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On a certain class of irreducible unitary representations of the infinite dimensional rotation group II

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Introduction

In the previous paper [3], we proved that the "regular" representation of the infinite dimensional rotation group G on the "infinite dimensional sphere" is decomposed into the "class one" representations with respect to the subgroup K of elements which fix a unit vector.

In this paper we shall prove an analogue of the Peter-Weyl theorem for the group $O(\boldsymbol{E})$ (for the definition, see § 1) which contains G.

As is well-known the group $O(\boldsymbol{E})$ as well as G admits no Haar measure. To formulate an analogue of the Peter-Weyl theorem we imbed $O(\boldsymbol{E})$ into a measure space Ω on which $O(\boldsymbol{E})$ acts on the left and right as measure-preserving transformations. Thus we obtain the left and right "regular" representations of $O(\boldsymbol{E})$ on the Hilbert space of all square integrable functions on Ω, the decomposition of which gives us an analogue of the Peter-Weyl theorem.

Now let M be a compact riemannian manifold and Diff M the group of all diffeomorphisms. In [5] A. M. Vershik, I. M. Gel'fand and M. I. Graev constructed a certain class of irreducible unitary representations of Diff M. For each irreducible representation ρ of the symmetric group $\mathfrak{S}_n$ $(n=1, 2,\dots)$ they assigned an irreducible unitary representation $U_{n,\rho}$. Putting $\boldsymbol{E}=C^\infty(M)$ one can prove that $U_{n,\rho}$ is extended to a representation $\pi_{n,\rho}$ of $O(\boldsymbol{E})$. The regular representation of $O(\boldsymbol{E})$ on the infinite dimensional sphere decomposes into the space of symmetric functions on $M\times\cdots\times M$ (n-times) which gives us the Fock space for Bose particles on M. Here appear only those representations $\pi_{n,\mathbf{1}}$ which correspond to the trivial representation $\mathbf{1}$ on $\mathfrak{S}_n$. One of the motivation of our study of the present article was to look for a scheme such that, by substituting a more general measure space for the infinite dimensional sphere, we may have representations $\pi_{n,\rho}$ as components of the irreducible decomposition. In § 4 and § 5 we shall prove that the Fock space for Fermi particles as well as Bose particles can be obtained as a subrepresentation of the left and regular representation of $O(\boldsymbol{E})$ on the Hilbert space of all square integrable functions on Ω.

Finally we would like to comment on the difference of the definition of the class one representation between the previous paper and the present paper. The pupose of the previous paper was to characterize those representations which appear in the irreducible decomposition of the regular representation of G on the infinite dimensional sphere. Our method was to generalize differential equations which are satisfied by the spherical functions and we considered the Casimir operator, so that we assumed the sufficient differentiability of K-fixed vectors. In this paper we shall give another characterization of these representations (McKean's conjecture) as an application of Theorems 1 and 2, specifying the kind of representations permitted, (see [4], p. 203).

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§ 1. Preliminaries

Let M be a compact riemannian manifold. We denote by Diff M the group of all diffeomorphisms. The group Diff M is assumed to be furnished with the natural C^∞-topology. Let $\mathfrak{S}_n$ be the group of all permutations of $\{1, 2,\dots, n\}$, and ρ be an irreducible representation of $\mathfrak{S}_n$ on a finite dimensional vector space V_ρ. Then one can choose an inner product on V_ρ such that for any σ in $\mathfrak{S}_n$ $\rho(\sigma)$ is a unitary operator on V_ρ. We denote by $\hat{\mathfrak{S}}_n$ the set of all equivalence classes of irreducible unitary representations of $\mathfrak{S}_n$. The group $\mathfrak{S}_n$ acts on $M\times\cdots\times M$ (n-times) on the right by $(p_1,\dots, p_n)\cdot\sigma=(p_{\sigma(1)},\dots, p_{\sigma(n)})$, where $(p_1,\dots, p_n)\in M\times\cdots\times M$, $\sigma\in\mathfrak{S}_n$. We denote by $L^2(M\times\cdots\times M)$ the Hilbert space of all square integrable functions on $M\times\cdots\times M$. For any irreducible representation (ρ, V_ρ) of $\mathfrak{S}_n$ we consider the Hilbert space $L^2(M\times\cdots\times M, V_\rho)$ of V_ρ-valued functions f on $M\times\cdots\times M$ such that

$$\|f\|^2 = \int_{M\times\cdots\times M}\|f(p_1,\dots, p_n)\|^2_{V_\rho}dp_1\cdots dp_n < +\infty.$$

We denote by $\mathscr{H}_{n,\rho}$ the subspace of functions f in $L^2(M\times\cdots\times M, V_\rho)$ such that

$$f(p_{\sigma(1)},\dots, p_{\sigma(n)}) = \rho(\sigma)^{-1}f(p_1,\dots, p_n)$$

for any σ in $\mathfrak{S}_n$. For any g in Diff M and f in $\mathscr{H}_{n,\rho}$ we define

$$(U_{n,\rho}(g)f)(p_1,\dots, p_n) = \left(\prod_{j=1}^n \left|\frac{dg^{-1}p_j}{dp_j}\right|\right)^{1/2} f(g^{-1}p_1,\dots, g^{-1}p_n)\,.$$

Then $U_{n,\rho}$ is a unitary representation of Diff M on $\mathscr{H}_{n,\rho}$. In case $n=0$ we put $\mathscr{H}_{n,\rho}=\boldsymbol{R}$ and $U_{n,\rho}(g)=I$ for any g in Diff M, where I denotes the identity operator.

