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Special concircular vector fields in Riemannian manifolds

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Introduction

The purpose of the present paper is to study Riemannian manifolds admitting some linearly independent special concircular vector fields and determine geometrical structures of such manifolds. Some results in this paper contain generalizations of results due to Y. Tashiro (see Proposition 7.3 in [4] and Corollaries 2 and 3 in this paper).

We shall define an almost everywhere warped product and give a few examples in § 1. We also state some properties of this kind of product. In § 2, we shall determine structures of n-dimensional Riemannian manifolds admitting n linearly independent special concircular vector fields and investigate some relations between these vector fields and their associated scalar fields. In § 3, we prove that any Riemannian manifold admitting some linearly independent special concircular vector fields is an almost everywhere warped product, a part of which is a space of constant curvature, and obtain some results on the given manifold. Finally, in § 4, we shall give geometrical structures of Riemannian manifolds mentioned in § 3.

Throughout this paper, we assume that manifolds and quantities are differentiable of class C^∞.

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§ 1. Almost everywhere warped products

Let M_1 and M_2 be Riemannian manifolds of dimension m and $n-m$ respectively, and f a positive-valued differentiable function on M_1. The *warped product* $M=M_1 \times {}_f M_2$ is by definition (see [1]) the product manifold $M_1 \times M_2$ endowed with Riemannian metric

$$(X, X) = (\pi_1 X, \pi_1 X) + f^2(\pi_1 x)(\pi_2 X, \pi_2 X)$$

for any vector $X \in T_x(M)$, $x \in M$, where π_α $(\alpha=1, 2)$ is the natural projection $M \to M_\alpha$, the tangential map of π_α is denoted by the same character, and $(\ ,\ )$ is the Riemannian inner product.

Let $(x^\kappa)=(x^h, x^p)$ be a local coordinate system of M, called a separate coordinate system, where (x^h) and (x^p) are those of M_1 and M_2 respectively. Here and hereafter Greek indices $\kappa, \lambda, \mu, \nu,\ldots$ run on the ranges $1, 2,\ldots, n$, and Latin indices run on the following ranges

$$h, i, j, k,\ldots = 1, 2,\ldots, m,$$
$$p, q, r, s,\ldots = \qquad m+1, m+2,\ldots, n$$

respectively, unless otherwise stated. If the components of the metric tensors of M, M_1 and M_2 are denoted by $g_{\mu\lambda}$, g_{ji} and $\bar{g}_{qp}$ respectively, then the metric form of the warped product $M=M_1\times {}_fM_2$ is expressed by

$$g_{\mu\lambda}dx^\mu dx^\lambda = g_{ji}dx^j dx^i + [f(x^h)]^2\bar{g}_{qp}dx^q dx^p \tag{1.1}$$

with respect to a separate coordinate system $(x^\kappa)=(x^h, x^p)$. The components $g_{\mu\lambda}$ of the metric tensor of M belonging to (x^p) are equal to

$$g_{qp} = f^2\bar{g}_{qp}. \tag{1.2}$$

Let M be an n-dimensional Riemannian manifold, M_1 a submanifold of M, f a differentiable function on M_1, N the zero-level hypersurface of M_1 defined by $f=0$ and M_1^0 a connected component of M_1-N. We assume that the gradient vector of f does not vanish on N. If $M-N$ is diffeomorphic to the product manifold $M_1^0\times M_2$ of M_1^0 with a certain Riemannian manifold M_2, and if the metric form of M is expressed by (1.1) on $M-N$, where $\bar{g}_{qp}dx^q dx^p$ is a metric form of M_2, then we say that M has an *almost everywhere warped product structure*, or simply, M is an *almost everywhere warped product* (briefly *AEWP*). Denoting M_1^0 by M_1 again, we also express M as $M_1\times {}_fM_2$.

We give two examples of AEWP structures on a space form $S^n(k)$ of curvature $k\neq 0$ as follows:

Example 1. Let $(X^0, X^1,\ldots, X^n)$ be a canonical coordinate system in R^{n+1} and M the hypersurface of R^{n+1} defined by

$$(\mathrm{sgn}\,(k))(X^0)^2 + (X^1)^2 +\cdots+ (X^n)^2 = 1/k, \tag{1.3}$$

where $k\neq 0$ and $X^0\geq |k|^{-1/2}$ when $k<0$. Then M with the induced metric from the metric form

$$(\mathrm{sgn}\,(k))(dX^0)^2 + (dX^1)^2 +\cdots+ (dX^n)^2$$

on R^{n+1} is a space form $S^n(k)$. We consider the following parametric equations

$$X^0 = [1-(k/4)R_1^2]\,[1-(k/4)R_2^2]/|k|^{1/2}S_1S_2,$$
$$X^i = x^i/S_1, \quad X^p = [1-(k/4)R_1^2]x^p/S_1S_2,$$

where we have put

$$R_1^2 = x^i x^i,\ R_2^2 = x^p x^p,\ S_1 = 1 + (k/4)R_1^2,\ S_2 = 1 + (k/4)R_2^2 \tag{1.4}$$

and summation convention has been also applied to repeated upper indices. Then the metric form of M is given by

$$ds^2 = (1/S_1^2)dx^i dx^i + \{[1-(k/4)R_1^2]/S_1\}^2(1/S_2^2)dx^p dx^p .$$

The parts

$$ds_1^2 = (1/S_1^2)dx^i dx^i,\quad \overline{ds_2^2} = (1/S_2^2)dx^p dx^p$$

are the metric forms of space forms $S^m(k)$ and $S^{n-m}(k)$ respectively, and hence M has an AEWP structure $S^m(k)\times {}_f S^{n-m}(k)$, where

$$f = [1-(k/4)R_1^2]/S_1 .$$

In particular, for $k<0$, the hyperbolic space $M=S^n(k)$ is an ordinary warped product of two hyperbolic spaces $S^m(k)$ and $S^{n-m}(k)$.

