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Semi-fine limits and semi-fine differentiability of Riesz potentials of functions in L^p

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1. Statement of results

In the n-dimensional Euclidean space R^n, we define the Riesz potential of order α, $0<\alpha<n$, of a non-negative measurable function f on R^n by

$$U_\alpha^f(x) = R_\alpha * f(x) = \int |x-y|^{\alpha-n} f(y)dy; \qquad R_\alpha(x) = |x|^{\alpha-n}.$$

For a set E in R^n and an open set G in R^n, we set

$$C_{\alpha,p}(E;\, G) = \inf \|f\|_p^p,$$

where $\|f\|_p$ denotes the L^p-norm in R^n, $1<p<\infty$, and the infimum is taken over all non-negative measurable functions f on R^n such that $f=0$ outside G and $U_\alpha^f(x) \geqq 1$ for every $x \in E$.

A set E in R^n is said to be (α, p)-semi-thin at $x^0 \in R^n$ if

$$\lim_{r \downarrow 0} r^{\alpha p-n} C_{\alpha,p}(E \cap B(x^0, r) - B(x^0, r/2);\, B(x^0, 2r)) = 0,$$

where $B(x^0, r)$ denotes the open ball with center at x^0 and radius r. We note here that E is (α, p)-semi-thin at x^0 if and only if

$$\lim_{i \to \infty} 2^{i(n-\alpha p)} C_{\alpha,p}(E_i;\, G_i) = 0,$$

where $E_i = \{x \in E;\, 2^{-i} \leqq |x-x^0| < 2^{-i+1}\}$ and $G_i = \{x \in R^n;\, 2^{-i-1} < |x-x^0| < 2^{-i+2}\}$.

THEOREM 1 (cf. [2; Theorem 2]). *Let $0<\beta<(n-\alpha p)/p$, and f be a non-negative measurable function on R^n such that $U_\alpha^f \not\equiv \infty$. If*

$$\lim_{r \downarrow 0} r^{(\alpha+\beta)p-n} \int_{B(x^0,r)} f(y)^p dy = 0, \tag{1}$$

then there exists a set E in R^n such that E is (α, p)-semi-thin at x^0 and

$$\lim_{x \to x^0, x \in R^n - E} |x-x^0|^\beta U_\alpha^f(x) = 0.$$

REMARK 1. (i) (cf. [2; Theorem 2]) If $\alpha p = n$ and f is a non-negative measurable function in $L^p(R^n)$ such that $U_\alpha^f \not\equiv \infty$, then there exists a set E in R^n with the following properties:

(a) $\sum_{i=1}^{\infty} C_{\alpha,p}(E \cap B(x^0, 2^{-i+1}) - B(x^0, 2^{-i}); B(x^0, 2^{-i+2})) = 0;$

(b) $\lim_{x \to x^0, x \in R^n - E} \left(\log \frac{1}{|x-x^0|}\right)^{1/p-1} U_\alpha^f(x) = 0.$

(ii) If $\alpha p > n$ and f is as above, then U_α^f is continuous on R^n.

REMARK 2. Let f be a non-negative function in $L^p(R^n)$, and set

$$A = \left\{x^0 \in R^n;\ \limsup_{r \downarrow 0} r^{\gamma-n} \int_{B(x^0,r)} f(y)^p dy > 0\right\}.$$

Then $H_{n-\gamma}(A) = 0$ in view of [1; p. 165], where H_ℓ denotes the ℓ-dimensional Hausdorff measure.

For $z \in R^n$ and a function u on R^n, we set

$$\Delta_z u(x) = u(x+z) - u(x)$$

if the right hand side has a meaning, and define $\Delta_z^m = \Delta_z(\Delta_z^{m-1})$ inductively with $\Delta_z^1 = \Delta_z$. Note that $\Delta_z^m u(x)$ is of the form

$$\sum_{k=0}^{m} a_{k,m} u(x+kz),$$

where each $a_{k,m}$ is an integer.

