L_i\,(3\leqq i\leqq m)$.

Then $G/K_1^\circ=(U_1\times U_2/V\circ L_1\circ L_2)\times U_3/L_3\times\cdots\times U_m/L_m$, and $m\leqq 3$ by Proposition 5.10 (CI) (o) and Lemma 7.3.7.

LEMMA 7.3.14. *In the case* (**b**), $m=2$ *or* 3. *If* $m=3$, *then we obtain* (5) *of* (7.1.2) *with* $r(H)=1$.

PROOF. By Proposition 5.10 (CI) (o) and the assumption, we see that $U_1\times U_2/V\circ L_1\circ L_2$ and U_3/L_3 are Q-cohomology spheres and one of their dimensions is k_2-1 and the other is $k=k_1+k_2-2$. Thus we may assume that $L_2=1$, $V\cong U_2$ and the U_1-action on $U_1\times U_2/V\circ L_1$ is transitive (cf. [8; Proof of Th. I]).

Now we show that V does not act transitively on $K_1^\circ/K^\circ\approx S^{k_1-1}$. To see this, assume the contrary. Then $V\cong U_2\cong S^3$, $k_1=3$, $k_2\geqq 4$ and $K^\circ=V'\circ L$ ($S^1\cong V'\subset V$). From (7.3.6) it follows that $M_2\not\subset U_1\times U_2$, $M_2\not\subset U_3$, and so $J'\not\subset U_1\times U_2$, $J'\not\subset U_3$ if $J'\neq 1$ (in (7.3.2)). Therefore, in $K^\circ=V'\circ L=N_2\circ M_2'\circ J'$, we see that $M_2'\circ J'=1$, and hence $M_2\cong S^3$, $k_2=4$ and $K_2=K^\circ\circ M_2$. Then $U_1\supset\Gamma_1\sim_\ell V\times L_1$ and $U_3\supset\xi_3(K_2)\sim_\ell M_2\times L_3$. By considering the Poincaré polynomials of U_1/Γ_1 and $U_3/\xi_3(K_2)$, Hirsch's formula shows $k_1+k_2-1\equiv 0 \bmod 4$. This leads a contradiction.

From this observation, the L-action on K_1°/K° is transitive. Hence the restricted $G'=U_1\times U_3$-action on G/K° is also transitive with $(G'\cap K_1)^\circ=L$, (this is the case (**a**)). Thus the lemma follows from Lemma 7.3.13, since (4), (6) do not occur by the condition $U_1\supset\Gamma_1\sim_\ell V\times L_1$. *q.e.d.*

LEMMA 7.3.15. *If* (**b**) *holds and* $m=2$, *then we obtain* (1), (2), (3) *with* $r(H)=1$, *and* (7), (8) *with* $H=1$ *of* (7.1.2).

PROOF. First, we recall that

$$(*)\qquad P(U_1/L_1)P(U_2/L_2)=(1+t^{k_2-1})(1+t^k)P(V)\quad(k=k_1+k_2-2\geqq 5)$$

by Proposition 5.10 (CI) (o). Thus we may set $r(U_1)=r(\Gamma_1)+1$ and $r(U_2)=r(\Gamma_2)$ since $r(G)=r(\Gamma)+1$.

(I) Suppose that U_1 is a toral group. Then we have $U_1\cong T^2$, $V\cong S^1$, $L_1=1$, $L_2=M_1$ by (7.3.9) and the above assumption. Moreover (*) implies $k_2=2$, $k_1\geqq 5$, and hence U_2 must satisfy

$$P(U_2)=(1+t^{k_1})P(L_2),\quad U_2\supset\Gamma_2\sim_\ell V\times L_2,$$

where $L_2=M_1\sim_\ell SO(k_1)$ or G_2 $(k_1=7)$. But any simple groups do not satisfy this condition (cf. [3], [14]). Therefore U_1 is not a toral group.

(II) Assume that U_2 is a toral group. Then $U_2\cong S^1$, $V\cong S^1$, $L_2=1$ and $L_1=M_1$. Hence we see easily that U_1 acts transitively on G/K_1° and so on G/K°. By Lemma 7.3.5, we obtain (1), (2), (3) with $H\cong S^1$.

(III) Finally, assume that $G=U_1\times U_2$ is semi-simple. Then (*) shows that $K_1^\circ=V\circ L$ is semi-simple and it has at most two simple factors. Furthermore we see that $V\sim_\ell S^3$ and L is simple $(k_2\geqq 6)$ or trivial $(k_2=4)$.

(i) If $L=1$ $(k_2=4)$, then $K_1^\circ=V$ $(k_1=3)$, $r(U_1)=2$, $r(U_2)=1$, and we obtain (7) with $H=1$, $k_2=4$ by (*).

(ii) Suppose $L=L_1$ $(k_2\geqq 6)$. Then $V\cong U_2\cong S^3$, and the U_1-action on G/K_1° is transitive. In $K_1^\circ=V\circ L_1=N_1\circ M_1$, we have $M_1=L_1$ or V $(k_1=3)$.

If $M_1=L_1$, then the U_1-action on G/K° is also transitive. Thus, by Lemma 7.3.5, we obtain (1), (2) and (3) with $H\cong S^3$.

If $M_1=V$ $(k_1=3)$, then $K^\circ=N_2\circ M_2'\circ J'=V'\circ L_1$ $(S^1\cong V'\subset V)$, where M_2' is simple by $k_2\geqq 6$. Then $M_2'=L_1$ and U_1 satisfies $P(U_1)=(1+t^{k_2-1})(1+t^{k_2+1})\cdot P(M_2')$ by (*). Thus, by (2.1), we have $U_1=SU(l+1)$ and $M_2'=SU(l-1)$ $(k_2=2l)$, and this is the case (7) with $H=1$.

(iii) Suppose $L=L_2$ $(k_2\geqq 6)$. Then $r(U_1)=2$, $r(U_2)=r(L_2)+1$, and (*) shows that for $\{l_1, l_2\}=\{k_2-1, k\}$ $(k=k_1+k_2-2)$, $P(U_1)=(1+t^3)(1+t^{l_1})$ and U_2/L_2 is a Q-cohomology l_2-sphere, where $l_1=5$, 7 or 11 according to $U_1=SU(3)$, $Sp(2)$ or G_2. Further, by considering the Poincaré polynomial of U_2/Γ_2 $(\Gamma_2\sim_\ell V\times L_2)$, we see that $l_2+1\equiv 0 \bmod 4$. By [15; (9.1)], M_2 is contained in U_2 since $r(G)=r(K_2)+1$, $r(M_2)\geqq 2$ $(k_2\geqq 6)$ and $L_1=1$. In $K_1^\circ=V\circ L_2=N_1\circ M_1$, we have $M_1=V$ $(k_1=3)$ or L_2.

If $M_1=V$ $(k_1=3)$, then $K^\circ=N_2\circ M_2'\circ J'=V'\circ L_2$ $(S^1\cong V'\subset V)$ and $L_2=M_2'$. Thus, by considering the Poincaré polynomial of U_2/M_2, we have $l_2+1\equiv 0 \bmod k_2$. On the other hand, there is no integer $k_2\geqq 0$ such that $\{l_1, l_2\}=\{k_2-1, k_2+1\}$, $l_2+1\equiv 0 \bmod \mathrm{lcm}(4, k_2)$, and $l_1=5$, 7 or 11. This leads a contradiction.

If $M_1=L_2$, then $M_2'\subset(U_2\cap K)^\circ=M_1'$, and the case ($\beta$) of (7.3.3) occurs. From $\{l_1, l_2\}=\{k_2-1, k\}$ and $l_2+1\equiv 0 \bmod 4$, only the case (ii) of (7.3.4) occurs. Thus. by (*) and $U_2\supset\Gamma_2\sim_\ell V\times L_2$, we obtain easily (8) with $H=1$. *q.e.d.*

Case (**c**). By the assumption and Lemma 7.3.11, we have $r(V)=1$ or 2 in $K_1^\circ=V\circ L$. Since $r(U_i)=r(\Gamma_i)$ and $L_i\not\sim 0$ in U_i $(1\leqq i\leqq m)$, Lemma 7.3.7 implies

(7.3.16) $\xi_i(V)\neq 1$ *for the projection* $\xi_i\colon G\to U_i$ $(1\leqq i\leqq m)$.

LEMMA 7.3.17. *In the case* (**c**), *we obtain*

(I) (9) *of* (7.1.2) *with* $H=1$, *if* V *is simple with* $r(V)=2$,

(II) (1), (2), (3), (5), (7) *and* (8) *of* (7.1.2), *otherwise.*

PROOF. (I) From the assumption and (7.3.16), it follows that $m=2$ and $G=U_1\times U_2$ is semi-simple. We recall that

$$(*) \qquad P(G) = (1+t^{k_2-1})(1+t^k)P(V)P(L) \quad (k = k_1+k_2-2 \geqq 5)$$

by Proposition 5.10 (CI) (o). This shows that $L=L_1\times L_2$ is simple ($k_2\geqq 6$) or trivial ($k_2=4$). Put $L_1=1$. Then $K_1^\circ=V\circ L_2$ and

$$(**) \qquad U_1 \subset \Gamma_1 \sim_Q V, \quad U_2 \supset \Gamma_2 \sim_Q V\times L_2,$$

where $r(U_i)=r(\Gamma_i)$ ($i=1, 2$) and V is simple with $r(V)=2$. By [3], this implies

$$(***) \qquad U_2 \neq SU(l),\ Spin(2l), \quad \text{and} \quad V\neq G_2.$$

First we show $U_1=\Gamma_1$. In fact, if $U_1\neq\Gamma_1$, then it is known that $U_1=G_2$ and $\Gamma_1=SU(3)$ (cf. [3]). Thus $H^5(U_2; Q)\neq 0$ by $(*)$, and hence $U_2=SU(l)$. This contradicts $(***)$.

Therefore the U_2-action on G/K_1° is transitive, and

$$(*)' \qquad P(U_2) = (1+t^{k_2-1})(1+t^k)P(L_2) \qquad (\text{by } (*)).$$

By $(***)$ and Lemma 7.3.5, we see that the U_2-action on G/K° is not transitive. Hence we have $V=M_1=Sp(2)$, $k_1=5$ and $U_1=Sp(2)$.

If $k_2=4$, then $(*)'$ shows that $L_2=1$ and $U_2=Sp(2)$. This is the case (9) with $H=1$.

Suppose that $k_2\geqq 6$. Then, by the same method as that in the proof (III)-(iii) of Lemma 7.3.15, we have $M_2\subset U_2$ and $M_2'=(U_2\cap K)^\circ=L_2$. By using (2.1) and [3], we see easily that the triple $(U_2, M_2, L_2=M_2')$ satisfying $(*)'$ and $(**)$ for $V=Sp(2)$ is given by $(Sp(l+1), Sp(l), Sp(l-1))$ ($k_2=4l$).

