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# On noetherian subrings of an affine domain

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

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## Introduction

Let $A$ be an affine domain over a field $k$ and let $R$ be a subring of $A$ containing $k$. It is well-known that $R$ is again an affine domain over $k$ when $\dim R=1$. But, when $\dim R \geqq 2$, $R$ is not necessarily noetherian, and not necessarily an affine domain over $k$ even when $R$ is noetherian. The purpose of the present paper is to find certain conditions for $R$ to be an affine domain over $k$.

In the first section we define an ideal $\mathscr{A}(R)$ of $R$ and by making use of this ideal we prove that $R$ is an affine domain over $k$ if and only if $R_{\mathfrak{m}}$ is a locality over $k$ for any maximal ideal $\mathfrak{m}$ of $R$, where a locality over $k$ is a local ring which is a localization of an affine domain over $k$ (cf. Theorem 1.6).

It is known that $R$ is an affine domain over $k$ if $R$ is a neotherian normal subring of dimension 2 and $\mathrm{tr.deg}_k R/\mathfrak{p}=1$ for any prime ideal $\mathfrak{p}$ of height 1 (cf. [2], [5]). If $R$ satisfies these conditions it is seen that $R$ is equidimensional, that is, we have $\dim R_{\mathfrak{m}}=2$ for any maximal ideal $\mathfrak{m}$ of $R$. In the third section we generalize this result as follows: if $R$ is a noetherian subring of an affine domain over a field $k$ such that the integral closure $R'$ of $R$ in its quotient field is equidimensional then $R$ is an affine domain over $k$ (cf. Theorem 3.2).

As a corollary of this theorem we prove that if $R$ is a universally catenary and equidimensional subring then $R$ is an affine domain over $k$. For the proof of this corollary we need the following theorem: the finiteness of the integral closure $R'$ of $R$ in its quotient field is a local property, that is, $R'$ is a finite $R$-module if and only if $R'_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of $R$. We prove this theorem in the second section (cf. Theorem 2.5).

Throughout this paper we fix a field $k$. All rings under consideration are commutative $k$-algebras and all affine domains are assumed to be defined over $k$.

## 1. The ideal $\mathscr{A}(R)$

Let $A$ be an affine domain over a field $k$, that is, an integral domain which is finitely generated over $k$. We are mainly interested in subrings $R$ of $A$, and we shall study when $R$ is again an affine domain over $k$. For this purpose we define an ideal $\mathscr{A}(R)$ of $R$ as follows:

PROPOSITION 1.1. *Let $R$ be a subring of an affine domain $A$ over $k$. Define $\mathscr{A}(R)$ by*

$$\mathscr{A}(R):=\{a;\ a\in R \text{ and } R[1/a] \text{ is an affine domain over } k\}\cup\{0\}.$$

*Then $\mathscr{A}(R)$ is a non-zero radical ideal of $R$.*

PROOF. By virtue of Propositon (2.1) in [3], we see that $\mathscr{A}(R)\neq 0$. We shall prove that $\mathscr{A}(R)$ is an ideal of $R$. Since $R[1/ax]=R[1/a][1/x]$, we have $ax\in\mathscr{A}(R)$ for any $a\in\mathscr{A}(R)$ and $x\in R$. We shall show $a+b\in\mathscr{A}(R)$ for any non-zero elelments $a$ and $b$ of $\mathscr{A}(R)$. Since $R[1/a]$ and $R[1/b]$ are affine domains over $k$, there exist elements $a_1,\dots,a_s$ and $b_1,\dots,b_t$ of $R$ such that $R[1/a]=k[1/a, a_1,\dots,a_s]$ and $R[1/b]=k[1/b, b_1,\dots,b_t]$. Let $C=k[a, b, a_1,\dots,a_s, b_1,\dots,b_t]$. Then $C$ is an affine domain over $k$ and we have $C\subseteq R$. Let $x$ be an arbitrary element of $R$. Since $R[1/a]\subseteq C[1/a]$ and $R[1/b]\subseteq C[1/b]$, we have $a^n x\in C$ and $b^n x\in C$ for a sufficiently large positive integer $n$. Then, as is easily seen, we have $(a+b)^{2n}x\in C$, whence we have $x\in C[1/(a+b)]$. Since $x$ is an arbitrary element of $R$, we have $C[1/(a+b)]\supseteq R[1/(a+b)]$ and hence $C[1/(a+b)]=R[1/(a+b)]$. Therefore $R[1/(a+b)]$ is an affine domain over $k$ and we have $a+b\in\mathscr{A}(R)$. Thus $\mathscr{A}(R)$ is an ideal of $R$. Finally we prove that $\mathscr{A}(R)$ is a radical ideal. Let $x$ be an element of $R$ with $x^n\in\mathscr{A}(R)$ for some positive integer $n$. Since $R[1/x]=R[1/x^n]$, we have $x\in\mathscr{A}(R)$. Therefore $\mathscr{A}(R)$ is a radical ideal. Q. E. D.