EXAMPLE 2. Let M be the hypersurface of R^{n+1} defined by in Example 1. We consider another parametric equations

$$X^0 = [1-(k/4)R_1^2]/|k|^{1/2}S_1,\quad X^i = x^i/S_1 \qquad (i = 1,\dots, m-1),$$
$$X^m = [1-(1/4)R_2^2]x^m/S_1\bar{S}_2,\quad X^p = x^m x^p/S_1\bar{S}_2 ,$$

where we have put

$$\bar{S}_2 = 1 + (1/4)R_2^2$$

and the others are the same as in (1.4). Then the metric form of M is expressed in the form

$$ds^2 = (1/S_1^2)dx^i dx^i + [(x^m)^2/S_1^2](1/\bar{S}_2^2)dx^p dx^p .$$

The second part

$$\overline{ds_2^2} = (1/\bar{S}_2^2)dx^p dx^p$$

is the metric form of the spherical space $S^{n-m}(1)$, and M has an AEWP structure $S^m(k)\times {}_f S^{n-m}(1)$, where

$$f = x^m/S_1 .$$

Return to a general AEWP $M=M_1\times {}_f M_2$. We denote the Christoffel symbols of M, M_1 and M_2 by $\Gamma^\kappa_{\mu\lambda}$, $\{^{\,h}_{ji}\}$ and $\{^{\,p}_{rq}\}$ respectively. In a separate coordinate system we have the relations

$$
(1.5)\qquad \begin{cases} \Gamma^h_{ji} = \left\{{h \atop ji}\right\}, & \Gamma^h_{jq} = 0, \quad \Gamma^h_{rq} = -ff^h\bar{g}_{rq}, \\ \Gamma^p_{ji} = 0, & \Gamma^p_{jq} = (1/f)f_j\delta^p_q, \quad \Gamma^p_{rq} = \left\{{p \atop rq}\right\}, \end{cases}
$$

where we have put $f_j = \nabla_j f$ and $f^h = g^{ih} f_i$.

For every point $x \in M - N$, we denote by $M_2(x)$ the copy of M_2 through x. Then we obtain immediately from (1.5) the following

THEOREM 1. *If M is an AEWP $M_1 \times {}_f M_2$, then M_1 is a totally geodesic submanifold of M and each copy $M_2(x)$, $x \in M - N$, is a totally umbilical submanifold of M.*

Denote the components of the curvature tensors of M, M_1 and M_2 by $K_{\nu\mu\lambda}{}^{\kappa}$, $R_{kji}{}^h$ and $R_{srq}{}^p$ respectively. Then we have the relations

$$
(1.6)\qquad \begin{cases} K_{kji}{}^h = R_{kji}{}^h, \quad K_{sji}{}^h = K_{krq}{}^p = K_{sri}{}^h = 0, \\ K_{krq}{}^h = -f(\nabla_k f^h)\bar{g}_{rq}, \quad K_{sji}{}^p = -(1/f)(\nabla_j f_i)\delta^p_s, \\ K_{srq}{}^p = R_{srq}{}^p - f_i f^i(\delta^p_s \bar{g}_{rq} - \delta^p_r \bar{g}_{sq}), \end{cases}
$$

∇ indicating the covariant differentiation with respect to $\left\{{h \atop ji}\right\}$.

We shall denote the magnitude of a tensor by $\| \ \|$. For example, that of the curvature tensor $K_{\nu\mu\lambda}{}^{\kappa}$ is defined by

$$
\|K_{\nu\mu\lambda}{}^{\kappa}\|^2 = K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa}.
$$

By means of the equations (1.6) we have

$$
\begin{aligned} \|K_{\nu\mu\lambda}{}^{\kappa}\|^2 = \|R_{kji}{}^h\|^2 &+ [4(n-m)/f^2]\|\nabla_j \operatorname{grad} f\|^2 \\ &+ (1/f^4)\|[R_{srq}{}^p - \|\operatorname{grad} f\|^2(\delta^p_s \bar{g}_{rq} - \delta^p_r \bar{g}_{sq})]\|^2. \end{aligned}
$$

If the function f has a zero-level surface N, then we make a point of M_1 tend to a point of N and obtain the following

