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On the group of fibre homotopy equivalences

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Introduction

Let (E, p, B, F) denote a Hurewicz fibration with projection $p: E \to B$ and fibre F. Then the set of all free fibre homotopy classes of free fibre homotopy equivalences of E to itself forms a group under the multiplication defined by the composition of maps. This group is called the group of fibre homotopy equivalences of a Hurewicz fibration (E, p, B, F), and we denote it by $\mathscr{L}(E)$.

The group $\mathscr{L}(E)$ has been studied by several authors, e.g., [5], [6], [15], [16], [19], [21], [24] and [33]. We notice that for any covering space, this is the group of all covering transformations.

The purpose of this paper is to study the group $\mathscr{L}(E)$ of a Hurewicz fibration (E, p, S^n, F) over the n-sphere S^n $(n \geqq 1)$, where the fibre F is assumed to be a locally compact CW-complex. Let aut F denote the H-space of all free homotopy equivalences of F to itself with the identity map $1: F \to F$ as the base point. Then we may consider a Hurewicz fibration

(1) $\qquad (E_k, p, S^n, F)$ with characteristic map $k \in \pi_{n-1}(\mathrm{aut}\, F)$,

because any fibration (E, p, S^n, F) is freely fibre homotopy equivalent to such a fibration by a classification theorem due to Stasheff [25, Th. 1.5–1.6] (for details, see §§ 1–2).

Now let $\mathscr{F}(F) = \pi_0(\mathrm{aut}\, F)$ be the group of all free homotopy classes of free homotopy equivalences of F to itself, and consider the action of $\mathscr{F}(F)$ on the homotopy group $\pi_i(\mathrm{aut}\, F)$ by the conjugation denoted by $\cdot$ (see § 1). Then, by using Gottlieb's theorem ([5, Th. 1]), we can prove the following basic theorem of this paper in Theorem 2.2 and Corollary 2.5:

THEOREM I. *For the group $\mathscr{L}(E_k)$ of fibre homotopy equivalences of a fibration* (1), *there holds the exact sequence*

$$\pi_1(\mathrm{aut}\, F) \xrightarrow{\partial_k} \pi_n(\mathrm{aut}\, F) \xrightarrow{G} \mathscr{L}(E_k) \xrightarrow{J_0} \mathscr{F}_k(F) \longrightarrow 1,$$

where ∂_k is given by the Samelson product: $\partial_k(x) = \langle k, x\rangle$, $\mathscr{F}_k(F) = \{\alpha \in \mathscr{F}(F) | \alpha \cdot k = k\}$, and J_0 is the homomorphism obtained by the restriction to the fibre F.

Especially, for the trivial fibration $(F \times S^n, p, S^n, F)$ which is the one of (1) *with $k = 0$, this sequence is the split exact sequence*

$$0 \longrightarrow \pi_n(\mathrm{aut}\, F) \xrightarrow{G} \mathscr{L}(F \times S^n) \xrightarrow{J_0} \mathscr{F}(F) \longrightarrow 1,$$

where G becomes the homorphism defined naturally and the action $\mathscr{F}(F)$ on $\pi_n(\mathrm{aut}\, F)$ is given by the conjugation.

Now we study the group $\mathscr{L}(E_k)$ in the case $F=S^q$, i.e., for a spherical fibration

(2) $(E_k,\, p,\, S^n,\, S^q)$ with characteristic map $k \in \pi_{n-1}(\mathrm{aut}\, S^q)(n,\, q \geqq 1)$,

by investigating in details the exact sequence

(3) $$\pi_1(\mathrm{aut}\, S^q) \xrightarrow{\partial_k} \pi_n(\mathrm{aut}\, S^q) \xrightarrow{G} \mathscr{L}(E_k) \xrightarrow{J_0} \mathscr{F}_k(S^q) \longrightarrow 1$$

in Theorem I for $F=S^q$. To study the group $\pi_i(\mathrm{aut}\, S^q)$ in (3) and (2), consider the evaluation map ω: aut $S^q \to S^q$, $\omega(f)=f(*)$ ($*$ is the base point), and set $\mathrm{aut}_0\, S^q=\omega^{-1}(*)$. Then, by G. W. Whitehead's theorem ([35, Th. 3.2]), we have the isomorphism

$$\tau\colon \pi_{i+q}(S^q) \cong \pi_i(\mathrm{aut}_0\, S^q)(i \geqq 1)$$

and the exact sequence (see (3.6))

(4) $$\cdots \longrightarrow \pi_{i+1}(S^q) \xrightarrow{[\iota_q,\,]} \pi_{i+q}(S^q) \xrightarrow{i_*\tau} \pi_i(\mathrm{aut}\, S^q) \xrightarrow{\omega_*} \pi_i(S^q) \longrightarrow \cdots$$

for $i \geqq 1$, where ι_q is the homotopy class of $1\colon S^q \to S^q$, $[\ ,\ ]$ is the Whitehead product, and i: $\mathrm{aut}_0\, S^q \subset \mathrm{aut}\, S^q$ is the inclusion. In the case $q=1$, 3 or 7, the canonical multiplication on S^q gives us a cross-section $t\colon S^q \to \mathrm{aut}\, S^q$, $t(x)(y)=xy$ $(x,\, y \in S^q)$, and (4) is the split exact sequence

(5) $$0 \longrightarrow \pi_{i+q}(S^q) \xrightarrow{i_*\tau} \pi_i(\mathrm{aut}\, S^q) \underset{t_*}{\overset{\omega_*}{\rightleftarrows}} \pi_i(S^q) \longrightarrow 0 \quad (q=1,\, 3,\, 7)$$

for $i \geqq 1$ (see (3.10)).

By using these results, we study in §4 the homorphism $\partial_k=\langle k,\ \rangle$ in (3). When $q \geqq 2$, $\pi_1(\mathrm{aut}\, S^q)=Z_2=\{i_*\tau(\eta_q)\}$ $(\pi_{q+1}(S^q)=\{\eta_q\})$ by (4) and we have to investigate the Samelson product

$$\langle k,\, i_*\tau(\eta_q)\rangle \quad \text{for} \quad k \in \pi_{n-1}(\mathrm{aut}\, S^q)(q \geqq 2).$$

We can determine it in the case

(A) when $n \geqq 2$, $q \geqq 2$ and $k=i_*\tau(k')$ for some $k' \in \pi_{n+q-1}(S^q)$, for instance, when $2 \leqq n \leqq q$ and k is any element (by (4)),

by Steer's formula ([27, Th. 5.76]) on Samelson products in $\pi_*(\mathrm{aut}_0 S^q)$ and by using several formulae in the homotopy groups of spheres; and in the case

(B) when $n \geqq 2$ and $q=3$ or 7, then any k can be represented as $k=i_*\tau(k')+t_*(k'')$ for some $k' \in \pi_{n+q-1}(S^q)$ and $k'' \in \pi_{n-1}(S^q)$ by (5), by the computations of certain Samelson products in $\pi_*(SO(q+1))$ based on the results of R. Bott (cf. [13], [14]) and I. M. James [14].[1] In the case

(C) when $n=1$, then $k\in\pi_0(\mathrm{aut}\, S^q)=\mathscr{F}(S^q)=Z_2=\{\pm 1\}$, or when $q=1$, we can determine ∂_k by studying the action $\cdot$ of $\mathscr{F}(S^q)$ on $\pi_1(\mathrm{aut}\, S^q)$ by conjugation in §5 and by noticing $\langle k, x\rangle = k\cdot x - x$ when $n=1$. Thus we have the following theorem in Theorems 4.3, 4.6, 4.8 and 4.11:

THEOREM II. *The homomorphism* $\partial_k=\langle k, \ \rangle$ *(the Samelson product) in* (3) *satisfies the following* (A)–(C) *in the above cases* (A)–(C) *respectively.*

(A) $$\langle k, i_*\tau(\eta_q)\rangle = i_*\tau\{k'\eta_{n+q-1} + (-1)^q\eta_q\Sigma k' + [\eta_q, \iota_q]\Sigma h_2(k')\},$$

where Σ is the suspension and h_2 is the generalization of the Hopf invariant due to Hilton.

(B) $$\langle k, i_*\tau(\eta_q)\rangle = \begin{cases} i_*\tau(k'\eta_{n+2}-\eta_3\Sigma k') + t_*(\eta_3\Sigma k'') & \text{if } q=3, \\ i_*\tau\{k'\eta_{n+6}-\eta_7\Sigma k' + (\bar\nu_7+\varepsilon_7)\Sigma^8 k''\} + t_*(\eta_7\Sigma k'') & \text{if } q=7, \end{cases}$$

where $\pi_{15}(S^7)=Z_2+Z_2+Z_2=\{\sigma'\eta_{14}\}+\{\bar\nu_7\}+\{\varepsilon_7\}$ *(cf.* [31, p. 61]).

(C) (i) *If* $n=q=1$, *then* $\mathrm{Coker}\,\partial_k=Z_2$ *for* $k=-1$, $=Z$ *for* $k=+1$.
(ii) $\mathrm{Coker}\,\partial_k=Z_2$ *if* $n=1$ *and* $q\geqq 2$, $=0$ *if* $n\geqq 2$ *and* $q=1$.

Furthermore, we can prove the following in Theorems 5.5, 5.4 and (5.3):

THEOREM III. *The group* $\mathscr{F}_k(S^q)=\{\alpha\in\mathscr{F}(S^q)\mid \alpha\cdot k=k\}\subset\mathscr{F}(S^q)=Z_2$ *in* (3) *is given by the following* (A)–(C) *in the above cases* (A)–(C) *respectively.*

(A) $$\mathscr{F}_k(S^q) = \begin{cases} Z_2 & \text{if } [\iota_q, \iota_q]H(k')=0, \text{ especially if } 2\leqq n\leqq q, \\ 1 & \text{otherwise,} \end{cases}$$

where H is the generalized Hopf invariant.

(B) $$\mathscr{F}_k(S^q) = \begin{cases} Z_2 & \text{if } \langle\iota_q, \iota_q\rangle\Sigma^q k''=0=2k'', \\ 1 & \text{otherwise.} \end{cases}$$

(C) $$\mathscr{F}_k(S^q) = \begin{cases} Z_2 & \text{if } n=1, \text{ or if } n\geqq 2,\ q=1 \text{ and } k=0, \\ 1 & \text{if } n\geqq 2,\ q=1 \text{ and } k\neq 0. \end{cases}$$

These two theorems together with (4) and (5) give some informations on the short exact sequence

$$0\longrightarrow A\longrightarrow \mathscr{L}(E_k)\xrightarrow{J_0}\mathscr{F}_k(S^q)\longrightarrow 1 \quad (A\cong \mathrm{Coker}\,\partial_k)$$

1) The author is indebted to Professor S. Oka for the improvement of the original manuscript in these computations; especially Theorem 4.8 for $q=7$ is due to him.

induced from (3). We consider in § 6 some conditions which imply that J_0 is a split epimorphism, and state in § 7 some results on the group $\mathscr{L}(E_k)$ for (E_k, p, S^n, S^q) of (2) with $n \leqq q$ or $q=1, 3, 7$ by giving the groups A and $\mathscr{F}_k(S^q)$ explicitly.

