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Mod p H-spaces and p-regularities

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Introduction

Let p be an odd prime or $p=0$, and X be a simply connected finite CW-complex whose integral cohomology group $H^*(X; \boldsymbol{Z})$ has no p-torsion if $p \neq 0$. Then we can consider the following conditions for X:

(ext) $H^*(X; \boldsymbol{Z}_p)$ is an exterior algebra $\Lambda(x_1, \ldots, x_k)$ over $\boldsymbol{Z}_p$ where $n_i = \deg x_i$ is odd $\geqq 3$.

(H) X is a mod p H-space, i.e., the p-localization $X_{(p)}$ is an H-space.

(reg) X is p-regular, i.e., X is p-equivalent to a product space $S^{n_1} \times \cdots \times S^{n_k}$ of spheres S^{n_i} with odd $n_i \geqq 3$.

It is well known that S^n (n: odd) is a mod p H-space, and we see that (reg) implies (H) (see Proposition 1.14). Further we see that (H) implies (ext) by Hopf's theorem (see Corollary 1.9). On the other hand, Arkowitz and Curjel [2] proved that these conditions for $p=0$ are equivalent; and Kumpel [7] studied some conditions that (H) implies (reg).

The purpose of this paper is to study the conditions that (ext) implies (H) and (H) implies (reg). By using the obstruction theory, we prove the following

THEOREM 2.3. (i) *The conditions* (ext), (H) *and* (reg) *for X are equivalent if $(n_1, \ldots, n_k)$ satisfies*

(*) ${}^p\pi_{t-1}(S^{n_i}) = 0$ *for any* $1 \leqq i \leqq k$ *and any* $t = n_{i_1} + \cdots + n_{i_s}$ $(1 \leqq i_1 < \cdots < i_s \leqq k)$,

where ${}^p\pi_{t-1}(S^n)$ denotes the p-primary component of the homotopy group $\pi_{t-1}(S^n)$ if $p \neq 0$ and ${}^0\pi_{t-1}(S^n) = 0$.

(ii) *The conditions* (H) *and* (reg) *for X are equivalent if $(n_1, \ldots, n_k)$ satisfies* (*) *with* $s=1$.

In this theorem, the assumptions on $(n_1, \ldots, n_k)$ are necessary. In fact, we see the following

THEOREM 2.6. (i) *If $(n_1, \ldots, n_k)$ does not satisfy* (*), *then there exists X which satisfies* (ext) *and is not p-regular.*

(ii) *If $(n_1, \ldots, n_k)$ does not satisfy* (*) *with* $s=1$ *and* $p \geqq 5$, *then there exists*

a mod *p H-space X which is not p-regular.*

(iii) *If* $(n_1,\ldots, n_k)$ *does not satisfy* (*) *with* $s\geqq 2$, *then there exists X which satisfies* (ext) *and is not a* mod *p H-space.*

We prepare some results on mod *p H*-spaces and *p*-regular spaces in § 1, and we prove these theorems in § 2. In § 3, we consider the complex and quaternion Stiefel manifolds $SU(n)/SU(n-k)$ and $Sp(n)/Sp(n-k)$, which are typical examples of spaces satisfying (ext); and study some conditions that these manifolds are *p*-regular (see Theorem 3.3).

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§ 1. Preliminaries

In this paper, we assume that all spaces have base points $*$, all maps and homotopies preserve base points, and all spaces have homotopy types of simply connected (*CW*-)complexes.

Furthermore, we assume that p is a prime or $p=0$, and we use the following terminologies and notations:

DEFINITION 1.1. A map $f: X\to Y$ is called a *p-equivalence* if the homomorphism

$$f_*: H_*(X; \boldsymbol{Z}_p)\longrightarrow H_*(Y; \boldsymbol{Z}_p) \quad \text{or} \quad f^*: H^*(Y; \boldsymbol{Z}_p)\longrightarrow H^*(X; \boldsymbol{Z}_p)$$

of the (co)homology groups induced by f is isomorphic, where $\boldsymbol{Z}_p$ is the cyclic group of order p if $p\neq 0$ and $\boldsymbol{Z}_0=\boldsymbol{Q}$ (the ring of rational numbers).

DEFINITION 1.2. A space X is called a mod p H-space if there exists a map μ: $X\times X\to X$ such that $\mu(\ ,*)$, $\mu(*,\ ): X\to X$ are p-equivalences, and μ is called a mod p *multiplication* of X.

DEFINITION 1.3. A finite complex K is said to be *p-universal* if for any map $\phi: K\to Y$ and any p-equivalence $f: X\to Y$ where X and Y are complexes of finite type, there exist a map $\psi: K\to X$ and a p-equivalence $h: K\to K$ such that $\phi h\sim f\psi$ ($\sim$ means "is homotopic to"), or equivalently (see [9; Th. 2.1]), if for any map $\phi': X'\to K$ and any p-equivalence $f': X'\to Y'$ where X' and Y' are finite complexes, there eixst a map $\psi': Y'\to K$ and a p-equivalence $h': K\to K$ such that $h'\phi'\sim\psi' f'$:

$$\begin{array}{ccc} K & \xrightarrow{h} & K \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{f} & Y, \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{h'} & K \\ \uparrow{\scriptstyle\phi'} & & \uparrow{\scriptstyle\psi'} \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

Then the following are known:

(1.4) (i) ([9; Cor. 4.3]) *Any finite co-H-space, e.g., any sphere, is p-universal for any prime p or* $p=0$.

(ii) ([11; Th. 1.2]) *If* $f: X \to Y$ *is a* 0*-equivalence between finite complexes and* X *is p-universal, then* Y *is also p-universal.*

(iii) ([11; Th. 1.7]) *Any finite* mod p *H-space is q-universal for any prime* q *or* $q=0$.

We also use the notion of the p-localization:

DEFINITION 1.5. (1) A space X is said to be *p-local* if the homotopy group $\pi_*(X)$ has a structure of $\boldsymbol{Z}_{(p)}$-module, where $\boldsymbol{Z}_{(p)}=\{b/a \in \boldsymbol{Q} \mid (a, p)=1\}$ if $p \neq 0$ and $\boldsymbol{Z}_{(0)}=\boldsymbol{Q}$.

(2) For any space X, there exist a p-local space $X_{(p)}$ and a map $e_{X,p}: X \to X_{(p)}$ uniquely up to homotopy satisfying the following condition: For any p-local space K and any map $f: X \to K$, there exists a map $\phi: X_{(p)} \to K$ uniquely up to homotopy such that $\phi e_{X,p} \sim f$, ([4; II, Th. 1A]). $(X_{(p)}, e_{X,p})$ or $X_{(p)}$ is called the p-localization of X.

(3) By definition, for any map $f: X \to Y$, there exists a map $f_{(p)}: X_{(p)} \to Y_{(p)}$ uniquely up to homotopy such that $f_{(p)} e_{X,p} \sim e_{Y,p} f$; and we have a map

$$l_{(p)}: [X, Y] \longrightarrow [X_{(p)}, Y_{(p)}], \quad l_{(p)}[f] = [f_{(p)}],$$

between homotopy sets. $l_{(p)}$ is said to be *quasi-epic* if for any map $\phi: X_{(p)} \to Y_{(p)}$, there exist a map $f: X \to Y$ and a homotopy equivalence $h: Y_{(p)} \to Y_{(p)}$ such that $f_{(p)} \sim h\phi$.

(1.6) ([4; II]) (i) $e_{X,p}: X \to X_{(p)}$ *is a p-equivalence.*

(ii) $f: X \to Y$ *is a p-equivalence if and only if* $f_{(p)}$ *is a homotopy equivalence.*

(iii) $(X_{(p)})_{(p)} \simeq X_{(p)}$, $(X_{(p)})_{(q)} \simeq X_{(0)}$ if $p \neq q$;

$(X \times Y)_{(p)} \simeq X_{(p)} \times Y_{(p)}$, $(X \vee Y)_{(p)} \simeq X_{(p)} \vee Y_{(p)}$, $(X \wedge Y)_{(p)} \simeq X_{(p)} \wedge Y_{(p)}$, *where* $\simeq$ *means "is naturally homotopy equivalent to".*

(1.7) ([8; Th. 5.3]) *Let* X *be a finite complex.*

(i) X *is p-universal if and only if* $l_{(p)}: [Y, X] \to [Y_{(p)}, X_{(p)}]$ *is quasi-epic for any finite complex* Y.

(ii) *If* X *is p-universal, then* $l_{(p)}: [X, Y] \to [X_{(p)}, Y_{(p)}]$ *is quasi-epic for any complex* Y *of finite type.*

By these properties and a theorem of Arkowitz and Curjel [2] we have the following

PROPOSITION 1.8. *For a finite complex* X, *the following* (1)–(3) *are equivalent*:

(1) *X is a* mod p *H-space.*

(2) *There exist an H-space Y and a p-equivalence $f: X \to Y$.*

(3) *The p-localization $X_{(p)}$ of X is an H-space.*

PROOF. (1)⇒(3): Let $\mu: X \times X \to X$ be a mod p multiplication of X. Then $\mu(\ , *)$, $\mu(*, \ ): X \to X$ induce homotopy equivalences $X_{(p)} \to X_{(p)}$ by definition and (1.6) (ii), and we denote their homotopy inverses by ϕ_1, $\phi_2: X_{(p)} \to X_{(p)}$ respectively. Then we see that $X_{(p)}$ is an H-space by a multiplication $\tilde{\mu}: X_{(p)} \times X_{(p)} \xrightarrow{\phi_1 \times \phi_2} X_{(p)} \times X_{(p)} \simeq (X \times X)_{(p)} \xrightarrow{\mu_{(p)}} X_{(p)}$.

(3)⇒(1): If $X_{(p)}$ is an H-space, then so is $X_{(0)} \simeq (X_{(p)})_{(0)}$. Hence Hopf's theorem shows that $H^*(X_{(0)}; \boldsymbol{Q})$ is an exterior algebra with finitely many odd dimensional generators. Therefore X is mod 0 H-space by [2], and X is p-universal by (1.4) (iii). Thus (1.7) (i) implies that for a multiplication $\tilde{\mu}$: $(X \times X)_{(p)} \simeq X_{(p)} \times X_{(p)} \to X_{(p)}$ of an H-space $X_{(p)}$, there are a map $\mu: X \times X \to X$ and a homotopy equivalence $h: X_{(p)} \to X_{(p)}$ such that $\mu_{(p)} \sim h\tilde{\mu}$. It is clear that μ is a mod p multiplication of X.