Thus we obtain (9) with $H=1$.

(II) It is sufficient to show that there exists a connected semi-simple normal subgroup G' of G such that the restricted G'-action on M satisfies (AI), the condition of the case (**a**) or (**b**), and $G/G'\sim_Q S^3$, S^1 or T^2.

(i) Suppose that U_1 is a toral group. Then, by (7.3.9) and the assumption, $L_1=1$ and $U_1=\xi_1(V)\cong S^1$ or T^2. Thus the semi-simple normal subgroup $G'=U_2\times\cdots\times U_m$ of G acts transitively on G/K_1°, and hence so on G/K° since $M_1\subset G'$.

(ii) Suppose that G is semi-simple. Since $K_1^\circ\not\sim 0$ in G, we see that K_1° is semi-simple, and $V\sim_Q S^3$ or $S^3\times S^3$. Then $\Gamma\sim_Q S^3\times S^3\times K_1^\circ$ by $r(\Gamma)=r(K_1^\circ)+2$. By Proposition 5.10 (CI) (o) and Hirsch's formula, we get $P(G/\Gamma)=(1-t^{k_2})(1-t^{k+1})/(1-t^4)^2$, and this shows k_2, $k+1\equiv 0 \bmod 4$. Thus $k_1\equiv 1 \bmod 4$, and hence L acts transitively on $K_1^\circ/K^\circ\approx S^{k_1-1}$ (i.e., $M_1\subset L$). Now consider

$$(*)'' \qquad \prod_{i=1}^m P(U_i/L_i) = (1+t^{k_2-1})(1+t^k)P(V) \quad (\text{by } (*)),$$

where $P(V)=(1+t^3)^j$ ($j=1, 2$), $L_i\subsetneqq U_i$, and $L_i\not\sim 0$ in U_i (cf. (7.3.9)).

If $m=2$, then $V\sim_{\mathbf{Q}} S^3\times S^3$ by $r(\Gamma)=r(K_1^\circ)+2$, and $(*)''$ implies $L_i=1$ $(i=1, 2)$. This contradicts $M_1\subset L=L_1\times L_2$. Therefore $m\geqq 3$ and $(*)''$ shows that $P(U_i/L_i)=1+t^3$ for some i, say $i=1$. Hence $L_1=1$ and $S^3\cong U_1=\xi_1(V)$ by (7.3.16). Then the normal subgroup $G'=U_2\times\cdots\times U_m$ acts transitively on G/K_1°, and so on G/K° since $M_1\subset L\subset G'$.

Clearly, for each case, the restricted G'-action on M satisfies the condition of the case (**a**) or (**b**). *q.e.d.*

By Lemmas 7.3.1, 7.3.5, 7.3.13–15 and 7.3.17, the proof of (7.1.2) is completed.

7.4 (Proof of Theorem 6.1 (CI)). In the last half of this section, we prove Theorem 6.1 (CI) by studying the existence and uniqueness of actions with (AI), (AII) and (7.1.1–2).

For this purpose, we consider the following assertions, where $[G, M]$ denotes the essential isomorphism class of (G, M), and $[G]$ denotes the local isomorphism class of G:

(R_0) $[G, M]$ *is determined by* $[G]$.

(R_s) $Z(G)^\circ\cap K_s\cong Z_{r_s}$ *and* $[G, M]$ *is determined by* $[G]$ *and* r_s $(s=1$ *or* $2)$.

(R_3) $Z(G)^\circ\cap K_s\cong Z_{r_s}$ $(s=1, 2)$ *and* $[G, M]$ *is determined by* $[G]$ *and* r_1, r_2.

Then we can show Theorem 6.1 (CI) by proving the following

Proposition 7.4.1. (I) *For the case* (CI) (e), (R_0) *holds.*

(II) *For the case* (CI) (o):

(i) $G=G'$ *and* (R_0) *holds in* (9) (*of* (7.1.2)).

(ii) $G=G'$ *or* $G'\times S^1$, *and* (R_0) *holds in* (1), (3), (7).

(iii) $G=G'$ *and* (R_2) *holds in* (4), (6).

(iv) (G, M) *with* (AI), (AII) *does not occur in* (2), (5), (8).

In fact, we can study the isotropy subgroups of the actions given in Theorem 6.1 (CI) by routine calculations, and we see that these actions realize the desired unique actions due to (7.1.1–2) and Proposition 7.4.1. Thus Theorem 6.1 (CI) holds.

We prove Proposition 7.4.1 in the following subsections §§ 7.5–15. In the proof of Proposition 7.4.1 for (CI) (o), we use G, K_s, K, $\tilde{G}=G\circ H$, $\tilde{K}_s$ and $\tilde{K}$ in place of G', $G'\cap K_s$, $G'\cap K$, G, K_s and K, respectively.

7.5 (Proof of Proposition 7.4.1 for (1), (2) in (7.1.1)). First, in the case (1), we note that a subgroup G_2 is unique up to conjugation in $Spin(7)$ by using Lemma 2.5 and the universal covering π: $Spin(7)\to SO(7)$.

Set $G=Spin(7)$ in (1), $=Sp(l)\times S^3$ in (2). By Lemma 5.4 all the isotropy subgroups are connected. Then we see easily that K_s (resp. K) is unique up to conjugation in G (resp. K_s) ($s=1, 2$), (except for the case $k_2=4$). Thus we may set

(1) $K_1 = Spin(7) \cap SO(7) = G_2$, $K_2 = Spin(6) = \pi^{-1}(SO(6))$ and

$K = Spin(7) \cap SO(6) = SU(3)\ (= K_1 \cap K_2)$,

where $G=Spin(7)$ is naturally imbedded in $SO(8)$,

(2) $K_1 = \left\{\left(\begin{pmatrix} p & 0 \\ 0 & X\end{pmatrix}, p\right) \in Sp(l)\times S^3 = G;\ p\in S^3 \subset H\right\}$,

$K_2 = \{(Y, z)\in G;\ z\in S^1 \subset C\}$ and

$K = \left\{\left(\begin{pmatrix} z & 0 \\ 0 & X\end{pmatrix}, z\right) \in G;\ z\in S^1 \subset C\right\}(= K_1 \cap K_2)$.

By easy calculation, we see that $N(K, G)/K$ has two components and $\alpha_0 K$ is not in the identity component for (1) $\alpha_0 \in SO(8)$, the diagonal matrix with the diagonal elements $1, -1,\dots, 1, -1$, and (2) $\alpha_0=\left(\begin{pmatrix} j & 0 \\ 0 & E\end{pmatrix}, j\right)$. Since α_0 is in K_1 and $\alpha_0^2\in K$, we see that $\beta=\alpha_0$ satisfies the condition (2) of Lemma 3.8 for $s=1$. Hence, by Lemma 3.7 (2), we get $M(1)\approx M(\alpha_0)$ in (3.3). Thus the assertion (R_0) holds for $G=Spin(7)$ and $Sp(l)\times S^3$.

7.6 (Proof of Proposition 7.4.1 for (3) in (7.1.1)). Now, we may assume that the G-action on M (hence on G/K) is effective by (BII). Thus the U_s-action on $U_s/U'_s\approx S^{k_s-1}$ is also effective for $K=U'_1\times U'_2\subset G=U_1\times U_2$, and such (U_s, U'_s) is the pair in (2.1) ($s=1, 2$). This implies that U'_s is connected, and K_1, K are also so.

It is clear that the connected subgroups K_1 and K_2 are unique up to automorphisms of G, and $K=K_1\cap K_2$. Clearly we get $N(K, G)/K=NU'_1/U'_1\times NU'_2/U'_2$ ($NU'_s=N(U'_s, U_s)$), where $NU'_1/U'_1\cong Z_2$ and $NU'_2/U'_2\cong Z_2$, S^1 or S^3 by Lemma 2.2. Here we choose an element $a_s\in NU'_s$ with $a_s^2=1$, $a_s\notin U'_s$ if $NU'_s/U'_s\cong Z_2$, and $a_2=1$ if $NU'_2/U'_2\cong S^1$ or S^3. Set $\alpha_1=(a_1, 1)$, $\alpha_2=(1, a_2)$ and $\alpha_3=(a_1, a_2)$. Since α_s ($s=1, 2$) is in K_s and of order two, we see that $\beta=\alpha_s$ satisfies Lemma 3.8 (2). Then, by Lemma 3.7 (2), we have $M(1)\approx M(\alpha_s)$ for $s=1, 2$. Also, by $\alpha_1=\alpha_2\alpha_3{}^{-1}$, we have $M(\alpha_2)\approx M(\alpha_3)$. Thus (R_0) holds for $G=U_1\times U_2$.

7.7 (Proof of Proposition 7.4.1 for (1) in (7.1.2)). All isotropy subgroups are connected by Lemma 5.4. Set $G=SU(5)$. By considering the representations $Sp(2)\to SU(5)$, there are, up to conjugation, just two connected subgroups $Sp(2)$ and $SO(5)$ of $SU(5)$ locally isomorphic to $Sp(2)$ by Lemma 2.5. Thus $K_1=Sp(2)$ or $SO(5)$. On the other hand, from $K_2/K\approx S^5$ it follows that $K(\sim_{\ell} S^3\times S^3)$

contains a normal subgroup $N_2(\sim_\ell S^3)$ of $K_2(\sim_\ell SU(3)\times S^3)$, and hence $Z(N_2, G)^\circ \supset SU(3)$ $(\subset K_2)$. This implies $K_1=Sp(2)$, and that K_2 is unique up to conjugation in G. Thus we may set

$$K_1=\left\{\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}\in SU(5);\ X\in Sp(2)\subset SU(4)\right\},$$

$$K=\left\{\begin{pmatrix}1 & & 0\\ & X_1 & \\ 0 & & X_2\end{pmatrix}\in SU(5);\ X_1,\ X_2\in SU(2)\right\}\quad\text{and}$$

$$K_2=\left\{\begin{pmatrix}X & 0\\ 0 & Y\end{pmatrix}\in SU(5);\ X\in SU(3),\ Y\in SU(2)\right\}.$$

By easy calculation, we get

(a) NK/K $(NK=N(K, G))$ has two components, and we can choose $\alpha_0\in NK-(NK)^\circ$ with $\alpha_0^2=1$ and $\alpha_0\in K_1$,

(b) $L=NK\cap NK_1\cap NK_2/K\cong S^1$ and $Z(L, NK/K)=NK/K$.

By the same method as that of § 7.5, (R_0) holds for $G=SU(5)$.