COROLLARY 1.2. *Let $R$ be a subring of an affine domain over $k$. Then we have* $\dim R=\mathrm{tr.deg}_k R$.

PROOF. Let $n=\mathrm{tr.deg}_k R$ and let $a$ be a non-zero element of $\mathscr{A}(R)$. Then $R[1/a]$ is an affine domain over $k$ and hence we have $\dim R[1/a]=n$ (cf. [4, (14.G)]). Therefore we have $\dim R\geqq \dim R[1/a]=n$. Since $\dim R\leqq n$ in general, we have $\dim R=n$. Q. E. D.

We call a local ring $S$ a locality over $k$ if $S$ is a localization of an affine domain over $k$ with respect to a prime ideal (cf. [6, Ch. VI]).

LEMMA 1.3. *Let $R$ be a subring of an affine domain $A$ over $k$ and let $\mathfrak{p}$ be a prime ideal of $R$. Then $R_{\mathfrak{p}}$ is a locality over $k$ if and only if $\mathscr{A}(R)\not\subseteq\mathfrak{p}$.*

PROOF. Let $x$ be a non-zero element of $\mathscr{A}(R)$. Replacing $A$ by $R[1/x]$, we may assume that $R$ and $A$ are birational. Assume that $\mathscr{A}(R)\not\subseteq\mathfrak{p}$ and take an element $a$ of $\mathscr{A}(R)\backslash\mathfrak{p}$. Then $R[1/a]$ is an affine domain over $k$ and $\mathfrak{p}[1/a]$ is a prime ideal of $R[1/a]$. Hence $R_{\mathfrak{p}}=R[1/a]_{\mathfrak{p}[1/a]}$ is a locality over $k$. Conversely, assume that $R_{\mathfrak{p}}$ is a locality over $k$. Then there exist an affine domain $B$ over $k$ and a prime ideal $P$ of $B$ such that $R_{\mathfrak{p}}=B_P$. We may assume that $B\subseteq R$. Let $K$

be the quotient field of $A$ and let $B'$ and $R'$ be the integral closures of $B$ and $R$ in $K$, respectively. Let $\mathfrak{p}'$ be a prime ideal of $R'$ lying over $\mathfrak{p}$ and let $P'=\mathfrak{p}'\cap B'$. Since the integral closure of $R_{\mathfrak{p}}=B_P$ in $K$ coincides with $R'_{\mathfrak{p}}=B'_P$, we have $R'_{\mathfrak{p}'}=B'_{P'}$. Let $F=\{Q;\ Q\in \operatorname{Spec} B',\ \operatorname{ht}Q=1 \text{ and } B'_Q\not\supseteq R'\}$. We claim that $F$ is a finite set. In fact, since $A$ is an affine domain, the integral closure $A'$ of $A$ in $K$ is also an affine domain. Therefore $A'$ is finitely generated over $B'$, whence there exists an element $b$ of $B'$ such that $B'[1/b]\supseteq A'\supseteq R'$. Then it is easy to see that $F$ is a subset of the set of the minimal prime divisors of $b$. Thus $F$ is a finite set. Let $F=\{P'_1,\dots, P'_n\}$ and let $P_i=P'_i\cap B$ for $1\leqq i\leqq n$. Suppose that $P_1\cap\cdots\cap P_n\subseteq P$. Then we have $P_i\subseteq P$ for some $i$, and hence $B'_{P_i}\supseteq B'_P=R'_{\mathfrak{p}}\supseteq R'$, which is a contradiction. Thus we have $P_1\cap\cdots\cap P_n\not\subseteq P$. Take an element $a$ of $P_1\cap\cdots\cap P_n\backslash P$ and let $\Lambda=\{Q;\ Q\in\operatorname{Spec} B',\ \operatorname{ht}Q=1 \text{ and } a\notin Q\}$. Then, as is easily seen, we have $B'_Q\supseteq R'$ for any element $Q$ of $\Lambda$, whence we have $B'[1/a]=\bigcap_{Q\in\Lambda}B'_Q\supseteq R'$. Therefore we have $B'[1/a]=R'[1/a]$, and $R'[1/a]$ is an affine domain over $k$. Hence $R[1/a]$ is also an affine domain over $k$ by the following well-known lemma. Since $a\in B\backslash P\subseteq R\backslash\mathfrak{p}$, we have $\mathscr{A}(R)\not\subseteq\mathfrak{p}$. Q. E. D.

