

On the behavior of solutions of generalized Emden-Fowler equations with deviating arguments

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1. Introduction

In the last twenty years there has been considerable interest in the problem of classifying the nonoscillatory solutions of both ordinary and functional differential equations in terms of their asymptotic behavior. Much of the work in this direction has been the derivation of both necessary and sufficient conditions for the existence of certain types of nonoscillatory solutions. As examples of such results we cite the recent papers of Kusano and Onose [7] and Odarič and Ševelo [8], and the references contained therein.

Here we are concerned with the classification of the solutions of the n -th order functional differential equation

$$(E) \quad (r(t)x^{(n-\nu)}(t))^{(\nu)} + \left(\prod_{j=1}^m |x(g_j(t))|^{\rho_j}\right) (F(t, x^2(g_1(t)), \dots, x^2(g_m(t)))) \prod_{k=1}^{q-1} \operatorname{sgn} x(g_{j_k}(t)) = 0$$

using a classification scheme similar to the one employed in [7] and [8]. However our interest is in obtaining conditions on the functions r , F , and g_j which ensure that all nonoscillatory solutions of (E) belong to certain specified classes rather than the existence of such a solution in a given class as was done in [7] and [8]. While results of the same type as ours have been obtained by other authors, e.g. [3; Th. 5], our results differ in a number of ways from those previously obtained. For example, the form of our integral conditions (see (7), (8), (12), and (22) below) differ from those previously required by other authors; moreover, when $r(t) \neq 1$, the fact that ν can be any integer satisfying $1 \leq \nu \leq n-1$ allows for numerous combinations of middle terms in (E) not considered before.

Notice that equation (E) may be viewed as a generalization of the well known Emden-Fowler equation. For a discussion of the physical and historical significance of the latter equation the reader is referred to the excellent survey paper of Wong [11].

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For simplicity our results are obtained for a special case of (E) in the next section, with their extensions to (E) being discussed in the last section.

2. Classification of solutions

Consider the n -th order differential equation

$$(1) \quad (r(t)x^{(n-\nu)}(t))^{(\nu)} + p(t)|x(g(t))|^\alpha \operatorname{sgn} x(g(t)) = 0$$

where $\alpha > 0$, $1 \leq \nu \leq n-1$, $g, p, r: [t_0, \infty) \rightarrow \mathbb{R}$ are continuous and satisfy

$$(2) \quad r(t) > 0 \quad \text{and} \quad \int_{t_0}^{\infty} [1/r(s)] ds = \infty,$$

$$(3) \quad g(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty$$

and

$$(4) \quad p(t) \geq 0 \quad \text{and} \quad p(t) \not\equiv 0 \quad \text{on} \quad [b, \infty) \quad \text{for any} \quad b \geq t_0.$$

At times we will also need that there exists a nondecreasing continuous function $h: [t_0, \infty) \rightarrow (0, \infty)$ which satisfies

$$(5) \quad h(t) \leq \inf_{s \geq t} (\min \{s, g(s)\}), \quad t \geq t_0,$$

and

$$(6) \quad h(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty.$$

Since we are only concerned with the asymptotic behavior of the solutions of (1), we will assume that every solution $x(t)$ considered here exists on $[t_x, \infty)$ for some $t_x \geq t_0$, and is nontrivial in the sense that $\sup \{|x(t)|: t \geq T\} > 0$ for every $T \geq t_x$. Such a solution will be called *oscillatory* if its set of zeros is unbounded, and will be called *nonoscillatory* otherwise. The following two lemmas, both of which can be found in [1], will be needed in the proofs of our results. Lemma 1 is an adaptation of Lemma 1 in [2] which in turn is an improved version of the well known lemma of Kiguradze [5, 6]; Lemma 2 is an adaptation of Lemma 2 of Staikos and Sficas [9].