Let $C^\infty(M)$ be the space of all C^∞-functions on M. Then we have a Gel'fand triple

$$C^\infty(M) \subset L^2(M) \subset C^\infty(M)^*,$$

where $C^\infty(M)^*$ is the dual space of $C^\infty(M)$. We write $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{E}^*$ instead of $C^\infty(M)$, $L^2(M)$ and $C^\infty(M)^*$, respectively. By the Bochner-Minlos theorem, there exists a probability measure μ on $\boldsymbol{E}^*$ such that for any ξ in $\boldsymbol{E}$ we have

$$e^{-\|\xi\|^2/2} = \int_{\boldsymbol{E}^*} e^{i\langle x,\xi\rangle} d\mu(x).$$

Let $\boldsymbol{N}$ be the set of all positive integers. We fix, once for all, an orthonormal basis $\{\xi_j; j\in\boldsymbol{N}\}$ of $\boldsymbol{H}$ such that $\xi_j\in\boldsymbol{E}$ for any $j\in\boldsymbol{N}$. We shall consider an Hermite polynomial;

$$H_k(t) = (-1)^k e^{t^2}\frac{d^k}{dt^k}e^{-t^2},\ k \geqq 0.$$

For any n in $\boldsymbol{N}\cup\{0\}$ we put

$$\mathfrak{B}_n = \{\textstyle\prod_{j=1}^\infty (n_j!\ 2^{n_j})^{-1/2} H_{n_j}(\langle x, \xi_j\rangle/2^{1/2});\ \sum_{j=1}^\infty n_j = n\}.$$

Then it is known that $\bigcup_{n=0}^\infty \mathfrak{B}_n$ is an orthonormal basis of $L^2(\boldsymbol{E}^*, \mu)$. We denote by $\mathscr{H}_n$ the closed subspace spanned by $\mathfrak{B}_n$. Then we get an orthogonal decomposition

$$L^2(\boldsymbol{E}^*, \mu) = \textstyle\sum_{n=0}^\infty \oplus \mathscr{H}_n \qquad \text{(Wiener-Itô decomposition)}.$$

We denote by P_n the projection operator of $L^2(\boldsymbol{E}^*, \mu)$ on $\mathscr{H}_n$. For any n in $\boldsymbol{N}$ we denote simply by $\mathbf{1}$ the trivial representation of $\mathfrak{S}_n$. In particular, if n is equal to 1, then we write simply $g\xi$ instead of $U_{1,\mathbf{1}}(g)\xi$ for each g in Diff M and ξ in $\boldsymbol{E}$. We use the same notation for the dual action of g on $\boldsymbol{E}^*$; $\langle gx, \xi\rangle = \langle x, g^{-1}\xi\rangle$. For any g in Diff M and f in $L^2(\boldsymbol{E}^*, \mu)$ we define

$$(U_*(g)f)(x) = f(g^{-1}x) \qquad \text{for a.e. } x \text{ in } \boldsymbol{E}^*.$$

Then U_* is a unitary representation of Diff M on $L^2(\boldsymbol{E}^*, \mu)$. Since $\mathscr{H}_n$ is $U_*(\text{Diff}\,M)$-invariant, we have the subrepresentation U_n of Diff M on $\mathscr{H}_n$. The following propositions were proved by A. M. Vershik, I. M. Gel'fand and M. I. Graev, (see [5]).

PROPOSITION 1. 1) *If ρ is irreducible, then $(U_{n,\rho}, \mathscr{H}_{n,\rho})$ is irreducible.* 2) *Two representations $(U_{n,\rho}, \mathscr{H}_{n,\rho})$ and $(U_{n',\rho'}, \mathscr{H}_{n',\rho'})$ are equivalent if and only if $n=n'$ and ρ is equivalent to ρ'.*

PROPOSITION 2. *For each non-negative integer n, the representation $(U_n, \mathscr{H}_n)$ is an irreducible unitary representation of* Diff M, *and is equivalent to the representation $(U_{n,\mathbf{1}}, \mathscr{H}_{n,\mathbf{1}})$.*

We denote by $O(\boldsymbol{E})$ the group of all linear homeomorphisms of $\boldsymbol{E}$ which are isometries of $\boldsymbol{H}$. For any g in $O(\boldsymbol{E})$ and f in $L^2(\boldsymbol{E}^*, \mu)$ we define

$$(\pi_*(g)f)(x) = f(g^{-1}x).$$

Then π_* is a unitary representation of $O(\boldsymbol{E})$. It is known [4] that $\mathscr{H}_n$ is $\pi_*(O(\boldsymbol{E}))$-invariant, so that we have the subrepresentation $(\pi_n, \mathscr{H}_n)$. According to the paper [1] we define a transformation $\mathscr{T}$ by

$$(\mathscr{T}f)(\xi) = \int_{\boldsymbol{E}^*} e^{i\langle x,\xi\rangle} f(x) d\mu(x), \quad f \in L^2(\boldsymbol{E}^*, \mu),\ \xi \in \boldsymbol{E}.$$

And we define a transformation $\mathscr{T}_*$ by

$$(\mathscr{T}_*f)(\xi) = e^{\|\xi\|^2/2} \sum_{n=0}^{\infty} (2^{1/2}i)^{-n} \mathscr{T}(P_n f)(\xi), \quad f \in L^2(\boldsymbol{E}^*, \mu),\ \xi \in \boldsymbol{E}.$$

Then $\mathscr{T}_*$ is injective. In case $\mathscr{T}_* f = \phi$, we write $f = \phi^\#$. We denote by $L^2(M \times \cdots \times M)^\wedge$ the Hilbert space of all square integrable symmetric functions on $M \times \cdots \times M$ (n-times). By the canonical isomorphism we have

$$L^2(M \times \cdots \times M) \cong L^2(M) \overline{\otimes} \cdots \overline{\otimes} L^2(M),$$

where $L^2(M) \overline{\otimes} \cdots \overline{\otimes} L^2(M)$ denotes the completion of the tensor product $L^2(M) \otimes \cdots \otimes L^2(M)$. Using this isomorphism, for any g in $O(\boldsymbol{E})$ we can define the unitary operator $\tilde{\pi}_n(g)$ on $L^2(M \times \cdots \times M)$ which corresponds to the mapping: $\eta_1 \otimes \cdots \otimes \eta_n \mapsto (g\eta_1) \otimes \cdots \otimes (g\eta_n)$, where $\eta_1 \otimes \cdots \otimes \eta_n \in L^2(M) \overline{\otimes} \cdots \overline{\otimes} L^2(M)$. Clearly $\tilde{\pi}_n$ is a unitary representation of $O(\boldsymbol{E})$ and $L^2(M \times \cdots \times M)^\wedge$ is $\tilde{\pi}_n(O(\boldsymbol{E}))$-invariant. Since $L^2(M \times \cdots \times M)^\wedge = \mathscr{H}_{n,1}$, we have the subrepresentation $(\pi_{n,1}, \mathscr{H}_{n,1})$ of $O(\boldsymbol{E})$. For any f in $\mathscr{H}_n$ there exists a unique F in $\mathscr{H}_{n,1}$ such that

$$\int_{\boldsymbol{E}^*} e^{i\langle x,\xi\rangle} f(x) d\mu(x) = e^{-\|\xi\|^2/2} i^n \int_{M \times \cdots \times M} F(p_1, \dots, p_n) \xi(p_1) \cdots \xi(p_n) dp_1 \cdots dp_n,$$

(see [1]). We put $A_n f = F$. Then for any g in $O(\boldsymbol{E})$ we have

$$A_n \cdot \pi_n(g) = \pi_{n,1}(g) \cdot A_n.$$

REMARK 1. The operator A_n gives the equivalence $(\pi_n, \mathscr{H}_n) \cong (\pi_{n,1}, \mathscr{H}_{n,1})$ and restricting these representations π_n and $\pi_{n,1}$ to Diff M we get the equivalence in Proposition 2.