THEOREM 2. *Let f be a non-negative measurable function in $L^p(R^n)$ such that $U_\alpha^f \not\equiv \infty$, and m be a positive integer. If $0 < \beta < m$ and*

$$\lim_{r \downarrow 0} r^{(\alpha+\beta-m)p-n} \int_{B(x^0,r)} |f(y) - f(x^0)|^p dy = 0, \tag{2}$$

then there exists a set E in R^n which is (α, p)-semi-thin at O and satisfies

$$\lim_{x \to O, x \in R^n - E} |x|^{\beta-m} \Delta_x^m U_\alpha^f(x^0) = 0. \tag{3}$$

For a point $x = (x_1, \ldots, x_n)$ and a multi-index $\lambda = (\lambda_1, \ldots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \cdots + \lambda_n, \quad \lambda! = \lambda_1! \cdots \lambda_n!,$$

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad \left(\frac{\partial}{\partial x}\right)^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

Finally we shall establish the following result (cf. [3; Theorem 2]).

THEOREM 3. *Let f be a non-negative measurable function on R^n such that $U_\alpha^f \not\equiv \infty$, and m be a non-negative integer not greater than α. If*

$$\lim_{r \downarrow 0} r^{(\alpha-m)p-n} \int_{B(x^0,r)} |f(y) - f(x^0)|^p dy = 0 \tag{4}$$

and

$$A_\lambda = \lim_{r\downarrow 0}\int_{R^n - B(x^0,r)} \left(\frac{\partial}{\partial x}\right)^\lambda R_\alpha(x^0 - y) f(y)dy$$

exists and is finite for each λ *with* $|\lambda| \leqq m$, *then there exists a set* E *which is* (α, p)*-semi-thin at* x^0 *and satisfies*

(5) $$\lim_{x\to x^0, x\in R^n - E} |x - x^0|^{-m}\{U_\alpha^f(x) - \textstyle\sum_{|\lambda|\leq m}(\lambda!)^{-1}C_\lambda(x - x^0)^\lambda\} = 0,$$

where $C_\lambda = A_\lambda$ *if* $|\lambda| < \alpha$ *and* $C_\lambda = A_\lambda + f(x^0)B_\lambda$ *if* $|\lambda| = \alpha$ *with* B_λ *which will be defined later (in Lemma 4).*

REMARK. Condition (4) implies the existence and finiteness of A_λ for $|\lambda| < m$.

If (5) holds for E which is (α, p)-semi-thin at x^0, then we say that U_α^f is m times (α, p)-semi-finely differentiable at x^0.

COROLLARY. *Let* f *be a function in* $L^p_{\mathrm{loc}}(R^n)$ *such that* $U_m^{|f|} \not\equiv \infty$. *Then* U_m^f *is* m *times* (m, p)*-semi-finely differentiable almost everywhere on* R^n.

This is an easy consequence of Theorem 3 and [4; Theorem 4 in § II]. According to [3; Theorem 2 and Remark 1 in § 3], U_m^f is k times (m, p)-finely differentiable on R^n except for a set whose Bessel capacity of index $(m-k, p)$ is zero; but in case $k = m$, this does not give any information.

2. Proof of Theorem 1

Before giving a proof of Theorem 1, we prepare several lemmas. Let us begin with

LEMMA 1. *Let* f *be a non-negative integrable function on* $B(O, 1)$, *and* β *and* γ *be real numbers. If*

$$\lim_{r\downarrow 0} r^{\gamma - n}\int_{B(O,r)} f(y)dy = 0,$$

then the following are satisfied:

i) *If* $\beta < 0$, *then* $\lim_{r\downarrow 0} r^\beta \int_{B(O,r)} |y|^{\gamma-\beta-n} f(y)dy = 0.$

ii) *If* $n - \gamma + 1 > 0$ *and* $\beta > 0$, *then* $\lim_{x\to O} |x|^\beta \int_{B(O,1)} (|x| + |y|)^{\gamma-\beta-n} f(y)dy = 0.$