(2)⇔(3): (2) implies that $Y_{(p)}$ is an H-space and $f_{(p)}: X_{(p)} \to Y_{(p)}$ is a homotopy equivalence. Thus (2) implies (3). Conversely (3) implies (2) by taking $Y = X_{(p)}$, because $e_{X,p}: X \to X_{(p)}$ is a p-equivalence by (1.6) (i). q.e.d.

COROLLARY 1.9. *Let X be a finite* mod p *H-space, and assume that $H^*(X)$ has no p-torsion if $p \neq 0$. Then $H^*(X; \boldsymbol{Z}_p)$ is an exterior algebra:*

$$H^*(X; \boldsymbol{Z}_p) = \Lambda_{\boldsymbol{Z}_p}(x_1, \ldots, x_k), \quad x_i \in H^{n_i}(X; \boldsymbol{Z}_p),\ n_i: \text{odd}\ (1 \leqq i \leqq k).$$

PROOF. $H^*(X; \boldsymbol{Z}_p) \cong H^*(X_{(p)}; \boldsymbol{Z}_p)$ by (1.6) (i), and $X_{(p)}$ is an H-space by the above proposition. Thus we have the result by Hopf's theorem. q.e.d.

COROLLARY 1.10. (i) *Let X be a finite complex and A is a subcomplex of X. If X is a* mod p *H-space and A is a retract of X, then A is a* mod p *H-space.*

(ii) *For finite complexes X and Y, $X \times Y$ is a* mod p *H-space if and only if X and Y are* mod p *H-spaces.*

PROOF. (i) By Proposition 1.8, $X_{(p)}$ is an H-space. Let $r: X \to A$ be a retraction and $i: A \subset X$ be the inclusion. Consider the homotopy fibre F of $r_{(p)}$: $X_{(p)} \to A_{(p)}$. Then by using the homotopy exact sequence of $r_{(p)}$ which is split by $i_{(p)}$, we see easily that the composition of

$$F \times A_{(p)} \xrightarrow{j \times i_{(p)}} X_{(p)} \times X_{(p)} \xrightarrow{\tilde{\mu}} X_{(p)} \quad (j: F \subset X_{(p)})$$

is a homotopy equivalence, where $\tilde{\mu}$ is a multiplication of an H-space $X_{(p)}$. Thus $A_{(p)}$ is an H-space and A is a mod p H-space by Proposition 1.8.

(ii) The necessity follows from (i) and the sufficiency is clear by definition. q.e.d.

DEFINITION 1.11. A finite complex X is said to be *p-regular* if there exists a p-equivalence of X to a product space $S^{n_1}\times\cdots\times S^{n_k}$ of n_i-spheres S^{n_i} for odd n_i $(1\leqq i\leqq k)$, and $\boldsymbol{n}=(n_1,\ldots,n_k)$ (where $3\leqq n_1\leqq\cdots\leqq n_k$) is called the *type* of X.

By (1.4) (i), (1.7) and (1.6) (ii), we see easily the following

LEMMA 1.12. *For a finite complex X, the following* (1)–(3) *are equivalent*:

(1) *There exists a p-equivalence of X to $S^{n_1}\times\cdots\times S^{n_k}$.*

(2) *There exists a p-equivalence of $S^{n_1}\times\cdots\times S^{n_k}$ to X.*

(3) *The p-localization $X_{(p)}$ is homotopy equivalent to $S^{n_1}_{(p)}\times\cdots\times S^{n_k}_{(p)}$.*

Finally we notice the following proposition which is an immediate consequence of Proposition 1.8 and

(1.13) (Adams [1]) *S^{2n+1} is a* mod p *H-space for any odd prime p or $p=0$.*

PROPOSITION 1.14. *If p is an odd prime or $p=0$, then every p-regular space is a* mod p *H-space.*

§2. The main theorems

In this section, we assume that p is an odd prime or $p=0$, and

(2.1) a sequence $\boldsymbol{n}=(n_1,\ldots,n_k)$ consists of odd integers with $3\leqq n_1\leqq\cdots\leqq n_k$;

and consider the following conditions for a finite complex X:

$(\mathrm{ext})_{\boldsymbol{n}}$ $H^*(X;\boldsymbol{Z})$ has no p-torsion and $H^*(X;\boldsymbol{Z}_p)$ is an exterior algebra:

$$H^*(X;\boldsymbol{Z}_p)=\Lambda_{\boldsymbol{Z}_p}(x_1,\ldots,x_k),\quad x_i\in H^{n_i}(X;\boldsymbol{Z}_p).$$

$(\mathrm{H})_{\boldsymbol{n}}$ X is a mod p H-space satisfying $(\mathrm{ext})_{\boldsymbol{n}}$ (see Corollary 1.9).

$(\mathrm{reg})_{\boldsymbol{n}}$ X is a p-regular space of type $(n_1,\ldots,n_k)$ (see Definition 1.11).

Then $(\mathrm{H})_{\boldsymbol{n}}$ implies $(\mathrm{ext})_{\boldsymbol{n}}$, and $(\mathrm{reg})_{\boldsymbol{n}}$ implies $(\mathrm{H})_{\boldsymbol{n}}$ by Proposition 1.14. When $p=0$, Arkowitz and Curjel [2] proved that $(\mathrm{ext})_{\boldsymbol{n}}$ implies $(\mathrm{H})_{\boldsymbol{n}}$ and $(\mathrm{H})_{\boldsymbol{n}}$ implies $(\mathrm{reg})_{\boldsymbol{n}}$.

Now our main theorems are stated as follows, where

(2.2) ${}^p\pi_t(S^n)$ denotes the p-primary component of the homotopy group $\pi_t(S^n)$ if $p\neq 0$ and ${}^0\pi_t(S^n)=0$.

THEOREM 2.3. (i) *The above conditions* $(\mathrm{ext})_{\boldsymbol{n}}$, $(\mathrm{H})_{\boldsymbol{n}}$ *and* $(\mathrm{reg})_{\boldsymbol{n}}$ *for a finite complex X are equivalent if $\boldsymbol{n}=(n_1,\ldots,n_k)$ in* (2.1) *satisfies*

(2.4) ${}^p\pi_{t-1}(S^{n_i})=0$ *for any* $1\leqq i\leqq k$ *and any* $t=n_{i_1}+\cdots+n_{i_s}$ $(1\leqq i_1<\cdots<i_s\leqq k)$.

(ii) *The above conditions* $(\mathrm{H})_n$ *and* $(\mathrm{reg})_n$ *for* X *are equivalent if*

$$(2.5) \qquad {}^p\pi_{t-1}(S^{n_i}) = 0 \quad \text{for any} \quad 1 \leqq i \leqq k \quad \text{and any} \quad t = n_j \ (i < j \leqq k).$$

THEOREM 2.6. *Let* p *be an odd prime.*

(i) *If* (2.4) *does not hold, then there is a finite complex* X *which satisfies* $(\mathrm{ext})_n$ *and does not satisfy* $(\mathrm{reg})_n$.

(ii) *If* (2.5) *does not hold and* $p \geqq 5$, *then there is a finite complex* X *which satisfies* $(\mathrm{H})_n$ *and does not satisfy* $(\mathrm{reg})_n$.

(iii) *If the condition* (2.4) *with* $s \geqq 2$ *does not hold, then there is a finite complex* X *which satisfies* $(\mathrm{ext})_n$ *and does not satisfy* $(\mathrm{H})_n$.

As a corollary of Theorem 2.3, we have the following

COROLLARY 2.7 (Kumpel [7]). *If* p *is an odd prime and*

$$n_k - n_1 + 4 \leqq 2p$$

for $\boldsymbol{n} = (n_1, \ldots, n_k)$ *in* (2.1), *then* $(\mathrm{H})_n$ *and* $(\mathrm{reg})_n$ *are equivalent.*

PROOF. According to Serre [12; V, Prop. 4], ${}^p\pi_{n+i}(S^n) = 0$ if n is odd and $i < 2p-3$. Thus $n_k - n_1 + 4 \leqq 2p$ implies (2.5), because $n_j - 1 - n_i < 2p-3$ by (2.1). q.e.d.

To prove Theorem 2.3 (i), we prepare the following lemma which may be known.

LEMMA 2.8. *Let* Y *be a finite complex such that* $H^*(Y; \boldsymbol{Z})$ *has no* p*-torsion if* p *is an odd prime. Let* n *be an odd integer* $\geqq 3$, *and assume that*

$$H^t(Y; {}^p\pi_{t-1}(S^n)) = 0 \qquad \text{for any integer} \quad t \geqq n+2.$$

Then for any element $y \in H^n(Y; \boldsymbol{Z}_p)$, *there is a map* $f: Y \to S^n$ *with* $y = f^*(v)$ *for some* $v \in H^n(S^n; \boldsymbol{Z}_p)$.

PROOF. When p is an odd prime, the mod p reduction $H^*(Y) \to H^*(Y; \boldsymbol{Z}_p)$ is epic since $H^*(Y)$ has no p-torsion by assumption. Thus we take an element $\tilde{y} \in H^n(Y)$ whose mod p reduction is a given element y. When $p = 0$, we take $\tilde{y} \in H^n(Y)$ such that $i_*(\tilde{y}) = qy$ in $H^n(Y; \boldsymbol{Z}_0)$ for some $q \neq 0$ in $\boldsymbol{Q}$, where $i: \boldsymbol{Z} \subset \boldsymbol{Q} = \boldsymbol{Z}_0$ is the inclusion.