REMARK 2. Proposition 2 shows that $L^2(\boldsymbol{E}^*, \mu)$ gives the Fock space for Bose particles. In §5 we shall show that the Fock space for Fermi particles as well as (the Fock space) for Bose particles can be obtained as a subrepresentation of $L^2(\Omega, \nu)$.

§2. Peter-Weyl theorem for $O(\boldsymbol{E})$

We shall consider a Gel'fand triple

$$C^\infty(M\times M)\subset L^2(M\times M)\subset C^\infty(M\times M)^*.$$

We can identify $C^\infty(M\times M)$, $L^2(M\times M)$ and $C^\infty(M\times M)^*$ with $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$, $\boldsymbol{H}\overline{\otimes}\boldsymbol{H}$ and $(\boldsymbol{E}\hat{\otimes}\boldsymbol{E})^*$ respectively, where $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$ and $\boldsymbol{H}\overline{\otimes}\boldsymbol{H}$ denote the completions of $\boldsymbol{E}\otimes\boldsymbol{E}$ and $\boldsymbol{H}\otimes\boldsymbol{H}$ respectively. Now, we get a probability measure ν on $(\boldsymbol{E}\hat{\otimes}\boldsymbol{E})^*$ such that for any ζ in $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$

$$e^{-\|\zeta\|^2/2}=\int_\Omega e^{i\langle x,\zeta\rangle}d\nu(x),$$

where $\Omega=(\boldsymbol{E}\hat{\otimes}\boldsymbol{E})^*$. Since $\{\xi_i\otimes\xi_j;\ i,j\in\boldsymbol{N}\}$ is an orthonormal basis contained in $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$, the collection $\{\prod_{i,j}(n_{ij}!\,2^{n_{ij}})^{-1/2}H_{n_{ij}}(\langle x,\xi_i\otimes\xi_j\rangle/2^{1/2});\ \sum_{i,j}n_{ij}<+\infty,\ i,j\in\boldsymbol{N}\}$ forms an orthonormal basis in $L^2(\Omega,\nu)$.

For any g in $O(\boldsymbol{E})$ let us consider two bilinear mappings of $\boldsymbol{E}\times\boldsymbol{E}$ into $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$;

$$(\xi,\eta)\longmapsto(g\xi)\otimes\eta,\qquad(\xi,\eta)\longmapsto\xi\otimes(g\eta).$$

Then there exist two linear mappings of $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$ into itself such that

$$L_g(\xi\otimes\eta)=(g\xi)\otimes\eta,\qquad R_g(\xi\otimes\eta)=\eta\otimes(g\xi).$$

We denote by gx and xg the dual actions of $O(\boldsymbol{E})$ on Ω defined by

$$\langle gx,\zeta\rangle=\langle x,L_{g^{-1}}\zeta\rangle,\qquad\langle xg,\zeta\rangle=\langle x,R_g\zeta\rangle,$$

where $x\in\Omega$, $\zeta\in\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$, $g\in O(\boldsymbol{E})$. It is clear that the measure ν is $O(\boldsymbol{E})$-biinvariant. For any g in $O(\boldsymbol{E})$ we define

$$(\pi_L(g)f)(x)=f(g^{-1}x),\qquad(\pi_R(g)f)(x)=f(xg).$$

Then π_L and π_R are unitary representations of $O(\boldsymbol{E})$. For any (g_1,g_2) in $O(\boldsymbol{E})\times O(\boldsymbol{E})$ we put

$$(\omega_*(g_1,g_2)f)(x)=f(g_1^{-1}xg_2).$$

Then ω_* is a unitary representation of $O(\boldsymbol{E})\times O(\boldsymbol{E})$. Fix any n in $\boldsymbol{N}\cup\{0\}$ and let $\mathfrak{H}_n$ be the closed subspace spanned by

$$\{\textstyle\prod_{i,j}(n_{ij}!\,2^{n_{ij}})^{-1/2}H_{n_{ij}}(\langle x,\xi_i\otimes\xi_j\rangle/2^{1/2});\ \sum_{i,j}n_{ij}=n,\ i,j\in\boldsymbol{N}\}.$$

Then it is clear that $\mathfrak{H}_n$ is $\omega_*(O(\boldsymbol{E})\times O(\boldsymbol{E}))$-invariant. Thus we obtain a unitary representation ω_n of $O(\boldsymbol{E})\times O(\boldsymbol{E})$ on $\mathfrak{H}_n$. Let ρ be an irreducible unitary representation of $\mathfrak{S}_n$ on V_ρ. By the canonical isomorphisms we have

$$L^2(M\times\cdots\times M,V_\rho)\cong L^2(M\times\cdots\times M)\otimes V_\rho\cong L^2(M)\overline{\otimes}\cdots\overline{\otimes}L^2(M)\otimes V_\rho.$$

Using these isomorphisms, we can define the unitary operator $\tilde{\pi}_{n,\rho}(g)$ on

$L^2(M\times\cdots\times M, V_\rho)$ which corresponds to the mapping: $\eta_1\otimes\cdots\otimes\eta_n\otimes v\mapsto(g\eta_1)\otimes\cdots\otimes(g\eta_n)\otimes v$. For any σ in $\mathfrak{S}_n$ and F in $L^2(M\times\cdots\times M, V_\rho)$ we define

$$\lambda(\sigma)F(p_1,\dots,p_n)=F((p_1,\dots,p_n)\cdot\sigma)=F(p_{\sigma(1)},\dots,p_{\sigma(n)}).$$

We put

$$\mathscr{H}^{\wedge}_{n,\rho}=\{\alpha\in L^2(M)\overline{\otimes}\cdots\overline{\otimes}L^2(M)\otimes V_\rho;\ \lambda(\sigma)\alpha=(I\otimes\cdots\otimes I\otimes\rho(\sigma)^{-1})\alpha,\ \sigma\in\mathfrak{S}_n\}.$$

Then $\mathscr{H}_{n,\rho}$ is isomorphic to $\mathscr{H}^{\wedge}_{n,\rho}$. As is easily seen $\mathscr{H}_{n,\rho}$ is $\tilde{\pi}_{n,\rho}(O(\boldsymbol{E}))$-invariant, so that we have the subrepresentation $(\pi_{n,\rho}, \mathscr{H}_{n,\rho})$. We remark that $\pi_{n,\rho}|_{\mathrm{Diff}M}=U_{n,\rho}$.