THEOREM 2. *For an AEWP $M = M_1 \times {}_f M_2$, if the function f has non-empty zero-level surface N and $\dim M_2 \geq 2$, then M_2 is a space of constant curvature $\|\operatorname{grad} f\|^2$, that is,*

$$
(1.7)\qquad R_{srq}{}^p = \|\operatorname{grad} f\|^2(\delta^p_s \bar{g}_{rq} - \delta^p_r \bar{g}_{sq}).
$$

If an AEWP $M = M_1 \times {}_f M_2$ is of constant curvature k, that is,

$$
(1.8)\qquad K_{\nu\mu\lambda}{}^{\kappa} = k(\delta^{\kappa}_{\nu} g_{\mu\lambda} - \delta^{\kappa}_{\mu} g_{\nu\lambda}),
$$

then we compare (1.6) with (1.8) and obtain the following

THEOREM 3. *An AEWP $M=M_1\times{}_fM_2$ is of constant curvature k if and only if the following conditions are satisfied:*

(1) *M_1 of dimension >1 is a space of constant curvature k, that is,*

$$R_{kji}{}^h = k(\delta_k^h g_{ji} - \delta_j^h g_{ki}); \tag{1.9}$$

(2) *Either M_2 of dimension >1 is a space of constant curvature $kf^2+\|\mathrm{grad} f\|^2$, that is,*

$$R_{srq}{}^p = (kf^2 + \|\mathrm{grad} f\|^2)(\delta_s^p \bar{g}_{rq} - \delta_r^p \bar{g}_{sq}), \tag{1.10}$$

or M_2 is 1-dimensional Euclidean space;

(3) *The function f satisfies the equation*

$$\nabla_j f_i = -kfg_{ji}. \tag{1.11}$$

A function f satisfying the equation (1.11) is called a *special concircular scalar field* with characteristic constant k. For such a function f, we easily see that $kf^2+\|\mathrm{grad} f\|^2$ is a constant. Thus the following is immediate from Theorems 2 and 3.

COROLLARY 1. *Suppose that the function f has a non-empty zero-level surface N on an AEWP $M=M_1\times{}_fM_2$. Then M is a space of constant curvature k if and only if the following conditions are satisfied:*

(1) *Provided* $\dim M_1>1$, *M_1 is of constant curvature k;*

(2) *The function f on M_1 is a special concircular scalar field with characteristic constant k.*

§2. Riemannian manifolds of dimension n admitting n special concircular vector fields

On an n-dimensional Riemannian manifold M, a vector field $V=(V^\kappa)$ is called a *special concircular vector field* (briefly *SCVF*) if it satisfies the equation

$$\nabla_X V = \phi X;\ \nabla_\mu V^\kappa = \phi\delta_\mu^\kappa \tag{2.1}$$

for any vector $X\in T_x(M)$, where ϕ is a scalar field on M. Locally an SCVF is a gradient vector field of a scalr field ρ, that is, $V=\nabla\rho$, which satisfies the equation

$$\nabla_\mu\nabla_\lambda\rho = \phi g_{\mu\lambda}. \tag{2.2}$$

If M is simply connected, then such a scalar field ρ exists globally for an SCVF V. If an SCVF has zero points, that is, stationary points of the scalar field ρ, then the points are isolated and the number of them is at most two. Geodesics issuing from a zero point of an SCVF V are trajectories of V.

SCVF's $V_{(i)}$ are simply said to be linearly independent in M if they are linearly independent except a border subset of M. The scalar fields ϕ and ρ for every SCVF $V_{(i)}$ are also marked off by suffix in parentheses.

For some SCVF's $V_{(i)}$, the following lemmas are known ([4]).

LEMMA 1. *If two SCVF's $V_{(1)}$ and $V_{(2)}$ have a zero point in common, then they are linearly dependent and one is a constant multiple of the other.*

LEMMA 2. *If more than two SCVF's $V_{(i)}$ are linearly independent in a Riemannian manifold M, then the scalar fields $\phi_{(i)}$ are written in the form*

$$\nabla_X \phi_{(i)} = -kg(X, V_{(i)}); \quad \phi_{(i)} = -k\rho_{(i)} + b_{(i)} \tag{2.3}$$

for any vector $X \in T_x(M)$, where k is a constant in common with all the SCVF's and $b_{(i)}$ are constants.

Consequently $\rho_{(i)}$'s are special concircular scalar fields with the same constant k, which is called the *characteristic constant* of $V_{(i)}$ too.

In the remaining of this section, we shall consider an n-dimensional Riemannian manifold M admitting n linearly independent SCVF's $V_{(i)}$ and the indices $h, i, j, k, \ldots$ will run on the range $1, 2, \ldots, n$. We prove the following

THEOREM 4. *If an n-dimensional Riemannian manifold M admits n linearly independent SCVF's $V_{(i)}$ with characteristic constant k, then M is of constant curvature k. In addition, if M is complete and simply connected, then M is a space form $S^n(k)$.*

PROOF. Computing $(\nabla_{V_{(k)}}\nabla_{V_{(j)}} - \nabla_{V_{(j)}}\nabla_{V_{(k)}} - \nabla_{[V_{(k)}, V_{(j)}]})V_{(i)}$ and using (2.1) and (2.3), we have

$$K(V_{(k)}, V_{(j)})V_{(i)} = k[g(V_{(j)}, V_{(i)})V_{(k)} - g(V_{(k)}, V_{(i)})V_{(j)}].$$

Since $V_{(i)}$ are linearly independent, M is a space of constant curvature k.