Let Y^l be the l-skelton of Y and $\iota_l: Y^l \subset Y$ be the inclusion. Then the projection $r: Y^n \to Y^n/Y^{n-1} = \vee_j S^n_j$ $(S^n_j = S^n)$ induces the epimorphism

$$r^*: \oplus_j H^n(S^n_j) = H^n(Y^n/Y^{n-1}) \longrightarrow H^n(Y^n),$$

and $\iota_n^*(\tilde{y}) = r^*(\sum_j \tilde{u}_j)$ for some $\tilde{u}_j \in H^n(S^n_j)$. For a generator $\tilde{u} \in H^n(S^n)$, take

maps $\phi_j: S^n_j \to S^n$ with $\phi_j^*(\tilde{u}) = \tilde{u}_j$, and set

$$f_n = (\vee_j \phi_j) r: Y^n \longrightarrow \vee_j S^n_j \longrightarrow S^n.$$

Then we have $\iota_n^*(\tilde{y}) = f_n^*(\tilde{u})$. Consider the cofibre sequence

$$\vee_k S^n_k \xrightarrow{\xi} Y^n \xrightarrow{\iota} Y^{n+1} \quad (S^n_k = S^n,\ \xi \text{ is the attaching map})$$

and the exact sequence

$$\oplus_k H^n(S^n_k) \xleftarrow{\xi^*} H^n(Y^n) \xleftarrow{\iota^*} H^n(Y^{n+1}) \longleftarrow 0.$$

Then $f_n^*(\tilde{u}) = \iota_n^*(\tilde{y}) = \iota^*(\iota_{n+1}^*(\tilde{y}))$ and hence $(f_n \xi)^*(\tilde{u}) = 0$ which shows that $f_n \xi \sim *: \vee_k S^n_k \to S^n$. Thus there is a map

$$f_{n+1}: Y^{n+1} \longrightarrow S^n \quad \text{with} \quad f_{n+1} \iota = f_n.$$

Since ι^* is monic, $f_{n+1}^*(\tilde{u}) = \iota_{n+1}^*(\tilde{y})$ in $H^n(Y^{n+1})$. Therefore

$$(*) \qquad \iota_{n+1}^*(y) = f_{n+1}^*(u) \quad \text{in} \quad H^n(Y^{n+1}; \boldsymbol{Z}_p)$$

where $u \in H^n(S^n; \boldsymbol{Z}_p)$ is the mod p reduction of $\tilde{u}$ when $p \neq 0$ and $u = (1/q) i_*(\tilde{u})$ when $p = 0$.

Now consider the map $e: S^n \to S^n_{(p)}$ of the p-localization. Then the obstructions for extending $ef_{n+1}: Y^{n+1} \to S^n_{(p)}$ to Y are in

$$H^t(Y/Y^{n+1}; \pi_{t-1}(S^n_{(p)})) \quad (t \geqq n+2),$$

where $\pi_{t-1}(S^n_{(p)}) \cong \pi_{t-1}(S^n) \otimes \boldsymbol{Z}_{(p)} \cong {}^p\pi_{t-1}(S^n)$ $(t \geqq n+2)$ (cf. [4; II, Th. 1B]). Thus this group is 0 if $t = n+2$, since p is an odd prime or 0. This group is also 0 if $t > n+2$ by the assumption, because the projection $r': Y \to Y/Y^{n+1}$ induces the isomorphism

$$r'^*: H^t(Y/Y^{n+1}; {}^p\pi_{t-1}(S^n)) \longrightarrow H^t(Y; {}^p\pi_{t-1}(S^n)) \quad (t > n+2).$$

Thus

$$(**) \qquad ef_{n+1}: Y^{n+1} \longrightarrow S^n_{(p)} \quad \text{has an extension} \quad \tilde{f}: Y \longrightarrow S^n_{(p)}.$$

For this map $\tilde{f}: Y \to S^n_{(p)}$, there are a map $f: Y \to S^n$ and a homotopy equivalence $h: S^n_{(p)} \to S^n_{(p)}$ such that $f_{(p)} \sim h\tilde{f}_{(p)}$ by (1.7) (i), because S^n is p-universal by (1.4) (i). Therefore $ef \sim f_{(p)} e_{Y,p} \sim h\tilde{f}_{(p)} e_{Y,p} \sim h\tilde{f}$. This and $(**)$ show that the diagram

$$\begin{array}{ccccc}
H^n(S^n; \boldsymbol{Z}_p) & \xrightarrow{f^*} & H^n(Y; \boldsymbol{Z}_p) & \xrightarrow{\iota_{n+1}^*} & H^n(Y^{n+1}; \boldsymbol{Z}_p) \\
\uparrow e^* & & \uparrow \tilde{f}^* & & \uparrow f_{n+1}^* \\
H^n(S^n_{(p)}; \boldsymbol{Z}_p) & \xrightarrow{h^*} & H^n(S^n_{(p)}; \boldsymbol{Z}_p) & \xrightarrow{e^*} & H^n(S^n; \boldsymbol{Z}_p)
\end{array}$$

is commutative, where h^*, e^* and ι_{n+1}^* are isomorphic. Set $v=e^*h^{*-1}e^{*-1}(u)\in H^n(S^n; \boldsymbol{Z}_p)$. Then $y=f^*(v)$ as desired, because $\iota_{n+1}^*f^*(v)=f_{n+1}^*(u)=\iota_{n+1}^*(y)$ by the above diagram and (*). q.e.d.

PROOF OF THEOREM 2.3. (i) It is sufficient to prove that $(\mathrm{ext})_n$ implies $(\mathrm{reg})_n$. Assume that X satisfies $(\mathrm{ext})_n$ and (2.4) holds for $\boldsymbol{n}=(n_1,\cdots, n_k)$. Then

$$H^*(X; \boldsymbol{Z}_p) = \Lambda(x_1,\cdots, x_k), \quad x_i\in H^{n_i}(X; \boldsymbol{Z}_p) \quad (1\leqq i\leqq k),$$

and $H^t(X; {}^p\pi_{t-1}(S^{n_i}))=H^t(X; \boldsymbol{Z})\otimes {}^p\pi_{t-1}(S^{n_i})=0$ for any $1\leqq i\leqq k$ and any $t\geqq n_i+2$. Therefore Lemma 2.8 shows that there are maps $f_i: X\to S^{n_i}$ and elements $v_i\in H^{n_i}(S^{n_i}; \boldsymbol{Z}_p)$ such that $x_i=f_i^*(v_i)$ for $1\leqq i\leqq k$. Consider the map

$$f: X\to S^{n_1}\times\cdots\times S^{n_k}, \quad f(x)=(f_1(x),\cdots, f_k(x)) \quad (x\in X).$$

Then f is clearly a p-equivalence and $(\mathrm{reg})_n$ holds.

(ii) We prove (ii) by the same way as the proof of Kumpel [7]. Assume that X satisfies $(\mathrm{H})_n$ and (2.5) holds. Let J_n be the subalgebra of $H^*(X; \boldsymbol{Z}_p)=\Lambda(x_1,\cdots, x_k)$ $(x_i\in H^{n_i}(X; \boldsymbol{Z}_p))$ generated by $\{x_i \mid n_i\leqq n\}$. Then we can prove that X satisfies $(\mathrm{reg})_n$ by constructing maps

$$(*) \qquad f_n: S_n=\prod_{n_i\leqq n} S^{n_i}\longrightarrow X \quad \text{such that} \quad f_n^*\mid J_n: J_n\cong H^*(S_n; \boldsymbol{Z}_p)$$

by induction on n; in fact, $f_{n_k}: S^{n_1}\times\cdots\times S^{n_k}\to X$ is a p-equivalence and X satisfies $(\mathrm{reg})_n$ by Lemma 1.12.

Take $f_0=*: S_0(=*)\to X$, and assume that f_n in (*) is constructed. If $n+1\notin\{n_1,\cdots, n_k\}$, then we may take $f_{n+1}=f_n$. Assume $n+1\in\{n_1,\cdots, n_k\}$. Then $n_{i-1}<n+1=n_i=\cdots=n_{j-1}<n_j$ for some $i<j$. Regarding $f_n: S^n\to X$ as the inclusion, consider the commutative diagram

$$\begin{array}{ccccccc}
\cdots\to\pi_*(S_n)\otimes\boldsymbol{Z}_p & \xrightarrow{f_{n*}\otimes 1} & \pi_*(X)\otimes\boldsymbol{Z}_p & \xrightarrow{\phi_*\otimes 1} & \pi_*(X, S_n)\otimes\boldsymbol{Z}_p\to\cdots \\
\downarrow h & & \downarrow h & & \downarrow h \\
\cdots\to H_*(S_n; \boldsymbol{Z}_p) & \xrightarrow{f_{n*}} & H_*(X; \boldsymbol{Z}_p) & \xrightarrow{\phi_*} & H_*(X, S_n; \boldsymbol{Z}_p)\to\cdots,
\end{array}$$

where ϕ is the natural inclusion and h's are the mod p Hurewitz maps. Then $H_*(X, S_n; \boldsymbol{Z}_p)=0$ for $*\leqq n$ by (*), and $h: \pi_{n+1}(X, S_n)\otimes\boldsymbol{Z}_p\to H_{n+1}(X, S_n; \boldsymbol{Z}_p)$ is an isomorphism. On the other hand, $S_n=\prod_{n_s\leqq n} S^{n_s}=\prod_{n_s<n} S^{n_s}$ since n is even, and $\pi_n(S_n)\otimes\boldsymbol{Z}_p\cong\oplus_{n_s<n}\pi_n(S^{n_s})\otimes\boldsymbol{Z}_p=0$ by the assumption (2.5). Thus the cokernel of $\phi_*: \pi_{n+1}(X)\to\pi_{n+1}(X, S_n)$ is a torsion group whose order is prime to p, and $\phi_*\otimes 1: \pi_{n+1}(X)\otimes\boldsymbol{Z}_p\to\pi_{n+1}(X, S_n)\otimes\boldsymbol{Z}_p$ is epic. Therefore we have elements $g_t\in\pi_{n+1}(X)$ such that

$$\phi_*h(g_t\otimes 1)=h(\phi_*\otimes 1)(g_t\otimes 1)=\phi_*(u_t) \quad (i\leqq t\leqq j-1),$$

where $u_t \in H_{n+1}(X; \boldsymbol{Z}_p)$ is the dual element of $x_t \in H^{n+1}(X; \boldsymbol{Z}_p)$ $(i \leqq t < j)$ with respect to the basis $\{x_{k_1} \cdots x_{k_l} \mid k_1 < \cdots < k_l\}$ of $H^*(X; \boldsymbol{Z}_p)$. Here $h(g_t \otimes 1)$ and u_t are primitive elements. Also $H_{n+1}(S_n; \boldsymbol{Z}_p)$ has no non-zero primitive elements, because any non-zero element in $H^{n+1}(S_n; \boldsymbol{Z}_p)$ is decomposable. Further $f_{n*}: H_{n+1}(S_n; \boldsymbol{Z}_p) \to H_{n+1}(X; \boldsymbol{Z}_p)$ is monic, because $f_n^*: H^*(X; \boldsymbol{Z}_p) \to H^*(S_n; \boldsymbol{Z}_p)$ is epic by (*). Thus the above equality and the lower exact sequence in the above diagram show that

$$h(g_t \otimes 1) = u_t \quad \text{in} \quad H_{n+1}(X; \boldsymbol{Z}_p) \quad (i \leqq t \leqq j-1).$$

Therefore by the definition of u_t, $g_t: S^{n_t} = S^{n+1} \to X$ satisfies that

(**) $\qquad g_t^*(x_t)$ is a generator of $H^{n+1}(S^{n_t}; \boldsymbol{Z}_p)$ $(i \leqq t \leqq j-1)$.

