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Dirichlet problem for a semi-linearly perturbed structure of a harmonic space

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Introduction

In [3], the author considered a semi-linear perturbation of a harmonic space and discussed Dirichlet problems of Perron-Brelot type with respect to the perturbed structure. In the present note, we further investigate such Dirichlet problems. In § 2, we are concerned with the problem whether a bounded boundary function, which is resolutive with respect to the original structure, remains resolutive with respect to the perturbed structure. Then, in § 3, we give sufficient conditions for a boundary point to be regular with respect to the Dirichlet problem for the perturbed structure. The results in § 3 are extensions of those in [2] where linear perturbations are treated.

As a simple but typical example to which our theory can be applied, consider a semi-linear equation

$$(*) \qquad \Delta u = q(x)\psi(u)$$

on a domain $\Omega \subset \mathbf{R}^n$ ($n \geqq 2$; Ω: hyperbolic if $n=2$), where q is a non-negative function belonging to $L^r_{\mathrm{loc}}(\Omega)$ with $r > n/2$ and ψ is a non-decreasing locally Lipschitz-continuous function on $\mathbf{R}$ such that $\psi(t_0)=0$ for some $t_0 \in \mathbf{R}$. For a compactification Ω^* of Ω and a bounded function φ on $\Omega^* \setminus \Omega$ which is resolutive with respect to $\Delta u = 0$, our theorems in § 2 imply the following results:

(i) Without any further assumptions on ψ, if $\varphi \geqq t_0$ or $\varphi \leqq t_0$, then φ is resolutive with respect to $(*)$;

(ii) If either ψ^+ or ψ^- is convex, then φ is always resolutive with respect to $(*)$.

As to regularity, our results in § 3 show that $\xi \in \Omega^* \setminus \Omega$ is regular with respect to the Dirichlet problem for $(*)$ if it is regular for $\Delta u = 0$ and if there exist an open neighborhood V of ξ in Ω^* and a potential p on $V \cap \Omega$ such that $p(x) \to 0$ as $x \to \xi$ and $\Delta p = -q$ on $V \cap \Omega$. Note that these conditions do not refer to the function ψ.

§ 1. Notation and basic assumptions

Let $(X, \mathscr{U})$ be a harmonic space in the sense of Constantinescu-Cornea [1] and assume that X has a countable base. The sheaf of harmonic functions will be denoted by $\mathscr{H}$, and that of continuous superharmonic functions by $\mathscr{U}_C$.

An open set U is called a P-set if there is a potential on U which is positive everywhere. U is called a PC-set if it is relatively compact and $\bar{U}\subset U'$ for some P-set U'. For a P-set U, let $\mathscr{P}_C(U)$ (resp. $\mathscr{P}_{BC}(U)$) be the set of all continuous (resp. bounded continuous) potentials on U.

By $\mathscr{R}$ we denote the sheaf of functions which are locally expressible as the difference of two continuous superharmonic functions. Let $\mathscr{M}$ denote the sheaf of signed Radon measures on X. A measure representation σ on X is a sheaf homomorphism of $\mathscr{R}$ into $\mathscr{M}$, with linear structures both in $\mathscr{R}(U)$ and $\mathscr{M}(U)$ for each open set U, such that $\sigma(f)\geqq 0$ on U if and only if f is superharmonic on U. We assume the existence of a measure representation σ and fix it once for all.

Let $\mathscr{M}_\sigma$ be the subsheaf of $\mathscr{M}$ consisting of measures which are locally images of σ. For a P-set U, let

$$\mathscr{M}_P(U) = \{\sigma(p)\,|\, p\in\mathscr{P}_C(U)\} \quad\text{and}\quad \mathscr{M}_{BP}(U) = \{\sigma(p)\,|\, p\in\mathscr{P}_{BC}(U)\}.$$

Note that $\mathscr{M}_{BP}(U)\subset\mathscr{M}_P(U)\subset \mathscr{M}_\sigma^+(U)=\{\mu\in\mathscr{M}_\sigma(U)\,|\,\mu\geqq 0\}$.

As in [3], we consider a sheaf morphism $F\colon \mathscr{R}\to\mathscr{M}_\sigma$ which satisfies the following two conditions:

(F.1) F is monotone, i.e., if $f_1, f_2\in\mathscr{R}(U)$ and $f_1\leqq f_2$ on U, then $F(f_1)\leqq F(f_2)$ on U;

(F.2) F satisfies condition (L) on every PC-set in X, i.e., for each PC-set U in X and for each $M>0$, there is $\pi_{U,M}\in\mathscr{M}_{BP}(U)$ such that

$$F(f_1)-F(f_2)\leqq (f_1-f_2)\pi_{U,M} \quad \text{on } U$$

whenever $f_1, f_2\in\mathscr{R}(U)$, $f_1\geqq f_2$ on U and $|f_i|\leqq M$ on U, $i=1, 2$.