Next, we consider the extension of this $SU(5)$-action to $\tilde{G}=SU(5)\times H$. From (b) and Lemma 4.5, we see that $H=S^1$ and ϕ of (4.6) is unique up to the diagram in (4.8) since $Z(G)^\circ=H$ acts effectively on M. Then the isotropy subgroups $(\tilde{K}, \tilde{K}_1, \tilde{K}_2)$ are unique up to automorphisms of $\tilde{G}$ by Lemma 4.7. By (b) and Lemmas 4.5 and 4.9, $N\tilde{K}/\tilde{K}$ $(N\tilde{K}=N(\tilde{K}, \tilde{G}))$ has two components, and $\tilde{\alpha}_0\in N\tilde{K}-(N\tilde{K})^\circ$, $\tilde{\alpha}_0^2=1$ and $\tilde{\alpha}_0\in\tilde{K}_1$ for $\tilde{\alpha}_0=(\alpha_0, 1)$. Therefore (R_0) also holds for $\tilde{G}=SU(5)\times S^1$.

7.8 (Proof of Proposition 7.4.1 for (2) in (7.1.2)). Set $G=Spin(8)$, and consider the commutative diagram

$$\begin{array}{llllll}
g_1: Sp(2) & \xrightarrow{\pi_1} & K_1 & & & \\
\quad\cup & & \cup\ \searrow & & & \\
S^3\times S^3 & \longrightarrow & K & \longrightarrow G=Spin(8) & \xrightarrow{\pi} SO(8) & \longrightarrow SU(8), \\
\quad\cap & & \cap\ \nearrow & & & \\
g_2: Sp(2)\times S^3 & \xrightarrow{\pi_2} & K_2 & & &
\end{array}$$

where K, K_1, K_2 are connected by Lemma 5.4, and π, π_s $(s=1, 2)$ are the universal coverings. From $K_1/K\approx S^4$ and $K_2/K\approx S^7$, we see that $S^3\times S^3$ in $Sp(2)$ is unique up to conjugation, and $S^3\times S^3$ in $Sp(2)\times S^3$ is given by

$$\text{(a)}\quad\left\{\left(\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}, q\right)\in Sp(2)\times S^3\right\}\quad\text{or}\quad\text{(b)}\ \left\{\left(\begin{pmatrix}p & 0\\ 0 & q\end{pmatrix}, q\right)\in Sp(2)\times S^3\right\}.$$

Denote by $\chi(g_s)$ the character of the representation g_s $(s=1,2)$. For the case (a), $\chi(g_1)=\chi(g_2)$ on $S^3\times S^3$ implies $g_1=(\nu_2)_C\oplus\theta$ and $g_2=(\nu_2)_C\oplus\mu_2\oplus\theta$, which are not the complexification of real representations. For the case (b), $\chi(g_1)\neq\chi(g_2)$ on $S^3\times S^3$ for any representations g_1 and g_2. Thus this case does not occur.

7.9 (PROOF OF PROPOSITION 7.4.1 FOR (3) IN (7.1.2)). Set $G=Spin(10)$. Then, from $K_s/K\approx S^{k_s-1}$, we see easily that $K_2=SU(5)$, $K=SU(4)=Spin(6)$, $K_1=Spin(7)$, and that K is unique up to conjugation in K_s $(s=1, 2)$. Consider the commutative diagram

$$\begin{array}{ccccc} g_1: Spin(7) = K_1 & & & & \\ \cup & \searrow & & & \\ SU(4) = K & \longrightarrow & G = Spin(10) & \xrightarrow{\pi} & SO(10), \\ \cap & \nearrow & & & \\ g_2: SU(5) = K_2 & & & & \end{array}$$

where π is the universal covering. By the similar argument to that in § 7.8, we see that the representations g_1 and g_2 are equivalent to $\Delta_7\oplus\theta_2$ and $(\mu_5)_R$, respectively. By Lemma 2.5, $\pi(K_2)$ is conjugate to $SU(5)\subset SO(10)$, and $\pi(K)$ is so to $SU(4)$ $(\subset SO(8))\subset SO(10)$. Since $\pi(K_1)/\pi(K)\approx S^6$, the center of $\pi(K_1)$ contains an element γ with $\gamma^2=1$ and $\gamma\in\pi(K)$. Hence $\pi(K_1)$ is in $SO(2)\times SO(8)$, and $\pi(K_1)$ is conjugate to $Spin(7)$ $(\subset SO(8))$. Therefore it follows that $K=\pi^{-1}(\pi K)^\circ$ and $K_s=\pi^{-1}(\pi K_s)^\circ$ $(s=1, 2)$ are unique up to conjugation in G.

The followings are seen by easy calculation:

(a) NK/K has two components, and we can choose $\alpha_0\in NK-(NK)^\circ$ with $\alpha_0^2=1$ and $\alpha_0\in K_1$.

(b) $L=NK\cap NK_1\cap NK_2/K\cong S^1$ and $Z(L, NK/K)=NK/K$.

Therefore the same discussion as that in § 7.7 shows that $\tilde{G}=Spin(10)$ or $Spin(10)\times S^1$, and (R_0) holds for these groups.

7.10 (PROOF OF PROPOSITION 7.4.1 FOR (4) IN (7.1.2)). By § 7.5, we may set

$$G = S^1 \times Spin(7) \subset S^1 \times SO(8), \quad K_1^\circ = Spin(7) \cap SO(7) = G_2 \quad \text{and}$$
$$K^\circ = Spin(7) \cap SO(6) = SU(3).$$

Since K° is a normal subgroup of $K_2(\sim_\ell S^1\times SU(3))$, we get

$$K_2 = \left\{\left(X^m, \begin{pmatrix} X^r & 0 \\ 0 & Y \end{pmatrix}\right)\in G \subset S^1\times SO(8);\ X\in SO(2) = S^1 \right\}$$

for some relatively prime non-negative integers r and m. Here $Z(G)^\circ\cap K_2\cong Z_r$, and we have $m=1$ because G/K_2 is simply connected by Lemma 5.4. If K and

K_1 are connected, then $f_1^*(H^7(G/K_1; Z_2)) \cap f_2^*(H^7(G/K_2; Z_2)) \neq 0$, and we see easily that M is not a Z_2-cohomology sphere by (5.5). Thus, by Lemma 5.4, K and K_1 are not connected, and $K = \bigcup b^i K^\circ$, $K_1 = \bigcup b^i K_1^\circ$ for some $b \in K_2 \cap K$. By using (BII) and $b \in N(K_1^\circ, G) = S^1 \times N(G_2, Spin(7))$, we get

(a) r is odd, and $K = K^\circ \cup bK^\circ$, $K_1 = K_1^\circ \cup bK_1^\circ$ for $b = (-1, -E)$,

(b) NK/K has two components, and $(1, A)K$ is not in the identity component for the diagonal matrix A with the diagonal elements $1, -1, \dots, 1, -1$,

Now the slice representation $\sigma_1 : K_1 \to O(7)$ in (3.3) is unique up to equivalence by Lemma 2.4. Therefore, by Lemmas 3.7 and 3.8, the assertion (R_2) holds for $G = S^1 \times Spin(7)$, as desired.

7.11 (Proof of Proposition 7.4.1 for (5) in (7.1.2)). Set $G = S^3 \times SU(3)$. Since $K_1(\sim_{\mathfrak{L}} S^3)$ is connected (by Lemma 5.4) and contained in $SU(3) \subset G$, we get $K_1 = 1 \times SU(2)$ or $1 \times SO(3)$. For each case, K is conjugate to a circle group in $1 \times SU(2)$. Thus $Z(K, SU(3))$ is a maximal torus of $SU(3)$. On the other hand, by using (7.3.6) and $K \subset SU(3)$, we see easily that

$$K_2 = \left\{ \left(X, \begin{pmatrix} \bar{z}^2 & 0 \\ 0 & zX \end{pmatrix} \right) \in G;\ X \in SU(2) = S^3 \right\} \quad \text{and} \quad K = K_2 \cap SU(3).$$

Then $U(2) \subset Z(K, SU(3))$ and this is contrary to $Z(K, SU(3)) \cong S^1 \times S^1$.

7.12 (Proof of Proposition 7.4.1 for (6) in (7.1.2)). Set $G = S^1 \times Spin(l+1)$ $(l = k_1 \geqq 3)$. When $l = 3$ and $G = S^1 \times S^3 \times S^3 (= S^1 \times Spin(4))$, we see easily that $K_1^\circ = S^3$ is not a normal subgroup of G by (7.3.6). Thus $K^\circ = \{(1, z, z) \in G;\ z \in S^1\}$, and we may assume that $G = S^1 \times SO(4)$, $K_1^\circ = SO(3)$ and $K^\circ = SO(2)$ by Lemma 3.1. When $l \geqq 5$ and $G = S^1 \times Spin(l+1)$, we see that K_2 is contained in $S^1 \times (S^1 \times Spin(l-1))$ by (7.3.6). Then $K^\circ = Spin(l-1)$ is naturally imbedded in $Spin(l+1) \subset G$. Therefore we may assume

$$G = S^1 \times SO(l+1), \quad K_1^\circ = SO(l) \quad \text{and} \quad K^\circ = SO(l-1) \quad \text{for } l \geqq 3,$$

where the inclusions $SO(l-1) \subset SO(l) \subset SO(l+1)$ are the canonical ones.

By the similar method to that of § 7.10, (R_2) holds for $G = S^1 \times SO(l+1)$.

When $l = 3$, we can not extend this G-action to any almost effective $\tilde{G}(= G \times H)$-actions for $H \neq 1$ by Lemma 4.5.

7.13 (Proof of Proposition 7.4.1 for (7) in (7.1.2)). Set $G = SU(l+1) \times S^3$ $(k_2 = 2l \geqq 4)$. We recall the result in the proof of Lemma 7.3.15 that

$$K_1 = M_2' \circ M_1, \quad K_2 = M_2 \circ S^1 \quad \text{and} \quad K = M_2' \circ M_1'$$

for $M_2 = SU(l)$, $M_2' = SU(l-1)$, $M_1' = S^1$ and $M_1(\sim_{\mathfrak{L}} S^3)$ is not contained in any simple normal subgroup of G.

Suppose $l=2$. Then $M_2 \cong S^3$ and $K_1 = M_1 \sim_L S^3$. It is easy to see that there are five conjugate classes of connected subgroups of $G = SU(3) \times S^3$ locally isomorphic to S^3. Under the condition $M_2 = S^3$, $M_2 \cap K = 1$ and (AII), we conclude that the isotropy subgroups are unique up to conjugation, and given by

$$K_1 = \left\{ \left(\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}, X \right) \in G = SU(3) \times S^3;\ X \in SU(2) = S^3 \right\},$$

$$K = \left\{ \left(\begin{pmatrix} Z & 0 \\ 0 & 1 \end{pmatrix}, Z \right) \in G;\ Z = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \in SU(2) \right\} \quad \text{and}$$

$$K_2 = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \right) \in G;\ X \in U(2) \right\},$$

where $M_2 = S^3$ is contained in $SU(3) \subset G$.