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§ 1. Preliminaries

For any CW-complex B and a space Y, let $L(B, Y)$ be the space of all (continuous) maps of B to Y with compact-open topology, and $L_0(B, Y)$ be its subspace consisting of all based maps.

We consider the evaluation map

$$(1.1) \qquad \omega: L(S^n, Y) \longrightarrow Y, \quad \omega(f) = f(*) \ (* \text{ denotes the base point}).$$

As is well-known, this is a Hurewicz fibration with fibre $L_0(S^n, Y) = \omega^{-1}(*)$, and for any based map $k \in L_0(S^n, Y)$, we have the homotopy exact sequence

$$(1.2) \quad \cdots \longrightarrow \pi_{i+1}(Y) \xrightarrow{\partial_k} \pi_i(L_0(S^n, Y), k) \xrightarrow{i_*} \pi_i(L(S^n, Y), k) \xrightarrow{\omega_*} \pi_i(Y) \longrightarrow \cdots.$$

Here we quote the following theorems:

THFOREM 1.3 (G. W. Whitehead [35, Th. 3.2], [37, (3.1)]). *In* (1.2), *there exist isomorphisms*

$$\tau: \pi_{n+i}(Y) = [S^i \wedge S^n, Y]_0 \cong \pi_i(L_0(S^n, Y), *) \cong \pi_i(L_0(S^n, Y), k) (i \geqq 1),$$

and the composition $\tau^{-1}\partial_k: \pi_{i+1}(Y) \rightarrow \pi_{n+i}(Y)$ *is given by*

$$\tau^{-1}\partial_k(x) = -[k, x] (\textit{the Whitehead proudct of } k \in \pi_n(Y) \textit{ and } x \in \pi_{i+1}(Y)).$$

THEOREM 1.4 (S. T. Hu [10, Th. 2.2]). *If* $i=1$ *in* (1.2), *then*

$$\mathrm{Im}\,\{\omega_*: \pi_1(L(S^n, Y), k) \longrightarrow \pi_1(Y)\} = \{\alpha \in \pi_1(Y) \mid \alpha \cdot k = k\},$$

where $\cdot$ *denotes the usual action of* $\pi_1(Y)$ *on* $\pi_n(Y)$.

By Theorem 1.3, (1.2) for $i \geqq 1$ turns out to the exact sequence

$$(1.2)' \quad \cdots \longrightarrow \pi_{i+1}(Y) \xrightarrow{[k,\ ]} \pi_{n+i}(Y) \xrightarrow{i_*\tau} \pi_i(L(S^n, Y), k) \xrightarrow{\omega_*} \pi_i(Y) \longrightarrow \cdots.$$

Now, we can classify Hurewicz fibrations by a theorem of J. Stasheff [25, Th. 1.5, 1.6] as follows: *Given any locally compact CW-complex F, there exists a universal Hurewicz fibration*

(1.5) $$p_\infty\colon E_\infty \longrightarrow B_\infty \text{ with fibre } F,$$

and any Hurewicz fibration (E, p, B, F) over a CW-complex B is freely fibre homotopy equivalent to the induced Hurewicz fibration

$(1.5)_k$ $$(E_k, p_k, B, F) \text{ with classifying map } k\colon B \longrightarrow B_\infty,$$

that is, we have the commutative diagram

$$\begin{array}{ccccc} E & \longrightarrow & E_k & \xrightarrow{\tilde{k}} & E_\infty \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_\infty} \\ B & = & B & \xrightarrow{k} & B_\infty, \end{array}$$

where the right square is a pull-back and the left upper map is a free fibre homotopy equivalence.

In this paper, we use the following theorem on the group $\mathscr{L}(E_k)$ of fibre homotopy equivalences of the Hurewicz fibration $(1.5)_k$:

THEOREM 1.6 (D. H. Gottlieb [5, Th. 1]). $\mathscr{L}(E_k) \overset{d}{\cong} \pi_1(L(B, B_\infty), k)$.

The proof of this theorem is given by the following process: Let $L(B, B_\infty, k)$ denote the path component of $L(B, B_\infty)$ containing k, and $L^*(E_k, E_\infty, k)$ the subspace of $L(E_k, E_\infty)$ consisting of all fibre preserving maps $\tilde{f}\colon E_k \to E_\infty$ with the properties that $\tilde{f}$ covers a map $f\colon B \to B_\infty$ with $f \in L(B, B_\infty, k)$ and that the restriction $\tilde{f}|p_k^{-1}(b)\colon p_k^{-1}(b) \to p_\infty^{-1}(f(b))\,(b \in B)$ to each fibre is a free homotopy equivalence. Then we have a map

(1.7) $$\Phi\colon L^* = L^*(E_k, E_\infty, k) \longrightarrow L = L(B, B_\infty, k), \quad \Phi(\tilde{f}) = f;$$

and $\Phi^{-1}(k)$ is naturally homeomorphic to the space $L^{**} = L^{**}(E_k, E_k)$ of all free fibre homotopy equivalences of E_k to itself with compact-open topology. Gottlieb proved that Φ satisfies the quasi-covering homotopy property, and obtained the exact sequence

$$\cdots \longrightarrow Q_i(L^{**}) \longrightarrow Q_i(L^*) \longrightarrow Q_i(L) \xrightarrow{d} Q_{i-1}(L^{**}) \longrightarrow \cdots$$

of the quasi-homotopy groups. Here $Q_i(L) = \pi_i(L)$ since B is a CW-complex. Furthermore he proved that $Q_i(L^*) = 0$ for any $i \geq 0$ and hence

(*) $$d\colon \pi_i(L(B, B_\infty, k)) \cong Q_{i-1}(L^{**}(E_k, E_k), 1) \quad (i \geqq 1).$$

When $i = 1$, $Q_0(L^{**}(E_k, E_k)) = \mathscr{L}(E_k)$ by definition, and we have the isomorphism

$$d\colon \pi_1(L(B, B_\infty, k)) \cong \mathscr{L}(E_k) \quad \text{in Theorem 1.6.}$$

The fact that this is a homomorphism is shown by the following

(1.8) *For $\alpha \in \pi_i(L(B, B_\infty,), k)$, take a representative $\alpha\colon I \times S^{i-1} \to L(B, B_\infty, k)$ with $\alpha(I \times S^{i-1} \cup I \times *) = k$ and a lifting $\tilde{\alpha}\colon I \times S^{i-1} \to L^*(E_k, E_\infty, k)$ with $\Phi\tilde{\alpha} = \alpha$ and $\tilde{\alpha}(1 \times S^{i-1} \cup I \times *) = \tilde{k}$. Then the image $d(\alpha)$ by d in $(*)$ is represented by $\tilde{\alpha}|0 \times S^{i-1}\colon S^{i-1} \to \Phi^{-1}(k) = L^{**}(E_k, E_k)$.*

In the above proof, we consider the special case that $B = *$ and $k = *$. Then $L(B, B_\infty, k) = B_\infty$, $E_k = F$ and $L^{**}(E_k, E_k) = \operatorname{aut} F$, where $\operatorname{aut} F$ is the H-space of all free homotopy equivalences of F to itself with the identity map 1 as base point, and (1.7) is the associated principal fibration

(1.9) $$\Phi\colon L^*(F, E_\infty, *) \longrightarrow B_\infty \text{ with fibre } \operatorname{aut} F$$

of (1.5), and $(*)$ is the isomorphism

(1.10) $$d\colon \pi_i(B_\infty) \cong \pi_{i-1}(\operatorname{aut} F) \quad \text{for} \quad i \geqq 1 \quad \text{(cf. [5, p. 49])}.$$

Furthermore, by applying the same proof as that of [3, pp. 813–814] to the associated principal fibration (1.9), we can prove the following

LEMMA 1.11. *By the isomorphism d of* (1.10), *the Whitehead product $[\ ,\ ]$ in $\pi_*(B_\infty)$ corresponds to the Samelson product $\langle\ ,\ \rangle$ in $\pi_{*-1}(\operatorname{aut} F)$, i.e.,*

$$d([x, y]) = (-1)^{i-1}\langle d(x), d(y)\rangle \quad \textit{for} \quad x \in \pi_i(B_\infty) \quad \textit{and} \quad y \in \pi_j(B_\infty).$$

Let $\alpha \in \operatorname{aut} F$ be a representative of an element α of the group

$$\mathscr{F}(F) = \pi_0(\operatorname{aut} F)$$

of all free homotopy classes of free homotopy equivalences of F to itself. Let

$$c_\alpha\colon \operatorname{aut} F \longrightarrow \operatorname{aut} F, \quad c_\alpha(f) = \alpha f \alpha^{-1} \quad \text{for} \quad f \in \operatorname{aut} F,$$

be the conjugation by α. Then the induced homomorphism $c_{\alpha*}\colon \pi_n(\operatorname{aut} F, 1) \to \pi_n(\operatorname{aut} F, \alpha\alpha^{-1}) \cong \pi_n(\operatorname{aut} F, 1)$ depends only on the homotopy class α. Hence we have an action

(1.12) $$\mathscr{F}(F) \times \pi_n(\operatorname{aut} F) \longrightarrow \pi_n(\operatorname{aut} F), \ \alpha \cdot \beta = c_{\alpha*}(\beta) \ (\alpha \in \mathscr{F}(F),\ \beta \in \pi_n(\operatorname{aut} F)).$$

We call this $\mathscr{F}(F)$-action on $\pi_n(\operatorname{aut} F)$ the action by conjugation.

Regarding α as an element of $\pi_0(\operatorname{aut} F)$, we see immediately from the definition of the Samelson product that

(1.13) $$\langle \alpha, \beta\rangle = \alpha\cdot\beta - \beta \quad \textit{for } \alpha \in \pi_0(\operatorname{aut} F) = \mathscr{F}(F), \quad \beta \in \pi_n(\operatorname{aut} F).$$

§ 2. Fibrations over the spheres

Now we consider Hurewicz fibrations over the n-sphere S^n $(n \geqq 1)$. Hereafter, by identifying $d^{-1}(k)$ with k by the isomorphism $d\colon \pi_n(B_\infty) \cong \pi_{n-1}(\operatorname{aut} F)$ in (1.10), we shall consider the Hurewicz fibration

(2.1) $\qquad (E_k, p, S^n, F)$ with characteristic map $k \in \pi_{n-1}(\operatorname{aut} F)$,

which is the fibration $(E_{k'}, p_{k'}, S^n, F)$ with classifying map $k' = d^{-1}(k) \in \pi_n(B_\infty)$ of $(1.5)_{k'}$.

The following theorem is basic in our study.