We define sheaves $\mathscr{H}^F$, $\mathscr{U}_C^F$ and $\mathscr{V}_C^F$ by

$$\mathscr{H}^F(U) = \{u\in\mathscr{R}(U)\,|\,\sigma(u)+F(u)=0 \quad \text{on } U\},$$

$$\mathscr{U}_C^F(U) = \{u\in\mathscr{R}(U)\,|\,\sigma(u)+F(u)\geqq 0 \quad \text{on } U\},$$

$$\mathscr{V}_C^F(U) = \{u\in\mathscr{R}(U)\,|\,\sigma(u)+F(u)\leqq 0 \quad \text{on } U\}.$$

§2. Resolutivity

In what follows, we assume that $1\in\mathscr{R}(X)$, X is a P-set and $|\sigma(1)|\in\mathscr{M}_{BP}(X)$. Let X^* be a compactification of X and let $\partial^*X=X^*\setminus X$. We know ([3; Proposition 4.1]) the following comparison principle:

PROPOSITION A. *If $u\in\mathscr{U}_C^F(X)$, $v\in\mathscr{V}_C^F(X)$, $p\in\mathscr{P}_C(X)$ and*

$$\liminf_{x\to\xi}\{u(x)-v(x)+p(x)\}\geqq 0$$

for all $\xi \in \partial^* X$, *then* $u \geqq v$ *on* X.

For a bounded (real) function φ on $\partial^* X$, we define

$$\bar{\mathscr{F}}_{\varphi}^{F,X^*} = \{u \in \mathscr{U}_C^F(X) \mid \liminf_{x\to\xi} u(x) \geqq \varphi(\xi) \ \text{ for all } \ \xi \in \partial^* X\},$$

$$\underline{\mathscr{F}}_{\varphi}^{F,X^*} = \{v \in \mathscr{V}_C^F(X) \mid \limsup_{x\to\xi} v(x) \leqq \varphi(\xi) \ \text{ for all } \ \xi \in \partial^* X\}.$$

If $\bar{\mathscr{F}}_{\varphi}^{F,X^*} \neq \phi$ (resp. $\underline{\mathscr{F}}_{\varphi}^{F,X^*} \neq \phi$), then we write $\bar{H}_{\varphi}^{F,X^*} = \inf \bar{\mathscr{F}}_{\varphi}^{F,X^*}$ (resp. $\underline{H}_{\varphi}^{F,X^*} = \sup \underline{\mathscr{F}}_{\varphi}^{F,X^*}$). By Proposition A, $\underline{H}_{\varphi}^{F,X^*} \leqq \bar{H}_{\varphi}^{F,X^*}$ if both exist. We say that φ is *F-resolutive* if $\bar{\mathscr{F}}_{\varphi}^{F,X^*}$ and $\underline{\mathscr{F}}_{\varphi}^{F,X^*}$ are both non-empty, $\bar{H}_{\varphi}^{F,X^*} = \underline{H}_{\varphi}^{F,X^*}$ and it belongs to $\mathscr{H}^F(X)$. In this case, we denote the common function by H_{φ}^{F,X^*}.

In case $F=0$, the index F will be omitted in terminologies and notation. Note that constant functions on $\partial^* X$ are resolutive for any compactification X^*; in fact, if we choose $p_1, p_2 \in \mathscr{P}_{BC}(X)$ such that $\sigma(1) = \sigma(p_1) - \sigma(p_2)$, then $1 + p_2 \in \bar{\mathscr{F}}_1^{X^*}$ and $1 - p_1 \in \underline{\mathscr{F}}_1^{X^*}$, which implies that 1 is resolutive and $H_1^{X^*} = 1 - p_1 + p_2$.

In the rest of this section, we fix a compactification X^* and omit the index X^*, i.e., $\bar{\mathscr{F}}_{\varphi}^F = \bar{\mathscr{F}}_{\varphi}^{F,X^*}$, $\bar{H}_{\varphi}^F = \bar{H}_{\varphi}^{F,X^*}$, etc.