LEMMA 1. *Let u be a positive $(n-\nu)$ -times continuously differentiable function on the interval $[a, \infty)$ and let μ be a positive continuous function on $[a, \infty)$ such that*

$$\int_a^{\infty} [1/\mu(t)] dt = \infty$$

and the function $w \equiv \mu u^{(n-\nu)}$ is ν -times continuously differentiable on $[a, \infty)$. Moreover, let

$$\omega_k = \begin{cases} u^{(k)}, & \text{if } 0 \leq k \leq n - \nu - 1, \\ w^{(k-n+\nu)}, & \text{if } n - \nu \leq k \leq n. \end{cases}$$

If $\omega_n(t) \equiv w^{(\nu)}(t)$ is of constant sign and not identically zero for all large t , then there exist $t_u \geq a$ and an integer l , $0 \leq l \leq n$, with $n+l$ even for ω_n nonnegative or $n+l$ odd for ω_n nonpositive, and such that for every $t \geq t_u$

$$l > 0 \text{ implies } \omega_k(t) > 0 \quad (k = 0, 1, \dots, l - 1)$$

and

$$l \leq n - 1 \text{ implies } (-1)^{l+k} \omega_k(t) > 0 \quad (k = l, l + 1, \dots, n - 1).$$

LEMMA 2. If the functions u , μ , w and ω_k are as in Lemma 1 and for some $k=0, 1, \dots, n-2$

$$\lim_{t \rightarrow \infty} \omega_k(t) = c, \quad c \in R,$$

then

$$\lim_{t \rightarrow \infty} \omega_{k+1}(t) = 0.$$

In order to simplify statements and proofs of results let

$$z(t) = r(t)x^{(n-\nu)}(t),$$

$$\omega_k(t) = \begin{cases} x^{(k)}(t), & k = 0, 1, \dots, n - \nu - 1, \\ z^{(k-n+\nu)}(t), & k = n - \nu, n - \nu + 1, \dots, n, \end{cases}$$

$$I(T, t) = \int_T^t [(t-s)^{n-\nu-1} s^{\nu-1} / (n-\nu-1)! r(s)] ds,$$

$$S(T, t) = \int_T^t [(t-s)^{\nu-1} s^{n-\nu-1} / (\nu-1)! r(s)] ds,$$

and

$$Q(T, t) = \int_T^t [(t-s)^{n-\nu-1} s^{\nu-2} / (n-\nu-1)! r(s)] ds.$$

We are now ready to prove the main results in this paper. In the next section we indicate how these results can be extended to the more general equation (E) by using essentially the same proofs as those given here. The theorem below divides the solutions of (1) into four classes; Corollary 2 further refines this classification.

THEOREM 1. Suppose that, in addition to (2)–(4), for sufficiently large T we have

$$(7) \quad \int^{\infty} S(T, s)p(s)ds = \infty$$

and

$$(8) \quad \int^{\infty} (I(T, g(s)))^2 p(s)ds = \infty.$$

If n is odd, then every solution $x(t)$ of (1) satisfies exactly one of the following:

- I. $x(t)$ is oscillatory;
- II. $\omega_k(t) \rightarrow 0$ monotonically as $t \rightarrow \infty$ for $k = 0, 1, \dots, n-1$;
- III. $\omega_{n-1}(t) \rightarrow 0$ and either $\omega_k(t) \rightarrow \infty$ or $\omega_k(t) \rightarrow -\infty$ as $t \rightarrow \infty$ for $k = 0, 1, \dots, n-2$;

or

- IV. There exists a constant $c \geq 0$ and an integer N satisfying $1 \leq N \leq n-2$, $\omega_k(t) \rightarrow 0$ as $t \rightarrow \infty$ for $k = N+1, N+2, \dots, n-1$, and either $\omega_N(t) \rightarrow c$ and $\omega_k(t) \rightarrow \infty$ or $\omega_N(t) \rightarrow -c$ and $\omega_k(t) \rightarrow -\infty$ as $t \rightarrow \infty$ for $k = 0, 1, \dots, N-1$.

If n is even, then $x(t)$ satisfies exactly one of I, III, or IV.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1). We may assume that $x(t) > 0$ on $[t_x, \infty)$ since $-x(t)$ is also a solution of (1). By (3) there exists $t_1 \geq t'_x = \max\{t_x, 1\}$ so that $g(t) \geq t'_x$ for $t > t_1$. This, together with (4) and equation (1), yields

$$(9) \quad \omega_n(t) = z^{(v)}(t) \leq 0$$

for $t \geq t_1$. Furthermore, (4) and (1) imply that $z^{(v)}(t) \neq 0$ for all large t , so we may assume that t_1 is large enough for Lemma 1 to imply that each of the functions ω_k , $k = 0, 1, \dots, n-1$, is of constant sign on $[t_1, \infty)$. Also since $x(t) > 0$, it is easy to see that $\omega_{n-1}(t) \equiv z^{(v-1)}(t) > 0$ on $[t_1, \infty)$.

