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Weakly serial subalgebras of Lie algebras

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Introduction

Recently Stitzinger [7] presented some equivalent conditions for a subalgebra to be an ω-step ascendant subalgebra in a locally solvable, ideally finite Lie algebra. Subsequently Tôgô, Honda and Sakamoto [9] generalized and sharpened the results of [7] by using the concepts of weakly ascendant subalgebras, E-pairs and E_∞-pairs of subalgebras. On the other hand, Stewart [6] investigated properties of serial subalgebras of a locally finite Lie algebra.

In this paper we shall introduce the concept of weakly serial subalgebras of a Lie algebra generalizing that of serial subalgebras. The purpose of this paper is first to investigate properties of weakly serial subalgebras of a locally finite Lie algebra, and secondly to generalize the results of [6] by using the concept of weakly serial subalgebras, and thirdly to develop the results analogous to those of [9, §§2 and 3] by using the concepts of weakly serial subalgebras and weakly descendant subalgebras.

In Section 2 we shall show that in a locally solvable, locally finite Lie algebra all the weakly serial subalgebras are precisely the serial subalgebras (Theorem 2.7). We shall also show that if H is a subalgebra of a locally finite Lie algebra L, then the condition $H \operatorname{wser} L$ is equivalent to each of the following conditions: (a) $H \operatorname{wser} \langle H, X\rangle$ for any finite subset X of L; (b) $H \operatorname{wser} \langle H, x\rangle$ for any $x \in L$; (c) $H \operatorname{wser} \langle H, [x,_n H]\rangle$ for any $x \in L$ ($n \in \mathbf{N}$); (d) For any $x \in L$ there exists an $n = n(x) \in \mathbf{N}$ such that $H \operatorname{wser} \langle H, [x,_n H]\rangle$ (Theorem 2.8). Furthermore, we shall show that for a subalgebra H of a locally finite Lie algebra L, $H \operatorname{wser} L$ if and only if $\lambda_{L\mathfrak{N}}(H) \lhd L$ and $H/\lambda_{L\mathfrak{N}}(H) \subseteq \mathfrak{e}(L/\lambda_{L\mathfrak{N}}(H))$ (Theorem 2.12). This generalizes [6, Theorem 5]. In Section 3 we shall generalize [9, Theorems 2.1 and 2.2] (Theorem 3.1). We shall also show that if H is a subalgebra of a locally solvable, ideally finite Lie algebra L, then the condition $H \lhd^{\omega} L$ is equivalent to each of the following conditions: (a) $H \operatorname{ser} L$; (b) $H \operatorname{wser} L$ (Theorem 3.3). In Section 4 we shall show that if L is an abelian-by-nilpotent Lie algebra and if σ is an infinite ordinal, then all the σ-step weakly descendant subalgebras of L are precisely the σ-step descendant subalgebras of L (Corollary 4.3). We shall also show that if L is an ideally finite Lie algebra such that $L/\zeta_1(L)$ is countable-dimensional and if H is a weakly serial nilpotent subalgebra of L, then H is an $\omega 2$-step weakly descendant subalgebra of L (Theorem 4.5 and Corollary 4.6). In Section 5 we shall present several examples in connection with the results in Sections 2 and 4.

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1.

Throughout the paper we always consider not necessarily finite-dimensional Lie algebras over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified.

Let L be a Lie algebra. The set of all left Engel elements of L is denoted by $e(L)$. For a subalgebra H of L, (H, L) is an E_∞-pair [3] iff for each $x \in L$ there exists an integer $n = n(x) > 0$ such that $[x,_n h] \in H$ for any $h \in H$. (H, L) is an E-pair [9] iff for any $x \in L$ and any $h \in H$ there exists an integer $n = n(x, h) > 0$ such that $[x,_n h] \in H$.

Let S be a non-empty set. A local system

$$\mathbf{L}$$

on S is a collection of subsets of S such that each finite subset of S lies within some member of $\mathbf{L}$ (cf. [4, p. 94]).

Let us recall some classes of Lie algebras:

$L \in \mathfrak{A}$ iff L is abelian.

$L \in \mathfrak{F}$ iff L is finite-dimensional.

$L \in \mathfrak{N}$ iff L is nilpotent.

$L \in \mathfrak{N}_n$ iff L is nilpotent of class $\leq n$.

$L \in \mathrm{E}\mathfrak{A}$ iff L is solvable.

Let $\mathfrak{X}$, $\mathfrak{Y}$ be any classes of Lie algebras. When $L \in \mathfrak{X}$, L is called an $\mathfrak{X}$-algebra.

$L \in \mathfrak{X}\mathfrak{Y}$ iff L has an ideal $I \in \mathfrak{X}$ such that $L/I \in \mathfrak{Y}$. When $L \in \mathfrak{X}\mathfrak{Y}$, L is called an $\mathfrak{X}$-by-$\mathfrak{Y}$-algebra.

$L \in \mathrm{L}\mathfrak{X}$ iff there exists a local system on L consisting of $\mathfrak{X}$-subalgebras of L. When $L \in \mathrm{L}\mathfrak{F}$, L is called a locally finite Lie algebra.

$L \in \mathrm{L}(\Delta)\mathfrak{X}$ [9] iff there exists a local system $\mathbf{L}$ on L such that $X \in \mathfrak{X}$ and $X \,\Delta\, L$ for all $X \in \mathbf{L}$, where Δ is any of the relations $\lhd$, si and so on. When $L \in \mathrm{L}(\lhd)\mathfrak{F}$, L is called an ideally finite Lie algebra.

Now we introduce the following notation: Let H be a subalgebra of L. We say L to lie in $\mathrm{L}(H\text{-perm})\mathfrak{F}$ if there exists a local system on L whose members are finite-dimensional subalgebras of L permuting with H. If $L \in \mathrm{L}(\lhd)\mathfrak{F}$ then clearly $L \in \mathrm{L}(H\text{-perm})\mathfrak{F}$ for any subalgebra H of L. However, the converse is not necessarily true (see Remark 2 of Theorem 3.1).

Let H be a subalgebra of L. For a totally ordered set Σ, a series from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subalgebras of L such that

(1) $H \leq \Lambda_\sigma$ and $H \leq V_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \leq V_\sigma$ if $\tau < \sigma$,
(3) $L \backslash H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,
(4) $V_\sigma \lhd \Lambda_\sigma$ for all $\sigma \in \Sigma$.

H is a serial subalgebra of L, denoted by $H \operatorname{ser} L$, if there exists a series from H to L. For an ordinal σ, H is a σ-step ascendant subalgebra of L, denoted by $H \lhd^\sigma L$, if there exists an ascending series $(H_\alpha)_{\alpha \leq \sigma}$ of subalgebras of L such that

(1) $H_0 = H$ and $H_\sigma = L$,
(2) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \bigcup_{\alpha < \lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is a σ-step descendant subalgebra of L, which we denote by $H \lhd_\sigma L$, if there exists a descending series $(H_\alpha)_{\alpha \leq \sigma}$ of subalgebras of L such that

(1) $H_0 = L$ and $H_\sigma = H$,
(2) $H_{\alpha+1} \lhd H_\alpha$ for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \bigcap_{\alpha < \lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant subalgebra (resp. a descendant subalgebra) of L, denoted by $H \operatorname{asc} L$ (resp. $H \operatorname{desc} L$), if $H \lhd^\sigma L$ (resp. $H \lhd_\sigma L$) for some ordinal σ. When σ is finite, H is a subideal of L and denoted by $H \operatorname{si} L$. It is well known that $H \operatorname{asc} L$ (resp. $H \operatorname{desc} L$, $H \operatorname{si} L$) if and only if there exists a series from H to L of type Σ where Σ is a well-ordered set (resp. a reversely well-ordered set, a finite set) (cf. [1, p. 27]).