PROOF. We shall prove only ii), because i) can be proved similarly. For δ, $0 < \delta \leqq 1$, set $\varepsilon(\delta) = \sup_{0<r\leq\delta} r^{\gamma-n}\int_{B(O,r)} f(y)dy$. Then we have

$$\limsup_{x\to O}|x|^{\beta}\int_{B(O,1)}(|x|+|y|)^{\gamma-\beta-n}f(y)dy$$
$$=\limsup_{x\to O}|x|^{\beta}\int_{B(O,\delta)}(|x|+|y|)^{\gamma-\beta-n}f(y)dy$$
$$=\limsup_{x\to O}(n-\gamma+\beta)|x|^{\beta}\int_0^{\delta}\left\{\int_{B(O,r)}f(y)dy\right\}(|x|+r)^{\gamma-\beta-n-1}dr$$
$$\leqq \text{const.}\ \varepsilon(\delta),$$

which implies ii).

For a non-negative measurable function f on R^n, we write

$$U_\alpha^f(x)=\int_{\{y;|x-y|\geqq|x|/2\}}|x-y|^{\alpha-n}f(y)dy$$
$$+\int_{\{y;|x-y|<|x|/2\}}|x-y|^{\alpha-n}f(y)dy = U'(x)+U''(x).$$

Since R_α is locally integrable on R^n, $U_\alpha^f\not\equiv\infty$ if and only if $\int(1+|y|)^{\alpha-n}f(y)dy<\infty$; in this case, $U'(x)$ is finite for $x\neq O$.

LEMMA 2. *Let $0<\beta<n-\alpha+1$ and $U_\alpha^f\not\equiv\infty$. Then the following are equivalent*:

i) $\lim_{x\to O}|x|^{\beta}U'(x)=0$; ii) $\lim_{r\downarrow 0}r^{\alpha+\beta-n}\int_{B(O,r)}f(y)dy=0.$

PROOF. Since $|x|^{\beta}U'(x)\geqq|x|^{\beta}\int_{B(O,|x|/2)}|x-y|^{\alpha-n}f(y)dy\geqq$ const. $|x|^{\alpha+\beta-n}\int_{B(O,|x|/2)}f(y)dy$, i) implies ii).

Suppose ii) holds. If $|x-y|\geqq|x|/2$, then $|x|+|y|\leqq 5|x-y|$, so that Lemma 1 gives

$$\limsup_{x\to O}|x|^{\beta}U'(x)\leqq\limsup_{x\to O}|x|^{\beta}\int 5^{n-\alpha}(|x|+|y|)^{\alpha-n}f(y)dy$$
$$=5^{n-\alpha}\limsup_{x\to O}|x|^{\beta}\int_{B(O,1)}(|x|+|y|)^{\alpha-n}f(y)dy=0.$$

Thus the lemma is proved.

LEMMA 3. *Let f be a non-negative measurable function on R^n satisfying* (1) *with $x^0=O$ and a real number β. Then there exists a set E in R^n which is (α,p)-semi-thin at O and satisfies*

$$\lim_{x\to O,x\in R^n-E}|x|^{\beta}U''(x)=0.$$

PROOF. Take a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty}a_i=\infty$ and

$$\lim_{i\to\infty} a_i 2^{i(n-(\alpha+\beta)p)} \int_{B(O,2^{-i+2})} f(y)^p dy = 0,$$

and define

$$E_i = \{x \in R^n;\ 2^{-i} \leqq |x| < 2^{-i+1},\ U''(x) \geqq a_i^{-1/p} 2^{i\beta}\}$$

for $i=1, 2,\ldots$. If $x \in E_i$ and $|x-y|<|x|/2$, then $|y|<2^{-i+2}$. Hence