LEMMA 1.4. *Let a ring $R'$ be an integral extension of a ring $R$. If $R'$ is an affine domain over $k$, then $R$ is also an affine domain over $k$.*

PROOF. See [1, Ch. V, § 1.9, Lemma 5].

COROLLARY 1.5. *Let $R$ be a subring of an affine domain over $k$. Then we have* $\mathrm{V}(\mathscr{A}(R))=\{\mathfrak{p};\ \mathfrak{p}\in\operatorname{Spec} R$ *and $R_{\mathfrak{p}}$ is not a locality over $k$*$\}$.

The following theorem is an immediate cosequence of Corollary 1.5 which asserts that affineness is a local property for subrings of an affine domain.

THEOREM 1.6. *Let $R$ be a subring of an affine domain over $k$. Then the following three conditions are equivalent to each other.*

(1) *$R$ is an affine domain over $k$.*
(2) *$R_{\mathfrak{p}}$ is a locality over $k$ for any prime ideal $\mathfrak{p}$ of $R$.*
(3) *$R_{\mathfrak{m}}$ is a locality over $k$ for any maximal ideal $\mathfrak{m}$ of $R$.*

## 2. Open properties of a ring

Let $\boldsymbol{P}$ be a property for domains. For the sake of brevity, we use the symbol $[\boldsymbol{P}]$ to denote the class of domains which have the property $\boldsymbol{P}$. We say that a property $\boldsymbol{P}$ is stable under localization if a domain $R$ belongs to $[\boldsymbol{P}]$ then $R_{\mathfrak{p}}$ belongs to $[\boldsymbol{P}]$ for any prime ideal $\mathfrak{p}$ of $R$. For a domain $R$, we define

$$\boldsymbol{P}(R) = \{a;\ a\in R \text{ and } R[1/a]\in[\boldsymbol{P}]\}\cup\{0\}$$

and

$$\Delta_{\boldsymbol{P}}(R) = \{\mathfrak{p}\,;\ \mathfrak{p} \in \operatorname{Spec} R \text{ and } R_{\mathfrak{p}} \notin [\boldsymbol{P}]\}.$$

We say that $\boldsymbol{P}$ is an open property for $R$ if $\Delta_{\boldsymbol{P}}(R)$ is a closed set of Spec $R$.

LEMMA 2.1. *Let $\boldsymbol{P}$ be a property stable under localization and let $R$ be a domain. Then*

(1) *If $\boldsymbol{P}(R)$ is an ideal of $R$ then $\boldsymbol{P}(R)$ is a radical ideal of $R$.*

(2) $\Delta_{\boldsymbol{P}}(R) \subseteq \mathrm{V}(\boldsymbol{P}(R))$.

(3) *If $\mathfrak{p} \in \Delta_{\boldsymbol{P}}(R)$ then $\mathfrak{q} \in \Delta_{\boldsymbol{P}}(R)$ for any specialization $\mathfrak{q}$ of $\mathfrak{p}$.*

PROOF. (1): Since $R[1/a]=R[1/a^n]$, we have $a \in \boldsymbol{P}(R)$ if and only if $a^n \in \boldsymbol{P}(R)$. Hence the assertion is obvious.

(2): Let $\mathfrak{p}$ be a prime ideal of $R$ with $\mathfrak{p} \not\supseteq \boldsymbol{P}(R)$ and let $a$ be an element of $\boldsymbol{P}(R) \backslash \mathfrak{p}$. Then $R[1/a] \in [\boldsymbol{P}]$ and $\mathfrak{p}[1/a] \in \operatorname{Spec} R[1/a]$, hence we have $R_{\mathfrak{p}} = R[1/a]_{\mathfrak{p}[1/a]} \in [\boldsymbol{P}]$ because $\boldsymbol{P}$ is stable under localization. Thus we have $\mathfrak{p} \notin \Delta_{\boldsymbol{P}}(R)$.

(3): Let $\mathfrak{p} \in \Delta_{\boldsymbol{P}}(R)$ and let $\mathfrak{q}$ be a specialization of $\mathfrak{p}$. Suppose that $\mathfrak{q} \notin \Delta_{\boldsymbol{P}}(R)$. Then we have $R_{\mathfrak{q}} \in [\boldsymbol{P}]$, and hence we have $R_{\mathfrak{p}} \in [\boldsymbol{P}]$ because $R_{\mathfrak{p}}$ is a localization of $R_{\mathfrak{q}}$ with respect to the prime ideal $\mathfrak{p}R_{\mathfrak{q}}$. Thus we have $\mathfrak{p} \notin \Delta_{\boldsymbol{P}}(R)$, which is a contradiction. Therefore we have $\mathfrak{q} \in \Delta_{\boldsymbol{P}}(R)$. Q.E.D.