The next lemma is an improvement of [3; Lemma 4.2]:

LEMMA 1. *If there exists a bounded function* $f \in \mathscr{R}(X)$ *such that* $F(f)^- \in \mathscr{M}_P(X)$ *(resp.* $F(f)^+ \in \mathscr{M}_P(X)$*), then* $\bar{\mathscr{F}}_{\varphi}^F \neq \phi$ *(resp.* $\underline{\mathscr{F}}_{\varphi}^F \neq \phi$*) for any bounded function* φ *on* $\partial^* X$. *If, moreover,* $F(f)^- \in \mathscr{M}_{BP}(X)$ *(resp.* $F(f)^+ \in \mathscr{M}_{BP}(X)$*), then* $\bar{\mathscr{F}}_{\varphi}^F$ *(resp.* $\underline{\mathscr{F}}_{\varphi}^F$*) contains bounded functions.*

PROOF. Choose $p \in \mathscr{P}_C(X)$ such that $\sigma(p) = F(f)^-$. Given φ, put $M_{\varphi} = \max(0, \sup_X f, \sup_{\partial^* X} \varphi)$. Choose a bounded function $s_0 \in \mathscr{U}_C(X)$ such that $s_0 \geqq 1$ (see [3]) and consider the function $u = M_{\varphi} s_0 + p$. Then $u \geqq M_{\varphi}$ on X and

$$\begin{aligned}\sigma(u) + F(u) &= M_{\varphi}\sigma(s_0) + \sigma(p) + F(M_{\varphi}s_0 + p)\\ &\geqq F(f)^- + F(M_{\varphi}s_0 + p)\\ &\geqq -F(f) + F(M_{\varphi}s_0 + p) \geqq 0.\end{aligned}$$

Hence $u \in \bar{\mathscr{F}}_{\varphi}^F$. Furthermore, if $F(f)^- \in \mathscr{M}_{BP}(X)$, then p is bounded, and hence u is bounded.

Now we prove

THEOREM 1. *Suppose there exist bounded functions* $f_1, f_2 \in \mathscr{R}(X)$ *such that* $F(f_1)^- \in \mathscr{M}_P(X)$ *and* $F(f_2)^+ \in \mathscr{M}_P(X)$. *If* φ *is a bounded resolutive function on* $\partial^* X$ *such that either* $H_{\varphi} \geqq f_1$ *or* $H_{\varphi} \leqq f_2$ *and if* $\bar{H}_{\varphi}^F, \underline{H}_{\varphi}^F \in \mathscr{H}^F(X)$, *then* φ *is F-resolutive. Furthermore, in case* $H_{\varphi} \geqq f_1$ *(resp.* $H_{\varphi} \leqq f_2$*),*

$$H_{\varphi}^F \leqq H_{\varphi} + p_1 \quad (\textit{resp. } H_{\varphi}^F \geqq H_{\varphi} - p_2)$$

with $p_1 \in \mathscr{P}_C(X)$ *such that* $\sigma(p_1)=F(f_1)^-$ *(resp.* $p_2 \in \mathscr{P}_C(X)$ *such that* $\sigma(p_2)=F(f_2)^+$*).*

PROOF. Let $u \in \bar{\mathscr{F}}_\varphi$. Since $u \geqq H_\varphi \geqq f_1$,

$$\begin{aligned}\sigma(u+p_1) + F(u+p_1) &= \sigma(u) + \sigma(p_1) + F(u+p_1)\\ &\geqq F(f_1)^- + F(u+p_1) \geqq -F(f_1) + F(u+p_1) \geqq 0.\end{aligned}$$

Hence, $u+p_1 \in \bar{\mathscr{F}}^F_\varphi$, so that $u+p_1 \geqq \bar{H}^F_\varphi$. Taking the infimum in u, we obtain $H_\varphi + p_1 \geqq \bar{H}^F_\varphi$.

We can take $s \in \mathscr{U}^+_C(X)$ such that $H_\varphi - \varepsilon s \in \underline{\mathscr{F}}_\varphi$ for all $\varepsilon > 0$ (cf. [1; Exercise 2.4.8]; note that we can choose $w_n \in \underline{\mathscr{F}}_\varphi$ such that $w_n \uparrow H_\varphi$ locally uniformly on X). Let

$$v_\varepsilon = \bar{H}^F_\varphi - p_1 - \varepsilon s \quad (\varepsilon > 0).$$

Then $v_\varepsilon \leqq H_\varphi - \varepsilon s$ and

$$\begin{aligned}\sigma(v_\varepsilon) + F(v_\varepsilon) &= \sigma(\bar{H}^F_\varphi) - \sigma(p_1) - \varepsilon\sigma(s) + F(\bar{H}^F_\varphi - p_1 - \varepsilon s)\\ &\leqq -F(\bar{H}^F_\varphi) + F(\bar{H}^F_\varphi - p_1 - \varepsilon s) \leqq 0.\end{aligned}$$

Hence $v_\varepsilon \in \underline{\mathscr{F}}^F_\varphi$, so that $v_\varepsilon \leqq \underline{H}^F_\varphi$ for all $\varepsilon > 0$. Therefore

$$\bar{H}^F_\varphi - p_1 \leqq \underline{H}^F_\varphi .$$

By assumption $\underline{H}^F_\varphi, \bar{H}^F_\varphi \in \mathscr{H}^F(X)$. Hence, by Proposition A, $\bar{H}^F_\varphi \leqq \underline{H}^F_\varphi$, so that $\bar{H}^F_\varphi = \underline{H}^F_\varphi$.