Now we locate ourselves in a local coordinate system (x^h) of a manifold M of constant curvature k, where the metric tensor is expressed as

$$g_{ji} = (1/S^2)\delta_{ji}, \quad S = 1 + (k/4)R^2, \quad R^2 = x^h x^h. \tag{2.4}$$

The Christoffel symbol is there given by

$$\Gamma^h_{ji} = -(k/2S)(x^j\delta_{ih} + x^i\delta_{jh} - x^h\delta_{ji}). \tag{2.5}$$

From (2.2) and (2.3) we have the equation

$$\nabla_j\nabla_i\rho = (-k\rho + b)g_{ji}, \tag{2.6}$$

which is reduced to

$$\partial_j\partial_i\rho + (k/2S)(x^j\partial_i\rho + x^i\partial_j\rho - \delta_{ji}x^h\partial_h\rho) = (1/S^2)(-k\rho + b)\delta_{ji}.$$

The general solution of the equation above is given by

$$\rho = (1/S)[A(1-(k/4)R^2) + B_h x^h + (b/2)R^2], \tag{2.7}$$

where A and B_h's are constants ([4]). We put

$$\rho_{(0)} = (1/S)[1-(k/4)R^2], \quad \rho_{(h)} = x^h/S, \quad \rho_{(\infty)} = R^2/2S. \tag{2.8}$$

The gradient vector fields corresponding to these scalar fields have covariant components as

$$\begin{cases} V_{(0)i} = -(k/S^2)x^i, \quad V_{(h)i} = (1/S^2)[S\delta_{hi} - (k/2)x^h x^i], \\ V_{(\infty)i} = (1/S^2)x^i, \end{cases} \tag{2.9}$$

and we have, from (2.4), (2.5), (2.8) and (2.9), the equations

$$\nabla_j V_{(\alpha)i} = (-k\rho_{(\alpha)} + b_{(\alpha)})g_{ji}, \tag{2.10}$$

where $\alpha = 0, 1,\dots, n, \infty$. This equation shows that these $n+1$ vector fields $V_{(\alpha)}$ are also SCVF's on M. In the case $k \neq 0$, the constant $b_{(\alpha)}$ in (2.10) is equal to zero for $V_{(0)}$ and $V_{(h)}$'s, and $V_{(\infty)}$ is parallel to $V_{(0)}$. In the case $k=0$, we have

$$V_{(h)i} = \delta_{hi}, \quad V_{(\infty)i} = x^i, \tag{2.11}$$

the first n vector fields of which are parallel and the last is a concurrent vector field. Any SCVF is represented by a linear combination of these $n+1$ vector fields with constant coefficients.

We shall investigate the zero-level hypersurface of $\rho_{(\alpha)}$ and the zero points of the SCVF's $V_{(\alpha)}$ corresponding to these functions $\rho_{(\alpha)}$. In the case $k>0$, the zero-level hypersurface of $\rho_{(0)}$ is the equatorial hypersphere and that of $\rho_{(h)}$ the longitudial hypersphere defined by $x^h=0$. The vector field $V_{(0)}$ vanishes at the north and south poles. Each vector field $V_{(h)}$ vanishes at the points belonging to an intersection of the equatorial hypersphere with one of the longitudial hyperspheres except the one defined by $x^h=0$. In the case $k<0$, $\rho_{(0)}$ vanishes nowhere and $\rho_{(h)}$ does on the hypersurface defined by $x^h=0$. The vector field $V_{(0)}$ vanishes at the point $x^h=0$ for all h and $V_{(h)}$ does nowhere. In the case $k=0$, $\rho_{(h)}$ vanishes on the hyperplane defined by $x^h=0$ and $\rho_{(\infty)}$ does at the origin. The parallel vector fields $V_{(h)}$ vanish nowhere and the concurrent vector field $V_{(\infty)}$ does at the origin.

§ 3. Riemannian manifolds admitting some special concircular vector fields

In this section, we consider an n-dimensional Riemannian manifold M admitting $m \geq 2$ linearly independent SCVF's $V_{(i)}$. Then, for $i=1, 2, \ldots, m$ and any vector $X \in T_x(M)$, $x \in M$, we have the equations (2.1), (2.3) and

$$g(X, V_{(i)}) = X\rho_{(i)}. \tag{3.1}$$

We define the distributions D_1 and D_2 by

$$\begin{cases} D_1 = \operatorname{span}\{V_{(1)}, V_{(2)}, \ldots, V_{(m)}\}, \\ D_2 = \{X \in T_x(M) \mid g(X, V_{(i)}) = 0\} \end{cases} \tag{3.2}$$

except at the zero points of $V_{(i)}$'s.

Lemma 3. *The distributions D_1 and D_2 are completely integrable.*

Proof. It follows from (2.1) that

$$[V_{(j)}, V_{(i)}] = \phi_{(i)}V_{(j)} - \phi_{(j)}V_{(i)} \in D_1 \tag{3.3}$$

and hence D_1 is completely integrable. For every two vectors $X, Y \in D_2$, we obtain

$$g([X, Y], V_{(i)}) = 0 \tag{3.4}$$

by means of (3.1). Hence the distribution D_2 is also completely integrable.