LEMMA 2.2. *Let $\boldsymbol{P}$ be a property stable under localization and let $R$ be a domain. Then the following two conditions are equivalent to each other.*

(1) $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$.

(2) *If $\mathfrak{p}$ is a prime ideal of $R$ with $R_{\mathfrak{p}} \in [\boldsymbol{P}]$, then $\boldsymbol{P}(R) \not\subseteq \mathfrak{p}$.*

PROOF. (1)⇒(2): Let $\mathfrak{p}$ be a prime ideal of $R$ with $R_{\mathfrak{p}} \in [\boldsymbol{P}]$. Then we have $\mathfrak{p} \notin \Delta_{\boldsymbol{P}}(R)$ by the definition. Since $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$, we have $\boldsymbol{P}(R) \not\subseteq \mathfrak{p}$.

(2)⇒(1): If the condition (2) holds, we have $R_{\mathfrak{p}} \notin [\boldsymbol{P}]$, i.e., $\mathfrak{p} \in \Delta_{\boldsymbol{P}}(R)$ for any $\mathfrak{p} \in \mathrm{V}(\boldsymbol{P}(R))$. Thus we have $\mathrm{V}(\boldsymbol{P}(R)) \subseteq \Delta_{\boldsymbol{P}}(R)$. On the other hand, we have $\mathrm{V}(\boldsymbol{P}(R)) \supseteq \Delta_{\boldsymbol{P}}(R)$ by Lemma 2.1, whence $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$. Q.E.D.

COROLLARY 2.3. *Let $\boldsymbol{P}$ be a property stable under localization and let $R$ be a domain. Assume that $R$ satisfies the following three conditions.*

(1) *$\boldsymbol{P}(R)$ is an ideal of $R$.*

(2) $\Delta_{\boldsymbol{P}}(R) = \mathrm{V}(\boldsymbol{P}(R))$.

(3) *$R_{\mathfrak{m}} \in [\boldsymbol{P}]$ for any maximal ideal $\mathfrak{m}$ of $R$.*

*Then $R$ has the property $\boldsymbol{P}$.*

PROOF. We have $\boldsymbol{P}(R) \not\subseteq \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $R$ by Lemma 2.2. Since $\boldsymbol{P}(R)$ is an ideal of $R$, we have $\boldsymbol{P}(R) = R$, hence $R \in [\boldsymbol{P}]$. Q.E.D.

We shall denote by $\boldsymbol{I}$ and $\boldsymbol{F}$ the properties for domains defined by

$R \in [\boldsymbol{I}]$ implies that $R$ is integrally closed.

$R\in[\boldsymbol{F}]$ implies that the integral closure $R'$ of $R$ in its quotient field is a finite $R$-module.

PROPOSITION 2.4. *Let $R$ be a noetherian subring of an affine domain over $k$ and let $\boldsymbol{P}$ be either $\boldsymbol{I}$ or $\boldsymbol{F}$.*

*Then we have the following*:

(1) *$\boldsymbol{P}(R)$ is a non-zero radical ideal of $R$.*

(2) $\Delta_{\boldsymbol{P}}(R)=\mathrm{V}(\boldsymbol{P}(R))$.

PROOF. (I) The case $\boldsymbol{P}=\boldsymbol{I}$.