Tôgô [8] introduced the following concept generalizing that of ascendant sublagebras: For an ordinal σ, H is a σ-step weakly ascendant subalgebra of L, denoted by $H \leq^\sigma L$, if there exists an ascending chain $(M_\alpha)_{\alpha \leq \sigma}$ of subspaces of L such that

(1) $M_0 = H$ and $M_\sigma = L$,
(2) $[M_{\alpha+1}, H] \subseteq M_\alpha$ for any ordinal $\alpha < \sigma$,
(3) $M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

The chain $(M_\alpha)_{\alpha \leq \sigma}$ is called a σ-step weakly ascending series from H to L. H is a weakly ascendant subalgebra of L, denoted by $H \operatorname{wasc} L$, if $H \leq^\sigma L$ for some ordinal σ. When σ is finite, H is a weak subideal of L and denoted by $H \operatorname{wsi} L$.

We analogously introduce the following concepts generalizing those of serial subalgebras and descendant subalgebras: For a totally ordered set Σ, a *weak series* from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subspaces of L such that

(1) $H \subseteq \Lambda_\sigma$ and $H \subseteq V_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \subseteq V_\sigma \subseteq \Lambda_\sigma$ if $\tau < \sigma$,
(3) $L \backslash H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,
(4) $[\Lambda_\sigma, H] \subseteq V_\sigma$ for all $\sigma \in \Sigma$.

H is a *weakly serial subalgebra* of L, which we denote by $H \operatorname{wser} L$, if there exists a weak series from H to L. For an ordinal σ, H is a *σ-step weakly de-*

scendant subalgebra of L, which we denote by $H \leq_\sigma L$, if there exists a descending chain $(M_\alpha)_{\alpha\leq\sigma}$ of subspaces of L such that

(1) $M_0 = L$ and $M_\sigma = H$,

(2) $[M_\alpha, H] \subseteq M_{\alpha+1}$ for any ordinal $\alpha < \sigma$,

(3) $M_\lambda = \bigcap_{\alpha<\lambda} M_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

We call the chain $(M_\alpha)_{\alpha\leq\sigma}$ a *σ-step weakly descending series* from L to H. H is a *weakly descendant subalgebra* of L, which we denote by H wdesc L, if $H \leq_\sigma L$ for some ordinal σ.

We can show the following fact as in [1, p. 27].

LEMMA 1.1. *Let H be a subalgebra of L.*

(1) *H* wasc *L if and only if there exists a weak series from H to L of type Σ where Σ is a well-ordered set.*

(2) *H* wdesc *L if and only if there exists a weak series from H to L of type Σ where Σ is a reversely well-ordered set.*

(3) *H* wsi *L if and only if there exists a weak series from H to L of type Σ where Σ is a finite set.*

Next we state elementary properties of weakly serial subalgebras.

LEMMA 1.2. *Let H, K be subalgebras of L.*

(1) *If H* wser *K and $X \leq L$, then $H \cap X$* wser *$K \cap X$.*

(2) *When θ is a homomorphism of L such that* $\mathrm{Ker}\,\theta \leq H$, *$H$* wser *$L$ if and only if $\theta(H)$* wser *$\theta(L)$.*

(3) *If H* wser *L and $L \in \mathfrak{F}$, then H* wsi *L.*

PROOF. (1) If $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ is a weak series from H to K, then $\{\Lambda_\sigma \cap X, V_\sigma \cap X: \sigma \in \Sigma\}$ is a weak series from $H \cap X$ to $K \cap X$.

(2) If $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ is a weak series from H to L, then $\{\theta(\Lambda_\sigma), \theta(V_\sigma): \sigma \in \Sigma\}$ is a weak series from $\theta(H)$ to $\theta(L)$. Conversely, if $\{\bar{\Lambda}_\sigma, \bar{V}_\sigma: \sigma \in \Sigma\}$ is a weak series from $\theta(H)$ to $\theta(L)$, then $\{\theta^{-1}(\bar{\Lambda}_\sigma), \theta^{-1}(\bar{V}_\sigma): \sigma \in \Sigma\}$ is a weak series from H to L.

(3) is trivial.

Let σ be any ordinal. All the above statements remain true, when we replace wser by $\leq_\sigma$.

Let H be a subalgebra of L. The *ideal closure series* of H in L is the descending series $(H^{L,\alpha})_{\alpha\geq 0}$ of subalgebras of L defined inductively by

$$H^{L,0} = L,$$

$$H^{L,\alpha+1} = H^{H^{L,\alpha}} \quad \text{for any ordinal } \alpha,$$

$$H^{L,\lambda} = \bigcap_{\alpha<\lambda} H^{L,\alpha} \quad \text{for any limit ordinal } \lambda.$$

This is the Lie-theoretic analogue of the standard series in group theory. In particular, the countable part $(H^{L,n})_{n<\omega}$ is the ideal closure series in the sense of [1] and $H^{L,\omega}$ is denoted by $\lim_L H$ in [1].

We inductively define the chain $(H_{L,\alpha})_{\alpha\geq 0}$ of subspaces of L by

$$H_{L,0} = L,$$
$$H_{L,\alpha+1} = [H_{L,\alpha}, H] + H \quad \text{for any ordinal } \alpha,$$
$$H_{L,\lambda} = \bigcap_{\alpha<\lambda} H_{L,\alpha} \quad \text{for any limit ordinal } \lambda.$$

Then we call the chain $(H_{L,\alpha})_{\alpha\geq 0}$ the *weak closure series* of H in L.

The following properties on these series are elementary.

LEMMA 1.3. *Let H be a subalgebra of L and let α be any ordinal. Then*
(1) $H \leq H^{L,\alpha+1} \lhd H^{L,\alpha}$.
(2) $H \subseteq H_{L,\alpha+1} \subseteq H_{L,\alpha}$ *and* $[H_{L,\alpha}, H] \subseteq H_{L,\alpha+1}$.
(3) $H_{L,\alpha} \subseteq H^{L,\alpha}$.

By set-theoretic considerations there exist ordinals σ, τ such that

$$H^{L,\sigma} = H^{L,\alpha} \qquad \text{for any} \quad \alpha \geq \sigma,$$
$$H_{L,\tau} = H_{L,\alpha} \qquad \text{for any} \quad \alpha \geq \tau.$$

That is, each of the series $(H^{L,\alpha})_{\alpha\geq 0}$ and $(H_{L,\alpha})_{\alpha\geq 0}$ terminates for some ordinal. But it is clear that none of these series necessarily terminates in H.

The following lemma states the relation between descendant subalgebras (resp. weakly descendant subalgebras) and the ideal closure series (resp. the weak closure series).

LEMMA 1.4. *Let H be a subalgebra of L and let σ be an ordinal.*
(1) $H \lhd_\sigma L$ *if and only if* $H^{L,\sigma} = H$.
(2) $H \leq_\sigma L$ *if and only if* $H_{L,\sigma} = H$.