Next suppose $l \geqq 3$. Then, by Lemma 2.5, M_2 and M_2' are unique up to conjugation in $SU(l+1)$ and M_2, respectively. Thus we may set

$$M_2 = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}, 1 \right) \in G;\ X \in SU(l) \right\} \quad \text{and}$$

$$M_2' = \left\{ \left(\begin{pmatrix} E_2 & 0 \\ 0 & X \end{pmatrix}, 1 \right) \in G;\ X \in SU(l-1) \right\}.$$

Since $M_1 \subset Z(M_2', G)$, $K \subset K_2 \subset N(M_2, G)$ and $K_2 = M_2 K$, we get

$$K_1 = \left\{ \left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, X \right) \in G;\ X \in SU(2) = S^3,\ Y \in SU(l-1) \right\},$$

$$K = \left\{ \left(\begin{pmatrix} Z & 0 \\ 0 & Y \end{pmatrix}, Z \right) \in G;\ Z = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \in SU(2),\ Y \in SU(l-1) \right\} \quad \text{and}$$

$$K_2 = M_2 K = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \right) \in G;\ X \in U(l) \right\}.$$

Clearly this also holds for $l=2$ from the first half of this subsection.

By easy calculation, we have

(a) NK/K has two components, and $\alpha_0 K$ is not in the identity comopnent for $\alpha_0 = \left(\begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, A \right) \left(A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$, which is in $K_1 \cap NK$ and of order two,

(b) $L = NK \cap NK_1 \cap NK_2/K \cong S^1$ and $Z(L, NK/K) = NK/K$.

Therefore $\tilde{G} = SU(l+1) \times S^3$ or $SU(l+1) \times S^3 \times S^1$ and (R_0) holds for these groups, by the same method as that of § 7.7.

7.14 (Proof of Proposition 7.4.1 for (8) in (7.1.2)). Set $G = Sp(2) \times Sp(3)$, and recall the result in the proof of Lemma 7.3.15 that

$$K_1 = V \circ M_1, \quad K_2 = M_2 \circ S^3 \circ S^3 \quad \text{and} \quad K = S^3 \circ S^3 \circ S^3$$

for $V \sim_{\ell} S^3$, $M_1 \sim_{\ell} Sp(2)$, $M_2 = Sp(2)$, $M_s \subset Sp(3) \subset G$ $(s=1, 2)$, and V is not contained in any simple normal subgroup of G. Thus we may set

$$K_1 = \left\{ \left(\varphi(p), \begin{pmatrix} p & 0 \\ 0 & X \end{pmatrix} \right) \in G = Sp(2) \times Sp(3);\ p \in S^3 \subset H \right\} \quad \text{and}$$

$$K = \left\{ \left(\varphi(p), \begin{pmatrix} p & 0 \\ 0 & P \end{pmatrix} \right) \in G;\ p \in S^3 \subset H,\ P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \in Sp(2) \right\}$$

for some non-trivial homomorphism φ: $S^3 \to Sp(2)$. Since $K_2/K \approx S^7$, one of the normal subgroup $W \cong S^3$ of K is also normal in K_2, and $K_2 = W \circ M_2 \circ S^3$. Then M_2 satisfies $M_2 \subset Z(W, G) \cap Sp(3)$ and $(M_2 \cap K)^\circ \cong S^3$. This implies $W \subset K \cap Sp(3)$. Then we see that $K_2 = M_2 K$ is conjugate to $\mathrm{Im}\,\varphi \times (Sp(2) \times S^3)$ in G, and this contradicts the condition $P(G/K_2) = (1+t^4)(1+t^{11})$ in Proposition 5.10 (CI) (o). Therefore this case does not occur.

7.15 (Proof of Proposition 7.4.1 for (9) in (7.1.2)). By using the similar method to that of § 7.13, we see that (R_0) holds for $G = Sp(l+1) \times Sp(2)$.

The proofs of Proposition 7.4.1 and Theorem 6.1 (CI) are now completed.

§ 8. The case (CII)

8.1. In the first place, we consider the case (CII) (o) of Proposition 5.10, and prepare the following

(8.1.1) *For the case* (CII) (o), *there exists a minimal connected normal subgroup G' of G such that the induced G'-action on G/K° is transitive. Then $G = G' \circ H$ for an essentially direct product H of some copies of S^3 and a toral group, and we have the following table (if $k_1 \leqq k_2$):*

	k_1	k_2	G'	$(G' \cap K_1)^\circ$	$(G' \cap K_2)^\circ$	$(G' \cap K)^\circ$
(1)	8	8	$Spin(8)$	$Spin(7)$	$Spin(7)$	G_2
(2)	6	8	$SU(4)$	$SU(3)$	$Sp(2)$	S^3
(3)	$(k_1, k_2) \neq (2, 2)$		$U_1 \times U_2$	$U_1 \times U_2'$	$U_1' \times U_2$	$U_1' \times U_2'$
(4)	2	2	$S^1 \times S^1$	S^1	S^1	1
(5)	2	$k_2 > 2$	$Q \times S^1$	$Q' \circ S^1$	Q	Q'
(6)	4	$4l$	$Sp(l) \times S^3$	$Sp(l-1) \circ S^3$	$Sp(l)$	$Sp(l-1)$

Here U_s $(s=1, 2)$ *is a simple group or a circle group with* $U_s/U'_s \approx S^{k_s-1}$, *and* $(Q, Q')=(SU(l), SU(l-1))$ $(k_2=2l)$ *or* $(Sp(l), Sp(l-1))$ $(k_2=4l)$. *In the cases* (5) *and* (6), $(G' \cap K_2)^\circ$ *is a normal subgroup of* G', *and the* G'*-action on* $G'/(G' \cap K_1)^\circ$ *is almost effective.*

8.2 (Proof of (8.1.1)).

Lemma 8.2.1. *For* $\alpha \in N(K, G)$ *in* (3.3), *we have* $K^\circ=(K_1 \cap \alpha^{-1}K_2\alpha)^\circ$.

Proof. In this proof, we use the notation K_2 in place of $\alpha^{-1}K_2\alpha$, and the cohomology with coefficient in Q.

Let us set $U=(K_1 \cap K_2)^\circ$, and consider the following commutative diagram $(0<i<n-1)$:

$$\begin{array}{ccccc}
H^i(G/K_1) \oplus H^i(G/K_2) & \xrightarrow{(\theta_1^*,\, \theta_2^*)} & & & H^i(G/K_1^\circ) \oplus H^i(G/K_2^\circ) \\
\downarrow{\scriptstyle f_1^*-f_2^*} & & \swarrow{\scriptstyle e_1^*-e_2^*} & & \downarrow{\scriptstyle v_1^*-v_2^*} \\
H^i(G/K) & \xrightarrow[\theta^*]{} & H^i(G/K^\circ) & \xleftarrow[v^*]{} & H^i(G/U),
\end{array}$$

where all the homomorphisms are induced from the natural projections. By (5.5) and (5.6), (θ_1^*, θ_2^*), θ^* and $f_1^*-f_2^*$ are isomorphic, and so is $e_1^*-e_2^*$.

Since $K^\circ \not\sim 0$ in U by Proposition 5.10 (CII) (o), $U=K^\circ$ if $r(U)=r(K^\circ)$. To prove the lemma, it is sufficient to show $r(U) \neq r(K^\circ)+1 (=r(K_s^\circ))$.

Suppose that $r(U)=r(K^\circ)+1$. Then $P(U)=(1+t^{p-1})P(K^\circ)$ for some even integer $p \geqq 2$ since $K^\circ \not\sim 0$ in G. Hence $P(K_s^\circ/U)=(1-t^{k_s})/(1-t^p)$ $(k_s=(m_s+1)p)$ by Hirsch's formula. By Leray-Hirsch's theorem for the fibering $K_s^\circ/U \to G/U \to G/K_s^\circ$, we get

$$P(G/U) = P(G/K_s^\circ)P(K_s^\circ/U) = (1+t^{k_{3-s}-1})(1-t^{k_s})/(1-t^p).$$

This implies $m_1=m_2$, $k_1=k_2=k$ and $H^{k-1}(G/U) \cong Q$. On the other hand, $H^{k-1}(G/K_s^\circ) \cong Q$ $(s=1, 2)$ by Proposition 5.10 (CII) (o), and this contradicts the commutativity of the above diagram. *q.e.d.*

Lemma 8.2.2. *If* G *is simple, then we obtain* (1) *and* (2) *of* (8.1.1).

Proof. By Proposition 5.10 (CII) (o) and the assumption, we see easily that K°, K_1° and K_2° are simple, and $P(G)=(1+t^{k_1-1})(1+t^{k_2-1})P(K^\circ)$ $(k_1, k_2 \geqq 6)$. Then the lemma follows immediately from (2.1). *q.e.d.*

Clearly we obtain (4) if $k_1=k_2=2$. From now on, we assume that $k_1 \leqq k_2$ and $k_2 \geqq 4$. To prove (8.1.1), we may also assume $K^\circ=(K_1 \cap K_2)^\circ$ by Lemma 8.2.1.

Let V_s $(s=1, 2)$ be the maximum connected normal subgroup of G acting trivially on G/K_s°. Since G/K_1° is a Q-cohomology (k_2-1)-sphere, we have (cf. [8; Proof of Th. I])

$$G = U_1 \times W_1 \times V_1 \quad \text{and} \quad K_1^\circ = (U_1' \circ Q_1) \times V_1,$$

where U_1 is a simple group acting transitively on G/K_1°, $U_1' = (U_1 \cap K_1)^\circ$ is simple or trivial, $W_1 \cong Q_1$ and $r(W_1) \leqq 1$.

Let M_1 be a connected simple (or a circle) normal subgroup of K_1° acting transitively on $K_1^\circ/K^\circ \approx S^{k_1-1}$. We divide our proof into three cases;

$$\text{(I)} \quad M_1 \subset U_1', \quad \text{(II)} \quad M_1 \subset V_1 \quad \text{and} \quad \text{(III)} \quad M_1 \not\subset U_1' \times V_1.$$

Case (I). In this case, we see easily that the simple group U_1 acts transitively on G/K°, and $r(W_1)$, $r(V_1) \leqq 1$. By setting $G' = U_1$, we obtain (1) and (2) by Lemma 8.2.2.