(1): Let $R'$ be the integral closure of $R$ in the quotient field $K$ of $R$ and let $a$ be a non-zero element of $\mathscr{A}(R)$. Since $R[1/a]$ is an affine domain, the integral closure $R'[1/a]$ of $R[1/a]$ in $K$ is a finite $R[1/a]$-module. Hence there exist elements $\alpha_1,\dots,\alpha_s$ of $R'$ such that $R'[1/a]=R[1/a]\alpha_1+\cdots+R[1/a]\alpha_s$. Take an element $b$ of $R$ such that $b\alpha_i\in R$ for $1\leqq i\leqq s$. Then we have $R'[1/a]\subseteq R[1/a][1/b]$, whence we have $R'[1/ab]=R[1/ab]$. Thus $R[1/ab]$ is integrally closed, and hence we have $0\neq ab\in\boldsymbol{I}(R)$. Therefore $\boldsymbol{I}(R)\neq 0$. Next we shall prove that $\boldsymbol{I}(R)$ is a radical ideal of $R$. It is easy to see that we have $ar\in\boldsymbol{I}(R)$ for any $a\in\boldsymbol{I}(R)$ and $r\in R$. Let $a$ and $b$ be two non-zero elements of $\boldsymbol{I}(R)$ and let $\mathfrak{p}$ be an element of $\mathrm{D}(a+b)$, where $\mathrm{D}(a+b)=\{\mathfrak{p}\,;\,\mathfrak{p}\in\operatorname{Spec}R$ and $a+b\notin\mathfrak{p}\}$. Then we have either $a\notin\mathfrak{p}$ or $b\notin\mathfrak{p}$, and we may assume that $a\notin\mathfrak{p}$. Then $R[1/a]$ is integrally closed and $\mathfrak{p}[1/a]$ is a prime ideal of $R[1/a]$, hence $R_{\mathfrak{p}}=R[1/a]_{\mathfrak{p}[1/a]}$ is integrally closed. Since $R[1/(a+b)]=\bigcap_{\mathfrak{p}\in\mathrm{D}(a+b)}R_{\mathfrak{p}}$, we see that $R[1/(a+b)]$ is integrally closed, i.e., $a+b\in\boldsymbol{I}(R)$. Therefore $\boldsymbol{I}(R)$ is an ideal of $R$, and hence $\boldsymbol{I}(R)$ is a radical ideal of $R$ by Lemma 2.1.

(2): Let $\mathfrak{p}$ be a minimal element of $\Delta_{\boldsymbol{I}}(R)$. We claim that $\operatorname{depth}R_{\mathfrak{p}}=1$. In fact, suppose that $\operatorname{depth}R_{\mathfrak{p}}>1$ and let $\Lambda=\{\mathfrak{q}\,;\,\mathfrak{q}\in\operatorname{Spec}R,\ \operatorname{depth}R_{\mathfrak{q}}=1$ and $\mathfrak{q}\subseteq\mathfrak{p}\}$. Then, for any element $\mathfrak{q}$ of $\Lambda$, we have $\mathfrak{q}\notin\Delta_{\boldsymbol{I}}(R)$, i.e., $R_{\mathfrak{q}}$ is integrally closed because $\mathfrak{p}$ is a minimal element of $\Delta_{\boldsymbol{I}}(R)$. Since $R_{\mathfrak{p}}=\bigcap_{\mathfrak{q}\in\Lambda}R_{\mathfrak{q}}$, we see that $R_{\mathfrak{p}}$ is integrally closed. Hence we have $\mathfrak{p}\notin\Delta_{\boldsymbol{I}}(R)$, which is a contradiction. By Lemma 2.1, we have $\Delta_{\boldsymbol{I}}(R)\subseteq\mathrm{V}(\boldsymbol{I}(R))$, hence we have $\mathfrak{p}\supseteq\boldsymbol{I}(R)$. Let $a$ be a non-zero element of $\boldsymbol{I}(R)$. Since $\operatorname{depth}R_{\mathfrak{p}}=1$ and $a\in\mathfrak{p}$, $\mathfrak{p}$ is a prime divisor of $aR$. Therefore we see that the number of the minimal elements of $\Delta_{\boldsymbol{I}}(R)$ is finite because $R$ is noetherian. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ be all the minimal elements of $\Delta_{\boldsymbol{I}}(R)$ and let $\mathfrak{a}=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_t$. Then we have $\Delta_{\boldsymbol{I}}(R)=\mathrm{V}(\mathfrak{a})$ by Lemma 2.1. We shall show that $\mathfrak{a}=\boldsymbol{I}(R)$. Since $\mathrm{V}(\mathfrak{a})=\Delta_{\boldsymbol{I}}(R)\subseteq\mathrm{V}(\boldsymbol{I}(R))$ and $\mathfrak{a}$ is a radical ideal, we have $\boldsymbol{I}(R)\subseteq\mathfrak{a}$. Conversely, let $x$ be an element of $\mathfrak{a}$ and let $\mathfrak{p}$ be an element of $\mathrm{D}(x)$, where $\mathrm{D}(x)=\{\mathfrak{p}\,;\,\mathfrak{p}\in\operatorname{Spec}R$ and $x\notin\mathfrak{p}\}$. Then we have $\mathfrak{p}\not\supseteq\mathfrak{a}$ and hence $\mathfrak{p}\notin\mathrm{V}(\mathfrak{a})=\Delta_{\boldsymbol{I}}(R)$, i.e., $R_{\mathfrak{p}}$ is integrally closed. Since $R[1/x]=\bigcap_{\mathfrak{p}\in\mathrm{D}(x)}R_{\mathfrak{p}}$, we see that $R[1/x]$ is integrally closed, whence we have $x\in\boldsymbol{I}(R)$. Thus we have $\mathfrak{a}\subseteq\boldsymbol{I}(R)$, and hence $\mathfrak{a}=\boldsymbol{I}(R)$.