$$\int_{B(O,2^{-i+2})} |x-y|^{\alpha-n} f(y)dy \geqq U''(x) \geqq a_i^{-1/p} 2^{i\beta}$$

for all $x \in E_i$, so that

$$C_{\alpha,p}(E_i;\ B(O,\ 2^{-i+2})) \leqq a_i 2^{-i\beta p} \int_{B(O,2^{-i+2})} f(y)^p dy,$$

which implies that $E=\cup_{i=1}^{\infty} E_i$ is (α, p)-semi-thin at O. Clearly,

$$\lim_{x\to O, x\in R^n-E} |x|^\beta U''(x) = 0.$$

Thus the proof of the lemma is complete.

Now we are ready to prove Theorem 1.

PROOF OF THEOREM 1. Without loss of generality, we may assume that x^0 is the origin O. By our assumption, f satisfies ii) in Lemma 2, so that i) in Lemma 2 holds. Now our theorem follows readily from Lemma 3.

We next give a characterization of (α, p)-semi-thin sets.

PROPOSITION. *Let $0<\beta<(n-\alpha p)/p$ and $E\subset R^n$. Then E is (α, p)-semi-thin at O if and only if there exists a non-negative function f in $L^p(R^n)$ such that $U_\alpha^f \not\equiv \infty$, $\lim_{r\downarrow 0} r^{(\alpha+\beta)p-n} \int_{B(O,r)} f(y)^p dy=0$ and $\lim_{x\to O, x\in E} |x|^\beta U_\alpha^f(x)=\infty$.*

PROOF. The "if" part follows readily from Theorem 1. Suppose E is (α, p)-semi-thin at O, and set $E_i=E\cap B(O, 2^{-i+1})-B(O, 2^{-i})$. Take a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i=\infty$ and

$$\lim_{i\to\infty} a_i^p 2^{i(n-\alpha p)} C_{\alpha,p}(E_i;\ G_i) = 0, \qquad G_i = \{x;\ 2^{-i-1} < |x| < 2^{-i+2}\}.$$

For each i, we can find a non-negative function f_i on R^n such that f_i vanishes outside G_i, $U_\alpha^{f_i}(x)\geqq 1$ for $x \in E_i$ and

$$\int f_i(y)^p dy \leqq C_{\alpha,p}(E_i;\ G_i) + a_i^{-p} 2^{-i(n-\alpha p+1)}.$$

Define $f=\sum_{i=1}^{\infty} a_i 2^{i\beta} f_i$. Then

$$\liminf_{x\to O, x\in E} |x|^\beta U_\alpha^f(x) \geqq \lim_{i\to\infty} a_i = \infty.$$

Moreover, $\lim_{i\to\infty} 2^{i(n-(\alpha+\beta)p)} \int [a_i 2^{i\beta} f_i(y)]^p dy = 0$, which implies

$$\lim_{i\to\infty} 2^{i(n-(\alpha+\beta)p)} \int_{B(O,2^{-i+1})-B(O,2^{-i})} f(y)^p dy = 0.$$

This is equivalent to

$$\lim_{i\to\infty} 2^{i(n-(\alpha+\beta)p)} \int_{B(O,2^{-i})} f(y)^p dy = 0.$$

Thus the proposition is proved.

3. Proof of Theorem 2

We first show the following lemma.

LEMMA 4. *Let* $U(x)=\int_{B(x^0,1)} |x-y|^{\alpha-n} dy$. *Then* $U \in C^\infty(B(x^0, 1))$. *If* λ *is a multi-index with* $|\lambda|=\alpha$, *then* $B_\lambda \equiv (\partial/\partial x)^\lambda U(x^0)$ *is independent of* x^0; *in fact,*

$$B_\lambda = \int_{\partial B(O,1)} \left(\frac{\partial}{\partial x}\right)^{\lambda'} R_\alpha(y) y^{\lambda''} dS(y),$$

where $\lambda=\lambda'+\lambda''$ *and* $|\lambda''|=1$.