THEOREM 2.2. *For the group $\mathscr{L}(E_k)$ of fibre homotopy equivalences of the fibration* (2.1), *the sequence*

$$\pi_1(\operatorname{aut} F) \xrightarrow{\partial_k} \pi_n(\operatorname{aut} F) \xrightarrow{G} \mathscr{L}(E_k) \xrightarrow{J_0} \mathscr{F}_k(F) \longrightarrow 1,$$

is exact, where

$$\partial_k(x) = \langle k, x\rangle \,(\text{the Samelson product}) \quad \text{for } x \in \pi_1(\operatorname{aut} F),$$

$$\mathscr{F}_k(F) = \{\alpha \in \mathscr{F}(F)(=\pi_0(\operatorname{aut} F)) \mid \alpha\cdot k - k\,(=\langle\alpha, k\rangle) = 0 \quad \text{in} \quad \pi_{n-1}(\operatorname{aut} F)\}$$

and J_0 is the homomorphism obtained by the restriction to the fibre F.

PROOF. Consider the diagram $(d(k')=k)$

$$(2.3)\qquad \begin{array}{ccccccc} \pi_2(B_\infty) & \xrightarrow{[k',\ ]} & \pi_{n+1}(B_\infty) & \xrightarrow{i_*\tau} & \pi_1(L(S^n, B_\infty), k') & \xrightarrow{\omega_*} & \pi_1(B_\infty) \\ d\big\downarrow\cong & & d\big\downarrow\cong & & d\big\downarrow\cong & & d\big\downarrow\cong \\ \pi_1(\operatorname{aut} F) & \xrightarrow{\partial_k} & \pi_n(\operatorname{aut} F) & & \mathscr{L}(E_k) & \xrightarrow{J_0} & \mathscr{F}(F)=\pi_0(\operatorname{aut} F), \end{array}$$

where the upper sequence is the exact sequence (1.2)′ for the evaluation map $\omega\colon L(S^n, B_\infty) \to B_\infty$, and d's are the isomorphisms in Theorem 1.6 and (1.10).

Then the left square is commutative up to sign $(-1)^n$ by Lemma 1.11, and so is the right one by [5, p. 52] (cf. [32]). Thus we have the desired exact sequence by taking

$$G = di_*\tau d^{-1}$$

and by proving $\operatorname{Im} J_0 = \mathscr{F}_k(F)$. The last equality is proved as follows.

By the well-known formula $[\alpha', k'] = \alpha'\cdot k' - k'$ $(\alpha' \in \pi_1(B_\infty))$ and by the commutativity of (2.3), Theorem 1.4 and Lemma 1.11, we have

$$\operatorname{Im} J_0 = d(\operatorname{Im} \omega_*) = \{\alpha \in \pi_0(\operatorname{aut} F) \mid \langle\alpha, k\rangle = 0\}.$$

Therefore $\operatorname{Im} J_0 = \mathscr{F}_k(F)$ by (1.13). *q.e.d.*

In the case $k=0$, we have $d^{-1}(k) = *$ by (1.8), and (2.1) is the trivial fibration

$$(F \times S^n, p, S^n, F) \text{ (p is the projection onto the second factor).}$$

For this trivial fibration, we can define naturally the homomorphism

$$i: \pi_n(\operatorname{aut} F) \longrightarrow \mathscr{L}(F \times S^n) \tag{2.4}$$

as follows: For any $\alpha \in \pi_n(\operatorname{aut} F)$, take a representative $\alpha: S^n \to \operatorname{aut} F$ and its adjoint map $\bar{\alpha}: F \times S^n \to F$ such that $\bar{\alpha}(x, y) = \alpha(y)(x)$. Then

$$(\bar{\alpha}, p): F \times S^n \longrightarrow F \times S^n, \quad (\bar{\alpha}, p)(x, y) = (\bar{\alpha}(x, y), y) \quad (x \in F, y \in S^n),$$

is a fibre homotopy equivalence, and its free fibre homotopy class $i(\alpha) = (\bar{\alpha}, p)$ is determined by the homotopy class α.

COROLLARY 2.5. *For the trivial fibration $(F \times S^n, p, S^n, F)$, the homomorphism G in Theorem 2.2 is equal to i in* (2.4) *and the exact sequence turns out to the split exact sequence*

$$0 \longrightarrow \pi_n(\operatorname{aut} F) \xrightarrow{i} \mathscr{L}(F \times S^n) \underset{j}{\overset{J_0}{\rightleftarrows}} \mathscr{F}(F) \longrightarrow 1,$$

where j is a right inverse of J_0 defined by $j(f) = f \times 1$ for $f \in \mathscr{F}(F)$ and the action of $\mathscr{F}(F)$ on $\pi_n(\operatorname{aut} F)$ is given by the conjugation $\cdot$ of (1.12).

PROOF. Consider the homomorphism $i_* \tau$ in (2.3) where $k=0$ and $k'=*$. Then $i_* \tau$ maps $a: (I \times S^n, \dot{I} \times S^n \cup I \times *) \to (B_\infty, *)$ to $b: (I, \dot{I}) \to (L(S^n, B_\infty), *)$ such that a is the adjoint map of b by Theorem 1.3. Consider the projection Φ of (1.9) and a lifting $\tilde{a}: I \times S^n \to L^*(F, E_\infty, *)$ with $\Phi \tilde{a} = a$ and $\tilde{a}(1 \times S^n \cup I \times *) = \tilde{*}$. Then $\tilde{a}$ is the adjoint map of some $\tilde{b}: I \to L^*(F \times S^n, E_\infty, *)$ which satisfies $\Phi \tilde{b} = b$ ($\Phi: L^*(F \times S^n, E_\infty, *) \to L(B, B_\infty, *)$ is the projection in (1.7) for $k=*$) and $\tilde{b}(1) = \tilde{*}$. Thus by (1.8) and the definition of i in (2.4), we see that $id(a) = d(b) = di_* \tau(a)$. Therefore $G = i$ by the equality $G = di_* \tau d^{-1}$ in the proof of Theorem 2.2, and we have the desired exact sequence by Theorem 2.2.

Clearly j is a right inverse of J_0. Thus the sequence is split, and we see immediately by definition that the action of $\mathscr{F}(F)$ on $\pi_n(\operatorname{aut} F)$ is given by the conjugation. *q.e.d.*

In the following sections, we study the case that the fibre F is a sphere. Here we give a few examples where F is not a sphere.

EXAMPLE 2.6. *Let F be an aspherical complex, and consider a fibration (E, p, S^n, F) over S^n ($n \geqq 2$) with fibre F. Then $\mathscr{L}(E)$ is a subgroup of $\mathscr{F}(F) = \operatorname{Aut} \pi_1(F)/\operatorname{Inn} \pi_1(F)$. For the trivial fibration $F \times S^n \to S^n$, we have $\mathscr{L}(F \times S^n) = \mathscr{F}(F)$.*

PROOF. Since $\pi_i(\mathrm{aut}\, F)=0$ for $i \geqq 2$ by [4, Th. III.2], we have the desired result by Theorem 2.2 and Corollary 2.5. *q.e.d.*

EXAMPLE 2.7. *Consider a fibration (E, p, S^2, CP^n) over S^2 with fibre CP^n (the complex projective space) for even n. Then we have the exact sequence*

$$0 \longrightarrow Z_2 \longrightarrow \mathscr{L}(E) \longrightarrow Z_2.$$

For the trivial fibration $CP^n \times S^2 \to S^2$, we have $\mathscr{L}(CP^n \times S^2)=Z_2+Z_2$ (n: even).

PROOF. Since $\pi_1(\mathrm{aut}\, CP^n)=Z_{n+1}$ and $\pi_2(\mathrm{aut}\, CP^n)=Z_2$ by [23, Prop. 1.2] and $\mathscr{F}(CP^n)=Z_2$, we see the desired result by Theorem 2.2 and Corollary 2.5. *q.e.d.*

REMARK 2.8. (i) *Theorem 2.2 remains true if F* is a *k-space* (*see* [34]).
(ii) *If we use the Whitehead and Samelson products in a general form* (*cf.* [1]), *then Theorem 2.2 still holds for fibrations over cogroup-like complexes* (*e.g. suspended complexes*).

§3. The group $\pi_i(\mathrm{aut}\, S^q)$

For a spherical Hurewicz fibration

(3.1) $\qquad (E_k, p, S^n, S^q)$ with characteristic map $k \in \pi_{n-1}(\mathrm{aut}\, S^q)$

of (2.1) for $F=S^q$ ($q \geqq 1$), we have the following exact sequence by Theorem 2.2:

$$\pi_1(\mathrm{aut}\, S^q) \xrightarrow{\partial_k} \pi_n(\mathrm{aut}\, S^q) \xrightarrow{G} \mathscr{L}(E_k) \xrightarrow{J_0} \mathscr{F}_k(S^q) \longrightarrow 1. \tag{3.2}$$

In the following sections, we shall investigate this sequence in details.

In this section, we study the group $\pi_i(\mathrm{aut}\, S^q)$ for $i \geqq 1$. Note that

$$\pi_0(\mathrm{aut}\, S^q) = \mathscr{F}(S^q) = Z_2 = \{\pm 1\}. \tag{3.3}$$

Consider the evaluation fibration

$$\omega\colon L(S^q, S^q) \longrightarrow S^q \quad \text{with fibre} \quad L_0(S^q, S^q). \tag{3.4}$$

Then the path component of $L(S^q, S^q)$ (resp. $L_0(S^q, S^q)$) of the identity map 1 is contained in $\mathrm{aut}\, S^q$ (resp. $\mathrm{aut}_0\, S^q=(\mathrm{aut}\, S^q) \cap L_0(S^q, S^q)$). Therefore the isomorphism τ in Theorem 1.3 is the isomorphism

$$\tau\colon \pi_{i+q}(S^q) \cong \pi_i(L_0(S^q, S^q), 1) = \pi_i(\mathrm{aut}_0\, S^q, 1), \tag{3.5}$$

and the exact sequence (1.2)′ is the exact sequence

$$\cdots \longrightarrow \pi_{i+1}(S^q) \xrightarrow{[\iota_q,\ ]} \pi_{i+q}(S^q) \xrightarrow{i_*\tau} \pi_i(\mathrm{aut}\, S^q) \xrightarrow{\omega_*} \pi_i(S^q) \longrightarrow \cdots \tag{3.6}$$

(ι_q is the homotopy class of $1: S^q \to S^q$ and i: $\mathrm{aut}_0 S^q \subset \mathrm{aut}\, S^q$).

By the exact sequence (3.6), we see immediately that

$$\pi_1(\mathrm{aut}\, S^q) = Z_2 = \{i_*\tau(\eta_q)\} \quad \textit{for} \quad q \geqq 2 \qquad (\text{cf. } [17]), \tag{3.7}$$

$$\pi_i(\mathrm{aut}\, S^q) = \begin{cases} i_*\tau(\pi_{i+q}(S^q)) & \textit{for} \quad i \leqq q-2, \\ i_*\tau(\pi_{2q-1}(S^q)/\{[\iota_q, \iota_q]\}) & \textit{for} \quad i = q-1, \end{cases} \tag{3.8}$$

where η_q is the generator of $\pi_{q+1}(S^q)$ $(q \geqq 2)$; in particular, we have used the relation $[\iota_2, \iota_2] = \pm 2\eta_2$.