Now we define a 1-form ω on M by

$$\begin{cases} \omega(V_{(i)}) = \phi_{(i)}, \\ \omega(X) = 0 \qquad \text{for} \quad X \in D_2. \end{cases} \tag{3.5}$$

Lemma 4. $d\omega = 0$.

Proof. For every two vectors $V_{(i)}, V_{(j)} \in D_1$, we have

$$\begin{aligned} 2d\omega(V_{(j)}, V_{(i)}) &= V_{(j)}(\phi_{(i)}) - V_{(i)}(\phi_{(j)}) \\ &\qquad - \omega(\phi_{(i)}V_{(j)} - \phi_{(j)}V_{(i)}) = 0 \end{aligned}$$

by use of (3.1), (3.3) and (3.5). For any vector $X \in D_2$ we obtain

$$\begin{aligned} &g(\nabla_{V_{(j)}}X, V_{(i)}) = V_{(j)}(g(X, V_{(i)})) = 0, \\ &g([X, V_{(j)}], V_{(i)}) = g(\phi_{(j)}X - \nabla_{V_{(j)}}X, V_{(i)}) = 0 \end{aligned}$$

by means of (2.1) and (3.5). Using these equations, we have

$$
\begin{aligned}
2d\omega(X, V_{(i)}) &= X(\omega(V_{(i)})) - \omega([X, V_{(i)}]) \\
&= -kg(X, V_{(i)}) = 0.
\end{aligned}
$$

For every two vectors $X, Y \in D_2$, we have $d\omega(X, Y)=0$ because of $[X, Y] \in D_2$. Hence we have $d\omega=0$ on M.

Let F be the set of all zero points of the SCVF's $V_{(i)}$. Then it is finite and a border subset of M as we have already seen in § 2. We prove the following

THEOREM 5. *Suppose that an n-dimensional Riemannian manifold M admits $m \geq 2$ linearly independent SCVF's with characteristic constant k and let F be the set of all zero points of the SCVF's. Then $M-F$ is locally an AEWP $M_1 \times {}_f M_2$ such that M_1 is an m-dimensional space of constant curvature k and the function f on M_1 satisfies the equation*

$$
\nabla_X \operatorname{grad} f = -kfX, \tag{3.6}
$$

where $X \in T_x(M_1)$ and ∇ is the covariant derivative on M_1.

PROOF. Lemma 4 shows that there exists a coordinate neighborhood $U(x)$ of $x \in M-F$ and a function ψ on U such that $\omega|_U = d\psi$. Then the definition (3.5) implies that

$$
V_{(i)}(\psi) = \phi_{(i)}, \quad X(\psi) = 0 \quad \text{for any vector} \quad X \in D_2 \tag{3.7}
$$

on U. Since the distributions D_1 and D_2 are completely integrable, we can choose $U(x)$ such that it is diffeomorphic to $U_1 \times U_2$, where U_1 and U_2 are the slices of D_1 and D_2 through x, and a separate coordinate system (x^h, x^p) in $U(x)$ such that (x^h) and (x^p) are local coordinate systems in U_1 and U_2 respectively. Since D_1 and D_2 are mutually orthogonal, the metric tensor g has components such as

$$
\begin{pmatrix} g_{ji} & 0 \\ 0 & g_{qp} \end{pmatrix}.
$$

Each $V_{(i)}$ of the SCVF's has components $(V_{(i)}{}^h, 0)$ with respect to this separate coordinate (x^h, x^p).

Putting $\kappa=p$ and $\mu=j$ in (2.1), we obtain

$$
\nabla_j V_{(i)}{}^p = \partial_j V_{(i)}{}^p + \Gamma^p_{jh} V_{(i)}{}^h + \Gamma^p_{jq} V_{(i)}{}^q = 0,
$$

from which it follows that $\Gamma^p_{jh}=0$ and hence $\partial_p g_{jh}=0$. This means that the components g_{ji} are independent of x^p and regarded as the metric tensors of U_1.

We shall denote the slice U_1 equipped with the metric tensors g_{ji} by M_1.

Putting $\kappa=p$ and $\mu=r$ in (2.1), we obtain

$$\Gamma^p_{rh} V_{(i)}{}^h = \phi_{(i)}\delta^p_r,$$

which implies the equation

$$V_{(i)}{}^h \partial_h g_{rq} = 2\phi_{(i)} g_{rq}.$$

By means of (3.7) and the linear independence of $V_{(i)}$, this equation is equivalent to

$$\partial_h g_{rq} = 2(\partial_h \psi) g_{rq}.$$

Therefore, putting $f=\exp(\psi)$, we see that the components g_{rq} are written in the form

$$g_{rq} = f^2 \bar{g}_{rq},$$

where $\bar{g}_{rq}$ are dependent on x^p only and regarded as the metric tensors of U_2. We also denote by M_2 the slice U_2 equipped with the metric tensors $\bar{g}_{qp}$.

