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On strong oscillation of retarded differential equations

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1. Introduction

In this paper we study the oscillatory and nonoscillatory behavior of solutions of the linear retarded differential equation

$$x^{(n)}(t) + p(t)x(g(t)) = 0, \quad t \geqq a, \tag{E}$$

where n is even and the following conditions are always assumed to hold:

(a) $p(t)$ is a positive continuous function on $[a, \infty)$;

(b) $g(t)$ is a continuously differentiable function on $[a, \infty)$ such that $g(t) \leqq t$, $g'(t) > 0$ for $t \geqq a$ and $\lim_{t\to\infty} g(t) = \infty$.

A solution $x(t)$ of (E) defined on $[T_x, \infty)$ is called oscillatory if $x(t)$ has an unbounded set of zeros, and otherwise it is called nonoscillatory. Equation (E) is said to be *oscillatory* if every solution of (E) is oscillatory, and *nonoscillatory* if at least one solution of (E) is nonoscillatory.

In the oscillation theory of differential equations one of the important problems is to find conditions on $p(t)$ which imply that (E) is oscillatory or (E) is nonoscillatory. For the second order ordinary differential equation

$$x'' + p(t)x = 0, \quad t \geqq a, \tag{1}$$

there is an extensive literature on this subject (see Swanson's book [11]). Especially, the following theorem is well known.

THEOREM A. (i) (Fite [2]) *Equation* (1) *is oscillatory if*

$$\int_a^\infty p(s)ds = \infty. \tag{2}$$

(ii) (Hille [4]) *Suppose* (2) *fails. Then equation* (1) *is oscillatory if*

$$\limsup_{t\to\infty} t \int_t^\infty p(s)ds > 1, \tag{3}$$

or if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > 1/4. \tag{4}$$

Equation (1) *is nonoscillatory if*

$$\limsup_{t\to\infty} t\int_t^\infty p(s)ds < 1/4. \tag{5}$$

For example, the equation

$$x'' + kt^\alpha x = 0, \quad t \geqq 1, \tag{6}$$

is oscillatory if either $\alpha > -2$, $k > 0$ or $\alpha = -2$, $k > 1/4$, and nonoscillatory if either $\alpha < -2$, $k > 0$ or $\alpha = -2$, $k < 1/4$. Note that in case $\alpha > -2$ (6) is oscillatory for any $k > 0$, and in case $\alpha < -2$ (6) is nonoscillatory for any $k > 0$.

In general, motivated by Nehari [9], we define as follows: Equation (E) is said to be *strongly oscillatory* if the related equation

$$x^{(n)}(t) + \lambda p(t)x(g(t)) = 0, \quad t \geqq a, \tag{E_λ}$$

is oscillatory for all positive values of λ. Equation (E) is said to be *strongly nonoscillatory* if (E_λ) is nonoscillatory for all positive λ. For the second order equation (1) necessary and sufficient conditions of strong oscillation and strong nonoscillation are established on the basis of (ii) of Theorem A.

THEOREM B (Nehari [9]). *Suppose* (2) *fails. Equation* (1) *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} t\int_t^\infty p(s)ds = \infty.$$

Equation (1) *is strongly nonoscillatory if and only if*

$$\lim_{t\to\infty} t\int_t^\infty p(s)ds = 0.$$

Note that, by (i) of Theorem A, (1) is strongly oscillatory if (2) holds. Thus equation (6) is strongly oscillatory iff $\alpha > -2$ and strongly nonoscillatory iff $\alpha < -2$.

The purpose of this paper is to extend Theorems A and B to equation (E). More precisely, we reduce oscillation and nonoscillation of (E) to those of associated second order equations, and as a consequence we are able to characterize completely the strong oscillation and strong nonoscillation for certain classes of (E) including the ordinary differential equation

$$x^{(n)} + p(t)x = 0, \quad t \geqq a. \tag{7}$$

Related results for (7) and (E) can be found in Chanturiya [1], Grimmer [3] and Lovelady [6, 7].

2. Results

We begin with lemmas which are needed in establishing our oscillation and nonoscillation criteria.

LEMMA 1 (Kiguradze [5]). *If $x(t)$ is an eventually positive solution of* (E), *then there are an odd integer $\ell \in \{1,\dots, n-1\}$ and a number $T \geqq a$ such that for $t \geqq T$*

$$(8)\qquad \begin{cases} x^{(i)}(t) > 0 \qquad (i = 0,\dots, \ell), \\ (-1)^{i-\ell} x^{(i)}(t) > 0 \qquad (i = \ell,\dots, n), \end{cases}$$

$$(9)\qquad x(t) \geqq \frac{1}{\ell!}(t-T)^{\ell-1} x^{(\ell-1)}(t).$$

LEMMA 2 (Onose [10]). *Equation* (E) *is nonoscillatory if and only if there exists an eventually positive function $y(t)$ satisfying the inequality*

$$y^{(n)}(t) + p(t)y(g(t)) \leqq 0, \quad t \geqq a.$$

LEMMA 3 (Mahfoud [8]). *Let $g^{-1}(t)$ be the inverse function of $g(t)$. If the ordinary differential equation*