THEOREM 1 (*an analogue of the Peter-Weyl theorem for $O(\boldsymbol{E})$*). *The unitary representation ω_* of $O(\boldsymbol{E})\times O(\boldsymbol{E})$ is decomposed as follows;*

$$L^2(\Omega, \nu)=\sum_{n=0}^{\infty}\oplus\sum_\rho\mathscr{H}_{n,\rho}\overline{\otimes}\mathscr{H}^*_{n,\rho},$$

where $\sum_\rho$ is taken over all ρ in $\hat{\mathfrak{S}}_n$, and $\omega_n(g_1, g_2)$ corresponds to $\pi_{n,\rho}(g_1)\otimes\pi^*_{n,\rho}(g_2)$ for each (g_1, g_2) in $O(\boldsymbol{E})\times O(\boldsymbol{E})$.

PROOF. We denote by $L^2((M\times M)\times\cdots\times(M\times M))^{\wedge}$ the Hilbert space of all square integrable symmetric functions on $(M\times M)\times\cdots\times(M\times M)$ (n-times). We put $\mathfrak{H}^{\wedge}_n=\{\beta\in L^2(M\times\cdots\times M)\overline{\otimes}L^2(M\times\cdots\times M);\ (\lambda(\sigma)\otimes\lambda(\sigma))\beta=\beta,\ \ \sigma\in\mathfrak{S}_n\}$. Then we have the canonical isomorphism B_n: $L^2((M\times M)\times\cdots\times(M\times M))^{\wedge}\to\mathfrak{H}^{\wedge}_n$. We put $B_nf=F$, where $f\in L^2((M\times M)\times\cdots\times(M\times M))^{\wedge}$ and $F\in\mathfrak{H}^{\wedge}_n$. Then it is easy to see that for any (g_1, g_2) in $O(\boldsymbol{E})\times O(\boldsymbol{E})$ and f in $L^2((M\times M)\times\cdots\times(M\times M))$ we have

$$\begin{aligned}&(B_n\omega_*(g_1, g_2)f)\,((p_1, q_1),\dots,(p_n, q_n))\\&=\left(\prod_j\left|\frac{dg_1^{-1}p_j}{dp_j}\right|\left|\frac{dg_2^{-1}q_j}{dq_j}\right|\right)^{1/2}B_nf((g_1^{-1}p_1, g_2^{-1}q),\dots,(g_1^{-1}p_n, g_2^{-1}q_n)).\end{aligned}$$

We put $(M\times\cdots\times M)'=\{(p_1,\dots,p_n)\in M\times\cdots\times M;\ p_i\neq p_j\ (i\neq j)\}$. Let F_n be a fundamental domain, so that the mapping:

$$F_n\times\mathfrak{S}_n\ni(u, \sigma)\longmapsto u\cdot\sigma\in(M\times\cdots\times M)'$$

is bijective. Let $L^2(\mathfrak{S}_n)$ be the space of all functions on $\mathfrak{S}_n$. We introduce an inner product defined by the normalized Haar measure on $\mathfrak{S}_n$. Then by the Peter-Weyl theorem for $\mathfrak{S}_n$, we have

$$L^2(\mathfrak{S}_n)=\sum_\rho V_\rho\otimes V^*_\rho.$$

We remark that the unitary operator defined by the right translation of σ in $\mathfrak{S}_n$ corresponds to $I\otimes\rho(\sigma)^*$. Since for any σ in $\mathfrak{S}_n$ $\lambda(\sigma)$ is a unitary operator, we get

$$L^2(M\times\cdots\times M)\cong L^2((M\times\cdots\times M)')\cong L^2(F_n\times\mathfrak{S}_n)$$
$$\cong L^2(F_n)\otimes L^2(\mathfrak{S}_n)\cong \textstyle\sum_\rho L^2(F_n)\otimes V_\rho\otimes V_\rho^*.$$

It follows that $\mathfrak{H}_n$ is identified with

$$\{\alpha\in \textstyle\sum_{\rho_1}\sum_{\rho_2} L^2(F_n)\otimes V_{\rho_1}\otimes V_{\rho_1}^*\otimes L^2(F_n)\otimes V_{\rho_2}\otimes V_{\rho_2}^*;$$
$$(I\otimes I\otimes \rho_1^*(\sigma)\otimes I\otimes I\otimes\rho_2^*(\sigma))\alpha=\alpha,\ \sigma\in\mathfrak{S}_n\}$$
$$=\textstyle\sum_\rho (L^2(F_n)\otimes V_\rho)\,\overline{\otimes}\,(L^2(F_n)\otimes V_{\rho^*}) = \sum_\rho \mathscr{H}_{n,\rho}\overline{\otimes}\mathscr{H}_{n,\rho}^*.$$

In the above we used the following. Schur's lemma implies that

$$\dim\{w\in V_{\rho_1}\otimes V_{\rho_2};\ (\rho_1(\sigma)\otimes\rho_2(\sigma))w=w\}=\begin{cases}0 & (\rho_1\not\simeq\rho_2^*),\\ 1 & (\rho_1\simeq\rho_2^*).\end{cases}$$

Finally we notice that $\omega_*(g_1, g_2)$ corresponds to $\pi_{n,\rho}(g_1)\otimes\pi_{n,\rho}^*(g_2)$.