Furthermore,

(3.9) *when* $q=1$, 3 or 7, *the fibering* (3.4) *has a cross-section*

$$t: S^q \longrightarrow \mathrm{aut}\, S^q \subset L(S^q, S^q), \quad t(x)(y) = xy \quad \textit{for} \quad x, y \in S^q,$$

given by the canonical multiplication xy *on* S^q. *Thus* (3.6) *is split and*

$$\pi_i(\mathrm{aut}\, S^q) = i_*\tau(\pi_{i+q}(S^q)) + t_*(\pi_i(S^q))(i \geqq 1) \quad \textit{for} \quad q = 1, 3, 7. \tag{3.10}$$

LEMMA 3.11. $\pi_q(\mathrm{aut}\, S^q) = \begin{cases} i_*\tau(\pi_{2q}(S^q)/\{[\iota_q, \eta_q]\}) + Z & \textit{for odd} \quad q, \\ i_*\tau(\pi_{2q}(S^q)/\{[\iota_q, \eta_q]\}) & \textit{for even} \quad q, \end{cases}$

where the second summand Z *for odd* q *is generated by* t *if* $q=1$, 3, 7 *and by the element* α *with* $\omega_*(\alpha) = 2\iota_q$ *otherwise.*

PROOF. By the exact sequence (3.6) for $i=q$, we have only to show that the kernel of $[\iota_q, \ ]: \pi_q(S^q) \to \pi_{2q-1}(S^q)$ is 0 for even q, Z generated by ι_q for $q=1$, 3, 7, and Z generated by $2\iota_q$ for odd $q \neq 1$, 3, 7. This fact is well-known by the EPH-sequence (cf. [31]). *q.e.d.*

§4. ∂_k in (3.2)

In this section, we shall study the homomorphism

$$\partial_k: \pi_1(\mathrm{aut}\, S^q) \longrightarrow \pi_n(\mathrm{aut}\, S^q) \quad (k \in \pi_{n-1}(\mathrm{aut}\, S^q)) \tag{4.1}$$

in (3.2), which is given by the Samelson product as follows:

$$\partial_k(x) = \langle k, x \rangle \quad \textit{for} \quad x \in \pi_1(\mathrm{aut}\, S^q) \quad (\text{see Theorem 2.2}). \tag{4.1$'$}$$

We first consider the case $n \geqq 2$ and $q \geqq 2$. B. Steer [27] represented the Samelson product on $\mathrm{aut}_0 S^q$ in terms of the Whitehead products on S^q and the Hilton-Hopf invariants by the following

THEOREM 4.2 (B. Steer [27, Th. 5.76]). *Under the isomorphism*

$$\tau\colon \pi_{i+q}(S^q) \cong \pi_i(\mathrm{aut}_0 S^q)(i \geqq 1) \quad of \quad (3.5),$$

the Samelson product $\langle\tau(\alpha), \tau(\beta)\rangle$ *for* $\alpha\in\pi_{i+q}(S^q)$ *and* $\beta\in\pi_{j+q}(S^q)$ $(i, j\geqq 1)$ *is given by*

$$\begin{aligned}\langle\tau(\alpha), \tau(\beta)\rangle = (-1)^q\tau\{&\beta\,\Sigma^j\alpha-(-1)^{ij}\alpha\,\Sigma^i\beta+(-1)^{j(q-1)}[\beta, \iota_q]\,\Sigma^j h_2(\alpha)\\ &-(-1)^{i(j+q-1)}[\alpha, \iota_q]\,\Sigma^i h_2(\beta)+[[\beta, \iota_q], \iota_q]\,\Sigma^j h_{\sigma_2}(\alpha)\\ &-(-1)^{ij}[[\alpha, \iota_q], \iota_q]\,\Sigma^i h_{\sigma_2}(\beta)\},\end{aligned}$$

where h_2 *and* h_{σ_2} *are generalizations of the Hopf-invariant due to Hilton.*

Consider the inclusion map $i\colon \mathrm{aut}_0 S^q\subset \mathrm{aut}\, S^q$ and the homomorphism

$$i_*\tau\colon \pi_{i+q}(S^q) \cong \pi_i(\mathrm{aut}_0 S^q) \xrightarrow{i_*} \pi_i(\mathrm{aut}\, S^q)(i \geqq 1)$$

in (3.6). Then we have the following

THEOREM 4.3. *Assume that* $q\geqq 2$, $n\geqq 2$ *and* $k\in\mathrm{Im}\, i_*$, *i.e.,* $k=i_*\tau(k')$ *for some* $k'\in\pi_{n+q-1}(S^q)$. *(This assumption is valid for any* k *if* $2\leqq n\leqq q$ *by (3.6).) Then the homomorphism* $\partial_k\colon \pi_1(\mathrm{aut}\, S^q)(=i_*\tau(\pi_{q+1}(S^q)))\to\pi_n(\mathrm{aut}\, S^q)$ *in* (4.1) *(see (3.7)) is given by*

$$\partial_k(i_*(\eta_q)) = \langle k, i_*\tau(\eta_q)\rangle = i_*\tau\{k'\eta_{n+q-1}+(-1)^q\eta_q\Sigma k'+[\eta_q, \iota_q]\Sigma h_2(k')\}.$$

PROOF. We first note that the Samelson product is natural under the H-map $i\colon \mathrm{aut}_0 S^q\subset \mathrm{aut}\, S^q$. Thus by the assumption $k=i_*\tau(k')$,

$$\partial_k(i_*\tau(\eta_q)) = \langle k, i_*\tau(\eta_q)\rangle = i_*\langle\tau(k'), \tau(\eta_q)\rangle.$$

Consider the equality in Theorem 4.2 for $\alpha=k'$ $(i=n-1)$ and $\beta=\eta_q (j=1)$:

$$(*)\qquad \begin{aligned}\langle\tau(k'), \tau(\eta_q)\rangle = (-1)^q\tau\{&\eta_q\Sigma k'+(-1)^n k\eta_{n+q-1}+(-1)^{q-1}[\eta_q, \iota_q]\Sigma h_2(k')\\ &-(-1)^{(n-1)}[k', \iota_q]\Sigma^{n-1}h_2(\eta_q)+[[\eta_q, \iota_q], \iota_q]\Sigma h_{\sigma_2}(k')\\ &-(-1)^{n-1}[[k', \iota_q], \iota_q]\Sigma^{n-1}h_{\sigma_2}(\eta_q)\}.\end{aligned}$$

The element h_{σ_2} lies in $\pi_{q+1}(S^{3q-2})=0$. $3[[\eta_q, \iota_q], \iota_q]=0$ by [7, Th. 6.10]. Thus $[[\eta_q, \iota_q], \iota_q]=0$, because $2\eta_q=0$ for $q\geqq 3$ and $[\eta_2, \iota_2]=0$ ([8, Cor. 2]). For $q\geqq 3$, $\eta_q=\Sigma\eta_{q-1}$ and $h_2(\eta_q)=0$ (cf. [31, p. 22]). For $q=2$, $[k', \iota_2]=0$, since all the Whitehead products vanish except $[\iota_2, \iota_2]$ in $\pi_*(S^2)$ by [8, Cor. 2]. These show that the last three terms in $(*)$ vanish. Since $2\eta_{n+q-1}=0$ and $2[\eta_q, \iota_q]=0$, we can drop the signs of the coefficiens of the second and the third terms in $(*)$. Thus we have the desired equality. *q.e.d.*

COROLLARY 4.4. *In addition to the assumption* $k=i_*\tau(k')$ *in Theorem* 4.3, *assume that* $q\equiv 3 \bmod 4$ *or* $q=2, 6$, *or* $k'=\Sigma k_1$ *for some* $k_1\in\pi_{n+q-2}(S^{q-1})$. *Then*

$$\partial_k(i_*\tau(\eta_q)) = i_*\tau(k'\eta_{n+q-1} + (-1)^q\eta_q\Sigma k').$$

Furthermore, when $q=3$ *or* 7, $\partial_k(i_*\tau(\eta_q))=0$ *if and only if* $k'\eta_{n+q-1}=\eta_q\Sigma k'$.

PROOF. By [9, Lemma 5.1] and [8, Cor. 2], $[\eta_q, \iota_q]=0$ $(q\geqq 2)$ if and only if $q\equiv 3 \bmod 4$ or $q=2, 6$. On the other hand, if $k'=\Sigma k_1$, then $h_2(k')=h_2(\Sigma k_1)=0$ (cf. [31, p. 22]). When $q=3$ or 7, $i_*\tau$ in (3.10) is monic. Thus the above theorem implies the corollary. *q.e.d.*

Now we consider the case $q=3$ or 7. Then by (3.10),

$$\pi_{n-1}(\mathrm{aut}\, S^q) = i_*\tau(\pi_{n+q-1}(S^q)) + t_*(\pi_{n-1}(S^q)) \quad \text{(direct sum)},$$

where $t\colon S^q\to\mathrm{aut}\, S^q$ is the cross-section given in (3.9). Thus any element $k\in\pi_{n-1}(\mathrm{aut}\, S^q)$ can be represented as

$$k = i_*\tau(k') + t_*(k'') \qquad (k'\in\pi_{n+q-1}(S^q),\ k''\in\pi_{n-1}(S^q)). \tag{4.5}$$

Therefore by the well-known formula $\langle k, \eta\rangle=\langle i_*\tau(k'), \eta\rangle+\langle t_*(k''), \eta\rangle$ and by Theorem 4.3, we can compute $\partial_k(\eta)$ for any k if we know

$$\partial_{t_*(k'')}(\eta) = \langle t_*(k''), \eta\rangle \qquad (\eta = i_*\tau(\eta_q)).$$

We can compute it by the following two theorems.