Now, by using a multiplication μ of a mod p H-space X, define a map

(2.9) $\quad \mu_l: X \times \cdots \times X$ (l copies) $\longrightarrow X$ by $\mu_2 = \mu$ and $\mu_l = \mu(\mu_{l-1} \times 1)$,

and put

$$f_{n+1} = \mu_{j-i+1}(f_n \times g_i \times \cdots \times g_{j-1}): S_{n+1} = S_n \times S^{n_i} \times \cdots \times S^{n_j - 1} \longrightarrow X.$$

Then by (*) and (**), we see immediately that $f_{n+1}^* \mid J_{n+1}: J_{n+1} \to H^*(S_{n+1}; \boldsymbol{Z}_p)$ is an isomorphism, as desired. q.e.d.

Thus we have proved Theorem 2.3 completely.

To prove Theorem 2.6, we use the following

LEMMA 2.10 ([19; 1.1.6]). *Let $F(A_1, \ldots, A_k)$ be the fat wedge of complexes $A_1, \ldots, A_k$, i.e.,*

$$F(A_1, \ldots, A_k) = \{(a_1, \ldots, a_k) \in A_1 \times \cdots \times A_k \mid a_i = * \text{ for some } i\}.$$

Then $\Sigma F(A_1, \ldots, A_k)$ is a retract of $\Sigma(A_1 \times \cdots \times A_k)$, where Σ denotes the reduced suspension.

In the following, the product space and the fat wedge of spheres are denoted simply by

$$S(\boldsymbol{n}) = S^{n_1} \times \cdots \times S^{n_k} \quad \text{and} \quad F(\boldsymbol{n}) = F(S^{n_1}, \ldots, S^{n_k}) \quad \text{for} \quad \boldsymbol{n} = (n_1, \ldots, n_k),$$

respectively. Then we have a cell decomposition

(2.11) $\quad S(\boldsymbol{n}) = (* \cup e^{n_1}) \times \cdots \times (* \cup e^{n_k}) = F(\boldsymbol{n}) \cup_{\xi} e^{|\boldsymbol{n}|}$ $(|\boldsymbol{n}| = n_1 + \cdots + n_k)$,

where the attaching map $\xi: S^{|\boldsymbol{n}|-1} \to F(\boldsymbol{n})$ is the Whitehead product of higher order.

LEMMA 2.12. *Assume that a countable complex X is an H-space. Then*

$$f_*(\xi) = 0 \text{ in } \pi_{|\boldsymbol{n}|-1}(X) \text{ for any map } f\colon F(\boldsymbol{n}) \longrightarrow X,$$

where $F(\boldsymbol{n})$, $|\boldsymbol{n}|$ *and* ξ *are the ones in* (2.11).

PROOF. Put $S=S(\boldsymbol{n})$ and $F=F(\boldsymbol{n})$ and let $r\colon \Sigma S\to\Sigma F$ be the retraction by Lemma 2.10, and $g\colon S\to\Omega\Sigma X$ (Ω denotes the loop space) be the adjoint map of $(\Sigma f)r\colon \Sigma S\to\Sigma F\to\Sigma X$. Then $g\,|\,F=\iota f\colon F\to\Omega\Sigma X$ ($\iota\colon X\subset\Omega\Sigma X$). On the other hand, according to James [5; 1.8], there is a retraction $q\colon \Omega\Sigma X\to X$ by the assumption. Thus we have a map

$$\tilde{f} = qg\colon S \longrightarrow \Omega\Sigma X \longrightarrow X \quad \text{with} \quad \tilde{f}\,|\,F = q\iota f = f.$$

Therefore $f_*(\xi)=0$ by the cell decomposition in (2.11). q.e.d.

Now we use the following notations:

(2.13) (1) For any sequence $\boldsymbol{n}=(n_1,\ldots, n_k)$ of (2.1), the set of finite complexes X satisfying the condition $(\mathrm{ext})_{\boldsymbol{n}}$, $(\mathrm{H})_{\boldsymbol{n}}$ or $(\mathrm{reg})_{\boldsymbol{n}}$ will be denoted by $\mathrm{ext}(\boldsymbol{n})$, $\mathrm{H}(\boldsymbol{n})$ or $\mathrm{reg}(\boldsymbol{n})$, respectively.

(2) For any sequence $\boldsymbol{n}=(n_1,\ldots, n_k)$, we set $|\boldsymbol{n}|=n_1+\cdots+n_k$. For any sequences $\boldsymbol{n}$ and $\boldsymbol{m}$, $\boldsymbol{n}\cup\boldsymbol{m}$ denotes the sequence consisting of integers in $\boldsymbol{n}$ or $\boldsymbol{m}$; and for a subsequence $\boldsymbol{m}$ of $\boldsymbol{n}$, $\boldsymbol{n}-\boldsymbol{m}$ denotes the complementary subsequence of $\boldsymbol{m}$ in $\boldsymbol{n}$.

LEMMA 2.14. *For any sequences* $\boldsymbol{n}$ *and* $\boldsymbol{m}$ *of* (2.1), *a finite complex* X *satisfies* $(\mathrm{ext})_{\boldsymbol{n}}$, $(\mathrm{H})_{\boldsymbol{n}}$ *or* $(\mathrm{reg})_{\boldsymbol{n}}$ *if and only if* $X\times S(\boldsymbol{m})$ *satisfies* $(\mathrm{ext})_{\boldsymbol{n}\cup\boldsymbol{m}}$, $(\mathrm{H})_{\boldsymbol{n}\cup\boldsymbol{m}}$ *or* $(\mathrm{reg})_{\boldsymbol{n}\cup\boldsymbol{m}}$, *respectively, where* $S(\boldsymbol{m})$ *is the product space of the spheres in* (2.11).

PROOF. By the definition of $(\mathrm{ext})_{\boldsymbol{n}}$, $(\mathrm{H})_{\boldsymbol{n}}$ or $(\mathrm{reg})_{\boldsymbol{n}}$, we see easily the lemma by Corollary 1.10 (ii) and (1.13). q.e.d.

PROPOSITION 2.15. *For an odd prime* p *and a sequence* $\boldsymbol{n}=(n_1,\ldots, n_k)$ *of* (2.1), *the following hold*:

(i) *If* $k\geqq 2$ *and* ${}^p\pi_{|\boldsymbol{n}|-1}(S^{n_i})\neq 0$ *for some* i, *then* $\mathrm{ext}(\boldsymbol{n})\supsetneqq\mathrm{H}(\boldsymbol{n})$.

(ii) *If* $k=2$ *and* ${}^p\pi_{n_2-1}(S^{n_1})\neq 0$, *then* $\mathrm{ext}(\boldsymbol{n})\supsetneqq\mathrm{reg}(\boldsymbol{n})$. *If* $p\geqq 5$ *in addition, then* $\mathrm{H}(\boldsymbol{n})\supsetneqq\mathrm{reg}(\boldsymbol{n})$.

(iii) *If* $k\geqq 3$ *and* ${}^p\pi_{|\boldsymbol{n}|-n_i-1}(S^{n_i})\neq 0$ *for some* i, *then* $\mathrm{ext}(\boldsymbol{n})\supsetneqq\mathrm{H}(\boldsymbol{n})$.

Before proving this proposition, we prove Theorem 2.6.

PROOF OF THEOREM 2.6. (iii) Assume that (2.4) with $s\geqq 2$ does not hold for $\boldsymbol{n}=(n_1,\ldots, n_k)$. Then we have a subsequence $\boldsymbol{m}=(m_1,\ldots, m_s)$ of $\boldsymbol{n}$ such that $s\geqq 2$ and ${}^p\pi_{|\boldsymbol{m}|-1}(S^{n_i})\neq 0$ for some i.

In case of $n_i=m_t$ for some t, Proposition 2.15(i) shows that there is a finite complex $Y\in\mathrm{ext}(\boldsymbol{m})-\mathrm{H}(\boldsymbol{m})$. Then $X=Y\times S(\boldsymbol{n}-\boldsymbol{m})\in\mathrm{ext}(\boldsymbol{n})-\mathrm{H}(\boldsymbol{n})$ as desired, by Lemma 2.14.

In case of $n_i \neq m_t$ for any t, consider the subsequence $\boldsymbol{m}' = \boldsymbol{m} \cup (n_i)$ of $\boldsymbol{n}$. Then Proposition 2.15 (iii) shows that there is $Y \in \mathrm{ext}(\boldsymbol{m}') - \mathrm{H}(\boldsymbol{m}')$, and $Y \times S(\boldsymbol{n}-\boldsymbol{m}') \in \mathrm{ext}(\boldsymbol{n}) - \mathrm{H}(\boldsymbol{n})$.

(i) Assume that (2.4) does not hold for $\boldsymbol{n}=(n_1,\ldots, n_k)$. Then we have a subsequence $\boldsymbol{m}=(m_1,\ldots, m_s)$ of $\boldsymbol{n}$ such that ${}^p\pi_{|\boldsymbol{m}|-1}(S^{n_i}) \neq 0$ for some i. If $s \geqq 2$, then $\mathrm{ext}(\boldsymbol{n}) \supsetneqq \mathrm{H}(\boldsymbol{n}) \supset \mathrm{reg}(\boldsymbol{n})$ by (iii). Assume $s=1$. Then $m_1 = n_j > n_i$ for some j and ${}^p\pi_{n_j-1}(S^{n_i}) \neq 0$. Therefore the first half of Proposition 2.15 (ii) shows that there is $Y \in \mathrm{ext}(\boldsymbol{m}') - \mathrm{reg}(\boldsymbol{m}')$ where $\boldsymbol{m}' = (n_i, n_j)$. Thus $X = Y \times S(\boldsymbol{n}-\boldsymbol{m}') \in \mathrm{ext}(\boldsymbol{n}) - \mathrm{reg}(\boldsymbol{n})$ by Lemma 2.14.