We put $U_L=\pi_L|_{\mathrm{Diff}\,M}$, $U_R=\pi_R|_{\mathrm{Diff}\,M}$ and $T_*=\omega_*|_{\mathrm{Diff}\,M\times\mathrm{Diff}\,M}$. Then we have the following

COROLLARY. *$T_*\simeq U_L\boxtimes U_R$, where $U_L\boxtimes U_R$ denotes the outer tensor product of U_L and U_R.*

§3. Polynomial representations of discrete class

In the following (§3~§5) we keep the notation; M, $C^\infty(M)$, $L^2(M)$, $C^\infty(M)^*$, $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{E}^*$, $\{\xi_j; j\in\boldsymbol{N}\}$, $O(\boldsymbol{E})$, Ω, $L^2(\boldsymbol{E}^*, \mu)$, $L^2(\Omega, \nu)$, π_*, ω_*, π_L, π_R, π_n, $\mathscr{H}_n$, $\pi_{n,\rho}$, $\mathscr{H}_{n,\rho}$, ϕ^*.

We shall identify every element g of $O(\boldsymbol{E})$ with the linear form on $\boldsymbol{E}\hat{\otimes}\boldsymbol{E}$ defined by

$$\xi_i\otimes\xi_j\longmapsto\langle\xi_i, g\xi_j\rangle\quad(i, j\in\boldsymbol{N}).$$

Thus we regard the group $O(\boldsymbol{E})$ as a subset of Ω. Let $R[X_{ij};\ i, j\in\boldsymbol{N}]$ be the polynomial ring of infinite variables X_{ij} $(i, j\in\boldsymbol{N})$ over $\boldsymbol{R}$. Let $C(\Omega)$ be the set of all continuous functions on Ω. We denote by $C(O(\boldsymbol{E}))$ the set of all functions given by the restriction of functions in $C(\Omega)$ to the group $O(\boldsymbol{E})$. We consider the mapping from $R[X_{ij};\ i, j\in\boldsymbol{N}]$ to $C(\Omega)$ defined by the map: $F\mapsto f$, where $F((X_{ij}))\in R[X_{ij};\ i, j\in\boldsymbol{N}]$, $f\in C(\Omega)$ and $f(x)=F((\langle x, \xi_i\otimes\xi_j\rangle))$. We shall denote by $F(\Omega)$ the image of this mapping. We call functions in $F(\Omega)$ polynomials on Ω. It is easy to see that the restriction map: $f\mapsto f|_{O(\boldsymbol{E})}$ is injective. We put $F(O(\boldsymbol{E}))=F(\Omega)|_{O(\boldsymbol{E})}$. We also call functions in $F(O(\boldsymbol{E}))$ polynomials on $O(\boldsymbol{E})$. Since the restriction mapping is injective, for each polynomial f on $O(\boldsymbol{E})$ there exists a unique polynomial $\tilde{f}$ on Ω such that $f=\tilde{f}|_{O(\boldsymbol{E})}$. In the following we use the same notation f instead of $\tilde{f}$. Let $(\pi, \mathfrak{H})$ be an irreducible unitary representation of $O(\boldsymbol{E})$. For v and w in $\mathfrak{H}$ we define a function $\phi^{\pi}_{v,w}$ on $O(\boldsymbol{E})$ by

$$\phi^{\pi}_{v,w}(g) = (v, \pi(g)w).$$

We call $(\pi, \mathfrak{H})$ a polynomial representation of $O(\boldsymbol{E})$ if there exists an orthonormal basis $\{v_j; j \in \boldsymbol{N}\}$ of $\mathfrak{H}$ such that $\phi^{\pi}_{i,j}(g) = (v_i, \pi(g)v_j)$ $(i, j \in \boldsymbol{N})$ are polynomials. We denote by $\mathfrak{H}_f$ the space of all finite linear combinations of v_j $(j \in \boldsymbol{N})$. Let $(\pi, \mathfrak{H})$ be an irreducible unitary polynomial representation of $O(\boldsymbol{E})$. We shall call $(\pi, \mathfrak{H})$ of discrete class if the multilinear functional B:

$$\mathfrak{H}_f \times \mathfrak{H}_f \times \mathfrak{H}_f \times \mathfrak{H}_f \ni (v, w, v', w') \longmapsto \int_{\Omega} \phi^{\pi\#}_{v,w}(x)\phi^{\pi\#}_{v',w'}(x)d\nu(x) \in \boldsymbol{R}$$

is continuous.

PROPOSITION 3. 1) *Let $(\pi, \mathfrak{H})$ be an irreducible unitary polynomial representation of discrete class. Then there exists a positive constant c such that*

$$\int_{\Omega} \phi^{\pi\#}_{v,w}(x)\phi^{\pi\#}_{v',w'}(x)d\nu(x) = c(v, v')(w, w') \quad (v, w, v', w' \in \mathfrak{H}_f).$$

2) *Let $(\pi, \mathfrak{H})$ and $(\pi', \mathfrak{H}')$ be irreducible unitary polynomial representations of discrete class. If π and π' are non-equivalent, then*

$$\int_{\Omega} \phi^{\pi\#}_{v,w}(x)\phi^{\pi'\#}_{v',w'}(x)d\nu(x) = 0 \quad (v, w \in \mathfrak{H}_f,\ v', w' \in \mathfrak{H}'_f).$$

PROOF. 1) We fix w and w'. Then

$$B(\cdot, w, \cdot, w') = \int_{\Omega} \phi^{\pi\#}_{\cdot,w}(x)\phi^{\pi\#}_{\cdot,w'}(x)d\nu(x)$$

is a continuous bilinear functional on $\mathfrak{H}_f \times \mathfrak{H}_f$. It is easy to see that $\phi^{\pi\#}_{\pi(g)v,w}(x) = \phi^{\pi\#}_{v,w}(g^{-1}x)$ $(g \in O(\boldsymbol{E}))$. Since the measure ν is $O(\boldsymbol{E})$-biinvariant, it follows that $B(\cdot, w, \cdot, w')$ is $\pi(O(\boldsymbol{E}))$-invariant. From this fact one can find a constant $c_{w,w'}$ such that

$$B(v, w, v', w') = c_{w,w'}(v, v') \quad (v, v' \in \mathfrak{H}_f). \tag{3.1}$$

Similarly, let us fix v and v'. Then there exists a constant $c'_{v,v'}$ such that

$$B(v, w, v', w') = c'_{v,v'}(w, w') \quad (w, w' \in \mathfrak{H}_f). \tag{3.2}$$

From (3.1) and (3.2) we conclude that there exists a constant c such that

$$B(v, w, v', w') = c(v, v')(w, w') \quad (v, w, v', w' \in \mathfrak{H}_f).$$

It is clear that c is positive.