THEOREM 4.6. *Assume that* $q=3$ *and* $n\geqq 2$. *For any* $k\in\pi_{n-1}(\mathrm{aut}\, S^q)$, *let* $k'\in\pi_{n+2}(S^3)$ *and* $k''\in\pi_{n-1}(S^3)$ *be elements in* (4.5). *Then*

$$\partial_k(i_*\tau(\eta_3)) = i_*\tau(k'\eta_{n+2}-\eta_3\Sigma k') + t_*(\eta_3\Sigma k'').$$

PROOF. By Corollary 4.4, it is sufficient to prove

$$\partial_k(i_*\tau(\eta_3)) = \langle k, i_*\tau(\eta_3)\rangle = t_*(\eta_3\Sigma k'') \quad \text{if} \quad k = t_*(k'').$$

By the definition of the Samelson product,

$$\langle t_*(k''), \eta\rangle = \langle t, \eta\rangle\Sigma k'' \quad (\eta = i_*\tau(\eta_3)).$$

Consider the natural inclusion $j\colon SO(4)\to\mathrm{aut}\, S^3$. Then $t=js$ by the definition of t in (3.9) where $s\colon S^3\to SO(4)$ is a cross-section. Consider the commutative diagram

$$\begin{array}{ccccc} \pi_i(SO(q-1)) & \xrightarrow{i_*} & \pi_i(SO(q)) & \xrightarrow{i_*} & \pi_i(SO(q+1)) \\ \downarrow{\scriptstyle -J} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle j_*} \\ \pi_{i+q-1}(S^{q-1}) & \xrightarrow{\Sigma} & \pi_{i+q}(S^q) & \xrightarrow{i_*\tau} & \pi_i(\mathrm{aut}\, S^q) \end{array} \tag{4.7}$$

for $i=1$ and $q=3$ (cf. [36, (9.1)]), where J is the J-homomorphism and the upper i is the inclusion $SO(l)\subset SO(l+1)$. Then $\eta_2=J(j_2)$ for a generator j_2 of $\pi_1(SO(2))=Z$, and $\eta=i_*\tau(\eta_3)=j_*(j_4)\,(j_4=i_*^2(j_2))$. Hence by the naturality of the Samelson product, we have

$$\langle t, \eta\rangle = \langle js, j_*(j_4)\rangle = j_*\langle s, j_4\rangle .$$

Now consider the natural inclusion $j'\colon SU(2)\subset SO(4)$. Then $s=j's\colon S^3\to SU(2)\subset SO(4)$ and $j_4=j'_*i'_*(j_2)$ where $i'\colon SO(2)=U(1)\subset SU(2)$, and therefore $\langle s, j_4\rangle =j'_*\langle s, i'_*(j_2)\rangle$. On the other hand, by using a theorem of R. Bott, we can show that

$$\langle s, i'_*(j_2)\rangle = s\eta_3 \quad \text{in} \quad \pi_4(SU(2)) \text{ (cf. [13, p. 167], [14, (19.1)])}.$$

Thus $\langle t, \eta\rangle =j_*j'_*(s\eta_3)=t_*(\eta_3)$ and $\langle t_*(k''), \eta\rangle =t_*(\eta_3\Sigma k'')$ as desired. *q.e.d.*

For the elements of the homotopy groups of spheres, we use the notations given in [31]. We note that $\pi_{15}(S^7)=Z_2+Z_2+Z_2=\{\sigma'\eta_{14}\}+\{\bar\nu_7\}+\{\varepsilon_7\}$.

THEOREM 4.8 (S. Oka). *Assume that $q=7$ and $n\geqq 2$. For any $k\in \pi_{n-1}(\mathrm{aut}\, S^7)$, let $k'\in\pi_{n+6}(S^7)$ and $k''\in\pi_{n-1}(S^7)$ be elements in* (4.5). *Then*

$$\partial_k(i_*\tau(\eta_7)) = i_*\tau\{k'\eta_{n+6}-\eta_7\Sigma k'+(\bar\nu_7+\varepsilon_7)\Sigma^8k''\} + t_*(\eta_7\Sigma k'').$$

To prove this theorem, we need some lemmas. We have to compute the Samelson product of a generator of $\pi_1(SO(8))=Z_2$ and the element $s\in\pi_7(SO(8))$ represented by the cross-section

$$s\colon S^7 \longrightarrow SO(8), \quad s(x)(y) = xy \ (x, y\in S^7),$$

as in the proof of Theorem 4.6 for $q=3$.

Let $\gamma\colon S^8\to SO(6)$ be a map such that $p_*(\gamma)(p\colon SO(6)\to S^5$ is the projection) is a generator of $\pi_8(S^5)=Z_{24}=\{\nu_5\}$ (see [28], [29]). We note that

$$\begin{aligned} &{}_2\pi_{14}(S^6) = Z_2+Z_8 = \{\varepsilon_6\}+\{\bar\nu_6\}, \quad {}_2\pi_{14}(S^{11}) = Z_8 = \{\nu_{12}\},\\ &{}_2\pi_{14}(S^7) = Z_8 = \{\sigma'\}, \quad \pi_{14}(S^5) = Z_2 = \{\varepsilon_5\},\\ &\pi_{18}(S^{10}) = Z_2+Z_2 = \{\bar\nu_{10}\}+\{\varepsilon_{10}\},\end{aligned}$$

where ${}_2\pi_i(S^n)$ denotes the 2-primary component of $\pi_i(S^n)$.

LEMMA 4.9. *Let $J\colon \pi_i(SO(n))\to\pi_{i+n}(S^n)$ be the J-homomorphism. Then*

$$J(s) = \sigma_8 \ (\in\pi_{15}(S^8), \textit{ the Hopf map}), \qquad J(\gamma) \equiv \bar\nu_6+\varepsilon_6 \bmod \{2\bar\nu_6\},$$

where the second equality is the one in the 2-primary component.

PROOF. The first equality is shown by definition. Consider the commutative diagram

$$\begin{array}{ccccc} & & \pi_8(SO(6)) & \xrightarrow[\cong]{p_*} & \pi_8(S^5) \\ & & \downarrow J & & \downarrow \Sigma^6 \\ \pi_{13}(S^5) & \xrightarrow{\Sigma} & \pi_{14}(S^6) & \xrightarrow{H} & \pi_{14}(S^{11}), \end{array}$$

where the lower sequence is exact (cf. [30, Cor. 3.6]). Then $HJ(\gamma)=\Sigma^6 p_*(\gamma)\equiv \Sigma^6\nu_5\equiv H(\bar\nu_6) \bmod \{2\nu_{11}\}$ in the 2-primary component ([31, p. 53]). Therefore

$$(*) \qquad J(\gamma)\equiv \bar\nu_6+\varepsilon_6 \quad \text{or} \quad \bar\nu_6 \quad \bmod\{2\bar\nu_6\} \quad \text{in the 2-primary component.}$$

On the other hand, consider the commutative diagram

$$\begin{array}{ccccc} \pi_7(SO(8)) & \xrightarrow{i_*^2} & \pi_7(SO(10)) & \xrightarrow{\eta_7^*} & \pi_8(SO(10)) = Z_2 = \{i_*^4\gamma\} \ ([28]) \\ \downarrow J & & \downarrow J & & \downarrow J \\ \pi_{15}(S^8) & \xrightarrow{\Sigma^2} & \pi_{17}(S^{10}) & \xrightarrow{\eta_{17}^*} & \pi_{18}(S^{10}) = Z_2+Z_2 = \{\bar\nu_{10}\}+\{\varepsilon_{10}\}, \end{array}$$

where $i\colon SO(l)\subset SO(l+1)$. Then $\Sigma^2 J(s)=\Sigma^2\sigma_8=\sigma_{10}$ and $\sigma_{10}\eta_{17}=\bar\nu_{10}+\varepsilon_{10}$ ([31, p. 54]). Thus

$$(**) \qquad \bar\nu_{10}+\varepsilon_{10}\in \operatorname{Im} J \quad (J\colon \pi_8(SO(10)) \longrightarrow \pi_{18}(S^{10})).$$

If $J(\gamma)\equiv\bar\nu_6$ in $(*)$, then $J(i_*^4\gamma)=\Sigma^4 J(\gamma)=\Sigma^4\bar\nu_6=\bar\nu_{10}$, because $2\bar\nu_{10}=0$ and $\pi_{18}(S^{10})$ has no odd torsion. This equlity and $(**)$ imply that the right J in the above diagram is surjective, which is a contradiction. Thus we see the second equality in the lemma by $(*)$. *q.e.d.*

Let j_2 be the generator of $\pi_1(SO(2))=Z$. Then

$\pi_1(SO(r))=Z_2=\{j_r\}$ for $r\geqq 3$, where $j_r=i_*^{r-2}(j_2)$ $(i\colon SO(l)\subset SO(l+1))$.

LEMMA 4.10. $\langle s, j_8\rangle = i_*^2\gamma + s\eta_7 \qquad (j_8 = i_*^6(j_2))$.

PROOF. Let $V_{8,2}=SO(8)/SO(6)$ be the real Stiefel manifold, and consider the commutative diagram

$$\begin{array}{ccccc} SO(7) & \xrightarrow[\subset]{i} & SO(8) & \underset{s}{\overset{p}{\rightleftarrows}} & S^7 \\ \downarrow & & \downarrow p_1 & & \| \\ S^6 & \xrightarrow[\subset]{i'} & V_{8,2} & \underset{v}{\overset{p'}{\rightleftarrows}} & S^7, \end{array}$$

where p, p_1 and p' are the projections, s is a cross-section of p and v is a cross-section of p' induced by the natural inclusion $SU(4)\subset SO(8)$. Then by the split exact sequence $\pi_7(S^6)\xrightarrow{i'_*}\pi_7(V_{8,2})\underset{v_*}{\overset{p'_*}{\rightleftarrows}}\pi_7(S^7)$, we have

$$p_1 s = v+\varepsilon i'\eta_6 \qquad \text{for some} \quad \varepsilon\in Z_2.$$

Therefore by using the relative Samelson product, we see that

$$\begin{aligned} p_{1*}\langle s, j_8\rangle &= p_{1*}\langle s, i^2_*(j_6)\rangle = \langle p_1 s, j_6\rangle \quad \text{(by [14, p. 98])} \\ &= \langle v, j_6\rangle + \varepsilon\langle i'\eta_6, j_6\rangle = \langle v, j_6\rangle + \varepsilon\langle i', j_6\rangle\eta_7 \quad \text{(by [14, (15.11)])} \\ &= v\eta_7 + \varepsilon i'\eta_6\eta_7 \quad \text{(by [14, (16.11)])} \quad = p_{1*}(s\eta_7). \end{aligned}$$

Thus by the exact sequence $\pi_8(SO(6)) \xrightarrow{i^2_*} \pi_8(SO)) \xrightarrow{p_{1*}} \pi_8(V_{8,2})$ and by noticing $\pi_8(SO(6)=Z_{24}=\{\gamma\}$ and $\operatorname{Im} i^2_*=Z_2=\{i^2_*\gamma\}$, we have

$$(*) \qquad \langle s, j_8\rangle = s\eta_7 + x i^2_*(\gamma) \qquad \text{for some} \quad x\in Z_2.$$

Now the image $i^2_*\langle s, j_8\rangle=i^2_*\langle s, i^6_*(j_2)\rangle$ in $\pi_8(SO(10))$ is 0 by [14, p. 123], that is, $i^2_*(s\eta_7)+xi^4_*(\gamma)=0$. Hence, by Lemma 4.9, we have $0=J(i^2_*(s\eta_7)+xi^4_*(\gamma))=\Sigma^2J(s)\eta_{17}+x\Sigma^4J(\gamma)=\sigma_{10}\eta_{17}+x(\bar{v}_{10}+\varepsilon_{10})$. But $\sigma_{10}\eta_{17}=\bar{v}_{10}+\varepsilon_{10}\neq 0$ by [31, p. 54]. Thus $x=1$, and we have the desired result by $(*)$. *q.e.d.*

Now we are ready to prove Theorem 4.8.