Since ψ and hence f are functions on M_1, the metric form of M are written in the form

$$ds^2 = ds_1^2 + [f(x^h)]^2 \overline{ds_2^2}$$

in U, where ds_1^2 and $\overline{ds_2^2}$ are the metric forms of M_1 and M_2 respectively. Therefore $M-F$ is locally an AEWP $M_1 \times {}_f M_2$.

Putting $\kappa=h$ and $\mu=r$ in (2.1), we have

$$\nabla_r V_{(i)}{}^h = \partial_r V_{(i)}{}^h = 0.$$

The equation (2.1) for $\kappa=h$ and $\mu=j$ yields

$$\nabla_j V_{(i)}{}^h = \partial_j V_{(i)}{}^h + \Gamma^h_{jk} V_{(i)}{}^k = (-k\rho_{(i)} + b_{(i)})\delta^h_j.$$

The above two equations show that the SCVF's $V_{(i)}$ are regarded as those of M_1. Therefore the m-dimensional part M_1 admits m linearly independent SCVF's and hence, by Theorem 4, M_1 is a space of constant curvature k.

The equation (3.7) is reduced to

$$g(V_{(i)}, \operatorname{grad} f) = f\phi_{(i)}.$$

Differentiating this equation covariantly along M_1, we obtain

$$g(\nabla_X V_{(i)}, \operatorname{grad} f) + g(V_{(i)}, \nabla_X \operatorname{grad} f) = (Xf)\phi_{(i)} + f\nabla_X \phi_{(i)}$$

for any vector $X \in T_x(M_1)$, $x \in M_1$, which implies

$$g(V_{(i)}, \nabla_X \operatorname{grad} f) = -kfg(V_{(i)}, X) \tag{3.8}$$

by means of (2.1) and (2.3). Since $V_{(i)}$'s are linearly independent, the equation (3.8) is reduced to (3.6). Thus the proof is completed.

We have already seen in § 1 that, if f satisfies the equation (3.6), the function $kf^2 + \|\operatorname{grad} f\|^2$ is a constant. The following are immediate consequences of Theorems 2, 3 and 5.

PROPOSITION 6. *Suppose that an n-dimensional Riemannian manifold M admits $m \geq 2$ linearly independent SCVF's with characteristic constant k and let F be the set of all zero points of the SCVF's. Then the manifold $M-F$ is locally an AEWP $M_1 \times {}_f M_2$, and M is of constant curvature k if and only if M_2 is a space of constant curvature $kf^2 + \|\operatorname{grad} f\|^2$.*

PROPOSITION 7. *Suppose that an n-dimensional Riemannian manifold M admits $n-1$ linearly independent SCVF's. Then M is a space of constant curvature.*

PROPOSITION 8. *Suppose that an n-dimensional Riemannian manifold M admits $2 \leq m < n-1$ linearly independent SCVF's and let F be the set of all zero points of the SCVF's. If the function f appearing in the AEWP $M-F = M_1 \times {}_f M_2$ has non-empty zero-level surface, then M is a space of constant curvature.*

§ 4. Structures of Riemannian manifolds admitting some linearly independent special concircular vector fields

Suppose that an n-dimensional Riemannian manifold M admits $m \geq 2$ linearly independent SCVF's $V_{(i)}$ with characteristic constant k. Then $M-F$ is locally an AEWP $M_1 \times {}_f M_2$, where F is the set of all zero points of the SCVF's $V_{(i)}$, and the integral manifold M_1 of the distribution D_1 defined by (3.2) is an m-dimensional space of constant curvature k by Theorem 5. As we have seen in the proof of Theorem 5, the SCVF's $V_{(i)}$ and the associated scalar fields $\rho_{(i)}$ are regarded as those of M_1.

We choose a local coordinate system (x^h, x^p) in M such that the metric tensors for the local coordinate (x^h) in M_1 are given by (2.4), and consider the equation (2.6) on M_1. The general solution (2.7) of (2.6) is a linear combination of the $m+1$ scalar fields $\rho_{(0)}$, $\rho_{(h)}$ and $\rho_{(\infty)}$ on M_1 given by (2.8) with constant coefficients. The part M_1 admits the $m+1$ SCVF's $V_{(0)}$ and $V_{(h)}$'s in the case $k \neq 0$ and $V_{(h)}$'s and $V_{(\infty)}$ in the case $k=0$ which are given by (2.9). When M admits m linearly independent SCVF's, there occurs a problem such as which of $m+1$ vectors given by (2.9) are the m linearly independent SCVF's in M_1 itself.

First, we consider the case $k\neq 0$. Then the equation (2.6) represented by the scalar fields $\rho_{(\alpha)}$ are written as

$$\nabla_{\mu}\rho_{(\alpha)\lambda}=\partial_{\mu}\rho_{(\alpha)\lambda}-\Gamma^{l}_{\mu\lambda}\rho_{(\alpha)l}=-k\rho_{(\alpha)}g_{\mu\lambda} \tag{4.1}$$

in M, where and in the sequel α indicates 0, 1,..., m and we have put $\rho_{(\alpha)\lambda}=\partial_{\lambda}\rho_{(\alpha)}$. Since $M-F$ is locally an AEWP, putting $\lambda=p$, $\mu=q$ into (4.1), we have the equation

$$g^{ji}(\partial_{j}g_{qp})\rho_{(\alpha)i}=-2k\rho_{(\alpha)}g_{qp}$$

or, substituting (2.4) into this equation,

$$S^{2}\rho_{(\alpha)i}(\partial_{i}g_{qp})=2k\rho_{(\alpha)}g_{qp}, \tag{4.2}$$

where and in the sequel summation convention is also applied to repeated lower indices.