2) We fix w in $\mathfrak{H}_f$ and w' in $\mathfrak{H}'_f$, and put

$$B_{w,w'}(v, v') = \int_{\Omega} \phi^{\pi\#}_{v,w}(x)\phi^{\pi'\#}_{v',w'}(x)d\nu(x) \quad (v \in \mathfrak{H}_f,\ v' \in \mathfrak{H}'_f).$$

From 1) it is easy to see that $B_{w,w'}(\cdot, \cdot)$ is an $O(\boldsymbol{E})$-invariant continuous bilinear functional on $\mathfrak{H}_f \times \mathfrak{H}'_f$ and so we have a continuous linear operator $A: \mathfrak{H}_f \to \mathfrak{H}'_f$ such that

$$B_{w,w'}(v, v') = (Av, v') \quad (v \in \mathfrak{H}_f,\ v' \in \mathfrak{H}'_f).$$

Obviously for any g in $O(\boldsymbol{E})$

$$A \cdot \pi(g) = \pi'(g) \cdot A.$$

Since π and π' are non-equivalent, we have $A=0$. If follows that

$$\int_{\Omega} \phi^{\pi\#}_{v,w}(x)\phi^{\pi'\#}_{v',w'}(x)d\nu(x) = 0.$$

THEOREM 2. *For any n in $\boldsymbol{N} \cup \{0\}$ and irreducible unitary representation (ρ, V_ρ) of $\mathfrak{S}_n$, $(\pi_{n,\rho}, \mathscr{H}_{n,\rho})$ is an irreducible unitary polynomial representation of discrete class. Conversely for any irreducible unitary polynomial representation of discrete class $(\pi, \mathfrak{H})$, there exist an n in $\boldsymbol{N} \cup \{0\}$ and an irreducible unitary representation (ρ, V_ρ) of $\mathfrak{S}_n$ such that $(\pi, \mathfrak{H})$ is equivalent to $(\pi_{n,\rho}, \mathscr{H}_{n,\rho})$.*

PROOF. From Proposition 1 the representation $(\pi_{n,\rho}, \mathscr{H}_{n,\rho})$ is an irreducible unitary representation of $O(\boldsymbol{E})$. Let $\{v_1,\dots, v_m\}$ be an orthonormal basis of V_ρ. Then we have an orthonormal basis $\{\xi_{i_1}\otimes\cdots\otimes\xi_{i_n}\otimes v_i;\ i \in \boldsymbol{N}\}$ of $L^2(M)\overline{\otimes}\cdots\overline{\otimes} L^2(M)\otimes V_\rho$, where $\xi_{i_1},\dots, \xi_{i_n}$ are orthonormal basis of $L^2(M)$. We put

$$v_{(i_1,\dots,i_n;i)} = \xi_{i_1}\otimes\cdots\otimes\xi_{i_n}\otimes v_i,$$

$$v_{(j_1,\dots,j_n;j)} = \xi_{j_1}\otimes\cdots\otimes\xi_{j_n}\otimes v_j, \quad g\xi_{j_k} = \sum_{l_k=1}^{\infty} g_{l_k j_k}\xi_{l_k} \quad (k=1,\dots, n).$$

We write simply (i) instead of $(i_1,\dots, i_n;\ i)$, and we put

$$\phi_{(i),(j)}(g) = (v_{(i)}, \pi_{n,\rho}(g)v_{(j)}).$$

Then we have

$$\phi_{(i),(j)}(g) = (v_{(i)}, (g\xi_{j_1})\otimes\cdots\otimes(g\xi_{j_n})\otimes v_j) = \delta_{i,j} g_{i_1 j_1}\cdots g_{i_n j_n},$$

where $\delta_{i,j}$ is Kronecker's δ. Thus $\phi_{(i),(j)}$ is a polynomial on $O(\boldsymbol{E})$.

Now we shall show that the functional B is continuous. For any v, w, v' and w' in $\mathfrak{H}_f$ we put

$$v = \sum_{(i)} a_{(i)}v_{(i)}, \quad v' = \sum_{(k)} a'_{(k)}v_{(k)},$$

$$w = \sum_{(j)} b_{(j)}v_{(j)}, \quad w' = \sum_{(l)} b'_{(l)}v_{(l)}.$$

Then we have

$$\phi_{v,w}(x) = \sum_{(i)} \sum_{(j)} \delta_{i,j} a_{(i)} b_{(j)} \langle x, \xi_{i_1} \otimes \xi_{j_1} \rangle \cdots \langle x, \xi_{i_n} \otimes \xi_{j_n} \rangle .$$

We put $f_{(i),(j)}(x) = \langle x, \xi_{i_1} \otimes \xi_{j_1} \rangle \cdots \langle x, \xi_{i_n} \otimes \xi_{j_n} \rangle$. Then

$$\int_{\Omega} f^{\#}_{(i),(j)}(x) f^{\#}_{(k),(l)}(x) d\nu(x) = 0$$

unless the followings hold; for any m and m', m occurs the same times in the series $i_1, \ldots, i_n$ and $k_1, \ldots, k_n$, so does m' in the series $j_1, \ldots, j_n$ and $l_1, \ldots, l_n$. Using the Schwarz inequality we have

$$\begin{aligned} &\left| \int_{\Omega} \phi^{\#}_{v,w}(x) \phi^{\#}_{v',w'}(x) d\nu(x) \right| \\ &\leqq \sum_{(i)} \sum_{(j)} \sum_{\sigma} \sum_{\tau} \delta_{i,j} \delta_{k,l} |a_{(i)} b_{(j)}| \, |a'_{\sigma(i)} b'_{\tau(j)}| \left| \int_{\Omega} f^{\#}_{(i),(j)}(x) f^{\#}_{\sigma(i),\tau(j)}(x) d\nu(x) \right| \\ &\leqq (n!)^2 \|v\| \, \|w\| \, \|v'\| \, \|w'\| . \end{aligned}$$

This shows that B is continuous.