Case (II). From the assumption, we have $V_1 = U_2 \times W_2$, where $r(W_2) \leqq 1$ and U_2 is simple ($k_1 \geqq 4$) or S^1 ($k_1 = 2$) acting transitively on $K_1^\circ/K^\circ \approx S^{k_1-1}$. Then the normal subgroup $G' = U_1 \times U_2$ of G acts transitively on G/K° with $(G' \cap K_1)^\circ = U_1' \times U_2$ and $r(W_s) \leqq 1$ $(s = 1, 2)$. To prove (8.1.1), we may assume that $G = U_1 \times U_2$ and $K_1^\circ = U_1' \times U_2$. Now we have $U_2 \not\subset V_2 (\subset K_2^\circ)$ by (BI) and $K^\circ = (K_1 \cap K_2)^\circ$.

(i) If $U_1 \subset V_2$, then $K_2^\circ = U_1 \times U_2' (U_2' \subset U_2)$, and hence $K^\circ = (K_1 \cap K_2)^\circ = U_1' \times U_2'$ and $U_s/U_s' \approx S^{k_{3-s}-1}$ $(s = 1, 2)$. Thus we obtain (3).

(ii) Suppose $U_1 \not\subset V_2$. Then $V_2 = 1$ and $r(U_s) = 1$ ($s = 1$ or 2), since G/K_2° is a Q-cohomology (k_1-1)-sphere and $U_2 \not\subset V_2$. By the assumption $k_1 \leqq k_2$ and $k_2 \geqq 4$, we get $U_2 = S^3$ ($k_1 = 4$) or S^1 ($k_1 = 2$). Since the G-action on G/K_2° is almost effective ($V_2 = 1$) and $P(G/K_{3-s}^\circ) = 1 + t^{k_s-1}(s = 1, 2)$, it is easy to see that $U_s = S^3$, $k_s = 4$ $(s = 1, 2)$ and $U_1' = 1$. Thus we obtain (6) for $k_1 = k_2 = 4$ (by exchanging K_1 and K_2).

Case (III). In this case, the Q_1-action on K_1°/K° is transitive, and so is the $G' = U_1 \times W_1$-action on G/K° with $(G' \cap K_1)^\circ = U_1' \circ Q_1$ and $r(V_1) \leqq 1$. Thus we may assume that $G = U_1 \times W_1$ and $K_1^\circ = U_1' \circ Q_1$.

(i) If $Q_1 \cong S^1$ ($k_2 > k_1 = 2$), then we see easily that $K_2^\circ = U_1$ and $K^\circ = (K_1 \cap K_2)^\circ = U_1'$. Hence $U_1/U_1' \approx S^{k_2-1}$ and we obtain (5) by (2.1).

(ii) Suppose $Q_1 \cong S^3$ ($k_2 \geqq k_1 = 4$). If $k_2 = 4$, then $U_1 \cong S^3$, $U_1' = 1$ and $K_2^\circ \cong S^3$. Clearly, K_2° is a normal subgroup of $G (= S^3 \times S^3)$ since $K^\circ = (K_1 \cap K_2)^\circ$. If $k_2 \geqq 6$, then U_1 is simple with $r(U_1) \geqq 2$, and hence U_1 acts trivially on G/K_2°. Thus we get $K_2^\circ = U_1$, $K^\circ = (K_1 \cap K_2)^\circ = U_1'$, $U_1/U_1' \approx S^{k_2-1}$, and we obtain (6) by (2.1).

This completes the proof of (8.1.1).

8.3 (Proof of Theorem 6.1 (CII)). By the same argument as that of § 7.4, Theorem 6.1 for (CII) is proved by Proposition 5.10 (CII) (e), (8.1.1) and the following

PROPOSITION 8.3.1. (I) *For the case* (CII) (e), (R_0) *holds.*

(II) *For the case* (CII) (o):

(i) *In* (1), (4) *of* (8.1.1), $G=G'$ *and* (R_0) *holds.*

(ii) *In* (2) *of* (8.1.1), $G=G'$ or $G'\times S^1$, *and* (R_0) *holds.*

(iii) *In* (3) *of* (8.1.1), $G=U_1\times U_2$, $Sp(l_1)\times Sp(l_2)\times S^3$ *or* $Q_1\times Q_2\times S^1$ (*see Theorem* 6.1); *and* (R_3) *holds if* $G=Q_1\times Q_2\times S^1$, *and* (R_0) *holds otherwise.*

(iv) *In* (5) *of* (8.1.1), $G=G'$ *or* $G'\times S^1$; *and* (R_3) *holds if* $G=G'$.

(v) *In* (6) *of* (8.1.1), $G=G'$, $G'\times S^1$ *or* $G'\times S^3$; *and* (R_0) *holds if* $G=G'$ *or* $G'\times S^1$.

In the cases (5) $G=G'\times S^1$ *and* (6) $G=G'\times S^3$, *there exists a normal subgroup* G'' *of* G *such that the restricted* G''*-action satisfies* (3) *of* (8.1.1), *and hence these cases are contained in* (iii).

This proposition is proved in the following §§ 8.4–10.

In the proof of Proposition 8.3.1 for (CII) (o), we use G, K_s, K, $\tilde{G}=G\circ H$, $\tilde{K}_s$ and $\tilde{K}$ as in §§ 7.7–14.

8.4 (PROOF OF PROPOSITION 8.3.1 FOR (CII) (e)). By Proposition 5.10 (CII) (e), we have $G=K_s$ and $n=k_s$ $(s=1, 2)$. We may assume that G acts effectively on M by Lemma 3.1, and hence so on $G/K\approx S^{n-1}$. Then such pair (G, K) is given in (2.1), and $NK/K\cong Z_2$, S^1 or S^3 by Lemma 2.2. Thus the assertion (R_0) is shown by the similar method to that of § 7.6.

8.5 (PROOF OF PROPOSITION 8.3.1 FOR (1) IN (8.1.1)). Let G be $Spin(8)$ imbedded in $SO(8)\times SO(8)\times SO(8)$ as follows (cf. [16; Chapter I]):

$$G = Spin(8) = \{(x_1, x_2, x_3);\ x_s\in SO(8)\ (1\leqq s\leqq 3)\quad\text{and}\quad (x_1u)\,(x_2v) = (\kappa x_3)\,(uv)\quad\text{for}\quad u, v\in Cay\},$$

where $(\kappa x)(u)=\overline{x(\bar{u})}$ for $x\in SO(8)$, $u\in Cay$. Let ν be the automorphism of $Spin(8)$ given by $\nu(x_1, x_2, x_3)=(x_2, x_3, x_1)$ $((x_1, x_2, x_3)\in Spin(8))$, and $I=\{(x, y, \kappa x)\in Spin(8)\}\cong Spin(7)$. Then, by using the representations $Spin(7)\rightarrow SO(8)$, we see that the subgroup $Spin(7)$ of $Spin(8)$ is conjugate to I, νI or $\nu^2 I$. Thus, up to automorphisms of G, we may set $(K_1, K_2)=(I, I)$ or $(I, \nu I)$. If $K_1=K_2=I$, then $\alpha^{-1}I\alpha=I$ for any $\alpha\in NK(=Z(G)K)$, and hence this contradicts Lemma 8.2.1. Hence we have $K_1=I$, $K_2=\nu I$ and $K=K_1\cap K_2=G_2$. Since $NK=Z(G)K$, $\tilde{G}=Spin(8)$ by Lemma 4.5, and (R_0) holds by Lemmas 3.7 (2) and 3.8 (1).

8.6 (PROOF OF PROPOSITION 8.3.1 FOR (2) IN (8.1.1)). Set $G=SU(4)$. Then it is clear that $K_1=SU(3)$, $K_2=Sp(2)$ and $K=S^3$ are unique up to conjugation in G. Hence we may set

$$K_1=\left\{\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}\in G;\ X\in SU(3)\right\},\ K_2=Sp(2)\subset SU(4)\ \text{and}\ K=K_1\cap K_2=S^3.$$

By routine calculation, we get

(a) NK is connected,

(b) $L^\circ\cong S^1$ and $Z(L, NK/K)\cong S^1\times S^1$ for $L=NK\cap NK_1\cap NK_2/K$.

These imply that $\tilde{G}=SU(4)$ or $SU(4)\times S^1$ by Lemma 4.5, and (R_0) holds for these groups by Lemma 3.7.

8.7 (Proof of Proposition 8.3.1 for (3) in (8.1.1)). Set $G=U_1\times U_2$ for U_s $(s=1, 2)$ in (8.1.1). Now we may assume that G acts effectively on M (hence on G/K) by Lemma 3.1. Then the pair (U_s, U'_s) $(s=1, 2)$ is given in (2.1). By the same method as that in § 7.6, we see easily that the assertion (R_0) holds for $G=U_1\times U_2$.

Consider the extension of this G-action to $\tilde{G}(=G\times H)$-actions. Clearly $NK\cap NK_1\cap NK_2/K=NK/K=N(U'_1, U_1)/U'_1\times N(U'_2, U_2)/U'_2$, where $N(U'_s, U_s)/U'_s$ $(s=1, 2)\cong S^3$, S^1 or Z_2 by Lemma 2.2. Except for the cases $(H, NK/K)=(S^1, S^1\times S^1)$, $(S^1, S^1\times S^3)$, $(S^1, S^3\times S^3)$ and $(S^3, S^3\times S^3)$, ϕ in (4.6) is unique up to the diagram in (4.8) and (R_0) holds by Lemmas 4.9, 3.7 and 3.8.

Now we show that (R_3) holds for the case $(H, NK/K)=(S^1, S^1\times S^1)$, since the proofs for the rest three cases are similar. Suppose $(H, NK/K)=(S^1, S^1\times S^1)$. Then ϕ is given by $\phi(z)=(z^{r_2}, z^{r_1})$ $(z\in S^1)$ for some relatively prime integers r_1 and r_2 (which means that r_1 or $r_2=1$ if $r_1r_2=0$), and $\tilde{G}=G\times S^1$, $G=SU(l_1)\times SU(l_2)$ $(k_s=2l_s)$ by Lemma 2.2. Hence $N(\tilde{K}, \tilde{G})/\tilde{K}\cong S^1\times S^1$ by Lemma 4.9. Thus $[\tilde{G}, M]$ is determined by the integers (r_1, r_2). Moreover, by Lemma 4.5, we have

$$\tilde{K}_1=\left\{\left(X, \begin{pmatrix}z^{r_1} & 0\\ 0 & Y\end{pmatrix}, z\right)\in\tilde{G}\right\}\quad\text{and}\quad \tilde{K}_2=\left\{\left(\begin{pmatrix}z^{r_2} & 0\\ 0 & X\end{pmatrix}, Y, z\right)\in\tilde{G}\right\}.$$

By considering the automorphisms of $\tilde{G}$, we may assume $r_s\geqq 0$ $(s=1, 2)$. Thus $Z(G)^\circ\cap\tilde{K}_s=Z_{r_s}$ $(s=1, 2)$, and (R_3) holds.