We shall say that F satisfies condition (P) (resp. (PB)) if there exist bounded functions $f_1, f_2 \in \mathscr{R}(X)$ such that $F(f_1)^- \in \mathscr{M}_P(X)$ (resp. $\mathscr{M}_{BP}(X)$) and $F(f_2)^+ \in \mathscr{M}_P(X)$ (resp. $\mathscr{M}_{BP}(X)$).

By the above theorem and [3; Proposition 4.2], we obtain the following improvement of [3; Theorem 4.1]:

COROLLARY. *Suppose there is a covering of X by regular PC-sets and suppose F satisfies condition* (P). *If φ is a bounded resolutive function on $\partial^* X$ such that either $F(H_\varphi)^- \in \mathscr{M}_P(X)$ or $F(H_\varphi)^+ \in \mathscr{M}_P(X)$, then φ is F-resolutive. If, in particular, $|F(H_\varphi)| \in \mathscr{M}_P(X)$, then $|H^F_\varphi - H_\varphi| \leqq p$ with $p \in \mathscr{P}_C(X)$ such that $\sigma(p) = |F(H_\varphi)|$.*

It is an open question whether every bounded resolutive function on $\partial^* X$ is F-resolutive if F satisfies condition (P). In this connection, we have the following

THEOREM 2. *Suppose there is a covering of X by regular PC-sets and F satisfies condition* (PB) *and the following condition* $(C)_+$ *or* $(C)_-$:

$(C)_+$ *(resp.* $(C)_-$*)* *For each $M > 0$, there exists $v_M \in \mathscr{M}_P(X)$ such that*

$$F(f_2)^+ - F(f_1)^+ \leqq F(f_2+g) - F(f_1+g) + \nu_M$$
$$(resp.\ F(f_1)^- - F(f_2)^- \leqq F(f_2-g) - F(f_1-g) + \nu_M)$$

for any $f_1, f_2, g \in \mathscr{R}(X)$ *satisfying* $-M \leqq f_1 \leqq f_2 \leqq M$ *and* $0 \leqq g \leqq M$.

Then any bounded resolutive function on $\partial^* X$ *is F-resolutive.*

PROOF. We assume $(C)_+$. Let F^+ be the sheaf morphism $\mathscr{R} \to \mathscr{M}_\sigma$ defined by $F^+(f) = F(f)^+$. It is easy to see that F^+ satisfies conditions (F.1), (F.2) and (PB). Let φ be a given bounded resolutive function on $\partial^* X$. Since $F^+(H_\varphi)^- = 0$, the above corollary implies that φ is F^+-resolutive. Put $f_0 = H_\varphi^{F^+}$. By Lemma 1, f_0 is bounded. Obviously the sheaf morphism $\tilde{F}$: $\mathscr{R} \to \mathscr{M}_\sigma$ defined by

$$\tilde{F}(f) = F(f_0+f) - F^+(f_0)$$

satisfies (F.1) and (F.2). Since $\tilde{F}(0) = F(f_0) - F^+(f_0) \leqq 0$, $\tilde{F}(0)^+ = 0 \in \mathscr{M}_{PB}(X)$. Let $F(\lambda_1)^- \in \mathscr{M}_{PB}(X)$ and put $\lambda_2 = \max(0, \lambda_1 - \inf_X f_0)$. Then $\tilde{F}(\lambda_2) = F(f_0+\lambda_2) - F^+(f_0) \geqq F(f_0+\lambda_2) - F^+(f_0+\lambda_2) = -F(f_0+\lambda_2)^- \geqq -F(\lambda_1)^-$, so that $\tilde{F}(\lambda_2)^- \leqq F(\lambda_1)^-$. Hence $\tilde{F}$ satisfies condition (PB). Since $\tilde{F}(0) \leqq 0$, 0 is $\tilde{F}$-resolutive by the above corollary and $H_0^{\tilde{F}} \geqq 0$. By Lemma 1, we can choose $v_0 \in \bar{\mathscr{F}}_\varphi^{F^+}$, $w_0 \in \underline{\mathscr{F}}_\varphi^{F^+}$ and $g_0 \in \bar{\mathscr{F}}_0^{\tilde{F}}$ which are all bounded. Let $M = \max(\sup_X |v_0|, \sup_X |w_0|, \sup_X g_0)$, and choose $p \in \mathscr{P}_C(X)$ such that $\sigma(p) = \nu_M$.