(II) The case $\boldsymbol{P}=\boldsymbol{F}$.

(1): Since $\boldsymbol{I}(R)\subseteq\boldsymbol{F}(R)$ and $\boldsymbol{I}(R)\neq 0$, we have $\boldsymbol{F}(R)\neq 0$. We shall prove that $\boldsymbol{F}(R)$ is a radical ideal of $R$. It is easy to see that we have $ar\in\boldsymbol{F}(R)$ for any $a\in\boldsymbol{F}(R)$ and $r\in R$. Let $a$ and $b$ be two non-zero elements of $\boldsymbol{F}(R)$. Then there exist elements $\alpha_1,\dots,\alpha_s$ and $\beta_1,\dots,\beta_t$ of $R'$ such that $R'[1/a]=R[1/a]\alpha_1+\cdots+R[1/a]\alpha_s$ and $R'[1/b]=R[1/b]\beta_1+\cdots+R[1/b]\beta_t$. Let $B=R[\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_t]$. Then $B$ is a finite $R$-module, and we have $R'[1/a]\subseteq B[1/a]$ and $R'[1/b]\subseteq B[1/b]$. Hence, for any element $x$ of $R'$, there exists a positive integer $n$ such that $a^n x\in B$ and $b^n x\in B$. Then we have $(a+b)^{2n}x\in B$, whence $x\in B[1/(a+b)]$. Therefore we have $R'[1/(a+b)]=B[1/(a+b)]$, and hence $R'[1/(a+b)]$ is a finite $R[1/(a+b)]$-module, i.e., $a+b\in\boldsymbol{F}(R)$. Thus $\boldsymbol{F}(R)$ is an ideal of $R$, and $\boldsymbol{F}(R)$ is a radical ideal by Lemma 2.1.

(2): By Lemma 2.1, it suffices to show that $\mathrm{V}(\boldsymbol{F}(R))\subseteq\Delta_{\boldsymbol{F}}(R)$. Let $\mathfrak{p}$ be an element of $\mathrm{V}(\boldsymbol{F}(R))$ and suppose that $\mathfrak{p}\notin\Delta_{\boldsymbol{F}}(R)$. Then we have $R_{\mathfrak{p}}\in[\boldsymbol{F}]$, hence the integral closure $R'_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ in its quotient field is a finite $R_{\mathfrak{p}}$-module. Thus there exist elements $\alpha_1,\dots,\alpha_r$ of $R'$ such that $R'_{\mathfrak{p}}=R_{\mathfrak{p}}\alpha_1+\cdots+R_{\mathfrak{p}}\alpha_r$. Let $C=R[\alpha_1,\dots,\alpha_r]$. Then $C$ is a finite $R$-module and we have $R'_{\mathfrak{p}}=C_{\mathfrak{p}}$. Let $P_1,\dots,P_n\in\operatorname{Spec}C$ be all the minimal elements of $\Delta_{\boldsymbol{I}}(C)$, and let $\mathfrak{p}_i=P_i\cap R$ for $1\leqq i\leqq n$. We shall show that $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\not\subseteq\mathfrak{p}$. In fact, if $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\subseteq\mathfrak{p}$ then we have $\mathfrak{p}_i\subseteq\mathfrak{p}$ for some $i$. Since $P_i\cap R=\mathfrak{p}_i\subseteq\mathfrak{p}$ and $C_{\mathfrak{p}}=R'_{\mathfrak{p}}$ is integrally closed, we see that $C_{\mathfrak{p}_i}$ is also integrally closed. Hence we have $P_i\notin\Delta_{\boldsymbol{I}}(C)$, which is a contradiction. Let $x$ be an element of $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\backslash\mathfrak{p}$, and let $P$ be a prime ideal of $C$ with $x\notin P$. Then we have $P_1\cap\cdots\cap P_n\not\subseteq P$, whence we have $P\notin\Delta_{\boldsymbol{I}}(C)$ because $P_1,\dots,P_n$ are all the minimal elements of $\Delta_{\boldsymbol{I}}(C)$. Thus $C_P$ is integrally closed, and hence we see that $C[1/x]$ is also integrally closed. Therefore the integral closure of $R[1/x]$ in its quotient field coincides with $C[1/x]$. Since $C[1/x]$ is a finite $R[1/x]$-module, we have $x\in\boldsymbol{F}(R)$, whence $\boldsymbol{F}(R)\not\subseteq\mathfrak{p}$. Thus we have $\mathfrak{p}\notin\mathrm{V}(\boldsymbol{F}(R))$, which is a contradiction. Hence we have $\mathfrak{p}\in\Delta_{\boldsymbol{F}}(R)$, and $\mathrm{V}(\boldsymbol{F}(R))\subseteq\Delta_{\boldsymbol{F}}(R)$. Q.E.D.