8.8 (Proof of Proposition 8.3.1 for (4) in (8.1.1)). Assume that $\tilde{G}=G\times H$ acts effectively on M. Then it is clear that $H=1$ and $G=S^1\times S^1$, $K_s=S^1$ $(s=1, 2)$, $K=1$. Here $K=(K_1\cap K_2)^\circ$ by Lemma 8.2.1. Thus we may set

$$K_1=1\times S^1\quad\text{and}\quad K_2=\{(z^{r_1}, z^{r_2})\in G;\ z\in S^1\}$$

for some relatively prime integers $r_1>0$ and $r_2\geqq 0$ (which means that $r_1=1$ if $r_2=0$). Further we have $r_1=1$ by (5.3) (i). By considering the automorphism

$\varphi(z, w)=(z, \bar{z}^{r_2}w)$ ($(z, w)\in G$) of G, we may set $K_1=1\times S^1$ and $K_2=S^1\times 1$. Hence (R_0) follows immediately from $NK/K\cong S^1\times S^1$ and Lemma 3.7 (1).

8.9 (Proof of Proposition 8.3.1 for (5) of (8.1.1)). Set $G=Q\times S^1$. Then $K_2^\circ=Q$ and $K^\circ=Q'$ for $(Q, Q')=(SU(l), SU(l-1))$ or $(Sp(l), Sp(l-1))$. Since the G-action on G/K_1° is almost effective, we have, up to automorphisms of G,

$$K_1^\circ=\left\{\left(\begin{pmatrix} z^{r_1} & 0 \\ 0 & X \end{pmatrix}, z^m\right)\in G;\ z\in S^1\subset C\right\}$$

for some relatively prime integers $r_1>0$ and $m\geqq 0$. Set $K_2/K_2^\circ\cong Z_{r_2}$. Then, by Lemma 5.4 (ii), we get $m=1$ and

$$K_1=K_1^\circ,\quad K=\cup_i b_1^i K^\circ,\quad K_2=\cup_i b_1^i K_2^\circ \qquad \text{for}\quad b_1=(A, \omega),$$

where $\omega=\exp(2\pi\boldsymbol{i}/r_2)$ and A is the diagonal matrix with the diagonal elements $\omega^{r_1}, \bar{\omega}^{r_1}, 1,\ldots, 1$. Further r_1 and r_2 are relatively prime integers by (BII), and $Z(G)^\circ\cap K_s\cong Z_{r_s}(s=1, 2)$. By easy calculation, we see that NK is connected. Thus (R_3) holds for $G=Q\times S^1$.

Now we consider the extension of this action to $\tilde{G}(=G\times H)$-actions, where $H=S^1$ or $S^1\times S^1$ since $NK\cap NK_1\cap NK_2/K\cong S^1\times S^1$.

If $H=S^1\times S^1$, then we see that the $\tilde{G}$-action is not almost effective by Lemma 4.5. If $H=S^1$, then we can take a normal subgroup $G''=Q\times S^1$ of $\tilde{G}$ such that the restricted G''-action satisfies (3) of (8.1.1) with $U_1=Q$ and $U_2=S^1$.

8.10 (Proof of Proposition 8.3.1 for (6) of (8.1.1)). Set $G=Sp(l)\times S^3$. Then the isotropy subgroups are connected, and we may set

$$K_2=Sp(l)\times 1,\ K=Sp(l-1)\times 1,\ K_1=\left\{\left(\begin{pmatrix} p & 0 \\ 0 & X \end{pmatrix}, p\right)\in G;\ p\in S^3\subset H\right\},$$

since the G-action on G/K_1 is almost effective. By routine calculation, we have $NK/K\cong S^3\times S^3$ and $NK\cap NK_1\cap NK_2/K\cong S^3\times Z_2$. Thus we see easily that $\tilde{G}=Sp(l)\times S^3\times H$, $H=1$, S^1 or S^3, and (R_0) holds for these groups by the same method as that of § 7.6. If $\tilde{G}=Sp(l)\times S^3\times S^3$, then there exists a normal subgroup $G''=Sp(l)\times S^3$ of $\tilde{G}$ such that the restricted G''-action satisfies (3) of (8.1.1) with $U_1=Sp(l)$ and $U_2=S^3$.

The proofs of Proposition 8.3.1 and Theorem 6.1 (CII) are completed.

§ 9. The cases (CIII) and (CIV)

9.1. In the first half of this section, we prepare the following (9.1.1–2):

(9.1.1) *The case* (CIII):

(a) *If* $k_2 \geqq 4$, *then the* G*-action on* G/K_1 *is almost effective, and*

n	k_2	G	$K_1 \sim_{\mathfrak{L}}$	$K_2^\circ \sim_{\mathfrak{L}}$	$K^\circ \sim_{\mathfrak{L}}$
(1) $2l+1 \geqq 9$	l	$Spin(l+1) \times S^1$	$Spin(l-1) \times S^1$	$Spin(l)$	$Spin(l-1)$
(2) 13	6	$G_2 \times S^1$	$S^3 \times S^1$	$SU(3)$	S^3

(b) *If* $k_2 = 2$, *then there exists a connected normal subgroup* $G' = S^3 \circ S^1$ *of* G *such that the induced* G'*-action on* G/K° *is transitive and* $r(G/G') \leqq 1$.

(9.1.2) *For the case* (CIV), *let* G' *be a minimal connected normal subgroup of* G *acting transitively on* G/K°. *Then*

$$\text{(e)} \quad G' \sim_{\mathfrak{L}} S^3 \quad \text{if} \quad n = 4, \qquad \text{(o)} \quad G' = S^3 \circ S^3 \quad \text{if} \quad n = 7.$$

9.2 (Proof of (9.1.2)). Let V be the maximum connected normal subgroup of G acting trivially on G/K_1°, and set

$$G = U \times V, \quad K_1^\circ = U' \times V \quad (U' \subset U),$$

where $V = 1$ or S^1 by (BI).

(e) Since $U/U' \approx S^2$ by Proposition 5.10 (CIV) (e), $U \sim_{\mathfrak{L}} S^3$ and $U' \cong S^1$. If the U'-action on $K_1^\circ/K^\circ \approx S^1$ is trivial, then $K^\circ = U' \times V'$ for $V' \subset V$. This contradicts the condition $K^\circ \not\sim 0$ in G. Therefore the U'-action on K_1°/K° is non-trivial, and hence transitive. By setting $G' = U$, (9.1.2) (e) holds.

(o) By Proposition 5.10 (CIV) (o), we see that

$$r(U) = r(U') + 1, \; c(U) = c(U') - 1 \quad \text{and} \quad \dim U/U' = 5.$$

Then U is an essentially direct product of some copies of S^3 and a toral group by Proposition 2.7, and so is G. By the same method as that in the proof of Lemma 7.3.1, there exists a normal subgroup $G' = S^3 \circ S^3$ of G acting transitively on G/K°, as desired.

9.3 (Proof of (9.1.1)). We recall

(9.3.1) K° and $K_1^\circ \not\sim 0$ in G, and

$$P(G/K_1^\circ) = 1 + t^{2k_2-1}, \quad P(G/K_2^\circ) = (1+t)(1+t^{k_2}),$$
$$P(G/K_2) = 1 + t, \quad P(G/K^\circ) = (1+t)(1+t^{2k_2-1}),$$

by Proposition 5.10 (CIII).

Let us consider the decomposition of G and its isotropy subgroups as in (7.2.1) and (7.3.2):

$$(9.3.2)\quad G = U\times W\times N,\quad K_1^\circ = (U'\circ V)\circ N = S^1\circ K^\circ\quad (U'\subset U),$$
$$K_2^\circ = N_2\circ M_2\circ J,\quad K^\circ = N_2\circ M_2'\circ J'\quad (M_2'\subset M_2),$$

where $W\cong V$, $r(W)\leqq 1$, $J\cong J'$ and $r(J)\leqq 1$. Here we see easily that U is a simple group by (9.3.1), and $N=1$ or S^1 by (BI).

LEMMA 9.3.3. *If $k_2\geqq 6$, then $M_2\subset U$, $M_2'=U'$ and*

$$(U, M_2) = (Spin(k_2+1), Spin(k_2))\quad or\quad (G_2, SU(3))\ (k_2 = 6).$$

PROOF. By the assumption, M_2 and M_2' are simple and $r(M_2)\geqq 2$. Thus in (9.3.2) we have $M_2\subset U$, and hence $M_2'=(M_2\cap K)^\circ\subset(U\cap K_1)^\circ=U'$. By (9.3.1) and (9.3.2), it is easy to see that U' is simple and $M_2'(\neq 1)$ is a normal subgroup of U'. Then $M_2'=U'$ and $r(M_2)=r(U)$. Therefore by (9.3.1) we see that $(1+t^{2k_2-1})P(M_2)=(1+t^{k_2-1})P(U)$, and $P(U/M_2)=1+t^{k_2}$ by Hirsch's formula. Thus $U/M_2\approx S^{k_2}$, and the lemma follows immediately from (2.1).

q.e.d.

LEMMA 9.3.4. (i) *If $k_2\geqq 4$, then $N=1$.*
(ii) *If $k_2\geqq 6$, then $W\cong V\cong S^1$ and $N_2\circ J'=1$.*

PROOF. Under the condition $k_2\geqq 4$, we note that U' is simple by (9.3.1), and K_1 is connected by Lemma 5.4.

(i) Suppose that $N\neq 1$ (i.e., $N\cong S^1$). Then the $U\times N$-action on G/K° is transitive, so that we may assume $G=U\times N$. Then, $K_1=U'\times N$ and $K=U'\times N'$ for some cyclic group $N'(\subset N)$. Since $K^\circ(=U')$ is simple and $K_2^\circ/K^\circ\approx S^{k_2-1}$ $(k_2\geqq 4)$, we see that K_2° is semi-simple with $K_2^\circ\subset U$. Therefore G/K_2 is homeomorphic to G/K_2° since $K_2=K_2^\circ K=K_2^\circ\times N'$. This contradicts the assumption that G/K_2 is non-orientable.