PROOF. Suppose $H \lhd_\sigma L$ and let $(H_\alpha)_{\alpha\leq\sigma}$ be a descending series from L to H. Using transfinite induction we have $H^{L,\alpha} \leq H_\alpha$ for any $\alpha\leq\sigma$. Hence $H^{L,\sigma}=H$. Conversely, suppose $H^{L,\sigma}=H$. By Lemma 1.3(1) $(H^{L,\alpha})_{\alpha\leq\sigma}$ is a descending series from L to H. Therefore $H \lhd_\sigma L$ and (1) is proved. (2) is similarly proved.

2.

In this section we shall investigate properties of weakly serial subalgebras of a locally finite Lie algebra.

In group theory, the notion of a serial subgroup can be expressed in functional form (cf. [2]). We here use the same method to characterize weakly serial subalgebras. First we show how to express the notion of a weakly serial subalgebra in functional form.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. Suppose that H is a weakly serial subalgebra of L and let $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ be a weak series from H to L. For each $x \in L \backslash H$ there exists a unique $\sigma(x) \in \Sigma$ such that $x \in \Lambda_{\sigma(x)} \backslash V_{\sigma(x)}$. Then the $\sigma(x)$ is simultaneously the least element of Σ such that $x \in \Lambda_{\sigma(x)}$ and the greatest element of Σ such that $x \notin V_{\sigma(x)}$. We can define a binary function $f_L: L \times L \rightarrow \{0, 1\}$ as follows: for any $x, y \in L$

$$f_L(x, y) = \begin{cases} 0 & \text{if } x \in H \text{ or if } x, y \notin H \text{ and } \sigma(x) \leq \sigma(y), \\ 1 & \text{otherwise.} \end{cases}$$

Let $x \in H$ and let $y \notin H$. We shall show that $f_L(y, [x, y]) = 1$. Since $y \in \Lambda_{\sigma(y)}$, $[x, y] \in [H, \Lambda_{\sigma(y)}] \subseteq V_{\sigma(y)}$. If $[x, y] \in H$ then clearly $f_L(y, [x, y]) = 1$. Hence we may assume that $[x, y] \notin H$. Since $\sigma([x, y])$ is the greatest element of Σ such that $[x, y] \notin V_{\sigma([x,y])}$, we have $\sigma([x, y]) < \sigma(y)$. Therefore $f_L(y, [x, y]) = 1$. Furthermore, we can easily see that the function f_L has the following properties, where $x, y, z \in L$ and $\alpha, \beta \in \mathfrak{k}$:

(i) If $f_L(x, y) = f_L(y, z) = 0$ then $f_L(x, z) = 0$.
(ii) Either $f_L(x, y) = 0$ or $f_L(y, x) = 0$.
(iii) If $x \in H$ then $f_L(x, y) = 0$.
(iv) If $f_L(x, z) = f_L(y, z) = 0$ then $f_L(\alpha x + \beta y, z) = 0$.
(v) If $x \in H$ and $y \notin H$ then $f_L(y, [x, y]) = 1$.

Conversely, suppose that there exists a binary function $f_L: L \times L \rightarrow \{0, 1\}$ satisfying the conditions (i)–(v). Let $x \sim y$ mean that $f_L(x, y) = f_L(y, x) = 0$. By (i) and (ii) the relation $\sim$ is an equivalence relation on L. By (iii) and (v) we have $H = \{x \in L : x \sim 0\}$. Let Σ denote the set of all $\sim$-equivalence classes except H. Let $\sigma, \tau \in \Sigma$. We write $\sigma < \tau$ if $\sigma \neq \tau$ and $f_L(x, y) = 0$ for any $x \in \sigma$ and any $y \in \tau$. It is a simple matter to check that $<$ is a total order on Σ. We define the terms of a weak series determined by f_L as follows: for each $\sigma \in \Sigma$

$$\Lambda_\sigma = \{x \in L : f_L(x, y) = 0 \text{ for all } y \in \sigma\},$$

$$V_\sigma = \begin{cases} \bigcup_{\tau < \sigma} \Lambda_\tau & \text{if } \{\tau \in \Sigma : \tau < \sigma\} \neq \phi, \\ H & \text{otherwise.} \end{cases}$$

It is not hard to show that $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ is a weak series from H to L. Here we verify only $[\Lambda_\sigma, H] \subseteq V_\sigma$. Let $x \in H$ and let $y \in \Lambda_\sigma$. Assume that $[x, y] \notin V_\sigma$. Then clearly $y \notin H$. Hence by (v) we have $f_L(y, [x, y]) = 1$. On the other hand, we can find a $\tau \in \Sigma$ such that $[x, y] \in \tau$. Since $[x, y] \notin V_\sigma$ and $[x, y] \in \tau \subseteq \Lambda_\tau$, we have $\sigma \leq \tau$. Since $y \in \Lambda_\sigma$ and $[x, y] \in \tau$, we have $f_L(y, [x, y]) = 0$. This is a contradiction. Hence $[x, y] \in V_\sigma$. Therefore $[\Lambda_\sigma, H] \subseteq V_\sigma$. Thus $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ is a weak series from H to L. Furthermore, the binary function on L determined by this weak series coincides with the original function.

We obtain the following

LEMMA 2.1. *Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. Then H wser L if and only if there exists a binary function f_L: $L \times L \to \{0, 1\}$ satisfying the conditions* (i)–(v).

The following result is essential to the argument in this section.

PROPOSITION 2.2. *Let H be a subalgebra of a Lie algebra L. Assume that there exists a local system $\mathbf{L}$ on L consisting of subalgebras of L. Then H wser L if and only if $H \cap X$ wser X for any $X \in \mathbf{L}$.*

PROOF. If H wser L, then by Lemma 1.2(1) $H \cap X$ wser X for any $X \in \mathbf{L}$. Conversely, suppose that $H \cap X$ wser X for any $X \in \mathbf{L}$. By making use of Lemma 2.1, for each $X \in \mathbf{L}$ there exists a binary function f_X: $X \times X \to \{0, 1\}$ satisfying the conditions (i)–(v) given by replacing L, H with X, $H \cap X$ respectively. Owing to [4, Lemma 8.22], there exists a binary function f: $L \times L \to \{0, 1\}$ such that, given any finite subset $\{(x_i, y_i) : 1 \leq i \leq n\}$ of $L \times L$, there exists an $X \in \mathbf{L}$ for which $\{(x_i, y_i) : 1 \leq i \leq n\} \subseteq X \times X$ and $f(x_i, y_i) = f_X(x_i, y_i)$, $1 \leq i \leq n$. Since each of the conditions (i)–(v) involves a finite number of elements of L, the function f also satisfies the conditions (i)–(v). Again using Lemma 2.1 we have H wser L.

As special cases of Proposition 2.2 we have the following two results.

COROLLARY 2.3. *Let $\mathfrak{X}$ be a class of Lie algebras and let $L \in \mathrm{L}\mathfrak{X}$. Then for a subalgebra H of L, H wser L if and only if $H \cap X$ wser X for any $\mathfrak{X}$-subalgebra X of L.*

COROLLARY 2.4. *Let H be a subalgebra of a locally finite Lie algebra L. Then H wser L if and only if $H \cap F$ wsi F for any finite-dimensional subalgebra F of L.*

REMARK. It is known that the notion of a serial subalgebra can be also expressed in functional form. Therefore the statements of Proposition 2.2, Corollaries 2.3 and 2.4 remain true, when we replace wser, wsi by ser, si respectively (cf. [1, Proposition 13.2.4]).