(ii) Assume that $p \geqq 5$ and (2.5) does not hold for $\boldsymbol{n}=(n_1,\ldots, n_k)$. Then ${}^p\pi_{n_j-1}(S^{n_i}) \neq 0$ for some $n_i < n_j$. Thus the second half of Proposition 2.15 (ii) shows that there is $Y \in \mathrm{H}(\boldsymbol{m}) - \mathrm{reg}(\boldsymbol{m})$ where $\boldsymbol{m} = (n_i, n_j)$. Hence $X = Y \times S(\boldsymbol{n}-\boldsymbol{m}) \in \mathrm{H}(\boldsymbol{n}) - \mathrm{reg}(\boldsymbol{n})$ by Lemma 2.14. q.e.d.

PROOF OF PROPOSITION 2.15. (i) Let $\alpha \in \pi_{|\boldsymbol{n}|-1}(S^{n_i})$ be an element of order p by the assumption. Consider the fat wedge $F(\boldsymbol{n}) = F(S^{n_i},\ldots, S^{n_k})$ in (2.11) and the inclusion $\iota: S^{n_i} \subset F(\boldsymbol{n})$. Then $\iota_*: \pi_*(S^{n_i}) \to \pi_*(F(\boldsymbol{n}))$ is monic and the order of $\beta = \iota_*(\alpha) \in \pi_*(F(\boldsymbol{n}))$ is also p. Consider the diagram

$$\begin{array}{ccccccc}
S^{|\boldsymbol{n}|-1} & \xrightarrow{\xi} & F(\boldsymbol{n}) & \longrightarrow & S(\boldsymbol{n}) & = & F(\boldsymbol{n}) \cup_{\xi} e^{|\boldsymbol{n}|} \\
\theta\uparrow & & \| & & \vdots\phi & & \\
S^{|\boldsymbol{n}|-1} & \xrightarrow{p\xi} & F(\boldsymbol{n}) & \longrightarrow & Y & = & F(\boldsymbol{n}) \cup_{p\xi} e^{|\boldsymbol{n}|} \\
\theta\downarrow & & \| & & \vdots\psi & & \\
S^{|\boldsymbol{n}|-1} & \xrightarrow{\xi+\beta} & F(\boldsymbol{n}) & \xrightarrow{\lambda} & X & = & F(\boldsymbol{n}) \cup_{\xi+\beta} e^{|\boldsymbol{n}|}
\end{array}$$

of cofiber sequences, where ξ is the map in (2.11) and θ is a map of degree p. Since $\xi_*(\theta) = p\xi$ and $(\xi+\beta)_*(\theta) = p(\xi+\beta) = p\xi$, there exist maps $\phi: Y \to S(\boldsymbol{n})$ and $\psi: Y \to X$ such that the above diagram is homotopy commutative. Therefore ϕ and ψ are 0-equivalences by the five lemma, since so is θ. Thus $H^*(X; \boldsymbol{Q}) \cong H^*(Y; \boldsymbol{Q}) \cong H^*(S(\boldsymbol{n}); \boldsymbol{Q})$. Furthermore $\dim H^*(S(\boldsymbol{n}); \boldsymbol{Q})$ is equal to the number of cells of $S(\boldsymbol{n})$ in (2.11), and the latter is equal to that of X by definition. Therefore $H^*(X; \boldsymbol{Z})$ is torsion free. Consider

$$H^*(S(\boldsymbol{n}); \boldsymbol{Z}) \xrightarrow{\phi^*} H^*(Y; \boldsymbol{Z}) \xleftarrow{\psi^*} H^*(X; \boldsymbol{Z}).$$

Then ϕ^* and ψ^* are isomorphisms if $* < |\boldsymbol{n}|$. If $* = |\boldsymbol{n}|$, then these groups are $\boldsymbol{Z}$, and ϕ^* and ψ^* send generators to p times of generators. Hence we see that $H^*(X; \boldsymbol{Z}_p) \cong H^*(S(\boldsymbol{n}); \boldsymbol{Z}_p)$, and $H^*(X; \boldsymbol{Z}_p) = \Lambda(x_1,\ldots, x_k)$ $(x_j \in H^{n_j}(X; \boldsymbol{Z}_p))$. Thus $X \in \mathrm{ext}(\boldsymbol{n})$. Furthermore the induced homomorphism $\lambda_j^*: H^*(X; \boldsymbol{Z}_p) \to H^*(S^{n_j}; \boldsymbol{Z}_p)$ of the restriction $\lambda_j = \lambda | S^{n_j}: S^{n_j} \to X$ of λ in the above diagram satisfies $\lambda_j^*(x_j) \neq 0$ $(1 \leqq j \leqq k)$. Therefore, if X is a mod p H-space with multi-

plication μ, then

(2.16) $\mu_k(\lambda_1 \times \cdots \times \lambda_k)\colon S(\boldsymbol{n}) \to X \times \cdots \times X$ (k copies) $\xrightarrow{\mu_k} X$ (μ_k is the map in (2.9))

is a p-equivalence, and there is a p-equivalence $f\colon X \to S(\boldsymbol{n})$ by Lemma 2.12. Consider the compositions

$$g\colon F(\boldsymbol{n}) \xrightarrow{\lambda} X \xrightarrow{f} S(\boldsymbol{n}) \xrightarrow{\mathrm{pr}} S^{n_i} \xrightarrow{e} S^{n_i}_{(p)} \quad \text{and} \quad g\iota\colon S^{n_i} \subset F(\boldsymbol{n}) \longrightarrow S^{n_i}_{(p)}.$$

Then $g\iota$ is also a p-equivalence. On the other hand, $g_*(\xi)=0$ by Lemma 2.12 and $\lambda_*(\xi+\beta)=0$ by the definition of λ. Thus $g_*(\beta)=0$ and $(g\iota)_*(\alpha)=g_*(\beta)=0$. This contradicts that the order of α is p. Therefore $X \notin \mathrm{H}(\boldsymbol{n})$.

(ii) The first half: For $l \geqq 2$, the induced homomorphism

$$\rho_*\colon \pi_*(SO(2l)) \longrightarrow \pi_*(S^{2l-1}) \quad (\rho\colon SO(2l) \longrightarrow S^{2l-1} \text{ is the projection})$$

is epic if $l=2$ or 4 and $\operatorname{Im}\rho_* = 2\pi_*(S^{2l-1})$ otherwise by [14; 23.4]. Let $\alpha \in \pi_{n_2-1}(S^{n_1})$ be an element of order $p(\neq 2)$ by the assumption. Then we can take $\beta \in \pi_{n_2-1}(SO(n_1+1))$ such that $\rho_*(\beta)=2\alpha$, and we have the n_1-sphere bundle X over S^{n_2} with characteristic class β. Since $n_2 > n_1+1$, it is clear that $X \in \mathrm{ext}(\boldsymbol{n})$. On the other hand, consider the homotopy exact sequence $\pi_{n_2}(S^{n_2}) \xrightarrow{\partial} \pi_{n_2-1}(S^{n_1}) \to \pi_{n_2-1}(X) \to 0$. Then $\operatorname{Im}\partial$ is generated by 2α, and ${}^p\pi_{n_2-1}(X) = {}^p(\pi_{n_2-1}(S^{n_1})/\operatorname{Im}\partial) \ncong {}^p\pi_{n_2-1}(S^{n_1})$ since the order of 2α is p. Thus $X \notin \mathrm{reg}(\boldsymbol{n})$ as desired.

The second half: The result is immediate consequence of Harper's result ([3; p. 554]) that for any $\alpha \in \pi_{n_2-1}(S^{n_1})$, there is a mod p H-space Y^α such that Y^α is p-equivalent to $S^{n_1} \cup_\alpha e^{n_2} \cup e^{n_1+n_2}$ ($p \geqq 5$). If we take α to be an element of order p by assumption, then ${}^p\pi_{n_2-1}(Y^\alpha) = {}^p\pi_{n_2-1}(S^{n_1} \cup_\alpha e^{n_2}) \ncong {}^p\pi_{n_2-1}(S^{n_1})$ and we see that $Y^\alpha \in \mathrm{H}(\boldsymbol{n}) - \mathrm{reg}(\boldsymbol{n})$.

(iii) Let $\alpha \in \pi_{|\boldsymbol{m}|-1}(S^{n_i})$ be an element of order p by the assumption, where $\boldsymbol{m} = \boldsymbol{n} - (n_i)$. Then by the same way as the proof of the first half of (ii), we have the sphere bundle

$$S^{n_i} \xrightarrow{\iota} Y \xrightarrow{\pi} S^{|\boldsymbol{m}|} \quad \text{with characteristic class } \beta \in \pi_{|\boldsymbol{m}|-1}(SO(n_i+1)),$$

where $\rho_*(\beta)=2\alpha$. Consider the pull-back diagram

$$\begin{array}{ccccc} S^{n_i} & \overset{\iota'}{\subset} & X & \xrightarrow{\tilde{\psi}} & Y \overset{\iota}{\supset} S^{n_i} \\ & & \pi' \downarrow & & \downarrow \pi \\ & & S(\boldsymbol{m}) & \xrightarrow{\psi} & S^{|\boldsymbol{m}|} \end{array}$$

where ψ is the map collapsing the fat wedge $F(\boldsymbol{m})$ to $*$. Then the induced homomorphism $\iota'^*\colon H^*(X; \boldsymbol{Z}_p) \to H^*(S^{n_i}; \boldsymbol{Z}_p)$ is epic, because $\tilde{\psi}\iota' = \iota$ and ι^* is epic. Thus S^{n_i} is totally non homologous to zero in X, and we see that $H^*(X; \boldsymbol{Z}_p) =$

$\Lambda(x_1,\dots,x_k)$ where $x_j=\pi'^*(u_j)$ if $j\neq i$ and x_i is an element with $\iota'^*(x_i)=u_i$ (u_j is a generator of $H^{n_j}(S^{n_j};\boldsymbol{Z}_p)$) (cf. [17; 15.47]). Therefore $X\in\mathrm{ext}(\boldsymbol{n})$.