By virtue of Proposition 2.4 and Corollary 2.3, we have the following:

THEOREM 2.5. *Let $R$ be a noetherian subring of an affine domain over $k$ and let $R'$ be the integral closure of $R$ in its quotient field. Then $R'$ is a finite $R$-module if and only if $R'_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of $R$.*

## 3. Affineness of noetherian subrings of an affine domain

In this section we shall prove that a noetherian subring $R$ of an affine domain over $k$ will be an affine domain over $k$ provided the integral closure $R'$ of $R$ in its quotient field is equidimensional. For the proof we need the following:

THEOREM 3.1. *Let $R$ be a $d$-dimensional subring of an affine domain over $k$ and let $R'$ be the integral closure of $R$ in its quotient field $K$. Let $\mathfrak{M}$ be a maximal ideal of $R'$ with* $\mathrm{ht}\,\mathfrak{M}=d$. *If $R$ is noetherian then $R'_{\mathfrak{M}}$ is a locality over $k$.*

PROOF. Let $\mathfrak{m}=\mathfrak{M}\cap R$. Since $R$ is noetherian, $\mathfrak{m}$ is finitely generated, say $\mathfrak{m}=(x_1,\dots, x_t)R$. Let $B$ be an affine domain over $k$ contained in $R$ such that $R$ and $B$ are birational and $x_1,\dots, x_t\in B$. Let $M=\mathfrak{m}\cap B$. Then we have $x_1,\dots, x_t\in M$ and hence $MR=\mathfrak{m}$. Since $\mathrm{ht}\,\mathfrak{M}=d$ and $\mathrm{tr.deg}_k R'/\mathfrak{M}\leqq \mathrm{tr.deg}_k R'-\mathrm{ht}\,\mathfrak{M}$, we have $\mathrm{tr.deg}_k B/M\leqq \mathrm{tr.deg}_k R'/\mathfrak{M}=0$. Thus $B/M$ is algebraic over $k$, hence $B/M$ is a field and $M$ is a maximal ideal of $B$. Let $B'$ be the integral closure of $B$ in $K$ and let $\bar{R}=R[B']$. Since $B$ is an affine domain over $k$, $B'$ is a finite $B$-module. Whence $\bar{R}$ is a finite $R$-module, especially $\bar{R}$ is noetherian. Let $\mathfrak{N}=\mathfrak{M}\cap \bar{R}$ and let $M'=\mathfrak{N}\cap B'$. Since $R'$ is integral over $\bar{R}$, we have $\mathrm{ht}\,\mathfrak{N}\geqq \mathrm{ht}\,\mathfrak{M}=d$, hence we have $\mathrm{ht}\,\mathfrak{N}=d$. On the other hand, $B'$ is integral over $B$ and $M'$ lies over the maximal ideal $M$ of $B$. Hence $M'$ is a maximal ideal of $B'$, and we have $\mathrm{ht}\,M'=d$ because $B'$ is an affine domain over $k$. Thus we have $\dim B'_{M'}=\dim \bar{R}_{\mathfrak{N}}$. Notice that $M'\bar{R}_{\mathfrak{N}}\supseteq M\bar{R}_{\mathfrak{N}}=\mathfrak{m}\bar{R}_{\mathfrak{N}}$ and $\mathfrak{m}\bar{R}_{\mathfrak{N}}$ is a $\mathfrak{N}\bar{R}_{\mathfrak{N}}$-primary ideal. Therefore there exists a positive integer $r$ such that $\mathfrak{N}^r\bar{R}_{\mathfrak{N}}\subseteq M'\bar{R}_{\mathfrak{N}}$. Let $k'=B'/M'$ and let $L=\bar{R}/\mathfrak{N}$. Then we have $\mathrm{length}_{k'}\bar{R}_{\mathfrak{N}}/M'\bar{R}_{\mathfrak{N}}\leqq(\mathrm{length}_{k'}L)(\mathrm{length}_{\bar{R}}\bar{R}/\mathfrak{N}^r)$. Since $\mathrm{ht}\,\mathfrak{N}=d$, we have $\mathrm{tr.deg}_k \bar{R}/\mathfrak{N}=0$, and hence $L=\bar{R}/\mathfrak{N}$ is a subfield of a certain affine domain over $k$ (cf. [7, Theorem 2]). Thus $L$ is a finite algebraic extension field of $k$, whence $\mathrm{length}_k L$ is finite, a fortiori, $\mathrm{length}_{k'}L$ is finite. On the other hand, since $\bar{R}$ is noetherian, we have $\mathrm{length}_{\bar{R}}\bar{R}/\mathfrak{N}^r$ is finite. Thus we have $\mathrm{length}_{k'}\bar{R}_{\mathfrak{N}}/M'\bar{R}_{\mathfrak{N}}$ is finite. Moreover, since $B'$ is a normal affine domain, $B'_{M'}$ is analytically normal by Theorem (37.5) in [6], and obviously $\bar{R}_{\mathfrak{N}}$ and $B'_{M'}$ are birational. Hence we have $B'_{M'}=\bar{R}_{\mathfrak{N}}$ by Theorem (37.4) in [6]. Thus $\bar{R}_{\mathfrak{N}}$ is integrally closed, whence we have $\bar{R}_{\mathfrak{N}}=R'_{\mathfrak{N}}$ because $R'_{\mathfrak{N}}$ is integral and birational over $\bar{R}_{\mathfrak{N}}$. Therefore $R'_{\mathfrak{N}}$ is a local ring, and hence we have $R'_{\mathfrak{N}}=R'_{\mathfrak{M}}$. Whence we have $R'_{\mathfrak{M}}=B'_{M'}$ and $R'_{\mathfrak{M}}$ is a locality over $k$. Q.E.D.