Conversely let $(\pi, \mathfrak{H})$ be an irreducible unitary polynomial representation of discrete class. Then, by definition, there exists an orthonormal basis $\{v_j; j \in \boldsymbol{N}\}$ of $\mathfrak{H}$ which satisfies the following conditions; $\phi^{\pi}_{i,j}(g) = (v_i, \pi(g) v_j)$ $(i, j \in \boldsymbol{N})$ are polynomials on $O(\boldsymbol{E})$ and B:

$$\mathfrak{H}_f \times \mathfrak{H}_f \times \mathfrak{H}_f \times \mathfrak{H}_f \ni (v, w, v', w') \longmapsto \int_{\Omega} \phi^{\pi\#}_{v,w}(x) \phi^{\pi\#}_{v',w'}(x) d\nu(x) \in \boldsymbol{R}$$

is continuous, where $\mathfrak{H}_f$ is the space of all finite linear combinations of v_j $(j \in \boldsymbol{N})$. From Proposition 3 there exists a positive constant c such that

$$\int_{\Omega} \phi^{\pi\#}_{v,w}(x) \phi^{\pi\#}_{v',w'}(x) d\nu(x) = c(v, v') \, (w, w').$$

Now we fix v_0 in $\mathfrak{H}_f$. For any v in $\mathfrak{H}_f$ we define a linear operator A by

$$(Av)(x) = \phi^{\pi\#}_{v,v_0}(x).$$

Since B is continuous, A defines a bounded linear operator of $\mathfrak{H}$ into $L^2(\Omega, \nu)$. We know that for any g in $O(\boldsymbol{E})$

$$(A\pi(g)v)(x) = \phi^{\pi\#}_{\pi(g)v,v_0}(x) = \phi^{\pi\#}_{v,v_0}(g^{-1}x) = (\pi_L(g)Av)(x).$$

This implies that A is an intertwining operator of $\mathfrak{H}$ into $L^2(\Omega, \nu)$. Thus $(\pi, \mathfrak{H})$ is equivalent to a subrepresentation of $(\pi_L, L^2(\Omega, \nu))$. On the other hand, from Theorem 1 we can prove that any subrepresentation of $(\pi_L, L^2(\Omega, \nu))$ is equivalent to $(\pi_{n,\rho}, \mathscr{H}_{n,\rho})$ for some n in $\boldsymbol{N} \cup \{0\}$ and ρ in $\hat{\mathfrak{S}}_n$. This completes the proof of the theorem.

§ 4. Class one representations

Let G be the subgroup (of $O(\boldsymbol{E})$) of all g in $O(\boldsymbol{E})$ such that $g\xi_j=\xi_j$ except finitely many j in $\boldsymbol{N}$. We put $K=\{g\in G;\ g\xi_1=\xi_1\}$. We denote by $\mathfrak{H}^{\mathrm{Bose}}$ the closed subspace spanned by $\{\prod_i H_{n_i}(\langle x,\ \xi_i\otimes\xi_1\rangle/2^{1/2});\ \sum_i n_i<+\infty\}$. Clearly $\mathfrak{H}^{\mathrm{Bose}}$ is $\pi_L(O(\boldsymbol{E}))$-invariant, so that we have the subrepresentation $(\pi^{\mathrm{Bose}},\ \mathfrak{H}^{\mathrm{Bose}})$. It is obvious that $(\pi_*,\ L^2(\boldsymbol{E}^*,\ \mu))$ is equivalent to $(\pi^{\mathrm{Bose}},\ \mathfrak{H}^{\mathrm{Bose}})$.

Let $(\pi,\ \mathfrak{H})$ be an irreducible unitary polynomial representation of discrete calss. We call $(\pi,\ \mathfrak{H})$ class one (with respect to K) if the space of all $\pi(K)$-fixed vectors is of one dimension.

THEOREM 3 (McKean's conjecture). *For any n in $\boldsymbol{N}\cup\{0\}$ $(\pi_n,\ \mathscr{H}_n)$ is an irreducible unitary polynomial representation of discrete class which is class one (with respect to K). Conversely, for any irreducible unitary polynomial representation of discrete class $(\pi,\ \mathfrak{H})$ which is class one (with respect to K), there exists an n in $\boldsymbol{N}\cup\{0\}$ such that $(\pi,\ \mathfrak{H})$ is equivalent to $(\pi_n,\ \mathscr{H}_n)$.*

PROOF. We can show in the same way as in [3] that $\mathfrak{H}^{\mathrm{Bose}}$ coincides with the space of all $\pi_R(K)$-fixed vectors in $L^2(\Omega,\ \nu)$. It follows from Theorem 1 that

$$\mathfrak{H}^{\mathrm{Bose}}\cong\sum_{n=0}^{\infty}\sum_{\rho}\mathscr{H}_{n,\rho}\ \overline{\otimes}\ (\mathscr{H}^*_{n,\rho})^K,$$

where $(\mathscr{H}^*_{n,\rho})^K$ denotes the space of all $\pi^*_{n,\rho}(K)$-fixed vectors. Since $(\pi^{\mathrm{Bose}},\ \mathfrak{H}^{\mathrm{Bose}})$ is equivalent to $(\pi_*,\ L^2(\boldsymbol{E}^*,\ \mu))$, it follows from Remark 1 that $(\mathscr{H}^*_{n,\rho})^K$ vanishes unless ρ is trivial and that $\dim(\mathscr{H}^*_{n,\rho})^K$ is equal to 1. Since

$$\mathscr{H}^*_{n,1}\cong\mathscr{H}_{n,1^*}\cong\mathscr{H}_{n,1}\cong\mathscr{H}_n\,,$$

the dimension of the space of all $\pi_n(K)$-fixed vectors in $\mathscr{H}_n$ is equal to 1. It follows from Theorem 2 that $(\pi_n,\ \mathscr{H}_n)$ is an irreducible unitary polynomial representation of discrete class.

Conversely, let $(\pi,\ \mathfrak{H})$ be an irreducible unitary polynomial representation of discrete class which is class one. Then from Theorem 2 there exist an n in $\boldsymbol{N}\cup\{0\}$ and ρ in $\hat{\mathfrak{S}}_n$ such that $(\pi,\ \mathfrak{H})$ is equivalent to $(\pi_{n,\rho},\ \mathscr{H}_{n,\rho})$. If ρ is not trivial, we have

$$(\mathscr{H}_{n,\rho})^K\cong(\mathscr{H}^*_{n,\rho^*})^K=\{0\}\,.$$

This completes the proof of the theorem.