If $v \in \bar{\mathscr{F}}_\varphi^{F^+}$, $v \leqq v_0$ and $g \in \bar{\mathscr{F}}_0^{\tilde{F}}$, $g \leqq g_0$, then $-M \leqq f_0 \leqq v \leqq M$ and $0 \leqq H_0^{\tilde{F}} \leqq g \leqq M$. Hence using condition $(C)_+$, we have

$$\begin{aligned}\sigma(v+g+p) + F(v+g+p) &\geqq -F^+(v) - \tilde{F}(g) + \nu_M + F(v+g)\\ &= -F^+(v) + F^+(f_0) - F(f_0+g) + F(v+g) + \nu_M \geqq 0.\end{aligned}$$

It follows that $v+g+p \in \bar{\mathscr{F}}_\varphi^F$, so that $v+g+p \geqq \bar{H}_\varphi^F$. Taking the infimums in v and g (note that $\bar{\mathscr{F}}_\varphi^{F^+}$, $\bar{\mathscr{F}}_0^{\tilde{F}}$ are lower directed; cf. [3]), we obtain

$$(1) \qquad H_\varphi^{F^+} + H_0^{\tilde{F}} + p \geqq \bar{H}_\varphi^F.$$

Next, let $w \in \underline{\mathscr{F}}_\varphi^{F^+}$, $w \geqq w_0$ and $\tilde{g} \in \underline{\mathscr{F}}_0^{\tilde{F}}$, $\tilde{g} \geqq 0$. Then $-M \leqq w \leqq f_0 \leqq M$ and $0 \leqq \tilde{g} \leqq H_0^{\tilde{F}} \leqq M$. Hence, again by $(C)_+$, we have $\sigma(w+\tilde{g}-p) + F(w+\tilde{g}-p) \leqq 0$, which implies

$$(2) \qquad H_\varphi^{F^+} + H_0^{\tilde{F}} - p \leqq \underline{H}_\varphi^F.$$

By (1) and (2),

$$0 \leqq \bar{H}_\varphi^F - \underline{H}_\varphi^F \leqq 2p.$$

Since $\bar{H}_\varphi^F$, $\underline{H}_\varphi^F \in \mathscr{H}^F(X)$ by [3; Proposition 4.2], Proposition A implies that $\bar{H}_\varphi^F = \underline{H}_\varphi^F$.

REMARK. In the above proof, $H_{\varphi}^{F^+} + H_0^{\tilde{F}} \in \mathscr{H}^F(X)$, and hence

$$H_{\varphi}^F = H_{\varphi}^{F^+} + H_0^{\tilde{F}}.$$

Now we give a sufficient condition for $(C)_{\pm}$.

PROPOSITION 1. *Let $\Psi: \mathscr{R} \to \mathscr{M}_{\sigma}$ be a sheaf morphism satisfying* (F.1) *and* (F.2) *and suppose $t \mapsto \Psi(t)$ is a convex mapping from* $\mathbf{R}$ *into $\mathscr{M}_{\sigma}(X)$. If for each $M>0$ there exists $v'_M \in \mathscr{M}_P(X)$ such that*

$$|F^+(f) - \Psi(f)| \leqq v'_M \quad (resp.\ |F^-(f) - \Psi(-f)| \leqq v'_M)$$

for any $f \in \mathscr{R}(X)$ with $|f| \leqq M$, then F satisfies condition $(C)_+$ *(resp.* $(C)_-$*).*

PROOF. First, we show that

$$\Psi(f_2) - \Psi(f_1) \leqq \Psi(f_2+g) - \Psi(f_1+g) \tag{3}$$

for any $f_1, f_2, g \in \mathscr{R}(X)$ such that $f_2 \geqq f_1$ and $g \geqq 0$. Let U be any PC-set in X and we show that (3) holds on U. This inequality is readily verified in case f_1, f_2, g are constant functions; in fact, $t \mapsto \int_U \varphi \, d\Psi(t)$ is a non-decreasing convex real function on $\mathbf{R}$ for any non-negative bounded Borel function φ on U. Let $M_1 = \sup_U |f_1| + \sup_U |f_2+g| + 1$. By condition (F.2) for Ψ, there is $\tau \in \mathscr{M}_{BP}(U)$ such that

$$\Psi(u) - \Psi(v) \leqq (u-v)\tau$$

for every $u, v \in \mathscr{R}(U)$ such that $u \geqq v$ and $|u| \leqq M_1$, $|v| \leqq M_1$. Let $\varepsilon>0$ $(\varepsilon<1/2)$ be arbitrarily given. For each $x_0 \in U$, we can find an open neighborhood U_{x_0} of x_0 such that $|f_i(x)-f_i(x_0)|<\varepsilon$ $(i=1, 2)$ and $|g(x)-g(x_0)|<\varepsilon$ for $x \in U_{x_0}$. Then, on U_{x_0},

$$\begin{aligned}
&\Psi(f_2+g) - \Psi(f_1+g) - \Psi(f_2) + \Psi(f_1) \\
&\quad \geqq \Psi(f_2(x_0)+g(x_0)-2\varepsilon) - \Psi(f_1(x_0)+g(x_0)+2\varepsilon) - \Psi(f_2(x_0)+\varepsilon) + \Psi(f_1(x_0)-\varepsilon) \\
&\quad \geqq \Psi(f_2(x_0)+g(x_0)) - \Psi(f_1(x_0)+g(x_0)) - \Psi(f_2(x_0)) + \Psi(f_1(x_0)) - 6\varepsilon\tau \\
&\quad \geqq -6\varepsilon\tau.
\end{aligned}$$