If the m SCVF's on M are the extensions of m linear combinations of $V_{(\alpha)}$'s on M_1 except $V_{(0)}$ with constant coefficients, then we may assume that the m SCVF's are $V_{(h)}$ themselves. Substituting

$$\rho_{(h)}=x^{h}/S,\quad \rho_{(h)i}=(1/S^{2})(S\delta_{hi}-(k/2)x^{h}x^{i})$$

into (4.2), we have the equation

$$S\partial_{h}g_{qp}-(k/2)x^{h}(x^{i}\partial_{i}g_{qp})=(2k/S)x^{h}g_{qp}. \tag{4.3}$$

By contraction of (4.3) with x^h, we have

$$x^{i}\partial_{i}g_{qp}=-\{2kR^{2}/S[1-(k/4)R^{2}]\}g_{qp}. \tag{4.4}$$

Substituting again (4.4) into (4.3), we have the solution of (4.3)

$$g_{qp}=\{[1-(k/4)R^{2}]/S\}^{2}\bar{g}_{qp}=\rho^{2}_{(0)}\bar{g}_{qp},$$

where $\bar{g}_{qp}$ is regarded as the metric tensor of M_2. The set F is contained in the zero-level surface N of $\rho_{(0)}$ for $k>0$ and is empty for $k<0$, as we have seen in the end of § 2. Thus the manifold M is locally an AEWP $M_1\times\rho_{(0)}M_2$. If $k>0$, the zero-level surface N of $\rho_{(0)}$ is the equatorial hypersphere of M_1. Therefore, by Theorem 2, the $(n-m)$-dimensional manifold M_2 is of positive constant curvature $\|V_{(0)}\|^2$ and hence M is a space of constant curvature k by Propositions 7 and 8. Moreover, we see that M is locally isometric to a space form $S^n(k)$ given by Example 1. If $k<0$, the function $\rho_{(0)}$ vanish nowhere. Therefore M is locally a warped product of an m-dimensional manifold M_1 of constant curvature k with an $(n-m)$-dimensional Riemannian manifold M_2. If M_2 is a space of constant curvature $k\rho_{(0)}{}^2+\|V_{(0)}\|^2$, then M is a space of constant curvature k by

Proposition 6 and hence it is locally isometric to a space form $S^n(k)$ given by Example 1.

If the m SCVF's on M are the extensions of m linear combinations of $V_{(0)}$ and $m-1$ $V_{(h)}$'s on M_1 with constant coefficients, then we may assume that the m SCVF's are $V_{(0)}$ and $V_{(h)}$'s $(h=1, 2,\dots, m-1)$. We have the equation (4.3) for $h=1, 2,\dots, m-1$. Substituting

$$\rho_{(0)} = (1/S)[1-(k/4)R^2], \quad \rho_{(0)i} = -(k/S^2)x^i$$

into (4.2) for $\alpha=0$, we also have

$$x^i\partial_i g_{qp} = (2/S)[1-(k/4)R^2]g_{qp}. \tag{4.6}$$

Substituting again (4.6) into (4.3) for $h=1,\dots, m-1$, we have

$$S\partial_h g_{qp} = -kx^h g_{qp}$$

or

$$\partial_h g_{qp} + (2/S)g_{qp}\partial_h S = 0.$$

Hence the components g_{qp} are written in the form

$$g_{qp} = (1/S^2)h_{qp},$$

where h_{qp} are functions of $x^m, x^{m+1},\dots, x^n$. Substituting these expressions into (4.6), we can easily see that h_{qp} is written as $(x^m)^2\bar{g}_{qp}$ with $\bar{g}_{qp}$ depending on $x^{m+1}, \dots, x^n$ only. Thus we have

$$g_{qp} = [(x^m)^2/S^2]\bar{g}_{qp} = \rho^2_{(m)}\bar{g}_{qp}, \tag{4.7}$$

where $\bar{g}_{qp}$ is regarded as a metric tensor of M_2. In this case, the set F is contained in the zero-level surface N of $\rho_{(m)}$ as we have seen in § 2. Therefore the manifold M is locally an AEWP $M_1\times \rho_{(m)}M_2$. By Theorem 2 the $(n-m)$-dimensional manifold M_2 is of positive constant curvature $\|V_{(m)}\|^2$ and hence the manifold M is of constant curvature k by Propositions 7 and 8. Moreover, the manifold M with the metric tensor given by (2.4) and (4.7) is locally isometric to a space form $S^n(k)$ given by Example 2.