The following two results, corresponding to [2, Theorem A], are deduced from Corollary 2.4.

PROPOSITION 2.5. *Let H be a weakly serial subalgebra of a locally finite*

Lie algebra L. If θ is a homomorphism of L, then $\theta(H)$ is a weakly serial subalgebra of $\theta(L)$.

Proof. It suffices to show that if I is an ideal of L then $H+I/I$ wser L/I. Let F/I be a finite-dimensional subalgebra of L/I. By modular law

$$(H+I/I)\cap(F/I)=(H+I)\cap F/I=(H\cap F)+I/I.$$

There exists a finitely generated subalgebra H_0 of $H\cap F$ such that $(H\cap F)+I=H_0+I$, and there exists a finitely generated subalgebra F_0 of F such that $F_0+I=F$ and $H_0\leq F_0$. Since $L\in L\mathfrak{F}$, $F_0\in\mathfrak{F}$. Hence $H\cap F_0$ wsi F_0. Therefore

$$(H\cap F_0)+I/I \text{ wsi } F_0+I/I=F/I.$$

Clearly $(H\cap F)+I=H_0+I\leq(H\cap F_0)+I\leq(H\cap F)+I$ and hence

$$(H\cap F)+I/I=(H\cap F_0)+I/I.$$

Therefore we have

$$(H+I/I)\cap(F/I) \text{ wsi } F/I.$$

This being true for all F/I we can appeal to Corollary 2.4 to deduce that $H+I/I$ wser L/I.

Proposition 2.6. *Let $\{H_\lambda:\lambda\in\Lambda\}$ be any collection of weakly serial subalgebras of a locally finite Lie algebra L. Then $\bigcap_{\lambda\in\Lambda}H_\lambda$ is a weakly serial subalgebra of L.*

Proof. Put $H=\bigcap_{\lambda\in\Lambda}H_\lambda$ and let F be any finite-dimensional subalgebra of L. Then for each $\lambda\in\Lambda$, $H_\lambda\cap F$ wsi F and hence there exists an $n(\lambda)\in\mathbf{N}$ such that $[F,_{n(\lambda)}H_\lambda\cap F]\subseteq H_\lambda\cap F$. For any $\lambda\in\Lambda$

$$\bigcap_{n<\omega}([F,_n H\cap F]+(H\cap F))\subseteq[F,_{n(\lambda)}H_\lambda\cap F]+(H_\lambda\cap F)=H_\lambda\cap F.$$

Therefore $(H\cap F)_{F,\omega}=\bigcap_{n<\omega}([F,_n H\cap F]+(H\cap F))=H\cap F$. By Lemma 1.4(2) we have $H\cap F\leq_\omega F$. Since $F\in\mathfrak{F}$, $H\cap F$ wsi F. By using Corollary 2.4 we have H wser L.

Under the assumption of Proposition 2.6, $\langle H_\lambda:\lambda\in\Lambda\rangle$ is not necessarily a weakly serial subalgebra of L (see Example 5.1).

Now we set about showing the main results of this section.

Theorem 2.7. *Let $L\in L(E\mathfrak{A}\cap\mathfrak{F})$ and let H be a subalgebra of L. Then H* wser *L if and only if H* ser *L.*

Proof. One implication is trivial. Suppose H wser L and let F be any finite-dimensional solvable subalgebra of L. Then $H\cap F$ wsi F. By making use of [8, Theorem 1], we have $H \cap F$ si F. Note that the statement of Corollary 2.3 remains true, when we replace wser by ser. Therefore H ser L and the theorem is proved.

As a special case of [9, Theorem 2.2] we have the following fact: Let H be a subalgebra of a finite-dimensional Lie algebra L. Then the following conditions are equivalent:

(1) H wsi L.
(2) H wsi $\langle H, X\rangle$ for any finite subset X of L.
(3) H wsi $\langle H, x\rangle$ for any $x \in L$.
(4) H wsi $\langle H, [x,_n H]\rangle$ for any $x \in L$ $(n \in \mathbf{N})$.
(5) For any $x \in L$ there exists an $n = n(x) \in \mathbf{N}$ such that H wsi $\langle H, [x,_n H]\rangle$.

We generalize this fact in the following

THEOREM 2.8. *Let H be a subalgebra of a locally finite Lie algebra L. Then the following conditions are equivalent*:

(1) H wser L.
(2) H wser $\langle H, X\rangle$ *for any finite subset X of L.*
(3) H wser $\langle H, x\rangle$ *for any $x \in L$.*
(4) H wser $\langle H, [x,_n H]\rangle$ *for any $x \in L$* $(n \in \mathbf{N})$.
(5) *For any $x \in L$ there exists an $n = n(x) \in \mathbf{N}$ such that H* wser $\langle H, [x,_n H]\rangle$.

PROOF. (1)⇒(2)⇒(3)⇒(4)⇒(5) is trivial. We have to show that (5) implies (1). Let F be any finite-dimensional subalgebra of L and let $x \in F$. By the condition (5) we can find an $n = n(x) \in \mathbf{N}$ such that H wser $\langle H, [x,_n H]\rangle$. By Lemma 1.2(1) and (3) we have

$$H \cap F \text{ wsi } \langle H, [x,_n H]\rangle \cap F.$$

Clearly $\langle H \cap F, [x,_n H \cap F]\rangle \leq \langle H, [x,_n H]\rangle \cap F$. Hence

$$H \cap F \text{ wsi } \langle H \cap F, [x,_n H \cap F]\rangle .$$

Using the previous fact we have $H \cap F$ wsi F. Therefore by Corollary 2.4 we have H wser L.

As a direct consequence of Theorems 2.7 and 2.8 we obtain the following result, in which the equivalence of (1) and (3) was shown by Stewart [6].

COROLLARY 2.9. *Let $L \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and let H be a subalgebra of L. Then the following conditions are equivalent*:

(1) H ser L.
(2) H ser $\langle H, X\rangle$ *for any finite subset X of L.*
(3) H ser $\langle H, x\rangle$ *for any $x \in L$.*

(4) H ser $\langle H, [x,_n H]\rangle$ *for any* $x \in L$ $(n \in \mathbf{N})$.

(5) *For any* $x \in L$ *there exists an* $n = n(x) \in \mathbf{N}$ *such that* H ser $\langle H, [x,_n H]\rangle$.

Let $\mathfrak{X}$ be a class of Lie algebras. The $\mathfrak{X}$-residual $\lambda_{\mathfrak{X}}(L)$ of L is the intersection of the ideals I of L such that $L/I \in \mathfrak{X}$. It is clear that

$$\lambda_{\mathrm{L}\mathfrak{N}}(L) \leq L^{\omega} \quad \text{and} \quad \lambda_{\mathrm{LE}\mathfrak{A}}(L) \leq L^{(\omega)},$$

where $L^{\omega} = \bigcap_{n<\omega} L^{n+1}$ and $L^{(\omega)} = \bigcap_{n<\omega} L^{(n)}$. We shall show a generalization of [6, Theorem 5]. To do this we need the following

LEMMA 2.10. *If* $H \leq^{\omega} L$ *then* $H^{\omega} \lhd L$ *and* $H^{(\omega)} \lhd L$.

PROOF. By [3, Lemma 1(b)] we have $H^{\omega} \lhd L$. Another one is shown as in the proof of [3, Lemma 1(b)].