$$z^{(n)} + \frac{p(g^{-1}(t))}{g'(g^{-1}(t))} z = 0, \quad t \geqq a,$$

is oscillatory, then equation (E) *is oscillatory.*

THEOREM 1. *Suppose that for every $T \geqq a$ the second order equation*

$$(10)\qquad w''(t) + \frac{1}{(n-1)!}(g(t)-T)^{n-2} p(t) w(g(t)) = 0, \quad t \geqq a,$$

is oscillatory. Then equation (E) *is oscillatory.*

PROOF. We shall prove that the existence of a nonoscillatory solution of (E) implies that for some $T \geqq a$ equation (10) has a nonoscillatory solution. Suppose $x(t)$ is a nonoscillatory solution of (E). We may assume with no loss of generality that $x(t)$ is eventually positive. By Lemma 1 there exist an odd integer $\ell \in \{1,\dots, n-1\}$ and a number $T \geqq a$ such that (8) and (9) hold for $t \geqq T$. We may suppose that $x(g(t)) > 0$ for $t \geqq T$. Applying Taylor's formula with remainder, we find that

$$x^{(\ell)}(t) = \sum_{j=0}^{n-\ell-1} \frac{x^{(\ell+j)}(\tau)}{j!}(t-\tau)^j + \frac{1}{(n-\ell-1)!}\int_\tau^t (t-s)^{n-\ell-1} x^{(n)}(s)\,ds$$

$$= \sum_{j=0}^{n-\ell-1} \frac{(-1)^j x^{(\ell+j)}(\tau)}{j!}(\tau-t)^j$$
$$+ \frac{1}{(n-\ell-1)!}\int_t^\tau (s-t)^{n-\ell-1}p(s)x(g(s))ds$$

for $\tau \geqq t \geqq T$. Taking (8) into account and letting $\tau \to \infty$, we obtain

$$x^{(\ell)}(t) \geqq \frac{1}{(n-\ell-1)!}\int_t^\infty (s-t)^{n-\ell-1}p(s)x(g(s))ds$$

for $t \geqq T$. From the above inequality it follows that

$$x^{(\ell-1)}(t) \geqq x^{(\ell-1)}(T) + \frac{1}{(n-\ell-1)!}\int_T^t \int_s^\infty (u-s)^{n-\ell-1}p(u)x(g(u))duds$$
$$= x^{(\ell-1)}(T) + \frac{1}{(n-\ell-1)!}\int_T^t \left(\int_T^u (u-s)^{n-\ell-1}ds\right)p(u)x(g(u))du$$
$$+ \frac{1}{(n-\ell-1)!}\int_t^\infty \left(\int_T^t (u-s)^{n-\ell-1}ds\right)p(u)x(g(u))du$$

for $t \geqq T$. Therefore, by virtue of the inequality

$$\int_T^t (u-s)^{n-\ell-1}ds \geqq \frac{1}{n-\ell}(t-T)(u-T)^{n-\ell-1} \qquad (T \leqq t \leqq u),$$

we conclude that

$$\text{(11)} \qquad x^{(\ell-1)}(t) \geqq x^{(\ell-1)}(T) + \frac{1}{(n-\ell)!}\int_T^t (u-T)^{n-\ell}p(u)x(g(u))du$$
$$+ \frac{1}{(n-\ell)!}(t-T)\int_t^\infty (u-T)^{n-\ell-1}p(u)x(g(u))du$$

for $t \geqq T$. Denote the right hand side of (11) by $y(t)$. In view of (9) we see that

$$x(g(t)) \geqq \frac{1}{\ell!}(g(t)-T)^{\ell-1}x^{(\ell-1)}(g(t)) \geqq \frac{1}{\ell!}(g(t)-T)^{\ell-1}y(g(t))$$

for all large t. Then by differentiation

$$y''(t) + \frac{1}{(n-\ell)!}(t-T)^{n-\ell-1}p(t)x(g(t)) = 0$$

and so

$$y''(t) + \frac{1}{(n-1)!}(g(t)-T)^{n-2}p(t)y(g(t)) \leqq 0$$

for all large t. It follows from Lemma 2 that equation (10) is nonoscillatory, contradicting the hypothesis. This completes the proof.

THEOREM 2. (i) *Equation* (E) *is oscillatory if*

$$\int_a^\infty [g(s)]^{n-2}p(s)ds = \infty. \tag{12}$$

(ii) *Suppose that* (12) *fails. Then equation* (E) *is oscillatory if*

$$\limsup_{t\to\infty} g(t)\int_t^\infty [g(s)]^{n-2}p(s)ds > (n-1)!, \tag{13}$$

or if

$$\liminf_{t\to\infty} g(t)\int_t^\infty [g(s)]^{n-2}p(s)ds > (n-1)!/4. \tag{14}$$

PROOF. According to Theorem 1 and Lemma 3, it is sufficient to show that the second order ordinary differential equation

$$z'' + \frac{(t-T)^{n-2}p(g^{-1}(t))}{(n-1)!g'(g^{-1}(t))}z = 0, \quad t \geqq a,$$

is oscillatory for every $T \geqq a$. With the aid of Theorem A we can easily observe that the above equation is oscillatory if any one of the conditions (12), (13), (14) is satisfied. The proof is complete.