PROOF. Take η, $0<\eta<1$, and $\varphi \in C_0^\infty(B(x^0, 1))$ which is equal to 1 on $B(x^0, \eta)$. Write

$$U(x) = \int |x-y|^{\alpha-n}\varphi(y)dy + \int_{B(x^0,1)} |x-y|^{\alpha-n}[1-\varphi(y)]dy.$$

Then one sees easily that $U \in C^\infty(B(x^0, \eta))$. Hence $U \in C^\infty(B(x^0, 1))$ by the arbitrariness of η.

Let $\lambda=\lambda'+\lambda''$, $|\lambda|=\alpha$ and $|\lambda''|=1$. Set $k_{\lambda'}(x)=(\partial/\partial x)^{\lambda'} R_\alpha$. Then $(\partial/\partial x)^{\lambda'} U(x)=\int_{B(x^0,1)} k_{\lambda'}(x-y)dy$ for $x \in B(x^0, 1)$. For the above φ, we have

$$\left(\frac{\partial}{\partial x}\right)^{\lambda''}\left(\int k_{\lambda'}(x-y)\varphi(y)dy\right)\Big|_{x=x^0} = -\int k_{\lambda'}(y)\left(\frac{\partial}{\partial y}\right)^{\lambda''}[\varphi(x^0-y)-1]dy$$

$$= \int_{B(O,1)} \left(\frac{\partial}{\partial y}\right)^{\lambda''} k_{\lambda'}(y)[\varphi(x^0-y)-1]dy$$

$$- \int_{\partial B(O,1)} k_{\lambda'}(y)[\varphi(x^0-y)-1]y^{\lambda''} dS(y)$$

$$= \left(\frac{\partial}{\partial x}\right)^{\lambda''} \int_{B(x^0,1)} k_{\lambda'}(x-y)[\varphi(y)-1]dy\Big|_{x=x^0} + \int_{\partial B(O,1)} k_{\lambda'}(y)y^{\lambda''} dS(y),$$

so that

$$\left(\frac{\partial}{\partial x}\right)^{\lambda} U(x^0) = \int_{\partial B(O,1)} k_{\lambda'}(y) y^{\lambda''} dS(y).$$

PROOF OF THEOREM 2. We write

$$\begin{aligned} U_\alpha^f(x) &= \int_{R^n - B(x^0,1)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{B(x^0,1)} |x-y|^{\alpha-n}[f(y)-f(x^0)]dy + f(x^0)\int_{B(x^0,1)} |x-y|^{\alpha-n}dy \\ &= U_1(x) + U_2(x) + f(x^0)U_3(x). \end{aligned}$$

In view of Lemma 4, U_1 and U_3 are infinitely differentiable on $B(x^0, 1)$, so that they satisfy (3) with E empty. Thus it remains to prove that U_2 satisfies (3) with E which is (α, p)-semi-thin at O. For this, we may assume that x^0 is the origin O, $f(O)=0$ and f vanishes outside $B(O, 1)$; in this case, $U_2 = U_\alpha^f(x)$. Note that $\lim_{r\downarrow 0} r^{\gamma-n}\int_{B(O,r)} f(y)dy = 0$ by (2) with $\gamma = \alpha+\beta-m$. Write

$$\begin{aligned} \Delta_x^m U_\alpha^f(O) &= \int_{R^n - B(O,(m+2)|x|)} (\Delta_x^m R_\alpha)(-y)f(y)dy \\ &\quad + \int_{B(O,(m+2)|x|)} (\Delta_x^m R_\alpha)(-y)f(y)dy = U'(x) + U''(x). \end{aligned}$$

If $y \notin B(O, (m+2)|x|)$, then we obtain by the mean value theorem,

$$|\Delta_x^m R_\alpha(-y)| \leqq \text{const.}\, |x|^m(|x|+|y|)^{\alpha-m-n}.$$

Hence Lemma 1 gives

$$\begin{aligned} &\limsup_{x\to O} |x|^{\beta-m}|U'(x)| \\ &\leqq \text{const.} \limsup_{x\to O} |x|^{\gamma-\alpha+m}\int_{B(O,1)} (|x|+|y|)^{\alpha-m-n} f(y)dy = 0. \end{aligned}$$