Next observe that x is monotonic since ω_1 has fixed sign. First suppose that $x(t) \rightarrow 2c_1$ as $t \rightarrow \infty$, where c_1 is a positive constant. Then there exists $T_1 \geq t_1$ such that $c_1 \leq x(t) \leq 3c_1$ for $t \geq T_1$. In view of condition (9), the integer l assigned to the solution x by Lemma 1 is such that $n+l$ is an odd integer. If $l \geq 2$, then by Lemma 1 we have ω_1 and ω_2 both positive, and two integrations would yield a contradiction to the boundedness of x . Therefore, $l=0$ if n is odd and $l=1$ if n is even. It is then a consequence of Lemma 1 that

$$(10) \quad n \text{ odd implies } (-1)^k \omega_k(t) > 0 \quad (k = 0, 1, \dots, n-1)$$

and

$$(11) \quad n \text{ even implies } (-1)^{k+1} \omega_k(t) > 0 \quad (k = 1, \dots, n-1).$$

Multiplying equation (1) by $S(T_1, t)$ and integrating yield

$$\int_{T_1}^t S(T_1, s)p(s)x^\alpha(g(s))ds = -\int_{T_1}^t S(T_1, s)z^{(v)}(s)ds.$$

But

$$\begin{aligned} \int_{T_1}^t S(T_1, s)z^{(v)}(s)ds &= S(T_1, t)z^{(v-1)}(t) - S'(T_1, t)z^{(v-2)}(t) + \dots \\ &+ (-1)^{v-1}S^{(v-1)}(T_1, t)z(t) + (-1)^v t^{n-v-1}x^{(n-v-1)}(t) + \dots \\ &+ (-1)^{n-1}(n-v-1)!x(t) + c_2 \end{aligned}$$

for some constant c_2 . We then conclude from (10) and (11) that

$$\int_{T_1}^\infty S(T_1, s)p(s)x^\alpha(g(s))ds < \infty$$

which (since $0 < c_1 \leq x(t)$) contradicts (7). Therefore, we have that every positive nonoscillatory solution $x(t)$ of equation (1) satisfies either $x(t) \rightarrow 0$ as $t \rightarrow \infty$ or $x(t) \rightarrow \infty$ as $t \rightarrow \infty$.

If $x(t) \rightarrow 0$ as $t \rightarrow \infty$, then Lemma 2 implies that $\omega_k(t) \rightarrow 0$ monotonically for $k=0, 1, 2, \dots, n-1$ and II holds in this case. Also notice that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ cannot occur when n is even. To see this we need only recall that for n even l is odd, and therefore Lemma 1 implies that $x'(t) > 0$ so that $x(t) \neq 0$ as $t \rightarrow \infty$. Thus if $x(t) \rightarrow 0$ as $t \rightarrow \infty$, then n is odd and II holds.

Now assume that $x(t) \rightarrow \infty$ as $t \rightarrow \infty$. Since x is monotonic, it follows that x is nondecreasing on $[t_1, \infty)$. Also, recall that it was established earlier in the proof that $z^{(v-1)}$ is positive and nonincreasing on $[t_1, \infty)$. Hence $z^{(v-1)}(t) \rightarrow L$ as $t \rightarrow \infty$ for some nonnegative constant L . We show next that $L=0$. To do this, assume the contrary, that $L>0$. Then from Lemma 2 we have $\omega_k(t) \rightarrow \infty$ as $t \rightarrow \infty$ for $k=0, 1, \dots, n-2$. Thus for $v \geq 2$ it follows that $x(t), x'(t), \dots, x^{(n-v-1)}(t), z(t), \dots, z^{(v-2)}(t)$ all increase without bound as $t \rightarrow \infty$, and, in view of (2), the same is true for $I(t_1, t), I'(t_1, t), \dots, I^{(n-v-1)}(t_1, t)$. Observing that $x^{(n-v)}(t)/I^{(n-v)}(t_1, t) = r(t)x^{(n-v)}(t)/t^{v-1} = z(t)/t^{v-1}$ and applying l'Hospital's rule repeatedly lead to

$$\begin{aligned} \lim_{t \rightarrow \infty} [x(t)/I(t_1, t)] &= \lim_{t \rightarrow \infty} [x'(t)/I'(t_1, t)] = \dots = \lim_{t \rightarrow \infty} [z(t)/t^{v-1}] \\ &= (1/(v-1)) \lim_{t \rightarrow \infty} [z'(t)/t^{v-2}] = \dots = [1/(v-1)!] \lim_{t \rightarrow \infty} z^{(v-1)}(t) = L/(v-1)!. \end{aligned}$$