§ 5. Fock space for Fermi particles

Let $\mathfrak{H}_n^{\mathrm{Fermi}}$ be the closed subspace spanned by $\{2^{-n/2}\sum_\sigma \mathrm{sgn}(\sigma)\prod_{i=1}^n H_1(\langle x,$

$\xi_{k_i}\otimes\xi_{\sigma(i)}\rangle/2^{1/2})$; $1\leqq k_1<\cdots<k_n\}$, where $\operatorname{sgn}(\sigma)$ denotes the signature of σ in $\mathfrak{S}_n$ and the summation $\sum_\sigma$ is taken over all σ in $\mathfrak{S}_n$. Clearly $\mathfrak{H}_n^{\mathrm{Fermi}}$ is $\pi_L(O(\boldsymbol{E}))$-invariant, so that we have the subrepresentation $(\pi_n^{\mathrm{Fermi}}, \mathfrak{H}_n^{\mathrm{Fermi}})$. We put $\mathfrak{H}^{\mathrm{Fermi}}=\sum_{n=0}^{\infty}\oplus\mathfrak{H}_n^{\mathrm{Fermi}}$. We write simply $v_{(k)}$ instead of $2^{-n/2}\sum_\sigma \operatorname{sgn}(\sigma)\prod_{i=1}^n H_1(\langle x, \xi_{k_i}\otimes\xi_{\sigma(i)}\rangle/2^{1/2})$. We shall calculate the $\mathscr{T}$-transformation of $v_{(k)}$. We put $c(\zeta)=e^{-\|\zeta\|^2}i^n$.

$$\begin{aligned}(\mathscr{T}v_{(k)})(\zeta) &= \int_\Omega e^{i\langle x,\zeta\rangle}2^{-n/2}\textstyle\sum_\sigma \operatorname{sgn}(\sigma)\prod_{i=1}^n H_1(\langle x, \xi_{k_i}\otimes\xi_{\sigma(i)}\rangle/2^{1/2})dv(x)\\ &= c(\zeta)\textstyle\sum_\sigma \operatorname{sgn}(\sigma)\prod_{i=1}^n\langle\zeta, \xi_{k_i}\otimes\xi_{\sigma(i)}\rangle\\ &= c(\zeta)\textstyle\sum_\sigma \operatorname{sgn}(\sigma)\displaystyle\int_{(M\times M)\times\cdots\times(M\times M)}(\textstyle\prod_{i=1}^n \xi_{k_i}(p_i)\xi_{\sigma(i)}(q_i)\zeta(p_i, q_i))\\ &\qquad\times dp_1dq_1\cdots dp_ndq_n\,.\end{aligned}$$

Since $\prod_{i=1}^n\xi_{\sigma(i)}(q_i)=\prod_{i=1}^n\xi_i(q_{\sigma^{-1}(i)})$, we have

$$\begin{aligned}(\mathscr{T}v_{(k)})(\zeta) = c(\zeta)\int_{(M\times M)\times\cdots\times(M\times M)}&(\textstyle\prod_{i=1}^n\xi_{k_i}(p_i))\det((\xi_l(q_m)))\\ &\times(\textstyle\prod_{i=1}^n\zeta(p_i, q_i))dp_1dq_1\cdots dp_ndq_n.\end{aligned}$$

Clearly the value of the integral is invariant by the action of $\mathfrak{S}_n$. It follows that

$$\begin{aligned}(\mathscr{T}v_{(k)})(\zeta) &= c(\zeta)\int_{(M\times M)\times\cdots\times(M\times M)}(n!)^{-1}\textstyle\sum_\sigma(\prod_{i=1}^n\xi_{k_i}(p_{\sigma(i)}))\det((\xi_l(q_{\sigma(m)})))\\ &\qquad\times(\textstyle\prod_{i=1}^n\zeta(p_{\sigma(i)}, q_{\sigma(i)}))dp_1dq_1\cdots dp_ndq_n\\ &= c(\zeta)\int_{(M\times M)\times\cdots\times(M\times M)}(n!)^{-1}\textstyle\sum_\sigma\operatorname{sgn}(\sigma)(\prod_{i=1}^n\xi_{k_i}(p_{\sigma(i)}))\det((\xi_l(q_m)))\\ &\qquad\times(\textstyle\prod_{i=1}^n\zeta(p_i, q_i))dp_1dq_1\cdots dp_ndq_n\\ &= c(\zeta)\int_{(M\times M)\times\cdots\times(M\times M)}(n!)^{-1}\det((\xi_{k_l}(p_m)))\det((\xi_l(q_m)))\\ &\qquad\times(\textstyle\prod_{i=1}^n\zeta(p_i, q_i))dp_1dq_1\cdots dp_ndq_n.\end{aligned}$$

Now we put

$$D_nv_{(k)}=(n!)^{-1/2}\det((\xi_{k_l}(p_m)))\,.$$

$\{v_{(k)}\}$ and $\{(n!)^{-1/2}\det((\xi_{k_l}(p_m)))\}$ are orthonormal bases of $\mathfrak{H}_n^{\mathrm{Fermi}}$ and $\mathscr{H}_{n,\mathrm{sgn}}$ respectively, where $\mathscr{H}_{n,\mathrm{sgn}}$ is the space of all skew-symmetric functions on $M\times\cdots\times M$ (n-times). It follows that D_n can be extended to an isometry of $\mathfrak{H}_n^{\mathrm{Fermi}}$ onto $\mathscr{H}_{n,\mathrm{sgn}}$. It is easy to see that for any g in $O(\boldsymbol{E})$

$$D_n\cdot\pi_n^{\mathrm{Fermi}}(g)=\pi_{n,\mathrm{sgn}}(g)\cdot D_n.$$

Thus we have the following

THEOREM 4. 1) $(\pi_n^{\mathrm{Fermi}}, \mathfrak{H}_n^{\mathrm{Fermi}})$ *is equivalent to* $(\pi_{n,\mathrm{sgn}}, \mathscr{H}_{n,\mathrm{sgn}})$.
2) $\mathfrak{H}^{\mathrm{Fermi}} = \sum_{n=0}^{\infty} \oplus \mathfrak{H}_n^{\mathrm{Fermi}}$ *(irreducible decomposition)*.

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