PROPOSITION 2.11. *If H is a weakly serial subalgebra of a locally finite Lie algebra L, then* $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ *and* $\lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L$.

PROOF. Here we prove only $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ by modifying the proof of [6, Theorem 5]. Another one is similarly proved. Let $\mathfrak{F}(L)$ denote the set of all finite-dimensional subalgebras of L. Put $K = \sum_{F \in \mathfrak{F}(L)} (H \cap F)^{\omega}$. Let $x \in L$ and let $F \in \mathfrak{F}(L)$. Since $L \in \mathrm{L}\mathfrak{F}$, $\langle F, x\rangle \in \mathfrak{F}(L)$. Therefore $H \cap \langle F, x\rangle$ wsi $\langle F, x\rangle$. By Lemma 2.10 $(H \cap F)^{\omega} \leq (H \cap \langle F, x\rangle)^{\omega} \lhd \langle F, x\rangle$. Hence $[(H \cap F)^{\omega}, x] \subseteq (H \cap \langle F, x\rangle)^{\omega} \subseteq K$. Therefore $K \lhd L$. On the other hand, we can show $K = \lambda_{\mathrm{L}\mathfrak{N}}(H)$ as in the proof of [6, Theorem 5]. Thus $\lambda_{\mathrm{L}\mathfrak{N}}(H) = K \lhd L$.

The following result, corresponding to [6, Corollary 6], characterizes weakly serial subalgebras of a locally finite Lie algebra.

THEOREM 2.12. *Let H be a subalgebra of a locally finite Lie algebra L. Then H* wser *L if and only if* $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ *and* $H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \subseteq \mathrm{e}(L/\lambda_{\mathrm{L}\mathfrak{N}}(H))$.

PROOF. Suppose H wser L. By Proposition 2.11 we have $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$. We denote images under the natural map $L \to L/\lambda_{\mathrm{L}\mathfrak{N}}(H)$ by bars. Let $\bar{h} \in \bar{H}$ and let $\bar{x} \in \bar{L}$. Then $\langle \bar{h}, \bar{x}\rangle \in \mathfrak{F}$. Since $\bar{H}$ wser $\bar{L}$, $\bar{H} \cap \langle \bar{h}, \bar{x}\rangle$ wsi $\langle \bar{h}, \bar{x}\rangle$. Clearly $\bar{H} \in \mathrm{L}\mathfrak{N}$ and therefore $\bar{H} \cap \langle \bar{h}, \bar{x}\rangle \in \mathfrak{N}$. Hence $\langle \bar{h}\rangle$ si $\bar{H} \cap \langle \bar{h}, \bar{x}\rangle$. Therefore we have $\langle \bar{h}\rangle$ wsi $\langle \bar{h}, \bar{x}\rangle$. It follows that $[\bar{x},_n \bar{h}] = 0$ for some $n \in \mathbf{N}$. Thus $\bar{H} \subseteq \mathrm{e}(\bar{L})$.

Conversely, suppose that $\lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L$ and $H/\lambda_{\mathrm{L}\mathfrak{N}}(H) \subseteq \mathrm{e}(L/\lambda_{\mathrm{L}\mathfrak{N}}(H))$. Let $\bar{F}$ be any finite-dimensional subalgebra of $\bar{L}$. Since $\bar{H} \subseteq \mathrm{e}(\bar{L})$, $(\bar{H} \cap \bar{F}, \bar{F})$ is an E-pair. By making use of [9, Theorem 2.1] we have $\bar{H} \cap \bar{F} \leq^{\omega} \bar{F}$. Since $\bar{F} \in \mathfrak{F}$, $\bar{H} \cap \bar{F}$ wsi $\bar{F}$. Using Corollary 2.4 we have $\bar{H}$ wser $\bar{L}$. Therefore by Lemma 1.2(2) H wser L.

3.

In this section we shall develop the results analogous to those of [9, §§2 and 3] by using the concept of weakly serial subalgebras.

We have the following result generalizing [9, Theorems 2.1 and 2.2].

THEOREM 3.1. *Let H be a subalgebra of a Lie algebra L. Assume that $L\in \mathfrak{L}(H\text{-perm})\mathfrak{F}$. Then the following conditions are equivalent*:

(1) H wser L.

(2) H wasc L.

(3) $H \leq^{\omega} L$.

(4) *(H, L) is an E-pair.*

(5) *(H, L) is an E_{∞}-pair.*

(6) H wser $\langle H, X\rangle$ *for any finite subset X of L.*

(7) H wser $\langle H, x\rangle$ *for any $x\in L$.*

(8) H wser $\langle H, [x,_n H]\rangle$ *for any $x\in L$ $(n\in \mathbf{N})$.*

(9) *For any $x\in L$ there exists an $n=n(x)\in \mathbf{N}$ such that* H wser $\langle H, [x,_n H]\rangle$.

PROOF. First we show (2)⇔(3) by modifying the proof of [9, Theorem 2.1]. One implication is trivial. Suppose H wasc L and let $(H_{\alpha})_{\alpha\leq\sigma}$ be a weakly ascending series from H to L. By the assumption we have $L=\bigcup_{\lambda\in\Lambda} A(\lambda)$, where each $A(\lambda)$ is a finite-dimensional subalgebra of L permuting with H. We consider L/H as an H-module by the adjoint action. Let $\lambda\in\Lambda$. Then for any $n\in\mathbf{N}$, $[A(\lambda),_n H]+H/H$ is a finite-dimensional H-submodule of L/H. Each H_{α}/H is also an H-submodule of L/H. Let $\mu(n)$ be the first ordinal such that $[A(\lambda),_n H]+H/H\subseteq H_{\mu(n)}/H$. Since $[A(\lambda),_n H]+H/H$ is finite-dimensional, $\mu(n)$ is not a limit ordinal. Therefore $\mu(n+1)<\mu(n)$ unless $\mu(n)=0$. Since the ordinals $\leq\sigma$ are well-ordered, there exists an $n\in\mathbf{N}$ such that $\mu(n)=0$. Then we have $[A(\lambda),_n H]\subseteq H$. Put

$$M_n = \{x\in L\colon [x,_n H]\subseteq H\} \qquad \text{for each } n\in\mathbf{N},$$

$$M_{\omega} = \bigcup_{n<\omega} M_n.$$

Then $M_{\omega}=\bigcup_{\lambda\in\Lambda} A(\lambda)=L$ and therefore $(M_{\alpha})_{\alpha\leq\omega}$ is an ω-step weakly ascending series from H to L. Thus we have $H\leq^{\omega} L$.

(3)⇒(5)⇒(4) is clear and (4)⇒(3) is shown as in the proof of [9, Theorem 2.1]. Therefore the conditions (2), (3), (4) and (5) are equivalent. On the other hand, the equivalence of (1), (6), (7), (8) and (9) is proved in Theorem 2.8. Furthermore, it follows from Theorem 2.12 that (1) implies (4). Therefore all the conditions (1)–(9) are equivalent.

REMARK 1. [9, Theorem 2.1] showed that if $L \in \mathrm{L}(H)\mathfrak{F}$, that is, there exists a local system on L consisting of finite-dimensional H-invariant subalgebras of L, then the conditions (2), (3), (4), (5) and another one are equivalent.