For positive integers i and k, $k \leqq m$, we set

$$E_{i,k} = \left\{x \in R^n;\ 2^{-i} \leqq |x| < 2^{-i+1},\ \int_{\{y;|kx-y|<|kx|/2\}} |kx-y|^{\alpha-n} f(y)dy \geqq a_i^{-1/p} 2^{i(\beta-\alpha)}\right\},$$

where $\{a_i\}$ is a sequence of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and $\lim_{i\to\infty} a_i 2^{i(n-\gamma p)}\int_{B(O,m2^{-i+2})} f(y)^p dy = 0$. If $x \in E_{i,k}$, then

$$k^\alpha \int_{\{z;|x-z|<|x|/2\}} |x-z|^{\alpha-n} f(kz)dz \geqq a_i^{-1/p} 2^{i(\beta-m)},$$

so that

$$C_{\alpha,p}(E_{i,k};\ B(O,\ 2^{-i+2})) \leqq k^{\alpha p} a_i 2^{i(m-\beta)p} \int_{B(O,2^{-i+2})} f(kz)^p dz$$
$$\leqq k^{\alpha p-n} a_i 2^{i(\alpha-\gamma)p} \int_{B(O,m2^{-i+2})} f(y)^p dy.$$

Hence $\lim_{i\to\infty} 2^{i(n-\alpha p)} C_{\alpha,p}(E_{i,k};\ B(O,\ 2^{-i+2}))=0$. Set $E=\cup_{k=1}^{m}\cup_{i=1}^{\infty} E_{i,k}$. Then it is easy to see that E is $(\alpha,\ p)$-semi-thin at O and

$$\lim_{x\to O, x\in R^n-E} |x|^{\beta-m} \int_{\{y;|kx-y|<|kx|/2\}} |kx-y|^{\alpha-n} f(y)dy = 0.$$

On the other hand,

$$|x|^{\beta-m} \int_{\{y\in B(O,(m+2)|x|);|kx-y|\geqq|kx|/2\}} |kx-y|^{\alpha-n} f(y)dy$$
$$\leqq \text{const.}\ |x|^{\gamma-n} \int_{B(O,(m+2)|x|)} f(y)dy \to 0 \quad \text{as} \quad x\to O$$

and by Lemma 1,

$$|x|^{\beta-m} \int_{B(O,(m+2)|x|)} |y|^{\alpha-n} f(y)dy \to 0 \quad \text{as} \quad x\to O.$$

Therefore $\lim_{x\to O, x\in R^n-E} |x|^{\beta-m} U''(x)=0$, and hence our theorem is obtained.

4. Proof of Theorem 3

We may assume that $x^0=O$, and set

$$K_m(x,\ y) = R_\alpha(x-y) - \sum_{|\lambda|\leqq m} (\lambda!)^{-1} x^\lambda \left(\frac{\partial}{\partial x}\right)^\lambda R_\alpha(-y).$$

For the sake of convenience, let $B_\lambda=0$ if $|\lambda|<\alpha$. For $x\in B(O,\ 1/2)$, write