Now consider a lifting $\tilde{\phi}\colon e^{|\boldsymbol{m}|}\to X$ of the characteristic map $\phi\colon e^{|\boldsymbol{m}|}\to S(\boldsymbol{m})$ of the $|\boldsymbol{m}|$-cell of $S(\boldsymbol{m})$ in (2.11) with $\phi\,|\,S^{|\boldsymbol{m}|-1}=\xi\colon S^{|\boldsymbol{m}|-1}\to F(\boldsymbol{m})$. Then $\tilde{\xi}=\tilde{\phi}|S^{|\boldsymbol{m}|-1}$ is a lifting of ξ, and $\tilde{\psi}\tilde{\xi}=pr_2\tilde{\xi}$ represents $\rho_*(\beta)=2\alpha\in\pi_{|\boldsymbol{m}|-1}(S^{n_i})$ by the constructions. Therefore X has a cell decomposition

$$X = F(\boldsymbol{m})\times S^{n_i}\cup(e^{|\boldsymbol{m}|}\times S^{n_i}) = F(\boldsymbol{m})\times S^{n_i}\cup_{\xi+2\alpha}(e^{|\boldsymbol{m}|}\times *)\cup(e^{|\boldsymbol{m}|}\times e^{n_i}).$$

Assume that X is a mod p H-space with multiplication μ. Then the restriction $\lambda_j=\lambda\,|\,S^{n_j}$ ($\lambda_i=\iota'$) of the inclusion $\lambda\colon F(\boldsymbol{m})\times S^{n_i}\subset X$ and μ define a p-equivalence $S(\boldsymbol{n})\to X$ by (2.16). Thus there is a p-equivalence $f\colon X\to S(\boldsymbol{n})$ by Lemma 1.12. Consider the composition

$$g\colon F(\boldsymbol{m})\times S^{n_i}\xrightarrow{\lambda} X\xrightarrow{f} S(\boldsymbol{n})\xrightarrow{\mathrm{pr}} S^{n_i}\xrightarrow{e} S^{n_i}_{(p)}.$$

Then $g\,|\,S^{n_i}$ is also a p-equivalence. On the other hand, $g_*(\xi)=0$ by Lemma 2.12, and $\lambda_*(\xi+2\alpha)=0$ by the above cell decomposition of X. Thus $2g_*(\alpha)=0$ and $(g\,|\,S^{n_i})_*(\alpha)=0$. This is a contradiction and X is not a mod p H-space.

q.e.d.

Thus Theorem 2.6 is proved completely.

In the conclusion of this section, we notice the following theorem which gives a sufficient condition that a complex in $\mathrm{ext}(\boldsymbol{n})$ belongs to $\mathrm{reg}(\boldsymbol{n})$.

THEOREM 2.17. *For $\boldsymbol{n}=(n_1,\dots,n_k)$ of (2.1), let $X\in\mathrm{ext}(\boldsymbol{n})$, i.e.,*

$$H^*(X;\boldsymbol{Z}_p) = \Lambda(x_1,\dots,x_k),\quad x_i\in H^{n_i}(X;\boldsymbol{Z}_p),$$

and suppose that there is a subsequence $\boldsymbol{m}$ of $\boldsymbol{n}$ satisfying

(2.18) ${}^p\pi_{|\boldsymbol{n}'|-1}(S^{n_i})=0$ *for any i and any subsequence $\boldsymbol{n}'$ of $\boldsymbol{n}$ with $n_i\notin\boldsymbol{m}$ and $n_i+2p-2<|\boldsymbol{n}'|$.*

Then $X\in\mathrm{reg}(\boldsymbol{n})$ if X satisfies the following two conditions:

(2.19) *There exists a map $f\colon X\to S(\boldsymbol{m})$ such that $f^*\colon H^*(S(\boldsymbol{m});\boldsymbol{Z}_p)\to H^*(X;\boldsymbol{Z}_p)$ sends a generator $u_j\in H^*(S^{n_j};\boldsymbol{Z}_p)$ to x_j for any $n_j\in\boldsymbol{m}$.*

(2.20) $\mathscr{P}^1x_i=0$ *in* $H^{n_i+2p-2}(X;\boldsymbol{Z}_p)$ *for any i with $n_i\notin\boldsymbol{m}$.*

PROOF. Under the conditions (2.20) and (2.18), we prove the following

(2.21) For any i with $n_i\notin\boldsymbol{m}$, there is a map $g_i\colon X\to S^{n_i}$ such that $g_i^*(u_i)=x_i$ for $g_i^*\colon H^*(S^{n_i};\boldsymbol{Z}_p)\to H^*(X;\boldsymbol{Z}_p)$ and a generator $u_i\in H^{n_i}(S^{n_i};\boldsymbol{Z}_p)$.

Then these maps g_i together with f in the condition (2.19) define a p-equivalence

$$f\times\prod_{n_i\in n-m} g_i\colon X\longrightarrow S(\boldsymbol{m})\times S(\boldsymbol{n}-\boldsymbol{m})=S(\boldsymbol{n}),$$

and the theorem is proved.

To prove (2.21), we take i with $n_i\notin\boldsymbol{m}$ and set $n=n_i$, $q=n_i+2p-3$ and $x=x_i$ for the simplicity. Since ${}^p\pi_m(S^n)=0$ for $m<q$ by [12; V, Prop. 4], we can construct

$$g\colon X^q\longrightarrow S^n_{(p)} \quad\text{with}\quad g^*(u)=\iota^*(x)$$
$$(u \text{ is a generator of } H^n(S^n_{(p)};\boldsymbol{Z}_p)=H^n(S^n;\boldsymbol{Z}_p))$$

by the same way as the proof of Lemma 2.8, where $\iota\colon X^q\subset X$.

Now take any $(q+1)$-cell of X with attaching map $\eta\colon S^q\to X^q$. Then, for $g\eta\colon S^q\to S^n_{(p)}$, we have a homotopy commutative diagram

$$\begin{array}{ccccc} S^q & \xrightarrow{e} & S^q_{(p)} & \xleftarrow{e} & S^q \\ \downarrow{\scriptstyle g\eta} & & \downarrow & & \downarrow{\scriptstyle \nu} \\ S^n_{(p)} & = & S^n_{(p)} & \xrightarrow{h} & S^n_{(p)} \xleftarrow{e} S^n \end{array}$$

for some homotopy equivalence h and some map ν, where e's are the p-localizations. Every horizontal map in this diagram is a p-equivalence, and hence

$$H^*(C_{g\eta};\boldsymbol{Z}_p)\cong H^*(C_\nu;\boldsymbol{Z}_p)\quad (C_\alpha \text{ is the mapping cone of } \alpha).$$

It is well known that $\mathscr{P}^1=0$ on $H^*(C_\nu;\boldsymbol{Z}_p)$ if and only if $\nu\in\pi_q(S^n)$ is 0 in ${}^p\pi_q(S^n)$ (e.g., cf. [16; 5.2]). Thus we see that

(*) $\mathscr{P}^1=0$ on $H^*(C_{g\eta};\boldsymbol{Z}_p)$ if and only if $g\eta=0$ in ${}^p\pi_q(S^n_{(p)})=\pi_q(S^n_{(p)})$.

Consider the induced homomorphisms

$$H^{q+1}(C_{g\eta};\boldsymbol{Z}_p)\xrightarrow{\tilde{g}^*}H^{q+1}(C_\eta;\boldsymbol{Z}_p)\xleftarrow{\iota''^*}H^{q+1}(X^{q+1};\boldsymbol{Z}_p)\xleftarrow{\iota'^*}H^{q+1}(X;\boldsymbol{Z}_p),$$

where $\tilde{g}\colon C_\eta\to C_{g\eta}$ is the map induced by g, and ι' and ι'' are the inclusions. Then $\tilde{g}^*$ is clearly isomorphic, and (2.20) and the equality $g^*(u)=\iota^*(x)$ show that $\mathscr{P}^1=0$ on $H^*(C_{g\eta};\boldsymbol{Z}_p)$. Thus $g\eta=0$ by (*); and we have an extension

$$g'\colon X^{q+1}\longrightarrow S^n_{(p)} \quad\text{of}\quad g \quad\text{with}\quad g'^*(u)=\iota'^*(x).$$

Now, by the same way as the proof of Lemma 2.8 by using the condition (2.18), we can get a map

$$\tilde{g}\colon X\longrightarrow S^n \quad\text{with}\quad \tilde{g}^*(u)=x.$$

Thus (2.21) is proved. q.e.d.

§ 3. The p-regularity of the Stiefel manifolds

In this section, we study the p-regularity (p: odd prime) of the complex (resp. quaternion) Stiefel manifold

$$W_{n,k} = SU(n)/SU(n-k) \text{ (resp. } Sp(n)/Sp(n-k)),$$

which is a typical example of a complex satisfying $(\mathrm{ext})_{\boldsymbol{n}}$ in § 2 for $\boldsymbol{n} = (2d(n-k+1)-1, \ldots, 2dn-1)$, in fact,

$$H^*(W_{n,k}; \boldsymbol{Z}_p) = \Lambda(\omega_{2d(n-k+1)-1}, \ldots, \omega_{2dn-1}), \quad \deg \omega_j = j, \tag{3.1}$$

where $d=1$ (resp. 2), (cf. [16; IV, 4.7]). We notice that

$$W_{n,n-1} = SU(n) \quad (\text{resp. } Sp(n))$$

is p-regular if and only if $p \geqq dn$ by [13; V, Prop. 7]. Furthermore, $W_{n,1} = S^{2dn-1}$ is p-regular.

The main result of this section is the following theorem, where

$$r(i) = d(n-i+1), \quad s(i) = i\{r(i)+r(1)-1\} = u(i, i), \tag{3.2}$$

$$u(i, j) = \textstyle\sum_{l=0}^{j-1} \{2r(i-l)-1\} = j\{r(i)+r(i-j+1)-1\}.$$

THEOREM 3.3. *Let $2 \leqq k \leqq n-1$ and consider the condition*

$$d(k-1)+1 < p, \text{ or } d(k-1)+1 = p \text{ and } n \equiv 0 \bmod p \quad (d=1 \text{ (resp. } 2)). \tag{3.4}$$

(i) *If* (3.4) *does not hold, then the complex (resp. quaternion) Stiefel manifold $W_{n,k}$ is not p-regular.*

(ii) *Under the condition* (3.4), *$W_{n,k}$ is p-regular if the following condition* (1) *or* (2) *holds*:

(1) $dn \leqq p$, *or* $(s(k)+2)/r(k) \leqq p$ (*e.g.*, $p \geqq 2k+1$ *and* $dn \geqq dk^2-k-d+2$).