2. For a subalgebra H of L, if $L \in \mathrm{L}(H)\mathfrak{F}$ then $L \in \mathrm{L}(H\text{-perm})\mathfrak{F}$. However, the converse is not necessarily true. In fact, let A be an abelian Lie algebra with basis $\{a_1, a_2 \cdots\}$ and let x be the derivation of A defined by $a_i x = a_i$ for each $i \geq 1$. Form $L = A \dotplus \langle x \rangle$, the split extension of A by $\langle x \rangle$. Then it is easy to see that $L \in \mathrm{L}(H\text{-perm})\mathfrak{F}$ for any subalgebra H of L. But $\langle x^A \rangle = L$ and therefore $L \notin \mathrm{L}(A)\mathfrak{F}$. In particular, $L \notin \mathrm{L}(\lhd)\mathfrak{F}$.

3. Theorem 3.1 states indirectly the fact that if $L \in \mathrm{L}(H\text{-perm})\mathfrak{F}$ then for any ordinals $\alpha_i \geq \omega$ $(1 \leq i \leq 5)$ the following conditions are equivalent:

(1) $H \leq^{\alpha_1} L$.

(2) $H \leq^{\alpha_2} \langle H, X \rangle$ for any finite subset X of L.

(3) $H \leq^{\alpha_3} \langle H, x \rangle$ for any $x \in L$.

(4) $H \leq^{\alpha_4} \langle H, [x,_n H] \rangle$ for any $x \in L$ $(n \in \mathbf{N})$.

(5) For any $x \in L$ there exists an $n = n(x) \in \mathbf{N}$ such that $H \leq^{\alpha_5} \langle H, [x,_n H] \rangle$.

This fact is a generalization of [9, Theorem 2.2].

The following result is an immediate consequence of Lemma 2.10 and Theorem 3.1.

COROLLARY 3.2. *Under the assumption of Theorem* 3.1, *if* H wser L *then* $H^\omega \lhd L$ *and* $H^{(\omega)} \lhd L$.

By Corollary 3.2 we see that if $L \in \mathrm{L}(H\text{-perm})\mathfrak{F}$ then for any ordinal α

$$(H^{L,\alpha})^\omega \lhd L \quad \text{and} \quad (H^{L,\alpha})^{(\omega)} \lhd L.$$

In particular, $(\lim_L H)^\omega = (H^{L,\omega})^\omega \lhd L$ and $(\lim_L H)^{(\omega)} = (H^{L,\omega})^{(\omega)} \lhd L$.

In [9, Theorem 3.1] several conditions for a subalgebra to be ω-step ascendant were given. We can add two weaker equivalent conditions in the following

THEOREM 3.3. *Let* $L \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ *and let* H *be a subalgebra of* L. *Then the following conditions are equivalent*:

(1) $H \lhd^\omega L$.

(2) H ser L.

(3) H wser L.

PROOF. By Theorem 2.7 the conditions (2) and (3) are equivalent. By Theorem 3.1 and [9, Theorem 3.1] the conditions (1) and (3) are equivalent. Therefore the equivalence of (1), (2) and (3) is shown.

Under the assumption of Theorem 3.3, we see that $H^{L,\alpha} \lhd^\omega L$ for any ordinal α.

Finally we show the following result which sharpens [9, Proposition 3.5].

PROPOSITION 3.4. *Let* $L \in \mathrm{L(si)}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ *and let* H *be a finitely generated subalgebra of* L. *Then the following conditions are equivalent*:

(1) H ser L.
(2) H desc L.
(3) H si L.
(4) H wser L.
(5) H wdesc L.
(6) H wsi L.

PROOF. It is sufficient to show that (4) implies (3). Suppose H wser L. Since $L \in \mathrm{L(si)}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$, there exists a finite-dimensional solvable subideal K of L containing H. Then H wsi K. By using [8, Theorem 1] we have H si K. Therefore H si L.

REMARK. Let H be a finitely generated subalgebra of L. If $L \in \mathrm{L(si)}\mathfrak{F}$ then the above conditions (1), (2) and (3) are equivalent. If $L \in \mathrm{L(wsi)}\mathfrak{F}$ then the above conditions (4), (5) and (6) are equivalent. The proofs are similar to the above one.

4.

In this section we shall investigate properties of weakly descendant subalgebras.

We begin with the following lemma corresponding to [8, Lemma 3].

LEMMA 4.1. *Let* H *be a subalgebra of* L *and let* K *be an* H*-invariant subalgebra of* L *such that* $K^2 \leq H$. *For an ordinal* σ, *if* $H \leq_\sigma L$ *then* $H \lhd_\sigma H+K$.

PROOF. We may assume that $L=H+K$. First we show that $H^{L,\alpha}=H_{L,\alpha}$ for any ordinal α. To do this we use transfinite induction on α. It is trivial for $\alpha=0$. Let $\alpha>0$ and suppose that $H^{L,\beta}=H_{L,\beta}$ for any $\beta<\alpha$. If α is a limit ordinal, then clearly $H^{L,\alpha}=H_{L,\alpha}$. Suppose that α is not a limit ordinal. Then by induction hypothesis we have $H^{L,\alpha-1}=H_{L,\alpha-1}$. Since $L=H+K$, $H_{L,\beta}=H+(H_{L,\beta} \cap K)$ for any ordinal β. Hence

$$\begin{aligned}[][H_{L,\alpha}, H^{L,\alpha-1}] &= [H_{L,\alpha}, H_{L,\alpha-1}] = [H + (H_{L,\alpha} \cap K), H + (H_{L,\alpha-1} \cap K)] \\ &\subseteq H^2 + [H_{L,\alpha}, H] + [H, H_{L,\alpha-1}] + K^2 \subseteq H_{L,\alpha}.\end{aligned}$$

Therefore $H^{L,\alpha}=H^{H^{L,\alpha-1}} \subseteq H_{L,\alpha}$. By Lemma 1.3(3) we have $H^{L,\alpha}=H_{L,\alpha}$. In particular, $H^{L,\sigma}=H_{L,\sigma}$. Using Lemma 1.4 we obtain $H \lhd_\sigma L$.

In general, a weakly descendant subalgebra is not necessarily a descendant subalgebra (see Example 5.2). However, we have the following result.

THEOREM 4.2. *Let $L\in\mathfrak{A}\mathfrak{N}_n$ with $n\in\mathbf{N}$ and let H be a subalgebra of L. For an ordinal σ, if $H\leq_\sigma L$ then $H\lhd_{n+\sigma} L$.*

PROOF. There exists an abelian ideal A of L such that $L/A\in\mathfrak{N}_n$. Then by Lemma 4.1 we have $H\lhd_\sigma H+A$. On the other hand, $H+A/A\leq L/A\in\mathfrak{N}_n$ and hence $H+A/A\lhd^n L/A$. It follows that $H+A\lhd^n L$. Therefore we have $H\lhd_{n+\sigma} L$.

If σ is an infinite ordinal, then $n+\sigma=\sigma$ for any $n\in\mathbf{N}$. Thus we obtain

COROLLARY 4.3. *Let $L\in\mathfrak{A}\mathfrak{N}$ and let H be a subalgebra of L. Then for an infinite ordinal σ, $H\leq_\sigma L$ if and only if $H\lhd_\sigma L$.*

REMARK. The above result remains true, when $L\in\mathfrak{A}\mathfrak{D}$, where $\mathfrak{D}$ is the class of Lie algebras in which every subalgebra is a subideal.

Next we present some conditions for a subalgebra to be a weakly descendant subalgebra in an ideally finite Lie algebra. To do this we need the following lemma due to Stewart.