$$|x|^{-m}\{U_\alpha^f(x) - \sum_{|\lambda|\leqq m} (\lambda!)^{-1} C_\lambda x^\lambda\}$$
$$= |x|^{-m} \int_{R^n-B(O,1)} K_m(x,\ y) f(y)dy$$
$$+ |x|^{-m} \int_{B(O,1)-B(O,2|x|)} K_m(x,\ y)[f(y)-f(O)]dy$$
$$- |x|^{-m} \sum_{|\lambda|\leqq m} (\lambda!)^{-1} x^\lambda \lim_{r\downarrow 0} \int_{B(O,2|x|)-B(O,r)} \left(\frac{\partial}{\partial x}\right)^\lambda R_\alpha(-y)[f(y)-f(O)]dy$$
$$+ f(O)|x|^{-m} \left\{\lim_{r\downarrow 0} \int_{B(O,1)-B(O,r)} K_m(x,\ y)dy - \sum_{|\lambda|\leqq m} (\lambda!)^{-1} B_\lambda x^\lambda\right\}$$
$$+ |x|^{-m} \int_{\{y\in B(O,2|x|);|x-y|\geqq|x|/2\}} |x-y|^{\alpha-n}[f(y)-f(O)]dy$$

$$+ |x|^{-m}\int_{\{y;|x-y|<|x|/2\}} |x-y|^{\alpha-n}[f(y)-f(O)]dy$$
$$= U_1(x) + U_2(x) - U_3(x) + f(O)U_4(x) + U_5(x) + U_6(x).$$

It is clear that $\lim_{x\to O} U_1(x)=0$. If $|y|\geqq 2|x|$, then

$$|K_m(x, y)| \leqq \text{const.}\ |x|^{m+1}(|x|+|y|)^{\alpha-n-m-1},$$

so that by Lemma 1,

$$\limsup_{x\to O} |U_2(x)|$$
$$\leqq \text{const.} \limsup_{x\to O} |x| \int_{B(O,1)} (|x|+|y|)^{\alpha-n-m-1}\, |f(y)-f(O)|dy = 0,$$

since $\lim_{r\downarrow 0} r^{\alpha-m-n}\int_{B(O,r)} |f(y)-f(O)|dy=0$.

If $|\lambda|<m$, then again by Lemma 1,

$$\limsup_{x\to O} |x|^{|\lambda|-m}\int_{B(O,2|x|)} \left|\left(\frac{\partial}{\partial x}\right)^{\lambda} R_\alpha(-y)\,[f(y)-f(O)]\right| dy$$
$$\leqq \text{const.} \limsup_{x\to O} |x|^{|\lambda|-m}\int_{B(O,2|x|)} |y|^{\alpha-n-|\lambda|}|f(y)-f(O)|dy = 0.$$

If $|\lambda|<\alpha$, then $(\partial/\partial x)^{\lambda}R_\alpha$ is locally integrable, and if $|\lambda|=\alpha$, then

$$\int_{B(O,r)-B(O,s)} \left(\frac{\partial}{\partial x}\right)^{\lambda} R_\alpha(-y)dy = 0$$

for any r and s, $r>s>0$. Hence if $|\lambda|=m$, then by the definition of A_λ,

$$\lim_{r\downarrow 0}\int_{B(O,2|x|)-B(O,r)} \left(\frac{\partial}{\partial x}\right)^{\lambda} R_\alpha(-y)\,[f(y)-f(O)]dy \to 0 \quad \text{as} \quad x\to O.$$

Therefore, $\lim_{x\to O} U_3(x)=0$. Since $U(x)=\int_{B(O,1)} |x-y|^{\alpha-n}dy \in C^\infty(B(O, 1))$,

$$U_4(x) = |x|^{-m}\left\{U(x) - \sum_{|\lambda|\leqq m} (\lambda!)^{-1}x^{\lambda}\left(\frac{\partial}{\partial x}\right)^{\lambda} U(O)\right\} \to 0 \quad \text{as} \quad x\to O.$$

As to U_5, we obtain

$$|U_5(x)| \leqq \text{const.}\ |x|^{\alpha-m-n}\int_{B(O,2|x|)} |f(y)-f(O)|dy \to 0 \quad \text{as} \quad x\to O.$$

In view of Lemma 3, one finds a set E in R^n which is (α, p)-semi-thin at O and satisfies

$$\lim_{x\to O,\, x\in R^n-E} U_6(x) = 0.$$

Thus the proof of Theorem 3 is complete.

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