Since x_0 is arbitrary, it follows that

$$\Psi(f_2+g) - \Psi(f_1+g) - \Psi(f_2) + \Psi(f_1) \geqq -6\varepsilon\tau$$

holds on U. Now, $\varepsilon>0$ is also arbitrary, so that we obtain (3) on U.

Now, if $f_1, f_2, g \in \mathscr{R}(X)$, $-M \leqq f_1 \leqq f_2 \leqq M$ and $0 \leqq g \leqq M$, then

$$\begin{aligned} F(f_2)^+ - F(f_1)^+ &\leqq \Psi(f_2) - \Psi(f_1) + 2\nu'_M \\ &\leqq \Psi(f_2+g) - \Psi(f_1+g) + 2\nu'_M \\ &\leqq F(f_2+g)^+ - F(f_1+g)^+ + 2(\nu'_M + \nu'_{2M}) \\ &\leqq F(f_2+g) - F(f_1+g) + 2(\nu'_M + \nu'_{2M}). \end{aligned}$$

Hence condition $(\mathrm{C})_+$ is satisfied with $\nu_M = 2(\nu'_M + \nu'_{2M})$.

EXAMPLE. Let ψ be a non-decreasing, locally Lipschitz-continuous function on $\mathbf{R}$ such that $[\psi - \psi(t_0)]^+$ is convex for some $t_0 \in \mathbf{R}$. Let $\mu \in \mathcal{M}_\sigma^+(X)$ and $\nu \in \mathcal{M}_\sigma(X)$ satisfy $\psi(t_0)\mu \leqq \nu \leqq \psi(t_1)\mu$ for some $t_1 \geqq t_0$. Let $G: \mathcal{R} \to \mathcal{M}_\sigma$ be a sheaf morphism satisfying (F.1) and (F.2) and suppose $|G(\lambda)| \in \mathcal{M}_P(X)$ for all $\lambda \in \mathbf{R}$ and $|G(\lambda_0)| \in \mathcal{M}_{BP}(X)$ for some $\lambda_0 \in \mathbf{R}$. Then

$$F(f) = \psi(f)\mu - \nu + G(f)$$

satisfies conditions (F.1), (F.2), (PB) and $(\mathrm{C})_+$.

PROOF. Obviously, F satisfies (F.1) and (F.2). Let $\lambda_1 = \max(t_1, \lambda_0)$ and $\lambda_2 = \min(t_0, \lambda_0)$. Then we see that $F(\lambda_1) \geqq G(\lambda_0) \geqq F(\lambda_2)$. Since $|G(\lambda_0)| \in \mathcal{M}_{BP}(X)$, it follows that (PB) is satisfied. Next, let $\Psi(f) = (\psi(f)\mu - \nu)^+$. Then Ψ satisfies (F.1) and (F.2). Since $\nu - \psi(t_0)\mu \geqq 0$,

$$\begin{aligned} \Psi(t) &= \sup\{\nu - \psi(t_0)\mu, [\psi(t) - \psi(t_0)]\mu\} - \nu + \psi(t_0)\mu \\ &= \sup\{\nu - \psi(t_0)\mu, [\psi(t) - \psi(t_0)]^+\mu\} - \nu + \psi(t_0)\mu \\ &= \{[\psi(t) - \psi(t_0)]^+\mu - \nu + \psi(t_0)\mu\}^+. \end{aligned}$$

Since $[\psi(t) - \psi(t_0)]^+$ is convex, it follows that $t \mapsto \Psi(t)$ is a convex mapping. Furthermore, if $f \in \mathcal{R}(X)$ and $|f| \leqq M$, then

$$\begin{aligned} |F^+(f) - \Psi(f)| &= |(\psi(f)\mu - \nu + G(f))^+ - (\psi(f)\mu - \nu)^+| \\ &\leqq |G(f)| \leqq \sup\{|G(-M)|, |G(M)|\}. \end{aligned}$$

Hence, by Proposition 1, condition $(\mathrm{C})_+$ is satisfied.