LEMMA 4.4 ([5, Proposition 3.5]). *If L is an ideally finite Lie algebra such that $L/\zeta_1(L)$ is countable-dimensional, then $L/\zeta_1(L)$ is embedded in a direct sum of countably many finite-dimensional Lie algebras.*

PROOF. Put $\bar{L}=L/\zeta_1(L)$. Then there exists an ascending chain $(L_i)_{i<\omega}$ of finite-dimensional ideals of $\bar{L}$ such that $L_0=0$ and $\bar{L}=\cup_{i<\omega}L_i$. For each $i\geq 1$, we can find an ideal K_i of $\bar{L}$ such that $L_i\cap K_i=0$ and $\bar{L}/K_i\in\mathfrak{F}$. Define $I_i=L_{i-1}+K_i$. Then $I_i\lhd\bar{L}$ and $\bar{L}/I_i\in\mathfrak{F}$. Furthermore, $L_i\cap I_i=L_{i-1}$. Hence $\cap_{i\geq 1}I_i=0$. Let θ denote the natural homomorphism of $\bar{L}$ into the Cartesian sum $\mathrm{Cr}_{i\geq 1}(\bar{L}/I_i)$. Then θ is injective and the image of $\bar{L}$ under θ is contained in the direct sum $\mathrm{Dr}_{i\geq 1}(\bar{L}/I_i)$.

THEOREM 4.5. *Let L be an ideally finite Lie algebra such that $L/\zeta_1(L)$ is countable-dimensional. Assume that H is a subalgebra of L such that $H+\zeta_1(L)/\zeta_1(L)\in\mathfrak{N}_n$ with $n\in\mathbf{N}$. If H* wser *L then $H\leq_{\omega+n+1} L$.*

PROOF. Put $\bar{L}=L/\zeta_1(L)$ and $\bar{H}=H+\zeta_1(L)/\zeta_1(L)$. By the proof of Lemma 4.4, $\bar{L}$ has a collection $\{I_i: i<\omega\}$ of ideals of finite codimension such that $\cap_{i<\omega}I_i=0$ and the image of $\bar{L}$ under the natural monomorphism $\theta\colon\bar{L}\to\mathrm{Cr}_{i<\omega}(\bar{L}/I_i)$ is contained in $\mathrm{Dr}_{i<\omega}(\bar{L}/I_i)$. Let π_i denote the projection map $\mathrm{Dr}_{i<\omega}(\bar{L}/I_i)\to\bar{L}/I_i$ and put $L_i=\pi_i\theta(\bar{L})$, $H_i=\pi_i\theta(\bar{H})$. By Proposition 2.5 H_i wser L_i. Since $L_i\in\mathfrak{F}$, there exists a $k=k(i)\in\mathbf{N}$ such that $[L_i,_k H_i]\subseteq H_i$. Hence

$$\begin{aligned}&\bigcap_{k<\omega}([\textstyle\sum_{i<\omega} L_i ,_k \sum_{i<\omega} H_i] + \sum_{i<\omega} H_i)\\ &= \bigcap_{k<\omega} \textstyle\sum_{i<\omega} ([L_i ,_k H_i] + H_i)\\ &= \textstyle\sum_{i<\omega} \bigcap_{k<\omega} ([L_i ,_k H_i] + H_i) = \sum_{i<\omega} H_i .\end{aligned}$$

By Lemma 1.4(2) we have $\sum_{i<\omega} H_i \leq_\omega \sum_{i<\omega} L_i$. Put $\bar{K} = \theta^{-1}(\theta(\bar{L}) \cap \sum_{i<\omega} H_i)$. Then $\bar{K} \leq_\omega \bar{L}$. On the other hand, it is easy to see that $\bar{K} = \bigcap_{i<\omega}(\bar{H} + I_i)$. Hence

$$[\bar{K},_n \bar{H}] \subseteq \bigcap_{i<\omega}(\bar{H}^{n+1} + I_i) = \bigcap_{i<\omega} I_i = 0.$$

It follows that $\bar{H} \leq_n \bar{K}$. Therefore $\bar{H} \leq_{\omega+n} \bar{L}$, whence $H + \zeta_1(L) \leq_{\omega+n} L$. Thus we have $H \leq_{\omega+n+1} L$.

By Theorems 3.1 and 4.5 we obtain

Corollary 4.6. *Let L be an ideally finite Lie algebra such that $L/\zeta_1(L)$ is countable-dimensional and let H be a nilpotent subalgebra of L. Then the following conditions are equivalent*:

(1) H wser L.
(2) H wdesc L.
(3) $H \leq_{\omega 2} L$.
(4) $H \leq^{\omega} L$.

In the above statement we cannot replace the condition (3) by the condition $H \leq_\omega L$ (see Example 5.4).

Corollary 4.7. *Let $L \in \mathrm{L}(\lhd)\mathfrak{F} \cap (\mathfrak{A}\mathfrak{N})$ such that $L/\zeta_1(L)$ is countable-dimensional and let H be a nilpotent subalgebra of L. Then the following conditions are equivalent*:

(1) H ser L.
(2) H desc L.
(3) $H \lhd_{\omega 2} L$.
(4) $H \lhd^{\omega} L$.
(5) H wser L.
(6) H wdesc L.
(7) $H \leq_{\omega 2} L$.

Proof. By Theorem 3.3 and Corollary 4.6 all the conditions except (2) and (3) are equivalent. By Corollary 4.3 the conditions (3) and (7) are equivalent. On the other hand, (3)⇒(2)⇒(1) is trivial. Therefore all the conditions (1)–(7) are equivalent.

5.

It is well known that the set of all serial subgroups of a locally finite group G is a complete sublattice of the lattice of all subgroups of G (cf. [2, Theorem A]). However, the following example shows that the set of all weakly serial subalgebras of a locally finite Lie algebra L is not necessarily a sublattice of the lattice of all subalgebras of L, even if L is finite-dimensional.

EXAMPLE 5.1. *Let $\mathfrak{k}$ be a field of characteristic zero. Then there exists a finite-dimensional Lie algebra L over $\mathfrak{k}$ containing x, y such that*

(1) $\langle x\rangle \operatorname{wsi} L$ *and* $\langle y\rangle \operatorname{wsi} L$,

(2) $\langle x, y\rangle$ *is not a weak subideal of* L.

In fact, let L be a split simple Lie algebra over $\mathfrak{k}$ of type A_2. Then it is a well-known fact that L has linearly independent generators $\{e_1, e_2, f_1, f_2, h_1, h_2\}$ satisfying the following relations, where $i, j \in \{1, 2\}$:

$$[e_i, f_j] = \delta_{ij}h_i,$$

$$[e_i, h_j] = \begin{cases} 2e_i & \text{if } i = j, \\ -e_i & \text{if } i \neq j, \end{cases}$$

$$[f_i, h_j] = \begin{cases} -2f_i & \text{if } i = j, \\ f_i & \text{if } i \neq j, \end{cases}$$

$$[e_{i},_2 e_j] = [f_{i},_2 f_j] = 0 \qquad \text{if } i \neq j.$$

Clearly $e_1, f_1 \in \mathfrak{e}(L)$. Therefore $\langle e_1\rangle \operatorname{wsi} L$ and $\langle f_1\rangle \operatorname{wsi} L$. If $\langle e_1, f_1\rangle \operatorname{wsi} L$, then we can find an $n \in \mathbf{N}$ such that $[L,_n h_1] \subseteq \langle e_1, f_1\rangle$. But $[f_2,_n h_1] = f_2 \notin \langle e_1\rangle + \langle f_1\rangle + \langle h_1\rangle = \langle e_1, f_1\rangle$, a contradiction. Thus $\langle e_1, f_1\rangle$ is not a weak subideal of L.