(ii) Since $N=1$ by (i), $K_1=U'\circ V$ acts transitively on $K_1/K^\circ\approx S^1$, where U' is simple by Lemma 9.3.3. Thus $V\cong S^1$ and $K^\circ=U'$. Then Lemma 9.3.3 implies $N_2\circ J'=1$ in (9.3.2), as desired. *q.e.d.*

For the case $k_2\geqq 6$, (9.1.1) follows immediately from the above two lemmas. Assume that $k_2=4$. Thus $N=1$ by Lemma 9.3.4, and

$$G = U\times W,\ K_1 = U'\circ V,\ K^\circ = U' = N_2\circ J'\quad\text{and}\quad K_2^\circ = N_2\circ M_2\circ J,$$

where $M_2\cong S^3$, U' is simple, and hence $W\cong V\cong S^1$ since $K_1/K^\circ\approx S^1$. This shows that K_2° is semi-simple, and $K_2^\circ\subset U$. Then $G/K_2^\circ=(U/K_2^\circ)\times W$, and $U/K_2^\circ\approx S^4$ by (9.3.1). Hence $(U, K_2^\circ)=(Spin(5), Spin(4))$ by (2.1). Further $U'\sim_2 S^3$ since $P(U)=(1+t^7)P(U')$ by (9.3.1). Thus we obtain (1) for $k_2=4$.

For the case $k_2=2$, U is simple with $P(U)=(1+t^3)P(U')$ by (9.3.1). Then $(U, U')=(S^3, 1)$. If $N=1$, then $W\cong V\cong S^1$, and hence $G=S^3\times S^1$. If $N\neq 1$, then the $N(\cong S^1)$-action on $K_1^\circ/K^\circ \approx S^1$ is transitive. Thus the $G'=U\times N$-action on G/K° is also transitive.

The proof of (9.1.1) is completed.

9.4 (Proof of Theorem 6.1 (CIII), (CIV)). By the similar discussion to that of § 7.4, we can prove Theorem 6.1 for (CIII) and (CIV) by (9.1.1), (9.1.2) and the following

Proposition 9.4.1. *For the case* (CIII), (R_1) *holds* $(k_1=2)$.

Proposition 9.4.2. *For the case* (CIV):

(e) *If* $n=4$, *then* $G=S^3$ *and* (R_0) *holds.*

(o) *If* $n=7$, *then* $G=S^3\times S^3$, *and* $[G, M]$ *is determined by* K_1° *and* K_2° *where*

$$K_s^\circ = \{(z^{l_s}, z^{m_s})\in G;\ z\in S^1\} \qquad (s=1, 2)$$

for relatively prime integers l_s *and* m_s *with*

$$l_s,\ m_s \equiv 1 \bmod 4, \quad 0 < l_1 - m_1 \equiv 4 \bmod 8 \quad \textit{and} \quad l_2 - m_2 \equiv 0 \bmod 8.$$

9.5 (Proof of Proposition 9.4.1). By the similar method to that in §§ 7.10 and 7.12, we see that (R_0) holds if $k_2 \geqq 4$.

Consider the case that $k_2=2$, and set $G=S^3\times S^1$. We recall

(9.5.1) $K_1^\circ \not\sim 0$ in G, and

$$P(G/K_1^\circ) = 1 + t^3, \quad P(G/K_2^\circ) = (1+t)(1+t^2), \quad P(G/K_2) = 1 + t,$$

by Proposition 5.10 (CIII). Then $K_1^\circ = S^1$, $K^\circ = 1$ and $K_2^\circ = S^1$. Consider $S^1(l, m) = \{(z^l, z^m)\in G;\ z\in S^1\}$ for relatively prime integers l and m (which means that l or $m=1$ if $lm=0$). Since $G/S^1(l, m)\approx S^3/Z_{|m|}$ (if $m\neq 0$) or $S^2\times S^1$ (if $m=0$), we see that K_1° and K_2° are conjugate to $S^1(l, m)$ $(m\neq 0)$ and $S^1(1, 0)$, respectively, by (9.5.1). Then, by using (9.5.1) and Lemma 5.4, we may set

$$K_2^\circ = S^1(1, 0),\ K_2 = \cup_s b_1^s K_2^\circ \quad \text{for} \quad b_1 = (\boldsymbol{j}, \gamma)\in K \cap NK_2^\circ,$$

where $\gamma^4=1$ by (BII). Furthermore, by Lemma 5.4, K_1° contains an element conjugate to b_1. Thus K_1° is conjugate to $S^1(l, m)$ for $lm\neq 0$, and this shows that K_1 is abelian since $K_1 \subset N(K_1^\circ, G)\cong S^1\times S^1$.

Now consider the slice representation σ_2: $K_2 \to O(2)$ in (3.3). Then, up to equivalence, we have $\sigma_2(b_1)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\sigma_2\,|\,K_2^\circ$ is of degree $k(\geqq 1)$. Hence we get

$$K = \cup_i b_1^i Z_k\langle(\omega_k, 1)\rangle \quad \text{for} \quad \omega_k = \exp(2\pi\boldsymbol{i}/k).$$

Here, $k=1$ or 2, since K_1 is abelian. By (5.3), it is easy to see that $k=2$ and $\gamma=-1$.

Next, from (ii) of (5.3), it follows that $b_1\in K_1^\circ\cap K$ and K_1 is connected. Then we have $|m|=1$ or 2 by (i) of (5.3), and m is even since K_1° contains an element conjugate to $b_1=(\boldsymbol{j}, -1)$. Up to automorphisms of G leaving K_2 and K invariant, we may assume l, $m>0$. Hence we get

$$K_1 = (\beta, 1)S^1(l, 2)(\beta^{-1}, 1) \quad \text{for some} \quad \beta\in S^3 \quad \text{with} \quad \beta\boldsymbol{i}\beta^{-1}=\boldsymbol{j},$$

where $Z(G)^\circ\cap K_1\cong Z_l$ (l: odd>0).

It is clear that NK/K has two components, and then the assertion (R_1) follows immediately from Lemmas 3.7 (2) and 3.8 (2).

In this case, $(NK\cap NK_1\cap NK_2/K)^\circ\cong S^1$. By the same method as that in § 8.9, this G-action can not be extended to any almost effective $G\times S^1$-actions.

The proofs of Proposition 9.4.1 and Theorem 6.1 (CIII) are completed.

9.6 (Proof of Proposition 9.4.2 (e)). Set $G=S^3$. Such G-actions are classified in [1; Th. 1.5]. It is easy to see that K_s is conjugate to $N(S^1, S^3)$, since G/K_s is non-orientable ($s=1, 2$). Further, under the condition $\pi_1(M(\alpha))=0$ ($\alpha\in NK$), the equivariant diffeomorphism class of $M(\alpha)$ is uniquely determined.

This G-action is not extendable to almost effective $G\times H$-actions for $r(H)\geqq 1$ by Lemma 4.5, because NK/K is finite. Thus we have Proposition 9.4.2 (e).

9.7 (Proof of Proposition 9.4.2 (o)). Set $G=S^3\times S^3$, and consider its subgroups

$$D^*(4h) = \{(z, z), (z\boldsymbol{j}, z\boldsymbol{j})\in G;\ z^{2h}=1,\ z\in S^1\subset C\},$$

$$S^1(l, m) = \{(z^l, z^m)\in G;\ z\in S^1\}\ (\cong S^1),$$

$$U(l, m) = S^1(l, m)\cup S^1(l, m)(\boldsymbol{j}, \boldsymbol{j}) \quad (l+m\text{: even})$$

for relatively prime integers l and m (which means that l or $m=1$ if $lm=0$).

Let ξ_1, ξ_2 and γ be the first Stiefel-Whitney classes of $S^3/Z_{4h}\to S^3/D^*(8h)$, $S^3/D^*(4h)\to S^3/D^*(8h)$ and $G/S^1(l, m)\to G/U(l, m)$, respectively. By using the Gysin sequences of these coverings and $G/S^1(l, m)\to G/U(l, m)$, $G/D^*(8h)\to G/U(l, m)$ (for $4h=|l-m|$), we see the following lemma by routine calculation, where the coefficient of the cohomology is in Z_2.

Lemma 9.7.1. (i) $H^*(G/U(l, m))=\Lambda(\delta)\otimes P[\gamma]/(\gamma^3)$ $(\deg\delta=3)$.
(ii) *$G/D^*(8h)\approx S^3\times(S^3/D^*(8h))$ and*

$$H^i(S^3/D^*(8h)) = \begin{cases} Z_2\langle\xi_1\rangle\oplus Z_2\langle\xi_2\rangle & \text{for} \quad i=1, \\ Z_2\langle\xi_1^2\rangle\oplus Z_2\langle\xi_1\xi_2\rangle & \text{for} \quad i=2, \\ Z_2\langle\xi_1^2\xi_2\rangle & \text{for} \quad i=3, \end{cases}$$

where $\xi_2^2=\xi_1^2+\xi_1\xi_2$ *if* h *is odd.*

(iii) *For the homomorphism* $g^*: H^*(S^3/D^*(8h))\to H^*(S^3/D^*(8))$ *induced by the projection* g,

$$g^*(\xi_1)=\xi_1 \quad and \quad g^*(\xi_2)=\begin{cases}\xi_2 & if \;\; h \text{ is odd,}\\ 0 & if \;\; h \text{ is even.}\end{cases}$$

(iv) *Let* h, m *and* l *satisfy* $4h=|l-m|$ *and* $lm\neq 0$. *Then for the homomorphism* $f^*: H^*(G/U(l, m))\to H^*(G/D^*(8))$ *induced by the projection* f, *and* $0\neq v\in H^3(S^3)\subset H^3(G/D^*(8))$, *we have*

$$f^*(\gamma)=\xi_1 \quad and \quad f^*(\delta)=\begin{cases}(v, \xi_1^2\xi_2) & if \;\; h \text{ is odd,}\\ (v, 0) & if \;\; h \text{ is even.}\end{cases}$$

Now we see easily that $K_s^\circ\cong S^1(s=1, 2)$ and $K^\circ=1$ by Proposition 5.10 (CIV) (o), and

(9.7.2) $K=\cup_{i,j} b_1^i b_2^j, \quad K_1=\cup_i b_2^i K_1^\circ,$

$\alpha^{-1}K_2\alpha=\cup_j b_1^j\alpha^{-1}K_2^\circ\alpha$ for $b_1\in K_1^\circ\cap K$ and $b_2\in\alpha^{-1}K_2^\circ\alpha\cap K$,

by Lemma 5.4 (iii).

LEMMA 9.7.3. K_s° *and* K_s *are conjugate to* $S^1(l_s, m_s)$ *and* $U(l_s, m_s)$ *for some* l_s, $m_s\equiv 1 \bmod 4$, *respectively*, $(s=1, 2)$.