The above example includes as a special case the following F: $F(f) = \psi(f)\mu$ with $\mu \in \mathcal{M}_\sigma^+(X)$ and a non-decreasing locally Lipschitz-continuous function ψ on $\mathbf{R}$ such that $\psi(t_0) = 0$ for some $t_0 \in \mathbf{R}$ and ψ^+ is convex on $\mathbf{R}$. Typical such ψ's are

$$\psi(t) = |t|^\alpha \operatorname{sgn} t \quad (\alpha \geqq 1), \qquad \psi(t) = e^t - 1.$$

§ 3. *F*-regularity of boundary points

Let $\xi \in \partial^* X$ be a regular point for the Dirichlet problem with respect to the original structure $\mathscr{H}$, i.e.,

$$\lim_{x\to\xi} H_\varphi^{X^*}(x) = \varphi(\xi)$$

whenever φ is a bounded resolutive function on $\partial^* X$ which is continuous at ξ. If φ is also F-resolutive, then can we assert that

$$\lim_{x\to\xi} H_\varphi^{F,X^*} = \varphi(\xi)?$$

In case F is linear, this problem was studied in [2]. In this section, we give an extension of results in [2].

First, we prepare two lemmas. For an open set U in X, let U^* denote the closure of U in X^*.

LEMMA 2. *Let U be an open set in X. If φ is a bounded F-resolutive function on $\partial^* X$ and if H_φ^{F,X^*} is bounded on X, then*

$$\psi = \begin{cases} \varphi & \text{on} \quad U^* \cap \partial^* X \\ H_\varphi^{F,X^*} & \text{on} \quad \partial U \end{cases}$$

is a bounded F-resolutive function on $\partial^* U$ with respect to the compactification U^* of U, and $H_\psi^{F,U^*} = H_\varphi^{F,X^*} \mid U$.

PROOF. If $u \in \bar{\mathscr{F}}_\varphi^{F,X^*}$, then $u \mid U \in \bar{\mathscr{F}}_\psi^{F,U^*}$; and if $v \in \underline{\mathscr{F}}_\varphi^{F,X^*}$, then $v \mid U \in \underline{\mathscr{F}}_\psi^{F,U}$. Hence

$$v \mid U \leqq \underline{H}_\psi^{F,U^*} \leqq \bar{H}_\psi^{F,U^*} \leqq u \mid U.$$

Taking the infimum in u and the supremum in v, we obtain

$$H_\varphi^{F,X^*} \mid U \leqq \underline{H}_\psi^{F,U^*} \leqq \bar{H}_\psi^{F,U^*} \leqq H_\varphi^{F,X^*} \mid U,$$

which means $\underline{H}_\psi^{F,U^*} = \bar{H}_\psi^{F,U^*} = H_\varphi^{F,X^*} \mid U \in \mathscr{H}^F(U)$.

LEMMA 3. *Let U be an open set in X. If φ is a bounded continuous function on ∂U, then*

$$\varphi^* = \begin{cases} \varphi & \text{on} \quad \partial U \\ 0 & \text{on} \quad U^* \cap \partial^* X \end{cases}$$

is resolutive with respect to the compactification U^* of U.

This lemma is essentially a consequence of [1; Theorem 2.4.2 and Corollary

2.4.1], and we omit the proof.

THEOREM 3. *Let $\xi \in \partial^* X$ and φ be a bounded resolutive function on $\partial^* X$ which is continuous at ξ. Suppose furthermore that φ is F-resolutive and there exists a neighborhood V of ξ in X^* with the following properties:*

(a) *H_φ^{F,X^*} is bounded on $U = V \cap X$;*

(b) *ξ is regular with respect to $\mathscr{H} \mid U$ and the compactification U^* of U;*

(c) *there exist $p_1, p_2 \in \mathscr{P}_C(U)$ such that*

$$\lim_{x\to\xi} p_i(x) = 0, \quad i = 1, 2,$$

$\sigma(p_1) = F(H_\varphi^{X^*} + M)^+ \mid U$ and $\sigma(p_2) = F(H_\varphi^{X^*} - M)^- \mid U$, where

$$M = (\sup_U H_1^{U^*})(\sup_U |H_\varphi^{F,X^*} - H_\varphi^{X^*}|).$$

Then

$$\lim_{x\to\xi} H_\varphi^{F,X^*}(x) = \varphi(\xi).$$

PROOF. Consider the function

$$\psi = \begin{cases} H_\varphi^{F,X^*} & \text{on } \partial U \\ \varphi & \text{on } U^* \cap \partial^* X. \end{cases}$$

By Lemma 2, ψ is F-resolutive with respect to U^* and $H_\psi^{F,U^*} = H_\varphi^{F,X^*} \mid U$. Note that $\psi = \psi_1 + \psi_2$, where

$$\psi_1 = \begin{cases} H_\varphi^{X^*} & \text{on } \partial U \\ \varphi & \text{on } U^* \cap \partial^* X \end{cases} \quad \text{and} \quad \psi_2 = \begin{cases} H_\varphi^{F,X^*} - H_\varphi^{X^*} & \text{on } \partial U \\ 0 & \text{on } U^* \cap \partial^* X. \end{cases}$$