(2) $dn > p$, $(s(k)+2)/r(k) > p$ *and*

$$^p\pi_{t-1}(S^{2r(i)-1}) = 0 \quad \text{for any} \quad t = 2da+u(i, j) \geqq 2r(i)+2p-2,$$

where i, j and a are integers with $2 \leqq j \leqq i \leqq k$, $0 \leqq a \leqq j(i-j)$ and $(s(i)+2)/r(i) > p$.

By using this theorem, we shall give some p-regular Stiefel manifolds in Examples 3.9, 3.11 and 3.12 below.

To prove Theorem 3.3, we prepare some results. The following proposition is well known and is verified easily by the comparison theorem of Zeeman [20; Th. 2].

PROPOSITION 3.5. *Suppose that a complex X of finite type satisfies*

$$H^*(X; \boldsymbol{Z}_p) = \Lambda(u_a \mid a \in A) \otimes \boldsymbol{Z}_p[v_b \mid b \in B] \quad \textit{for} \quad * < N,$$

where the degrees of u_a *and* v_b *are odd and even, respectively, and* $\boldsymbol{Z}_p[\ ]$ *denotes the polynomial algebra. Then*

$$H^*(\Omega X; \boldsymbol{Z}_p) = \Lambda(\sigma v_b \mid b \in B) \otimes \boldsymbol{Z}_p[\sigma u_a \mid a \in A]$$
$$\textit{for} \quad * < \min\{N-2,\ p\deg(\sigma u_a) - 1 \mid a \in A\},$$

where σ *denotes the cohomology suspension.*

The following theorem is a generalization of Stasheff's result [14; Prop. 4]:

THEOREM 3.6. *Let* X *and* Y *be complexes and* $f: X \to \Omega^{2l}Y$ *be a map for some* $l \geqq 1$, *and suppose that*

$$H^*(X; \boldsymbol{Z}_p) = \Lambda(x_1, \ldots, x_r), \quad H^*(Y; \boldsymbol{Z}_p) = \Lambda(y_1, \ldots, y_r)$$

and $f^*(\sigma^{2l} y_i) = x_i$ *for any* $1 \leqq i \leqq r$, *where* $n_i = \deg x_i = \deg y_i - 2l$ *is odd and* $n_1 \leqq \cdots \leqq n_r$. *If*

$$p \geqq (4+2N)/(1+n_1) \quad (N = \textstyle\sum_{i=1}^{r} n_i),$$

then X *is a* mod p *H-space.*

PROOF. By the assumption on Y and by the repeated use of the above proposition, we see that

$$H^*(\Omega^{2l}Y; \boldsymbol{Z}_p) = \Lambda(\sigma^{2l} y_1, \ldots, \sigma^{2l} y_r) \qquad \text{for} \quad * < p(n_1+1) - 3.$$

Therefore the homotopy fibre F of $f_{(p)}: X_{(p)} \to (\Omega^{2l}Y)_{(p)}$ is $(p(n_1+1)-5)$-connected. Furthermore F is homotopy eqivalent to the p-localization of the homotopy fibre of f ([4; II, 1.10]), and hence $\pi_*(F)$ is p-local.

Consider the homotopy commutative diagram

$$\begin{array}{ccccc} X_{(p)} \vee X_{(p)} & \xrightarrow{\nabla} & X_{(p)} & \xrightarrow{f_{(p)}} & (\Omega^{2l}Y)_{(p)} \\ \downarrow{\scriptstyle \iota} & & & & \uparrow{\scriptstyle \mu_{(p)}} \\ X_{(p)} \times X_{(p)} & \xrightarrow{f_{(p)} \times f_{(p)}} & (\Omega^{2l}Y)_{(p)} \times (\Omega^{2l}Y)_{(p)} & \simeq & (\Omega^{2l}Y \times \Omega^{2l}Y)_{(p)}, \end{array}$$

where ι is the inclusion, ∇ is the folding map and μ is the loop multiplication of $\Omega^{2l}Y$. Then the obstruction for extending ∇ to a multiplication $X_{(p)} \times X_{(p)} \to X_{(p)}$ are in

$$(*) \qquad H^*(X_{(p)} \wedge X_{(p)}; \pi_{*-1}(F)).$$

This is 0 for $* \leqq p(n_1+1)-4$, since F is $(p(n_1+1)-5)$-connected. On the other

hand, because $H^*(X_{(p)}; \boldsymbol{Z}_p) \cong H^*(X; \boldsymbol{Z}_p) = \Lambda(x_1, \ldots, x_r)$, $H^*(X_{(p)}; \boldsymbol{Z})$ $(*>N)$ is a torsion group whose order is prime to p and hence so is $H^*(X_{(p)} \wedge X_{(p)}; \boldsymbol{Z})$ $(*>2N)$. Thus the group of $(*)$ is 0 for $*>p(n_1+1)-4$ by the universal coefficient theorem, since $\pi_*(F)$ is p-local and $p(n_1+1)-4 \geqq 2N$ by the assumption. Therefore $X_{(p)}$ is an H-space and X is a mod p H-space by Proposition 1.8.

q.e.d.

Now we can prove Theorem 3.3 by using the following

LEMMA 3.7. (i) (James [6; Th. 1.4]) *There exists a positive integer $m>k$ such that the projection $\pi: W_{m,k} \to W_{m,1} = S^{2dm-1}$ has a cross-section $\theta: S^{2dm-1} \to W_{m,k}$.*

(ii) *Let $J: W_{n,k} \to \Omega^{2dm} W_{m+n,k}$ be the adjoint map of the composition of*

$$S^{2dm} \wedge W_{n,k} = S^{2dm-1} * W_{n,k} \xrightarrow{\theta * 1} W_{m,k} * W_{n,k} \xrightarrow{h} W_{m+n,k},$$

where $*$ denotes the join and h is the intrinsic join due to James. Then

$$J^*(\sigma^{2dm}\omega_{2dt-1}) = \omega_{2dt-2dm-1} \quad \text{for any} \quad m+n-k+1 \leqq t \leqq m+n.$$

PROOF OF THEOREM 3.3. (i) According to [10; Th. 1.1], (3.4) is equivalent to the condition that $\mathscr{P}^1 = 0$ on $H^*(W_{n,k}; \boldsymbol{Z}_p)$. It is clear that $\mathscr{P}^1 = 0$ on $H^*(X; \boldsymbol{Z}_p)$ for any p-regular space X. Thus we see (i).

(ii) We prove (ii) for $W_{n,k} = SU(n)/SU(n-k)$ and $d=1$. The result for $Sp(n)/Sp(n-k)$ and $d=2$ can be proved similarly.

(1) The case $n \leqq p$: In this case, there is a p-equivalence $\phi: S^3 \times S^5 \times \cdots \times S^{2n-1} \to SU(n)$ by Corollary 2.7 and (3.1). Let $\pi: SU(n) \to W_{n,k}$ be the projection and put

$$f = \pi(\phi \mid S^{2(n-k)+1} \times \cdots \times S^{2n-1}): S^{2(n-k)+1} \times \cdots \times S^{2n-1} \longrightarrow W_{n,k}.$$

Then we see easily that f is p-equivalence.

The case $(s(k)+2)/r(k) \leqq p$: Since $p \geqq (s(k)+2)/r(k) = \{k(2n-k)+2\}/(n-k+1)$ by (3.2), we can apply Theorem 3.6 to J in Lemma 3.7 (ii) and we see that $W_{n,k}$ is a mod p H-space. Thus $W_{n,k}$ is p-regular by Corollary 2.7 becauae $p \geqq (s(k)+2)/r(k) \geqq k+1$.

(2) Let $k'(<k)$ be the maximum number with $(s(k')+2)/r(k') \leqq p$. Then (3.4) holds also for $k=k'$. Thus $W_{n,k}$ is p-regular by (1). Assume inductively that $W_{n,i}$ is p-regular for $k' \leqq i < k$, and consider $X = W_{n,i+1}$ and the composition

$$f = \phi\pi: X = W_{n,i+1} \longrightarrow W_{n,i} \longrightarrow S(\boldsymbol{m}) \quad (\boldsymbol{m} = (2n-2i+1, 2n-2i+3, \ldots, 2n-1))$$

of the projection π and a p-equivalence ϕ. Then $\boldsymbol{n}$ in Theorem 2.17 is $(2n-2i-1) \cup \boldsymbol{m}$, and (2.19) holds clearly. Further (2.20) holds by [10; Th. 1.1] since (3.4) holds for $k=i+1$. The condition (2.18) is contained in the last con-

dition in (2). Thus $X=W_{n,i+1}$ is p-regular by Theorem 2.17, and so is $W_{n,k}$ by the induction. q.e.d.

PROOF OF LEMMA 3.7. We prove (ii) by using (i) which is proved in [6; Th. 1.4].

We regard $W_{a,b}$ as the set of all normal systems $(\lambda_1,\dots,\lambda_b)$ of vectors λ_i in $\boldsymbol{F}^a$ ($\boldsymbol{F}$ is the complex or quaternion field) with $(\lambda_i,\lambda_j)=\delta_{ij}$. For any $a>b>c$ and l, let

$$\varepsilon\colon W_{a,b}\longrightarrow W_{a+l,b+l} \quad\text{and}\quad \pi\colon W_{a,b}\longrightarrow W_{a,b-c} \tag{3.8}$$

be the inclusion and the projection given by $\varepsilon(\lambda_1,\dots,\lambda_b)=(\lambda_1,\dots,\lambda_b,e_{a+1},\dots,e_{a+l})$ (e_i is the i-th unit vector in $\boldsymbol{F}^{a+l}$) and $\pi(\lambda_1,\dots,\lambda_b)=(\lambda_{c+1},\dots,\lambda_b)$. The intrinsic join $h\colon W_{m,b}*W_{a,b}\to W_{m+a,b}$ is defined by

(*) $h((\lambda_1,\dots,\lambda_b),(\mu_1,\dots,\mu_b),t)=(v_1,\dots,v_b)$, $v_j=\lambda_j\cos(\pi t/2)+\mu_j\sin(\pi t/2)\in\boldsymbol{F}^m\times\boldsymbol{F}^a=\boldsymbol{F}^{m+a}$, for $(\lambda_1,\dots,\lambda_b)\in W_{m,b}$, $(\mu_1,\dots,\mu_b)\in W_{a,b}$ and $0\leqq t\leqq 1$.