If $v=1$, then $z(t) = z^{(v-1)}(t) \rightarrow L$ as $t \rightarrow \infty$, and we have as above that $[x(t)/I(t_1, t)] \rightarrow L$ as $t \rightarrow \infty$. Therefore, in either case, it follows from (3) that there are constants $L_1 > 0$ and $T \geq t_1$ so that for each $t \geq T$, $x(g(t)) \geq L_1 I(t_1, g(t))$. Integrating equation (1) over $[t_1, t]$ we obtain

$$\begin{aligned} z^{(v-1)}(t_1) &= z^{(v-1)}(t) + \int_{t_1}^t p(s)x^\alpha(g(s))ds \\ &\geq \int_T^t p(s)x^\alpha(g(s))ds \geq \int_T^t p(s)(L_1 I(t_1, g(s)))^\alpha ds \end{aligned}$$

which contradicts (8). Thus we conclude that $z^{(v-1)}(t) \equiv \omega_{n-1}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Next we examine the behavior of $z_1(t)$ as $t \rightarrow \infty$ where $z_1(t) = z^{(v-2)}(t)$ if $v > 1$, and $z_1(t) = x^{(n-v-1)}(t) = x^{(n-2)}(t)$ if $v = 1$. Notice that z_1 has fixed sign by Lemma 1, and is nondecreasing since $z^{(v-1)}(t) > 0$. Now if $z_1(t) < 0$ for $t \geq t_1$, then $z_1(t) \rightarrow 0$ as $t \geq \infty$, for otherwise there would exist a positive constant L_2 such that $z_1(t) < -L_2$ for $t \geq t_1$, which is impossible since $x(t) > 0$ on $[t_1, \infty)$. Continuing in this way it is not difficult to see that $x(t)$ satisfies IV in case $z_1(t) < 0$. Finally, if $z_1(t) > 0$ on $[t_1, \infty)$, then x satisfies one of III or IV and the proof of the theorem is complete.

COROLLARY 2. *In addition to the hypotheses of Theorem 1, let conditions (5), (6) and*

$$(12) \quad \int^{\infty} sQ^{\alpha}(T, h(s))p(s)ds = \infty$$

hold. If $v \geq 2$ and x is a nonoscillatory solution of equation (1) such that $x(t)[r(t)x^{(n-v)}(t)]^{(v-2)}$ is eventually positive, then x satisfies part III of the conclusion of Theorem 1.

PROOF. Let x be a nonoscillatory solution of equation (1), say $x(t) > 0$ on $[t_x, \infty)$. From the proof of Theorem 1 there exists $t_1 \geq t_x$ so that on $[t_1, \infty)$ we have $g(t) \geq \max\{t_x, 1\}$, each ω_k has fixed sign, and $z^{(v-1)}(t) > 0$. Moreover, $\omega_{n-1}(t) \equiv z^{(v-1)}(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence if $z^{(v-2)}(t) > 0$, then $\omega_{n-2}(t) \equiv z^{(v-2)}(t)$ is bounded below on $[t_1, \infty)$. Thus Lemma 2, together with the fact that $x(t) > 0$ on $[t_1, \infty)$, implies that $\omega_k(t) \rightarrow \infty$ monotonically for $k = 0, 1, \dots, n-3$. Therefore if the conclusion of the corollary does not hold, then $z^{(v-2)}(t)$ tends monotonically to a positive constant as $t \rightarrow \infty$. Assume that this is the case. Then there exists $T_1 \geq t_1$ such that

$$K/2 \leq z^{(v-2)}(t) \leq 3K/2$$

for $t \geq T_1$ and some $K > 0$. Successive integrations yield

$$K_1Q(T, t) \leq x(t) \leq K_2Q(T, t)$$

for $t \geq T_1$ and some positive constants K_1 and K_2 . Hence in view of (5) and (6) there exists $T \geq T_1$ such that

$$(13) \quad K_1Q(T, h(t)) \leq x(h(t)) \leq K_2Q(T, h(t))$$

for $t \geq T$.