By Lemma 2 (with $F=0$), ψ_1 is resolutive with respect to U^* and by Lemma 3, ψ_2 is resolutive with respect to U^*, so that ψ is resolutive with respect to U^*. Since $H_{\psi_1}^{U^*} = H_\varphi^{X^*} | U$ by Lemma 2 and since $|H_{\psi_2}^{U^*}| \leqq M$, we have

$$H_\varphi^{X^*} \mid U - M \leqq H_\psi^{U^*} \leqq H_\varphi^{X^*} | U + M.$$

Hence

$$-\sigma(p_2) \leqq F(H_\psi^{U^*}) \leqq \sigma(p_1),$$

so that

$$F(H_\psi^{U^*})^- \leqq \sigma(p_2) \in \mathscr{M}_P(U) \quad \text{and} \quad F(H_\psi^{U^*})^+ \leqq \sigma(p_1) \in \mathscr{M}_P(U).$$

Therefore, by Theorem 1,

$$|H_\psi^{U^*} - H_\psi^{F,U^*}| \leqq p_1 + p_2.$$

Since $\lim_{x\to\xi} p_i(x)=0$, $i=1, 2$, and $\lim_{x\to\xi} H_\psi^{U*}(x)=\varphi(\xi)$ by condition (b), it follows that

$$\lim_{x\to\xi} H_\psi^{F,U*}(x) = \varphi(\xi),$$

which implies the desired result.

COROLLARY 1. *Suppose $\xi\in\partial^*X$ is locally regular, i.e., regular with respect to $\mathscr{H}\,|\,V\cap X$ for any neighborhood V of ξ in X^*. Suppose furthermore that for each $\alpha\in\mathbf{R}$ there exists a neighborhood V_α of ξ in X^* such that $|F(\alpha)|\,|\,V_\alpha \cap X\in\mathscr{M}_P(V_\alpha\cap X)$ and*

$$\lim_{x\to\xi} p_\alpha(x) = 0$$

for $p_\alpha\in\mathscr{P}_C(V_\alpha\cap X)$ satisfying $\sigma(p_\alpha)=|F(\alpha)|\,|\,V_\alpha\cap X$.

Then, for any bounded resolutive function φ on ∂^*X which is continuous at ξ, F-resolutive and for which $H_\varphi^{F,X*}$ is bounded in a neighborhood of ξ in X^*,

$$\lim_{x\to\xi} H_\varphi^{F,X*}(x) = \varphi(\xi).$$

PROOF. Let W be a neighborhood of ξ in X^* such that $H_\varphi^{F,X*}$ is bounded on $W\cap X$. Choose $q\in\mathscr{P}_{BC}(X)$ such that $\sigma(q)=\sigma(1)^-$ and put $\beta=\sup_X(1+q)$. Then $\beta\geqq H_1^{U*}$ for any open subset U of X. Let

$$M = \beta\sup_{W\cap X}|H_\varphi^{F,X*} - H_\varphi^{X*}|,$$

$$\alpha_1 = \sup_{W\cap X} H_\varphi^{X*} + M \quad\text{and}\quad \alpha_2 = \inf_{W\cap X} H_\varphi^{X*} - M.$$

Consider $V=W\cap V_{\alpha_1}\cap V_{\alpha_2}$. Then

$$F(H_\varphi^{X*}+M)^+ \leqq F(\alpha_1)^+ \quad\text{and}\quad F(H_\varphi^{X*}-M)^- \leqq F(\alpha_2)^-$$

on $V\cap X$. Hence, there exist $q_1, q_2\in\mathscr{P}_C(V\cap X)$ such that $\sigma(q_1)=F(H_\varphi^{X*}+M)^+$ and $\sigma(q_2)=F(H_\varphi^{X*}-M)^-$, and $q_i\leqq p_{\alpha_i}$, $i=1, 2$. Hence, condition (c) of Theorem 3 is satisfied with this V. Conditions (a) and (b) of Theorem 3 are clearly satisfied by our assumptions. Hence, we obtain the assertion of the corollary.

COROLLARY 2 (cf. Example 4.1 in [3]). *Suppose $F(f)=\psi(f)\mu$ with a locally Lipschitz-continuous non-decreasing function ψ on $\mathbf{R}$ and $\mu\in\mathscr{M}_\sigma^+(X)$. If $\xi\in\partial^*X$ is locally regular and if there exists a neighborhood V of ξ in X^* such that $\mu\,|\,V\cap X\in\mathscr{M}_P(V\cap X)$ and $\lim_{x\to\xi} p(x)=0$ for $p\in\mathscr{P}_C(V\cap X)$ satisfying $\sigma(p)=\mu\,|\,V\cap X$, then the same assertion as in Corollary 1 holds.*

References

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