Then the diagram

$$\begin{array}{ccccc} W_{m,b}*W_{a,b} & \xleftarrow{\pi*1} & W_{m,b+1}*W_{a,b} & \xrightarrow{1*\varepsilon} & W_{m,b+1}*W_{a+1,b+1} \\ \downarrow h & & & & \downarrow h \\ W_{m+a,b} & & \xrightarrow{\qquad\varepsilon\qquad} & & W_{m+a+1,b+1} \end{array} \tag{**}$$

is homotopy commutative. In fact, $((\lambda_1,\dots,\lambda_{b+1}),(\mu_1,\dots,\mu_b),t)\in W_{m,b+1}*W_{a,b}$ is mapped to

$$(v_1,\dots,v_b,v)\ (v_j \text{ is the one in (*)} \quad\text{and}\quad v=\lambda_{b+1}\cos(\pi t/2)+e_{a+1}\sin(\pi t/2))$$

by $h(1*\varepsilon)$, and to $(v_1,\dots,v_b,e_{m+a+1})$ by $\varepsilon h(\pi*1)(\phi*1)$ where $\phi\colon W_{m,b+1}\to W_{m,b+1}$ is the map given by $\phi(\lambda_1,\dots,\lambda_b,\lambda_{b+1})=(\lambda_{b+1},\lambda_1,\dots,\lambda_b)$. Therefore $h(1*\varepsilon)\sim\varepsilon h(\pi*1)(\phi*1)\sim\varepsilon h(\pi*1)$ and (**) is homotopy commutative, because $\phi\sim 1$.

Now let $\theta\colon S^{2dm-1}\to W_{m,k}$ be a cross section of $\pi\colon W_{m,k}\to W_{m,1}=S^{2dm-1}$ given in (i) and put

$$\theta_b=\pi\theta\colon S^{2dm-1}\longrightarrow W_{m,k}\longrightarrow W_{m,b} \qquad \text{for any} \quad 1\leqq b\leqq k,$$

which is a cross section of $\pi\colon W_{m,b}\to W_{m,1}=S^{2dm-1}$. Furthermore let $J_b\colon W_{a,b}\to\Omega^{2dm}W_{m+a,b}$ be the adjoint map of the composition of

$$S^{2dm}\wedge W_{a,b}=S^{2dm-1}*W_{a,b}\xrightarrow{\theta_b*1}W_{m,b}*W_{a,b}\xrightarrow{h}W_{m+a,b}\quad(b-a=n-k).$$

Then by the homotopy commutativity of (**), we have the homotopy commutative diagram

$$\begin{array}{ccccc} W_{a,b} & \xrightarrow{\varepsilon} & W_{a+1,b+1} & \xrightarrow{\pi} & S^{2d(a+1)-1} \\ \downarrow J_b & & \downarrow J_{b+1} & & \downarrow J_1 \\ \Omega^{2dm}W_{m+a,b} & \xrightarrow{\Omega^{2dm}\varepsilon} & \Omega^{2dm}W_{m+a+1,b+1} & \xrightarrow{\Omega^{2dm}\pi} & \Omega^{2dm}S^{2d(m+a+1)-1}. \end{array}$$

By noticing that J_1 is the adjoint map of $1\colon S^{2dm}\wedge S^{2d(a+1)-1}\to S^{2d(m+a+1)-1}$ and by the induction on b, we see easily that

$$J_b^*\colon H^*(\Omega^{2dm}W_{m+a,b};\, \boldsymbol{Z}_p) \longrightarrow H^*(W_{a,b};\, \boldsymbol{Z}_p) \quad (b-a=n-k)$$

satisfies $J_b^*(\sigma^{2dm}\omega_{2dt-1})=\omega_{2dt-2dm+1}$ for $m+a-b+1\ (=m+n-k+1)\leqq t\leqq m+a$. Thus $J=J_k$ satisfies the desired equality. q.e.d.

Thus Theorem 3.3 is proved completely.

In the rest of this section, we give some examples satisfying the conditions of Theorem 3.3.

EXAMPLE 3.9. *Assume that* $dn>p$, $(s(k)+2)/r(k)>p$, $s(k-1)<2p^2-4$ *and* (3.4). *Then the condition* (2) *in* (ii) *of Theorem* 3.3 *holds, if one of the following* (1)–(3) *holds for any integers* i *and* j *with* $2\leqq j\leqq i\leqq k$ *and* $(s(i)+2)/r(i)>p$:

(1) $b(i,j)<2(p-1)$ *when* $j\equiv 1 \bmod 2d$.

(2) $b(i,j)<\min\{p(p-1),\, r(i)(p-1)\}$ *when* $j\equiv 2 \bmod 2d$.

(3) $a(i,j)\not\equiv 0 \bmod p-1$ *and* $[a(i,j)/(p-1)]=[b(i,j)/(p-1)]$ *when* $j\equiv 1, 2 \bmod 2d$, *where* $a(i,j)=[u(i,j)/2]-r(i)+1$, $b(i,j)=[s(j)/2]-r(i)+1$.

PROOF. For any i and j with $2\leqq j\leqq i\leqq k$ and $(s(i)+2)/r(i)>p$, put

$$t = 2da+u(i,j) \quad \text{and} \quad l = t-2r(i) \quad \text{where} \quad 0\leqq a\leqq j(i-j).$$

Then $u(i,j)\leqq t\leqq 2dj(i-j)+u(i,j)=s(j)$ by (3.2) and

$$(*) \qquad l_0 = u(i,j)-2dr(i) \leqq l \leqq l_1 = s(j)-2dr(i), \quad l-l_0\equiv 0 \bmod 2d.$$

Furthermore $l\leqq l_1\leqq s(k)-2dr(k)=s(k-1)-1<2p^2-5$ by the assumption. On the other hand, by a result of Toda [18; Th. 7.1],

$$(3.10) \qquad {}^p\pi_{t-1}(S^{2dr(i)-1}) = 0 \quad (l=t-2dr(i)<2p^2-5),$$

if l *is not equal to* $2c(p-1)-1$ $(1\leqq c\leqq p)$, $2c(p-1)-2$ $(dr(i)\leqq c\leqq p)$ *or* $2p(p-1)-2$.

Assume that $j\equiv 1 \bmod 2d$. Then l is odd, and (1) implies that $l_1<4(p-1)-1$. Further (3) implies that $2c(p-1)-1<l_0\leqq l_1<2(c+1)(p-1)-1$ for $c=[a(i,j)/(p-1)]$. Thus (3.10) holds for $t\geqq 2dr(i)+2p-2$ by (*). Assume that $j\equiv 2 \bmod 2d$. Then l is even, and (2) implies that $l_1=2b(i,j)-2<\min\{2p(p-1)-2,$

$2dr(i)(p-1)-2\}$. Further (3) implies $2c(p-1)-2<l_0\leqq l_1<2(c+1)(p-1)-2$ for $c=[a(i,j)/(p-1)]$. Thus (3.10) holds by (*). Assume that $d=2$ and $j\equiv 0, 3 \bmod 4$. Then $l-l_0\equiv 0 \bmod 4$ and $l_0\equiv 0, 1 \bmod 4$, and (3.10) holds. q.e.d.

EXAMPLE 3.11. (i) *$SU(n)/SU(n-k)$ $(2\leqq k\leqq 6)$ is p-regular in the following cases*:

$(k=2)$ $p\geqq 5$, *or* $p=3$ *and* $n=4, 5, 7$.
$(k=3)$ $p\geqq 7$, *or* $p=5$ *and* $n=5, 6, 8, 10, 12$.
$(k=4)$ $p\geqq 11$, *or* $p=7$ *and* $n=6, 7, 9, 12, 15$.
$(k=5)$ $p\geqq 13$, *or* $p=11$ *and* $7\leqq n\leqq 11$, $n=14, 19$, $n\geqq 21$, *or* $p=7$ *and* $n=7$.
$(k=6)$ $p\geqq 17$, *or* $p=13$ *and* $8\leqq n\leqq 13$, $n\geqq 31$, *or* $p=11$ *and* $8\leqq n\leqq 11$.

(ii) *$Sp(n)/Sp(n-k)$ $(2\leqq k\leqq 6)$ is p-regular in the following cases*:

$(k=2)$ $p\geqq 5$, *or* $p=3$ *and* $n=6, 9$.
$(k=3)$ $p\geqq 7$, *or* $p=5$ *and* $n=5, 15, 20, 25$.
$(k=4)$ $p\geqq 13$, *or* $p=11$ *and* $n\geqq 7$.
$(k=5)$ $p\geqq 17$, *or* $p=13$ *and* $n\geqq 11$, *or* $p=11$ *and* $n\geqq 23$.
$(k=6)$ $p\geqq 19$, *or* $p=17$ *and* $n=8$, $n\geqq 11$, *or* $p=13$ *and* $n\geqq 33$.

This example follows from Theorem 3.3 by using Example 3.9. Furthermore, by using a result of Toda [18] on ${}^p\pi_{n+l}(S^n)$ (n: odd) for $2p^2-5\leqq l<2(p^2+p)(p-1)-5$, we see the following

EXAMPLE 3.12. (i) *$SU(n)/SU(n-k)$ is also p-regular in the following cases*:

$(k=2)$ $p=3$ *and* $8\leqq n\leqq 18$ *with* $n\neq 11, 14, 16$.
$(k=3)$ $p=5$ *and* n *is even with* $14\leqq n\leqq 58$, $n\neq 20, 22, 34, 38, 40, 44, 48$.

(ii) *$Sp(n)/Sp(n-k)$ is also p-regular in the following cases*:

$(k=2)$ $p=3$ *and* $n=6, 9$. $(k=3)$ $p=5$ *and* $n=15, 20, 25$.

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