PROOF OF THEOREM 4.8. By Corollary 4.4, it is sufficient to prove

$$\langle k, i_*\tau(\eta_7)\rangle = i_*\tau\{(\bar{v}_7+\varepsilon_7)\Sigma^8 k''\}+t_*(\eta_7\Sigma k'') \quad \text{if} \quad k = t_*(k'').$$

In the same way as the proof of Theorem 4.6, we have

$$\langle t_*(k''), i_*\tau(\eta_7)\rangle = \langle t, i_*\tau(\eta_7)\rangle\Sigma k'', \quad i_*\tau(\eta_7) = j_*(j_8) \quad \text{and} \quad t = js,$$

where $j\colon SO(8)\subset \operatorname{aut} S^7$; and hence by Lemma 4.10,

$$\langle t, i_*\tau(\eta_7)\rangle = j_*\langle s, j_8\rangle = j_*(i^2_*(\gamma))+t_*(\eta_7).$$

By the commutative diagram (4.7) for $i=8$ and $q=7$ and by Lemma 4.8, $j_*(i^2_*\gamma)=-i_*\tau\Sigma J(\gamma)=i_*\tau(\bar{v}_7+\varepsilon_7)$, since $2\pi_{15}(S^7)=0$. Thus $\langle k, i_*\tau(\eta_7)\rangle=\{i_*\tau(\eta_7+\varepsilon_7)+t_*(\eta_7)\}\Sigma k''=i_*\tau\{(\bar{v}_7+\varepsilon_7)\Sigma^8k''\}+t_*(\eta_7\Sigma k'')$ as desired. *q.e.d.*

The following theorem is the results in the case $n=1$ or $q=1$.

THEOREM 4.11. (i) *If $n=q=1$, then $k\in\pi_0(\operatorname{aut} S^1)=Z_2=\{\pm 1\}$, and*

$$\partial_k(x) = \begin{cases} -2x & \text{if} \quad k = -1, \\ 0 & \text{if} \quad k = +1, \end{cases} \qquad \text{for} \quad x\in\pi_1(\operatorname{aut} S^1) = Z.$$

Thus Coker $\partial_k=Z_2$ *if* $k=-1$, $=Z$ *if* $k=+1$.

(ii) *If $n=1$ and $q\geqq 2$, then $\partial_k=0$ and* Coker $\partial_k=\pi_1(\operatorname{aut} S^q)=Z_2$.

(iii) *If $n\geqq 2$ and $q=1$, then $\pi_n(\operatorname{aut} S^1) = 0$ and $\partial_k=0$.*

PROOF. (iii) is seen by (3.10). Consider the case $n=1$. Then $k\in\pi_0(\operatorname{aut} S^q)$

$=\mathscr{F}(S^q)=Z_2=\{\pm 1\}$. By (1.13), $\partial_k(x)=k\cdot x-x$, where $x\in\pi_1(\mathrm{aut}\,S^q)=Z$ or Z_2 according to $q=1$ or $q\geqq 2$ by (3.10) or (3.7) respectively. Since the action $k\cdot$ is an isomorphism, $k\cdot x=x$ and $\partial_k=0$ for $q\geqq 2$. We shall prove in Theorem 5.4(i) that $(-1)\cdot x=-x$ for $q=+1$. This shows the result for $n=q=1$. *q.e.d.*

In conclusion of this section, we notice the following theorem on sphere bundles over spheres. It is well-known that

(4.12) *a fibration (E_k, p, S^n, S^q) with characteristic map $k\in\pi_{n-1}(\mathrm{aut}\,S^q)$ of* (3.1) *is fibre homotopy equivalent to an $SO(q+1)$-bundle if and only if*

$$k\in\mathrm{Im}\,j_* \quad (j\colon SO(q+1)\subset \mathrm{aut}\,S^q).$$

THEOREM 4.13. *For any sphere bundle (E_k, p, S^n, S^q) with $k\in\mathrm{Im}\,j_*$ and $q\geqq 2$, we have*

$$\omega_*\{\partial_k(i_*\tau(\eta_q))\} = \eta_q\Sigma\omega_*(k) + [\eta_q, \iota_q]\Sigma H(\omega_*(k)),$$

where $\omega\colon \mathrm{aut}\,S^q\to S^q$ is the restriction of the evaluation map ω in (3.4) *and H is the generalized Hopf invariant.*

PROOF. We have the equality by taking $\gamma=\omega_*(k)$ in [14, (16.8)]. *q.e.d.*

§5. $\mathscr{F}_k(S^q)$ in (3.2)

In this section, se shall study the group

$$\mathscr{F}_k(S^q)(=\mathrm{Im}\,J_0) = \{\alpha\in\mathscr{F}(S^q)\,|\,\alpha\cdot k = k\} \quad (k\in\pi_{n-1}(\mathrm{aut}\,S^q))$$

in (3.2), where $\cdot$ is the action by conjugation of (1.12) (see Theorem 2.2).

By noticing $\mathscr{F}(S^q)=Z_2=\{\pm 1\}$, we have immediately

$$\text{(5.1)} \qquad \mathscr{F}_k(S^q) = \begin{cases} Z_2 & \text{if } (-1)\cdot k = k, \\ 1 & \text{otherwise.} \end{cases}$$

We note that $-1\in\mathscr{F}(S^q)$ is represented by a map of degree -1. Then by the definition of the action $\cdot$ given in (1.12), we see the following

(5.2) *For the adjoint map $\bar{k}\colon S^q\times S^{n-1}\to S^q$ of a representative of $k\in\pi_{n-1}(\mathrm{aut}\,S^q)$, $f\bar{k}(f^{-1}\times 1)\colon S^q\times S^{n-1}\xrightarrow{f^{-1}\times 1} S^q\times S^{n-1}\xrightarrow{\bar{k}} S^q\xrightarrow{f} S^q$ (f is a map of degree -1) is the adjoint map of a representative of $(-1)\cdot k=k\in\pi_{n-1}(\mathrm{aut}\,S^q)$.*

For $k\in\pi_0(\mathrm{aut}\,S^q)=\mathscr{F}(S^q)$, this shows that $(-1)\cdot k=k$. Therefore

(5.3) *if $n=1$, then $\mathscr{F}_k(S^q)=Z_2$ for any $k\in\pi_0(\mathrm{aut}\,S^q)$.*

Now, we assume $n\geqq 2$ and consider the case that S^q is an H-space.

THEOREM 5.4. *Assume that* $n \geqq 2$ *and* $q=1, 3$ *or* 7, *and take any element*

$$k = i_*\tau(k') + t_*(k'') \in \pi_{n-1}(\mathrm{aut}\, S^q) \ (k' \in \pi_{n+q-1}(S^q),\ k'' \in \pi_{n-1}(S^q))$$

(*see the direct sum decomposition of* (3.10)). *Then we have the following* (i) *and* (ii).

(i) $$(-1)\cdot k = i_*\tau(k' + (-1)^n \langle \iota_q, \iota_q \rangle \Sigma^q k'') - t_*(k''),$$

where $\langle \iota_1, \iota_1 \rangle = 0$, *and* $\langle \iota_3, \iota_3 \rangle = \omega$ *and* $\langle \iota_7, \iota_7 \rangle = \lambda_7$ *are the generators of* $\pi_6(S^3) = Z_{12}$ *and* $\pi_{14}(S^7) = Z_{120}$, *respectively* ([12, p. 175]).

(ii) $$\mathscr{F}_k(S^q) = \begin{cases} Z_2 & \text{if } \langle \iota_q, \iota_q \rangle \Sigma^q k'' = 0 = 2k'', \\ 1 & \text{otherwise.} \end{cases}$$

PROOF. (ii) is an immediate consequence of (i) and (5.1). We shall prove (i). Since the action $\cdot$ is linear, it is sufficient to prove (i) for the case $k'=0$ or $k''=0$.

(a) The case $k'=0$: Consider $f\bar{k}(f^{-1} \times 1)$ in (5.2). Then the equality $k = t_*(k'')$ and (3.9) imply that $\bar{k}$ is given by $\bar{k}(x, y) = (k''(y)) \cdot x$ (the canonical multiplication on S^q) for $x \in S^q$ and $y \in S^{n-1}$. We can take f to be an inversion map with respect to the multiplication on S^q. Thus

$$f\bar{k}(f^{-1} \times 1)(x, y) = x(k''(y))^{-1} = (k''(y))^{-1} x [x^{-1}, k''(y)] \ (x \in S^q,\ y \in S^{n-1}),$$

where [,] denotes the commutator in S^q.

If $q=1$, then the commutator vanishes, and $f\bar{k}(f^{-1} \times 1)$ maps (x, y) to $(k''(y))^{-1} x$. Thus $f\bar{k}(f^{-1} \times 1)$ is the adjoint map of a representative of $t_*(-k'') = -t_*(k'')$, because the group structures of $\pi_*(S^q)$ coincides with the one given by the multiplication on S^q. Therefore $(-1) \cdot t_*(k'') = -t_*(k'')$ by (5.2).

If $q=3$, then we have to add one more term represented by the map

$$g\colon S^{n-1} \to \mathrm{aut}\, S^3,\ g(y)(x) = [x^{-1}, k''(y)] = \omega(f \wedge 1)(1 \wedge k'')\pi(x, y) \ (x \in S^3,\ y \in S^{n-1}),$$

where $\pi\colon S^3 \times S^{n-1} \to S^3 \wedge S^{n-1}$ is the projection and $\omega = \langle \iota_3, \iota_3 \rangle$, by the definition of the Samelson product. Since $\omega(f \wedge 1) = -\omega$ and $1 \wedge k'' = (-1)^{n-1} \Sigma^3 k''$, g represents $i_*\tau((-1)^n \omega \Sigma^3 k'')$ and hence $(-1) \cdot t_*(k'') = (-1)^n i_*\tau(\omega \Sigma^3 k'') - t_*(k'')$.

When $q=7$, we need the associator as well as the commutator. But any subalgebra of the Cayley algebra generated by two elements is associative (cf. [26, p. 108]). Therefore we may consider in the same way as in the case $q=3$, and we obtain the desired equality.

(b) The case $k''=0$: Then $(-1) \cdot i_*\tau(k')$ is $i_*\tau(fk'(f^{-1} \wedge 1))$ by (5.2). Since the degree of $f^{-1} \wedge 1$ is -1 and $f_*(x) = -x$ for $x \in \pi_i(S^q)$, $fk'(f^{-1} \wedge 1)$ is homotopic to k'. Therefore $(-1) \cdot i_*\tau(k') = i_*\tau(k')$. *q.e.d.*

The following theorem holds under the assumption of Theorem 4.3:

THEOREM 5.5. *Assume that* $n\geqq 2$ *and* $k=i_*\tau(k')$ *for some* k', *where* $i_*\tau$: $\pi_{n+q-1}(S^q)\to\pi_{n-1}(\mathrm{aut}\, S^q)$ *is the homomorphism in* (3.6); *for instance, assume* $2\leqq n\leqq q$. *Then*

(i) $(-1)\cdot k=k-i_*\tau([\iota_q, \iota_q]H(k'))$ (*H is the generalized Hopf invariant*). *Especially, if* $2\leqq n\leqq q$, *then* $(-1)\cdot k=k$.