Next observe that $z^{(v-2)}$ is a bounded nonoscillatory solution of the second order linear delay equation

$$(14) \quad y''(t) + P(t)y(h(t)) = 0$$

where $P(t) = p(t)x^\alpha(g(t))/z^{(\nu-2)}(h(t))$. Then we have from [10] that

$$(15) \quad \int^{\infty} sP(s)ds < \infty$$

since (14) has a bounded nonoscillatory solution. But condition (13) implies that

$$(16) \quad P(t) = p(t)x^\alpha(g(t))/z^{(\nu-2)}(h(t)) \geq M_1 p(t)(Q(T, h(t)))^\alpha$$

where M_1 is a positive constant. Then (12) and (16) imply that $\int^{\infty} sP(s)ds = \infty$ contradicting (15) and the proof is complete.

REMARK. The classification of a solution $x(t)$ depending on the sign of the product $x(t)x^{(n-1)}(t)$ has also been done by Kartsatos [4]. If $r(t) \equiv 1$ in Corollary 2, then $x(t)[r(t)x^{(n-\nu)}(t)]^{(\nu-2)} = x(t)x^{(n-2)}(t)$ so that the results of Kartsatos and Corollary 2 are quite different.

3. Generalizations and examples

By simple modifications of their proofs the results of the previous section are easily extended to the equation

$$(E) \quad (r(t)x^{(n-\nu)}(t))^{(\nu)} + \prod_{j=1}^m |x(g_j(t))|^{\rho_j} F(t, x^2 \langle g(t) \rangle) \prod_{k=1}^{2q-1} \text{sgn } x(g_{j_k}(t)) = 0$$

where each $g_j: [t_0, \infty) \rightarrow R$ is continuous, each ρ_j is a nonnegative constant with $\sum_{j=1}^m \rho_j = 1$, q is a positive integer such that $2q-1 \leq m$, $\langle y \rangle = (y_1, \dots, y_m)$, $x^2 \langle g(t) \rangle = (x^2(g_1(t)), \dots, x^2(g_m(t)))$, $F: [t_0, \infty) \times [0, \infty)^m \rightarrow [0, \infty)$, and r, ν and n are as before. Moreover we shall ask that the function G defined by

$$(17) \quad G(t, \langle y \rangle) = \left\{ \prod_{j=1}^m (y_j)^{\rho_j/2} \right\} F(t, \langle y \rangle)$$

be continuous on the set $U = [t_0, \infty) \times [0, \infty)^m$, and that

$$(18) \quad G(t, \langle u(t) \rangle) \neq 0$$

for all large t where $\langle u(t) \rangle = (u_1(t), \dots, u_m(t))$ is any vector such that each $u_j(t) > 0$ on $[t_0, \infty)$. Furthermore we assume that for each j

$$(19) \quad g_j(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

and that there exist continuous functions $h_j: [t_0, \infty) \rightarrow (0, \infty)$ satisfying

$$(20) \quad h_j(t) \leq \inf_{s \geq t} (\min \{s, g_j(s)\})$$

and

$$(21) \quad h_j(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

According to the definitions in [1-3], we will say that equation (E) is:

- (i) g -distorted superlinear if for any fixed $t \geq t_0$ the function $F(t, \langle y \rangle)$ is nondecreasing with respect to $\langle y \rangle$ on $(0, \infty)^m$;
- (ii) g -distorted sublinear if for any fixed t the function $F(t, \langle y \rangle)$ is non-increasing with respect to $\langle y \rangle$ on $(0, \infty)^m$.

If we define $h(t) = \min_{1 \leq j \leq m} \{h_j(t)\}$, then we obtain the following two results for equation (E).

THEOREM 3. *Let condition (2) and (17)–(19) hold and equation (E) be either g -distorted superlinear or sublinear. If in addition for every constant $c > 0$ and all large T*

$$\int_0^\infty S(T, s)F(s, c, \dots, c)ds = \infty$$

and

$$\int_0^\infty \prod_{j=1}^m I^{\rho_j}(T, g_j(s))F(s, cI^2(T, g_1(s)), \dots, cI^2(T, g_m(s)))ds = \infty$$

hold, then the solutions of (E) satisfy the conclusion of Theorem 1.

COROLLARY 4. *In addition to the hypotheses of Theorem 3 and conditions (20) and (21), assume that for every positive constant c*

$$(22) \quad \int_0^\infty sQ(T, h(s))F(s, cQ^2(T, h(s)), \dots, cQ^2(T, h(s)))ds = \infty$$

holds. If $v \geq 2$ and x is a nonoscillatory solution of (E) such that $x(t)[r(t)x^{(n-v)}(t)]^{(v-2)}$ is eventually positive, then x satisfies part III of the conclusion of Theorem 1.