The following example shows that a weakly descendant subalgebra is not necessarily a descendant subalgebra.

EXAMPLE 5.2. *Let $\mathfrak{k}$ be a field of characteristic zero. Then there exists a countable-dimensional abelian-by-simple Lie algebra L over $\mathfrak{k}$ containing x such that*

(1) $\langle x\rangle \leq_{\omega} L$,

(2) $\langle x\rangle$ *is not a descendant subalgebra of* L.

In fact, let A be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{a_1, a_2, \ldots\}$ and let x, y, z be respectively the derivations of A defined by

$$a_i x = a_{i+1} \qquad (i \geq 1),$$
$$a_1 y = 0 \quad \text{and} \quad a_i y = i(i-1)a_{i-1} \qquad (i \geq 2),$$
$$a_i z = 2ia_i \qquad (i \geq 1).$$

Put $S = \langle x, y, z \rangle \leq \mathrm{Der}(A)$. Then S is a 3-dimensional split simple Lie algebra with multiplication

$$[x, z] = 2x, \quad [y, z] = -2y, \quad [x, y] = z.$$

We construct the split extension $L = A \dotplus S$, which is the example described in [8, § 3]. Then L is abelian-by-simple. It is easy to see that

$$\langle x \rangle_{L,1} = \langle a_2, a_3, \cdots \rangle + \langle x, z \rangle,$$
$$\langle x \rangle_{L,n} = \langle a_{n+1}, a_{n+2}, \cdots \rangle + \langle x \rangle \qquad (n \geq 2).$$

Hence $\langle x \rangle_{L,\omega} = \bigcap_{n<\omega} \langle x \rangle_{L,n} = \langle x \rangle$. By Lemma 1.4(2) we have $\langle x \rangle \leq_{\omega} L$. On the other hand, clearly $\langle x \rangle^{L,1} = \langle x^L \rangle = L$ and therefore $\langle x \rangle^{L,\alpha} = L$ for any ordinal α. Using Lemma 1.4(1) we have (2).

The following example shows that in Theorem 4.5, Corollaries 4.6 and 4.7 we cannot remove the assumption that L is ideally finite.

EXAMPLE 5.3. *There exists a countable-dimensional Lie algebra L containing x such that*

(1) $L \in \mathrm{L}\mathfrak{F} \cap \mathfrak{A}^2$ *and* $L \notin \mathrm{L}(\lhd)\mathfrak{F}$,

(2) $\langle x \rangle \lhd^{\omega+1} L$,

(3) $\langle x \rangle$ *is not a weakly descendant subalgebra of L.*

In fact, let A be an abelian Lie algebra with basis $\{a_0, a_1, \dots\}$ and let x be the derivation of A defined by

$$a_0 x = 0 \quad \text{and} \quad a_i x = a_{i-1} \qquad (i \geq 1).$$

We construct the split extension $L = A \dotplus \langle x \rangle$, which is the example described in [1, p. 119]. Then $L \in \mathfrak{Z}_{\omega+1} \cap \mathfrak{A}^2 \leq \mathrm{L}\mathfrak{F} \cap \mathfrak{A}^2$. Therefore $\langle x \rangle \lhd^{\omega+1} L$. On the other hand, $\langle x \rangle_{L,1} = [L, x] + \langle x \rangle = A + \langle x \rangle = L$ and hence $\langle x \rangle_{L,\alpha} = L$ for any ordinal α. By using Lemma 1.4(2) we have (3). Since $\langle x \rangle_{L,1} = L$, $\langle x^L \rangle = \langle x \rangle^{L,1} = L$ and therefore $L \notin \mathrm{L}(\lhd)\mathfrak{F}$.

Finally we present an example showing that an ω-step weakly ascendant subalgebra of an ideally finite Lie algebra is not necessarily an ω-step weakly descendant subalgebra.

EXAMPLE 5.4. *There exist a countable-dimensional Lie algebra L and an abelian subalgebra H of L such that*

(1) $L \in \mathrm{L}(\lhd)\mathfrak{F} \cap \mathfrak{A}^2$,
(2) $H \leq^{\omega} L$,
(3) $H \not\leq_{\omega} L$.

In fact, let A be an abelian Lie algebra with basis $\{a_0, a_1, \dots\}$. For each $n \geq 1$, put $A_n = \langle a_i \colon 0 \leq i \leq n \rangle \leq A$ and let x_n be the derivation of A_n defined by

$$a_0 x_n = 0 \quad \text{and} \quad a_i x_n = a_{i-1} \qquad (1 \leq i \leq n).$$

Form the split extension $L_n = A_n \dotplus \langle x_n \rangle$ of A_n by $\langle x_n \rangle$. Then it is easy to see that for any $n \geq 1$

$$\langle x_n \rangle \leq^{n+1} L_n \quad \text{and} \quad \zeta_1(L_n) = \langle a_0 \rangle .$$

First we show that there exist a Lie algebra L containing a and monomorphisms $\theta_n \colon L_n \to L$, $n \geq 1$, such that

(a) $L = \sum_{n \geq 1} \theta_n(L_n)$,
(b) $[\theta_m(L_m), \theta_n(L_n)] = 0$ if $m \neq n$,
(c) $a = \theta_n(a_0)$ for all $n \geq 1$.

Put $D = \mathrm{Dr}_{n \geq 1} L_n$ and let ι_n denote the inclusion map of L_n into D. Put $I = \langle -\iota_n(a_0) + \iota_{n+1}(a_0) \colon n \geq 1 \rangle \leq D$. Since $I \leq \zeta_1(D)$, $I \lhd D$. Define $L = D/I$. Let ρ denote the natural homomorphism of D onto L and let $\theta_n = \rho \circ \iota_n$ for each $n \in \mathbf{N}$. We can easily see that $\iota_n(L_n) \cap I = 0$. Hence each θ_n is injective. Clearly we have (a) and (b). Furthermore, for any $n \geq 1$

$$\iota_{n+1}(a_0) = \iota_n(a_0) + (-\iota_n(a_0) + \iota_{n+1}(a_0)) \in \iota_n(a_0) + I$$

and therefore $\theta_n(a_0) = \theta_{n+1}(a_0)$. (c) is also proved.

Each $\theta_n(L_n)$ is a finite-dimensional metabelian ideal of L. Hence L is an ideally finite metabelian Lie algebra. Also L is countable-dimensional. Define $H = \sum_{n \geq 1} \langle \theta_n(x_n) \rangle \leq L$. Then H is abelian. Since $\langle \theta_n(x_n) \rangle \leq^{n+1} \theta_n(L_n)$ for any $n \geq 1$, we have $H \leq^{\omega} L$. On the other hand, for any $n \geq 1$

$$a = \theta_n(a_0) = [\theta_n(a_n),_n \theta_n(x_n)] \in [L,_n H] .$$

Hence $a \in \bigcap_{n < \omega} ([L,_n H] + H) = H_{L,\omega}$. But clearly $a \notin H$. Therefore $H \subsetneqq H_{L,\omega}$. Using Lemma 1.4(2) we have $H \not\leq_{\omega} L$. (It is actually shown that $H \leq_{\omega+1} L$.)

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