PROOF. Since $K_s^\circ\cong S^1$, it is clear that K_s° is conjugate to $S^1(l_s, m_s)$. By using (9.7.2) and $N(S^1(l, m), G)\cong N(S^1, S^3)\times S^3$ (if $lm=0$) or $S^1\times S^1\cup S^1\times S^1(j, j)$ (if $lm\neq 0$), we see the following since G/K_1 and G/K_2 are non-orientable:

(a) If K_1° is conjugate to $S^1(l_1, m_1)$ for $l_1m_1\neq 0$, then K_2° is so to $S^1(l_2, m_2)$ for some odd integers l_2 and m_2.

(b) If K_1° is conjugate to $S^1(1, 0)$, then K_2° is so to $S^1(1, 0)$.

By using (5.5), it is easy to see that K_1° and K_2° are not conjugate to $S^1(1, 0)$. Therefore, from (a) and (b) it follows that $K_s^\circ(s=1, 2)$ is conjugate to $S^1(l_s, m_s)$ for some odd integers l_s and m_s. Further K_s is conjugate to $U(l_s, m_s)$ since G/K_s is non-orientable. Here we may assume that l_s, $m_s\equiv 1 \bmod 4$, because l_s and m_s are odd integers and $U(l_s, m_s)$ is conjugate to $U(\varepsilon_1 l_s, \varepsilon_2 m_s)\,(\varepsilon_1, \varepsilon_2=\pm 1)$. *q.e.d.*

First we set $K_1=U(l_1, m_1)$ by Lemma 9.7.3. Then the slice representation $\sigma_1: K_1\to O(2)$ is of degree k on K_1° and $\sigma_1(j, j)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, up to equivalence. Thus $K=Z_k\cup Z_k(j, j)$, where Z_k is generated by $(\omega^{l_1}, \omega^{m_1})$ $(\omega=\exp(2\pi i/k))$. Since any element in $K_2-K_2^\circ$ is of order 4, we have $k=4$ by (9.7.2). From these observations, we may set

$$K = D^*(8) \quad \text{and} \quad K_s = U(l_s, m_s) \qquad (s = 1, 2)$$

for some relatively prime integers l_s and m_s with $l_s, m_s \equiv 1 \bmod 4$.

By the same method as that of [1; Lemma 5.10], Lemma 3.9 shows that for any $\alpha \in NK(\cong O^* \times Z_2)$ $M(\alpha)$ is equivariantly diffeomorphic to $M(1)$ or $M(\beta)$, where $\beta = (\beta', \beta')$ $(\beta' = (1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})/2)$. From Van-Kampen's theorem it follows that $\pi_1(M(1)) \cong Z_2$ and $\pi_1(M(\beta)) = 0$. Further, by (5.5) and Lemma 9.7.1, we see that $M(\beta)$ is a Z_2-cohomology sphere if and only if $(l_1 - m_1 + l_2 - m_2)/4$ is odd.

This G-action can not be extended to almost effective $G \times H$-actios for $r(H) \geqq 1$, because NK/K is finite.

Thus the proofs of Proposition 9.4.2 (o) and Theorem 6.1 (CIV) are completed.

§ 10. The case (CV)

10.1. In the first half of this section, we prepare the following

(10.1.1) (cf. [15; (7.4)]) *For the case* (CV):

	n	k_s	G	K_s	K	χ
(1)	n	n	G	G	K	1
(2)	7	3	$SU(3)$	$S^3 \circ S^1$	$S^1 \times S^1$	3
(3)	13	5	$Sp(3)$	$Sp(2) \circ S^3$	$S^3 \circ S^3 \circ S^3$	3
(4)	25	9	F_4	$\sim_{\ell} Spin(9)$	$\sim_{\ell} Spin(8)$	3
(5)	9	3	$Sp(2)$	$S^3 \circ S^1$	$S^1 \times S^1$	4
(6)	13	3	G_2	$S^3 \circ S^1$	$S^1 \times S^1$	6
(7)	$k_1 + k_2 - 1$	k_s	$U_1 \times U_2$	$U_s \times U'_{3-s}$	$U'_1 \times U'_2$	2

where U_s *is a simple group with* $U_s/U'_s \approx S^{k_s - 1}$ $(U'_s \subset U_s)$.

10.2 (Proof of (10.1.1)).

(10.2.1) ([15; (6.2)]) *For* $\alpha \in N(K, G)$ *in* (3.3), K_1 *and* $\alpha^{-1} K_2 \alpha$ *generate the entire group* G.

Lemma 10.2.2. *If* G *is simple and* $K_s \subsetneqq G$ $(s = 1, 2)$, *then we obtain* (2)–(6) *of* (10.1.1).

Proof. Since K_s/K is an even sphere, we see that K_s contains a connected normal subgroup locally isomorphic to $SO(k_s)$ or G_2 $(k_s = 7)$ (see (2.1)). Compare the Poincaré polynomials in (1)–(10) of Lemma 2.6 and Proposition 5.10 (CV). Then we obtain (2)–(6) of (10.1.1) from (1) $(l=2)$, (5) $(l=3)$, (6), (4) $(l=2)$ and (10) in Lemma 2.6, respectively. *q.e.d.*

To prove (10.1.1), we may assume that G is generated by K_1 and K_2 by (10.2.1). If $K_1=G$, then we have $K_2=G$ by Proposition 5.10 (CV), and we obtain (1) of (10.1.1).

From now on, we assume $K_s \subsetneqq G$ $(s=1, 2)$. From (BI) and $r(G)=r(K_s)$, it follows that G is semi-simple and

$$G = U_1 \times \cdots \times U_m, \quad K_1 = Q_1 \times \cdots \times Q_m, \quad K_2 = R_1 \times \cdots \times R_m,$$

where U_i is simple with $Q_i \cup R_i \subset U_i$ $(1 \leqq i \leqq m)$. Further one of the following two cases occurs since $K_s/K \approx S^{k_s-1}$:

(I) $\quad K = Q_1' \times Q_2 \times \cdots \times Q_m = R_1' \times R_2 \times \cdots \times R_m \quad (Q_1' \subset Q_1, R_1' \subset R_1)$,

(II) $\quad K = Q_1' \times Q_2 \times \cdots \times Q_m = R_1 \times R_2' \times \cdots \times R_m \quad (Q_1' \subset Q_1, R_2' \subset R_2)$.

Here $m=1$ in (I) and $m=2$ in (II), because G is generated by K_1 and K_2, and the G-action on G/K is almost effective. In the case (I), G is simple, and hence we obtain (2)–(6) by Lemma 10.2.2. In the case (II), we get $Q_1=U_1$, $R_2=U_2$ by $K_1 \cup K_2 \subset Q_1 \times R_2$, and so (7) of (10.1.1).

These complete the proof of (10.1.1).

10.3 (Proof of Theorem 6.1 (CV)). By the similar argument to that of § 7.4, Theorem 6.1 for (CV) is proved by the following

Proposition 10.3.1. *For the case* (CV), (R_0) *holds.*

10.4 (Proof of Proposition 10.3.1 for (1), (7) in (10.1.1)). In the case (1) (resp. (7)), we can show the assertion (R_0) by the same method as that of § 8.4 (resp. §§ 7.6 and 8.7).

10.5 (Proof of Proposition 10.3.1 for (2), (3), (4) in (10.1.1)). In these cases, we see easily the following:

(a) K is unique up to conjugation.

(b) There are just three connected subgroups of G, containing K and being locally isomorphic to (2) $S^3 \times S^1$, (3) $Sp(2) \times S^3$ and (4) $Spin(9)$. Further, they are conjugate to each other by the element of NK.

(c) The factor group NK/K is isomorphic to the symmetric group of three elements.

From (a) and (b), we may assume that $K_1=K_2$, and K and K_s $(s=1, 2)$ are naturally imbedded in G. By (c) and Lemmas 3.7 and 3.8, we see that there are two essential isomorphism classes of $M(\alpha)$, where α varies in NK, and $M(1)$ is not a Z_2-cohomology sphere by (10.2.1). Therefore (R_0) holds for $G=SU(3)$, $Sp(3)$ and F_4. Here we note that G/K_s $(s=1, 2)$ is (2) $P_2(C)$, (3) $P_2(H)$ and (4) $P_2(Cay)$, respectively.

10.6 (PROOF OF PROPOSITION 10.3.1 FOR (5), (6) IN (10.1.1)). In the case (5), we can show that (R_0) holds by the same method as the proof for (6) given below.

$G=G_2$ is the group of linear automorphisms $x \in SO(8)$ of Cay satisfying

$$x(u)x(v) = x(uv) \qquad (u, v \in Cay).$$

Let $A(\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $(\theta \in R)$ and set

$$t(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} E_2 & & & 0 \\ & A(\theta_1) & & \\ & & A(\theta_2) & \\ 0 & & & A(\theta_3) \end{pmatrix} \in G_2 \subset SO(8) \quad (\theta_1+\theta_2+\theta_3=0)$$

and $T(l_1, l_2, l_3)=\{t(l_1\theta, l_2\theta, l_3\theta) \in G_2;\ \theta \in R\} \cong S^1$ $(l_1+l_2+l_3=0)$.

Since $K(\cong S^1 \times S^1)$ is the maximal torus of G, we may set

$$K = \{t(\theta_1, \theta_2, \theta_3) \in G;\ \theta_1 + \theta_2 + \theta_3 = 0\}.$$

Then, by routine calculation, we have

(a) There are just six connected subgroups H_s $(1 \leqq s \leqq 6)$ of G, which contain K and are locally isomorphic to $S^3 \times S^1$;

$$H_1 = Z(T(0, 1, -1), G), \qquad H_2 = Z(T(1, 0, -1), G),$$
$$H_3 = Z(T(1, -1, 0), G), \qquad H_4 = Z(T(-2, 1, 1), G),$$
$$H_5 = Z(T(1, -2, 1), G), \qquad H_6 = Z(T(1, 1, -2), G).$$

Here H_s, H_{s+1} and H_{s+2} $(s=1, 4)$ are conjugate to each other, but H_1 and H_4 are not so.

(b) $NK/K \cong N(K, SU(3))/K \times Z_2\langle AK\rangle$ for $SU(3)=G_2 \cap SO(6)$ and the diagonal matrix A with the diagonal elements $1, -1, \dots, 1, -1$.

If K_1 is conjugate to K_2, then we see easily that the G-manifolds $M(\alpha)$ are not Z_2-cohomology spheres by (10.2.1) and [15; (7.5)]. Thus we may set $K_1=H_1$ and $K_2=H_4$. Then by (b) and Lemma 3.9, there are two essential isomorphism classes of $M(\alpha)$ where α varies in NK, and $M(1)$ is not a Z_2-cohomology sphere by (10.2.1). Thus the assertion (R_0) holds.

The proof of Proposition 10.3.1 is now completed. Thus Theorem 6.1 is proved completely.

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