(ii) $$\mathscr{F}_k(S^q)=\begin{cases} Z_2 & \text{if } [\iota_q, \iota_q]H(k')=0, \text{ especially if } 2\leqq n\leqq q, \\ 1 & \text{otherwise.}\end{cases}$$

PROOF. It is sufficient to prove (i) by (5.1). Since $k=i_*\tau(k')$, (5.2) shows that

$$(-1)\cdot k = i_*\tau\{(-\iota_q)k'((-\iota_q)\wedge 1)\} = -i_*\tau((-\iota_q)k').$$

Now $(-\iota_q)k'=-k'+[\iota_q, \iota_q]H(k')$ by [7, Th. 6.7 and Th. 6.9]. Thus we have the desired equality. If $n<q$, then $H(k')=0$ by definition. If $n=q$, then $H(k')\in \pi_{2q-1}(S^{2q-1})$ and $i_*\tau([\iota_q, \iota_q])=0$ by (3.6). Thus $(-1)\cdot k=k$ if $n\leqq q$. *q.e.d.*

§6. The case that J_0 in (3.2) is split

In this section, we shall study some cases that the epimorphism

$$(6.1)\qquad J_0\colon \mathscr{L}(E_k)\longrightarrow \mathscr{F}_k(S^q) \quad \text{in} \quad (3.2),$$

obtained by the restriction to the fibre S^q (see Theorem 2.2), is split. But in these cases, we can not determine the group extension in the short exact sequence $0\to \mathrm{Coker}\,\partial_k \xrightarrow{G} \mathscr{L}(E_k) \xrightarrow{J_0} \mathscr{F}_k(S^q)\to 1$ induced from (3.2) except for the trivial fibration (see Corollary 2.5), because the homomorphism G can not be given explicitly for us.

Let (E_k, p, S^n, S^q) $(k\in\pi_{n-1}(\mathrm{aut}\, S^q))$ be a Hurewicz fibration of (3.1). Then J. Stasheff [25, Prop. 1] (cf. [20]) proved that E_k *has the homotopy type of*

$$S^q \cup_{\bar{k}}(D^n\times S^q)\ (\bar{k}\colon S^{n-1}\times S\longrightarrow S^q \text{ is the adjoint map of } k).$$

More precisely, consider the map

$$(6.2)\qquad p'\colon S^q\cup_{\bar{k}}(D^n\times S^q)\longrightarrow S^q \text{ with } p'|S^q=* \text{ and } p'|D^n\times S^q=\chi p_1,$$

where $p_1\colon D^n\times S^q\to D^n$ is the projection and $\chi\colon D^n\to S^n$ is the map collapsing the boundary S^{n-1} of D^n to $*$. Then

(6.3) *there are suitable maps* α *and* β *in the diagram*

$$\begin{array}{ccl} E_k & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & E'_k = S^q \cup_{\bar{k}} (D^n \times S^q) \\ \downarrow p & & \downarrow p' \\ S^n & = & S^n \end{array}$$

with $p'\alpha = p$, $p\beta = p'$ and $\alpha|S^q = 1 = \beta|S^q$ on $S^q = p^{-1}(*) = p'^{-1}(*)$, and there are homotopies K_t: $E_k \to E_k$ and H_t: $E'_k \to E'_k$ with $K_0 = \beta\alpha$, $K_1 = 1$, $pK_t = p$ and $H_0 = \alpha\beta$, $H_1 = 1$, $p'H_t = p'$.

LEMMA 6.4. *Consider the diagram ($n \geqq 2$, $q \geqq 1$)*

$$\begin{array}{ccc} S^{n-1} \times S^q & \xrightarrow{1 \times f} & S^{n-1} \times S^q \\ \downarrow \bar{k} & & \downarrow \bar{k} \\ S^q & \xrightarrow{f} & S^q, \end{array} \tag{6.5}$$

where f is a map of degree -1 and $\bar{k}$ is the adjoint map of k. Then $\mathscr{F}_k(S^q)$ in (6.1) is Z_2 if and only if (6.5) is homotopy commutative. Moreover, if we can take k and f so that (6.5) is strictly commutative, then J_0: $\mathscr{L}(E_k) \to \mathscr{F}_k(S^q)$ of (6.1) is a split epimorphism.

PROOF. The first half is proved by (5.1) and (5.2). We prove the second half. Assume that (6.5) is strictly commutative. Then we can define a map

$$T\colon E'_k (= S^q \cup_{\bar{k}} (D^n \times S^q)) \longrightarrow E'_k \quad \text{by} \quad T|D^n \times S^q = 1 \times f,\ T|S^q = f,$$

which satisfies $p'T = p'$ for p' in (6.2). Thus we have a fibre map

$$\gamma = \beta T \alpha\colon E_k \xrightarrow{\alpha} E'_k \xrightarrow{T} E'_k \xrightarrow{\beta} E_k$$

by using the maps α and β in (6.3). Then $\alpha|S^q = f$ and γ^2: $E_k \to E_k$ is fibre homotopic to the identity map. Thus we obtain a homomorphism.

$$s\colon \mathscr{F}_k(S^q) \longrightarrow \mathscr{L}(E_k), \quad \text{defined by} \quad s(-1) = \gamma,$$

which is a right inverse of J_0. *q.e.d.*

By the above lemma, we have the following

THEOREM 6.6. *$\mathscr{F}_k(S^q) = Z_2$ and J_0: $\mathscr{L}(E_k) \to \mathscr{F}_k(S^q)$ of (6.1) is a split epimorphism, if one of the following conditions (1)–(3) holds:*

(1) *$k \in \operatorname{Im} \Sigma_*$, where Σ_*: $\pi_{n-1}(\operatorname{aut} S^{q-1}) \to \pi_{n-1}(\operatorname{aut} S^q)$ is the induced homomorphism of the suspension map Σ: $\operatorname{aut} S^{q-1} \to \operatorname{aut} S^q$.*

(2) *(E_k, p, S^n, S^q) is fibre homotopy equivalent to an $SO(q+1)$-bundle, i.e., $k \in \operatorname{Im} j_*$, where j: $SO(q+1) \subset \operatorname{aut} S^q$ (cf. (4.12)), and in addition q is even or $n \leqq q$.*

(3) (E_k, p, S^n, S^q) *is fibre homotopy equivalent to an* $SO(q)$*-bundle, i.e.,* $k \in \mathrm{Im}\,(j_* i_*)$, *where* $i: SO(q) \subset SO(q+1)$, *which is equivalent to* $k \in \mathrm{Im}\,(i_* \tau J)$, *where* $\pi_{n-1}(SO(q)) \xrightarrow{J} \pi_{n+q-1}(S^q) \xrightarrow{i_*\tau} \pi_{n-1}(\mathrm{aut}\, S^q)$ *are the J-homomorphism and the homomorphism in* (3.6).

Proof. When $k \in \mathrm{Im}\, \Sigma_*$, (6.5) is strictly commutative by taking $f: S^q \to S^q$ with $f(x_0, \dots, x_{q-1}, x_q) = (x_0, \dots, x_{q-1}, -x_q)$. When $k \in \mathrm{Im}\, j_*$ and q is even, (6.5) is so by taking $k \in \pi_{n-1}(SO(q+1))$ and the antipodal map f with $f(x) = -x$. Thus we have the theorem for these cases by Lemma 6.4. If $n \leqq q$, then $\pi_{n-1}(S^q) = 0$ and $i_*: \pi_{n-1}(SO(q)) \to \pi_{n-1}(SO(q+1))$ is epic. Thus, if $k \in \mathrm{Im}\, j_*$ and $n \leqq q$, then $k \in \mathrm{Im}\,(j_* i_*)$ and (3) holds. Since $\mathrm{Im}\,(j_* i_*) \subset \mathrm{Im}\, \Sigma_*$, (3) implies (1). The equality $j_* i_* = i_* \tau J$ is seen by the right commutative square in (4.7). *q.e.d.*

For the J-homomorphism in (3) of the above theorem, we notice the following lemma which are used in Examples 7.12–17:

Lemma 6.7. *The J-homomorphism* $J: \pi_{n-1}(SO(q)) \to \pi_{n+q-1}(S^q)$ *is epic, if* $n = 4$, 5 *or* 7 *when* $q = 3$, *and if* $n = 8$ *when* $q = 7$.

Proof. The result for the case $n = q+1$ is shown in [12, p. 176] and it implies the results for the other cases. *q.e.d.*

§7. The group $\mathscr{L}(E_k)$ for spherical fibrations over spheres

In this section, we shall study the group $\mathscr{L}(E_k)$ of fibre homotopy equivalences of a Hurewicz fibration

(7.1) (E_k, p, S^n, S^q) with characteristic map $k \in \pi_{n-1}(\mathrm{aut}\, S^q) (n, q \geqq 1)$

of (3.1), by using the results obtained in the previous sections.

Consider the short exact sequence

$$(7.2) \qquad 0 \longrightarrow A \longrightarrow \mathscr{L}(E_k) \xrightarrow{J_0} \mathscr{F}_k(S^q) \longrightarrow 1$$

induced from the exact sequence (3.2), where

$$A \cong \mathrm{Coker}\, \partial_k = \pi_n(\mathrm{aut}\, S^q)/\partial_k(\pi_1(\mathrm{aut}\, S^q)).$$

Then the results in this section are stated by giving the groups A and $\mathscr{F}_k(S^q)$ and by indicating the case that (7.2) is split.

In the first place, we consider the special case that $n = 1$ or $q = 1$.

Example 7.3. $\pi_0(\mathrm{aut}\, S^q) = Z_2$, *and there are two types of* S^q*-fibrations over* S^1: *the trivial one* $(S^q \times S^1, p, S^1, S^q)$ *and the non-trivial one* (U_q, p, S^1, S^q), *where the generalized Klein bottle* U_q *is the quotient space of* $S^q \times I$ *obatined* *by identifying* $((x_0, x_1,\dots, x_q), 1)$ *with* $((-x_0, x_1,\dots, x_q), 0)$.

EXAMPLE 7.4. $\pi_{n-1}(\operatorname{aut} S^1)=t_*(\pi_{n-1}(S^1))$ *is* Z *if* $n=2$ *and* 0 *if* $n>2$ *(see* (3.10)), *and there are countable types of S^1-fibrations over S^2 besides the trivial one* $S^1\times S^2\to S^2$, *e.g., the Hopf bundle* $S^3\to S^2$ *or* $SO(3)\to S^2$, *and any S^1-fibration over* $S^n\,(n>2)$ *is fibre homotopy equivalent to the trivial one* $S^1\times S^n\to S^n$.