THEOREM 3. *Suppose that for some* $T \geqq a$ *the second order equation*

$$w''(t) + \frac{1}{(n-2)!}(t-T)^{n-2}p(t)w(g(t)) = 0, \quad t \geqq a, \tag{15}$$

is nonoscillatory. Then equation (E) *is nonoscillatory.*

PROOF. Let $w(t)$ be an eventually positive solution of (15). Find $T_0 \geqq T$ such that $w(t)>0$ and $w(g(t))>0$ for $t \geqq T_0$. It is easily verified that

$$w(t) \geqq w(T_0) + \frac{1}{(n-2)!}\int_{T_0}^t \int_s^\infty (u-T)^{n-2}p(u)w(g(u))duds$$

for $t \geqq T_0$, so

$$w(t) \geqq w(T_0) + \frac{1}{(n-2)!}\int_{T_0}^t \int_s^\infty (u-s)^{n-2}p(u)w(g(u))duds \tag{16}$$

for $t \geqq T_0$. Denote the right hand side of (16) by $y(t)$. In view of (16) we see that

$$y^{(n)}(t) + p(t)y(g(t)) \leqq y^{(n)}(t) + p(t)w(g(t)) = 0$$

for all large t. Now from Lemma 2 it follows that equation (E) is nonoscillatory. This completes the proof.

THEOREM 4. *Suppose that*

$$\int_a^\infty s^{n-2}p(s)ds < \infty.$$

Then equation (E) *is nonoscillatory if*

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds < (n-2)!/4. \tag{17}$$

PROOF. It is enough to prove that equation (15) is nonoscillatory for some $T \geqq a$. Applying Theorem A, we see that the ordinary differential equation

$$y'' + \frac{1}{(n-2)!} t^{n-2}p(t)y = 0, \quad t \geqq a,$$

has an eventually positive solution $y(t)$ under the condition (17). Since $y(t)$ is increasing for all large t (see Lemma 1), it follows that

$$y''(t) + \frac{1}{(n-2)!}(t-T)^{n-2}p(t)y(g(t)) \leqq 0$$

for all large t, where $T \geqq a$ is a positive constant. Thus by Lemma 2 we conclude that equation (15) is nonoscillatory. The proof is complete.

On the basis of Theorems 2 and 4, a characterization of strong oscillation and strong nonoscillation of (E) is established.

THEOREM 5. *Assume that*

$$\liminf_{t\to\infty} g(t)/t > 0. \tag{18}$$

(i) *Equation* (E) *is strongly oscillatory if and only if either*

$$\int_a^\infty s^{n-2}p(s)ds = \infty \tag{19}$$

or

$$\limsup_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds = \infty. \tag{20}$$

(ii) *Equation* (E) *is strongly nonoscillatory if and only if*

$$\int_a^\infty s^{n-2}p(s)ds < \infty \tag{21}$$

and

$$\lim_{t\to\infty} t\int_t^\infty s^{n-2}p(s)ds = 0. \tag{22}$$

PROOF. Condition (18) implies that there is a positive constant c such that $g(t) \geqq ct$ for all large t.

(i) Suppose (E) is strongly oscillatory. Since (E_λ) is oscillatory for every $\lambda>0$, if (19) does not hold, then

$$\limsup_{t\to\infty} \lambda t \int_t^\infty s^{n-2}p(s)ds \geqq (n-2)!/4$$

for every $\lambda>0$ by Theorem 4, so that (20) must be satisfied. Conversely, suppose either (19) or (20). If (19) holds, then by (i) of Theorem 2 it is clear that (E) is strongly oscillatory. If (20) holds, then

$$\limsup_{t\to\infty} \lambda g(t) \int_t^\infty [g(s)]^{n-2}p(s)ds = \infty$$

for all positive λ, which shows the oscillation of (E_λ) for all positive λ by (ii) of Theorem 2.

(ii) If (E) is strongly nonoscillatory, then (21) holds by (i) of Theorem 2 and the inequality

$$\limsup_{t\to\infty} \lambda g(t) \int_t^\infty [g(s)]^{n-2}p(s)ds \leqq (n-1)!$$

is satisfied for all $\lambda>0$ by (ii) of Theorem 2 and hence (22) holds. Conversely, if (21) and (22) hold, then

$$\limsup_{t\to\infty} \lambda t \int_t^\infty s^{n-2}p(s)ds < (n-2)!/4$$

for all $\lambda>0$ is obvious and so (E) is strongly nonoscillatory by Theorem 4. The proof is complete.

EXAMPLE 1. Let r be a nonnegative number and consider the equation

(23) $$x^{(n)}(t) + kt^\alpha x(t-r) = 0, \quad t \geqq 1,$$

where k is a positive constant. Then equation (23) is strongly oscillatory if and only if $\alpha>-n$, and strongly nonoscillatory if and only if $\alpha<-n$.

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