Thus we can state the following

THEOREM 9. *Suppose that an n-dimensional Riemannian manifold M admits $m\geq 2$ linearly independent SCVF's $V_{(i)}$ with associated scalar fields $\rho_{(i)}$ and characteristic constant $k\neq 0$. In the case where each $\rho_{(i)}$ is one of the functions*

$$[1-(k/4)R^2]/S \quad \textit{for} \quad k>0 \quad \textit{and} \quad x^i/S,$$

the manifold M is locally isometric to a space form $S^n(k)$. In the case where one of $\rho_{(i)}$'s is the function

$$[1-(k/4)R^2]/S \qquad \text{for} \quad k<0,$$

the manifold M is locally a warped product $M_1 \times \rho_{(i)} M_2$ of m-dimensional manifold M_1 of constant curvature k with an $(n-m)$-dimensional Riemannian manifold M_2.

COROLLARY 2. *Suppose that M is complete and simply connected in addition to the assumption of Theorem 9. In the case of $k>0$, the manifold M is isometric to a sphere $S^n(k)$. In the case of $k<0$, the manifold M is either isometric to a hyperbolic space $S^n(k)$ or a warped product $M_1 \times {}_f M_2$ with $f=[1-(k/4)R^2]/S$ on M_1.*

Next, we consider the case $k=0$. Then the equations of $\rho_{(\beta)}$ are given by

$$\nabla_\mu \rho_{(\beta)\lambda} = \partial_\mu \rho_{(\beta)\lambda} - \Gamma^l_{\mu\lambda}\rho_{(\beta)l} = b_{(\beta)} g_{\mu\lambda},$$

which are reduced to

$$\rho_{(\beta)i}(\partial_i g_{qp}) = 0, \tag{4.8}$$

where β indicates $1, 2,\dots, m, \infty$.

If the m SCVF's on M are the extensions of m linear combinations of $V_{(\beta)}$'s on M_1 except $V_{(\infty)}$ with constant coefficients, then we may assume that the m SCVF's are $V_{(h)}$ themselves. The part M_1 is locally Euclidean. Since

$$\rho_{(h)i} = \delta_{hi} \tag{4.9}$$

and hence the equations (4.8) are reduced to $\partial_i g_{qp}=0$, we see that g_{qp} are independent of x^h. Since the parallel vector fields (4.9) vanish nowhere, the set F is empty. Thus we obtain the well-known result that a manifold admitting m parallel vector fields $V_{(h)}$ is locally the product space $M_1 \times M_2$ of an m-dimensional Euclidean space M_1 and an $(n-m)$-dimensional Riemannian manifold M_2.

If the m SCVF's on M are the extensions of m linear combinations of $V_{(h)}$'s and $V_{(\infty)}$ with constant coefficients, then we may assume that the SCVF's are $m-1$ $V_{(h)}$'s, say $h=1, 2,\dots, m-1$, and $V_{(\infty)}$. Then we have the equations

$$\partial_h g_{qp} = 0$$

for $h=1, 2,\dots, m-1$. For the vector $V_{(\infty)}$ we have

$$x^i \partial_i g_{qp} = 2g_{qp}.$$

It follows from the above two equations that the components g_{qp} are expressed in the form

$$g_{qp} = (x^m)^2 \bar{g}_{qp} = \rho_{(m)}^2 \bar{g}_{qp},$$

where $\bar{g}_{qp}$ are independent of x^h and regarded as a metric tensor of an $(n-m)$-dimensional manifold M_2. As we have seen in § 2, the set F is contained in the zero-level surface N of $\rho_{(m)}$. Thus the manifold M is locally an AEWP $M_1 \times \rho_{(m)} M_2$. From Theorem 2 we see that M_2 is a space of constant curvature 1, and from Propositions 7 and 8 that the manifold M is locally isometric to a Euclidean space.

Thus we state the following

THEOREM 10. *Suppose that an n-dimensional Riemannian manifold M admits* $m \geq 2$ *linearly independent SCVF's* $V_{(i)}$ *with characteristic constant* $k=0$. *If one of the SCVF's is concurrent and hence the others are parallel, then the manifold M is locally isometric to an Euclidean space. If the SCVF's are parallel, then M is locally the product space* $M_1 \times M_2$ *of an m-dimensional Euclidean space* M_1 *and an* $(n-m)$*-dimensional Riemannian manifold* M_2.

In the case where M is a complete and simply connected manifold, we obtain the well-known result.

COROLLARY 3. *Suppose that M is complete and simply connected in addition to the assumption of Theorem* 10. *If M admits a concurrent vector field V, then M is Euclidean. If M admits m parallel vector fields* $V_{(i)}$, *then M is the product space* $M_1 \times M_2$ *of an m-dimensional Euclidean space* M_1 *and an* $(n-m)$*-dimensional Riemannian manifold* M_2.

Bibliography

[1] R. L. Bishop and B. O'Neil, Manifolds of negative curvature, Trans. Amer. Math. Soc. **145** (1969), 1–49.

[2] H. Gauchman, On locally symmetric vector fields on Riemannian manifolds, Israel J. Math. **33** (1979), 37–51.

[3] J. A. Schouten, Ricci-Calculus, second edition, Spring-Verlag, Berlin, 1954.

[4] Y. Tashiro, Conformal transformations in complete Riemannian manifolds, Publ. of the Study Group of Geom. **3**, Kyoto Univ. 1967.

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