The proofs of Theorem 3 and Corollary 4 parallel those of Theorem 1 and Corollary 2 respectively and will not be given.

Each of the examples

$$(23) \quad x^{(4)} + (15/16t^4)x^5(t^{1/5}) = 0, \quad t > 1$$

and

$$(24) \quad x''' + (3/8t^3)(x(t^3))^{1/3} = 0, \quad t > 1$$

satisfies the hypotheses of Theorem 3 with $r(t) \equiv 1$, and $v = m = q = 1$. This is easy to verify by taking $\rho_1 = 1$, $h(t) = h_1(t) = g_1(t) = t^{1/5}$ and $F(t, y) = 15/16t^4y^{-2}$ in (23), and by taking $\rho_1 = 1$, $g_1(t) = t^3$, $h(t) = h_1(t) = t$, and $F(t, y) = 3/8t^3y^{1/3}$ in

(24). Observe that $x_1(t)=t^{5/2}$ and $x_2(t)=t^{3/2}$ are solutions of (23) and (24) respectively and that both satisfy part III of the conclusion of Theorem 1. This is in agreement with the results in [3] and [7] for the first example. None of the results in [7] apply to the second example since it is not a retarded equation.

Another difference in our results and those in [3] and [7] is that the type of solution described in part IV of the conclusion of Theorem 1 when $c=0$ is not included in the classification given in these papers. That such solutions may exist is demonstrated by the example

$$(t^{1/4}x''(t))''' + (135/256t^{19/4})x(t) = 0$$

which possesses the solution $x(t)=t^{3/2}$ of type IV with $c=0$. Both Theorem 3 and Corollary 4 apply to this example, but none of the results in either [3] or [7] apply since $r(t)=t^{1/4}$. Finally, notice that it follows from Corollary 4 that there is no solution $x(t)$ of this example of type IV such that $x(t)(t^{1/4}x''(t))'$ is eventually positive.

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References

- [1] M. K. Grammatikopoulos, On the effect of deviating arguments on the behavior of bounded solutions of nonlinear differential equations, *Ukrain. Mat. Ž.* **30** (1978), 462–473 (Russian).
- [2] M. K. Grammatikopoulos, Y. G. Sficas and V. A. Staikos, Oscillatory properties of strongly superlinear differential equations with deviating arguments, *J. Math. Anal. Appl.* **67** (1979), 171–187.
- [3] M. K. Grammatikopoulos, Y. G. Sficas and V. A. Staikos, On the type of nonoscillatory solutions of differential equations with deviating arguments, *Bull. Fac. Sci. Ibaraki Univ. Ser. A* (1977), 65–79.
- [4] A. G. Kartsatos, On positive solutions of perturbed nonlinear differential equations, *J. Math. Anal. Appl.* **47** (1974), 58–68.
- [5] I. T. Kiguradze, On the oscillation of solution of the equation $d^m u/dt^m + \alpha(t)|u|^m \operatorname{sgn} u = 0$, *Mat. Sb.* **65** (1964), 172–187 (Russian).
- [6] I. T. Kiguradze, The problem of oscillation of solutions of nonlinear differential equations, *Differencial'nye Uravnenija* **1** (1965), 995–1006 (Russian).
- [7] T. Kusano and H. Onose, Nonoscillatory solutions of differential equations with retarded arguments, *Bull. Fac. Sci. Ibaraki Univ. Ser. A.* (1975), 1–11.
- [8] O. N. Odarič and V. N. Ševelo, Some questions concerning the asymptotic properties of solutions of nonlinear differential equations with delay, *Differencial'nye Uravnenija* **9** (1973), 637–646 (Russian).
- [9] V. A. Staikos and Y. G. Sficas, Oscillatory and asymptotic behavior of functional differential equations, *J. Differential Equations* **12** (1972), 426–437.

- [10] J. S. W. Wong, Second order oscillation with retarded arguments, in "Ordinary Differential Equations, 1971 NRL-MRC Conference," Academic, New York, 1972, 581–596.
- [11] J. S. W. Wong, On the generalized Emden-Fowler equation, SIAM Review **17** (1975), 339–360.

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