THEOREM 3.2. *Let $R$ be a $d$-dimensional subring of an affine domain over $k$ and let $R'$ be the integral closure of $R$ in its quotient field. If $R$ is noetherian and $R'$ is equidimensional, that is,* $\dim R'_{\mathfrak{M}}=d$ *for any maximal ideal $\mathfrak{M}$ of $R'$, then $R$ is an affine domain over $k$.*

PROOF. Since $R'$ is equidimensional, $R'_{\mathfrak{M}}$ is a locality over $k$ for any maximal ideal $\mathfrak{M}$ of $R'$ by Theorem 3.1. Thus $R'$ is an affine domain over $k$ by Theorem 1.6, and hence $R$ is also an affine domain over $k$ by Lemma 1.4. Q.E.D.

Recall that a ring $R$ is called catenary if, for any pair of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ with $\mathfrak{p}\supseteq\mathfrak{q}$, we have $\mathrm{ht}\,\mathfrak{p}=\mathrm{ht}\,\mathfrak{q}+\mathrm{ht}(\mathfrak{p}/\mathfrak{q})$. A ring $R$ is called universally catenary if $R$ is noetherian and if every $R$-algebra of finite type is catenary (cf. [4, (14.B)]).

COROLLARY 3.3. *Let $R$ be a subring of an affine domain over $k$. If $R$ is universally catenary and equidimensional then $R$ is an affine domain over $k$.*

PROOF. Let $\mathfrak{m}=(x_1,\dots, x_t)R$ be an arbitrary maximal ideal of $R$ and let $B$ be an affine domain over $k$ contained in $R$ such that $R$ and $B$ are birational and $x_1,\dots, x_t \in B$. Let $B'$ be the integral closure of $B$ in its quotient field and let $\bar{R}=R[B']$. Then $\bar{R}$ is a finite $R$-module. Let $\mathfrak{M}_1,\dots, \mathfrak{M}_n$ be all the maximal ideals of $\bar{R}$ lying over $\mathfrak{m}$. Since $R$ is universally catenary and $\bar{R}$ is a finite $R$-module, the dimension formula holds between $R$ and $\bar{R}$, whence we have $\operatorname{ht}\mathfrak{M}_i=\operatorname{ht}\mathfrak{m}$ for each $i$ (cf. [4, (14.C)]). Thus, as is shown in the proof of Theorem 3.1, we have $\bar{R}_{\mathfrak{M}_i}=B'_{M_i}$ for each $i$, where $M_i=\mathfrak{M}_i\cap B'$. Therefore $\bar{R}_{\mathfrak{M}_i}$ is integrally closed for each $i$, hence $\bar{R}_{\mathfrak{m}}$ is integrally closed. Thus the integral closure of $R_{\mathfrak{m}}$ in its quotient field is equal to $\bar{R}_{\mathfrak{m}}$ which is a finite $R_{\mathfrak{m}}$-module. Whence, by Theorem 2.5, the integral closure $R'$ of $R$ in its quotient field is a finite $R$-module, and hence the dimension formula holds between $R$ and $R'$. Since $R$ is equidimensional, we see that $R'$ is also equidimensional. Thus the assertion follows from Theorem 3.2. Q.E.D.

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