THEOREM 7.5. (i) *If* $n=1$ *and* $q\geqq 1$, *then*

$$\mathscr{L}(U_q) = Z_2+Z_2, \qquad \mathscr{L}(S^q\times S^1) = \begin{cases} D(Z) & \text{if } \ q=1 \text{ ([15])}, \\ Z_2+Z_2 & \text{if } \ q\geqq 2, \end{cases}$$

where $D(Z)$ *is the split extension* $Z\to D(Z)\to Z_2$ *with* Z_2 *acting on* Z *as inversion.*

(ii) *If* $n=2$ *and* $q=1$, *then* $\mathscr{L}(E_k)=0$ *for* $k\neq 0$ *and* $\mathscr{L}(S^1\times S^2)=Z_2$.

(iii) *If* $n\geqq 3$ *and* $q=1$, *then* $\mathscr{L}(S^1\times S^n)=Z_2$.

PROOF. (i) For the trivial fibration $S^q\times S^1$ over S^1, (7.1) is a split exact sequence

$$0\to A\to \mathscr{L}(S^q\times S^1)\to Z_2\to 1, \quad \text{where} \quad A=\pi_1(\operatorname{aut} S^q) = \begin{cases} Z & \text{if } \ q=1, \\ Z_2 & \text{if } \ q\geqq 2, \end{cases}$$

by Theorem 4.11, (5.3) and Corollary 2.5, and Z_2 acts on $A=t_*(\pi_1(S^1))=Z$ if $q=1$ as $(-1)\cdot n=-n$ by Theorem 5.4 (i), and on $A=Z_2$ if $q\geqq 2$ trivially by Theorem 5.5 (i). Thus we see the results for $\mathscr{L}(S^q\times S^1)$. For the non-trivial fibration $U_q\to S^1$, (7.2) is $0\to Z_2\to\mathscr{L}(U_q)\to Z_2\to 1$ by Theorem 4.11 and (5.3). The map $f\colon S^q\times I\to S^q\times I$, $f((x_0, x_1,\dots, x_q), t)=((-x_0, x_1,\dots, x_q), t)$, induces a fibre map $g\colon U_q\to U_q$ such that $g^2=1$ and $J_0(g)=-1$. Thus $\mathscr{L}(U_q)=Z_2+Z_2$. The other results are shown by Theorems 4.11 (iii) and 5.4 (ii). *q.e.d.*

THEOREM 7.6. *Assume that* $2\leqq n\leqq q$.

(i) *Then* (7.2) *is the exact sequence*

$$0\longrightarrow A\longrightarrow \mathscr{L}(E_k)\longrightarrow Z_2\longrightarrow 1,$$

where

$$A=\begin{cases} \pi_{n+q}(S^q) & \text{if } \ n\leqq q-2, \\ \pi_{2q-1}(S^q)/\{[\iota_q, \iota_q]\} & \text{if } \ n=q-1, \\ (\pi_{2q}(S^q)/\{[\iota_q, \eta_q]\})+Z & \text{if } \ n=q \ \text{ is odd}, \\ \pi_{2q}(S^q)/\{[\iota_q, \eta_q]\} & \text{if } \ n=q \ \text{ is even} \neq 4, 8. \end{cases}$$

(ii) *If E_k is a S^q-bundle over S^n, then the above sequence is split.*

(iii) *In particular, for the trivial fibration $E_0 = S^q \times S^n$ over S^n,*

$\mathscr{L}(S^q \times S^n) = A + Z_2$ *(direct sum) if* $n < q$ *or* $n = q$ *is even* $\neq 4, 8$.

PROOF. (i) Since $n \leqq q$, the second equality in (3.6) implies

$$k = i_*\tau(k') \qquad \text{for some} \quad k' \in \pi_{n+q-1}(S^q).$$

Thus $\mathscr{F}_k(S^q)$ in (7.2) is Z_2 by Theorem 5.5 (ii). If we show that $\partial_k = 0$, then $A = \pi_n(\mathrm{aut}\, S^q)$ is given by (3.8) and Lemma 3.11 as desired.

We prove that $\partial_k = 0$ for $n \leqq q-1$ or $n = q$ separately.

(a) The case $n \leqq q-1$: By the Freudenthal suspension theorem, $k' = \Sigma k_1$ for some $k_1 \in \pi_{n+q-1}(S^{q-1})$. Then by Corollary 4.4 and [2, Prop.],

$$(*) \qquad (-1)^q \partial_k(i_*\tau(\eta^q)) = i_*\tau(\Sigma k_1 \Sigma^{n+q-3}\eta_2 - \Sigma^{q-2}\eta_2 \Sigma^2 k_1)$$
$$= i_*\tau\{[\iota_q, \iota_q]\Sigma^2 H(k_1)\Sigma^{q+n-3}H(\eta_2)\} = i_*\tau\{[\iota_q, \iota_q]\Sigma^2 H(k_1)\}.$$

If $n \leqq q-2$, then k_1 is also a suspension and $H(k_1) = 0$. If $n = q-1$, then $\Sigma^2 H(k_1) \in \pi_{2q-1}(S^{2q-1})$ and hence $\partial_k(i_*\tau(\eta_q))$ is a multiple of $i_*\tau[\iota_q, \iota_q]$, which is 0 by (3.6). Thus $\partial_k = 0$ by (3.7).

(b) The case $n = q \geqq 2$ and $q \neq 4, 8$: Assume $q \neq 2$ in addition. Then, in the exact sequence $\pi_{2q-2}(S^{q-1}) \xrightarrow{\Sigma} \pi_{2q-1}(S^q) \xrightarrow{H} \pi_{2q-1}(S^{2q-1})$, $H = 0$ if q is odd and $\mathrm{Im}\, H = \{2\iota_{2q-1}\} = \{[\iota_q, \iota_q]\}$ if q is even (cf. [31]). Therefore by this exact sequence and (3.6), we may replace $k' \in (i_*\tau)^{-1}(k)$ so that $k' = \Sigma k_1$ for some $k_1 \in \pi_{2q-2}(S^{q-1})$. Thus, in the same way as above, we have $(*)$, where $H(k_1) \in \pi_{2q-2}(S^{2q-3})$ and $[\iota_q, \iota_q]\eta_{2q-1} = [\iota_q, \eta_q]$. Thus, $\partial_k(i_*\tau(\eta_q))$ is a multiple of $i_*\tau[\iota_q, \eta_q]$, which is 0 by (3.6).

On the other hand, if $n = q = 2$, then $k \in \pi_1(\mathrm{aut}\, S^2) = Z_2 = \{i_*\tau(\eta_2)\}$ by (3.7), and we may take $k' = 0$ or η_2. Thus the equality $\partial_k(i_*\tau(\eta_2)) = i_*\tau(k'\eta_3 + \eta_2 \Sigma k')$ of Corollary 4.4 implies $\partial_k = 0$ as desired.

(ii) is shown in Theorem 6.6. (iii) follows from Corollary 2.5 and Theorem 5.5 (i), since $\pi_n(\mathrm{aut}\, S^q) = \mathrm{Im}\, i_*\tau$ if $n < q$ or $n = q$ is even $\neq 4, 8$. *q.e.d.*

EXAMPLE 7.7. *If $n = 2$ and $q \geqq 2$, then $\pi_1(\mathrm{aut}\, S^q) = Z_2$, and there are two types of S^q-fibrations over S^2; and*

$$\mathscr{L}(E_k) = Z_2 + Z_2 \qquad \textit{for any } k.$$

PROOF. By Theorem 7.6 (i), the following sequence is exact:

$$0 \longrightarrow \pi_{q+2}(S^q) \longrightarrow \mathscr{L}(E_k) \longrightarrow Z_2 \longrightarrow 1.$$

Now $\pi_{q+2}(S^q) = Z_2$, and this is split for any k by Theorem 7.6 (ii), since j_*: $\pi_1(SO(q+1)) \to \pi_1(\mathrm{aut}\, S^q)$ is an isomorphism by [35, (5.2)] and (4.7). Hence we have the desired result. *q.e.d.*

THEOREM 7.8. *Let (E_k, p, S^n, S^q) $(q \geqq 2)$ be an $SO(q+1)$-bundle such that the structure group is reduced to $SO(q-2)$, i.e., $k \in \pi_{n-1}(\operatorname{aut} S^q)$ belongs to the image of $\pi_{n-1}(SO(q-2))$ under the homomorphism induced by the inclusion $SO(q-2) \subset SO(q+1) \subset \operatorname{aut} S^q$. Then we have the split exact sequence*

$$0 \longrightarrow \pi_n(\operatorname{aut} S^q) \xrightarrow{G} \mathscr{L}(E_k) \xrightarrow{J_0} Z_2 \longrightarrow 1.$$

PROOF. Consider the commutative diagram (cf. [36, (9.1)])

$$\begin{array}{ccccc}
\pi_{n-1}(SO(q-2)) & \xrightarrow{i_*^2} & \pi_{n-1}(SO(q)) & \xrightarrow{i_*} & \pi_{n-1}(SO(q+1)) \\
\downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle j_*} \\
\pi_{n+q-3}(S^{q-2}) & \xrightarrow{\Sigma^2} & \pi_{n+q-1}(S^q) & \xrightarrow{i_*\tau} & \pi_{n-1}(\operatorname{aut} S^q).
\end{array}$$

Then $k \in \operatorname{Im}(i_*\tau\Sigma^2)$ by the assumption. Therefore, we see that $\partial_k = 0$ in the same way as (a) for $n \leqq q-2$ in the proof of Theorem 7.6 (i). Thus we have the desired split exact sequence by (7.2) and Theorem 6.6. *q.e.d.*

REMARK 7.9. *Let (E, p, B, F) be any Hurewicz fibration. Then*

$$(*) \qquad p^E\colon L(E, E) \longrightarrow L(E, B), \quad p^E(f) = pf,$$

is a Hurewicz fibration with fibre $(p^E)^{-1}(p) = L'(E, E)$, which is the space of all fibre preserving maps of E to itself. Therefore we have the homotopy exact sequence

$$\cdots \to \pi_{i+1}(L(E, B), p) \to \pi_i(L'(E, E), 1) \to \pi_i(L(E, E), 1) \to \pi_i(L(E, B), p) \to \cdots.$$

If E is a k-space, *then $\mathscr{L}(E)$ (resp. $\mathscr{F}(E)$) is the group of consisting of invertible elements of $\pi_0(L'(E, E), 1)$ (resp. $\pi_0(L(E, E), 1)$, and the above sequence is transformed into the exact sequence*

$$(7.10) \quad \pi_1(L(E, E), 1) \to \pi_1(L(E, B), p) \xrightarrow{\partial} \mathscr{L}(E) \xrightarrow{\nu} \mathscr{F}(E) \to \pi_0(L(E, B), p),$$

where ∂ is shown to be a homomorphism (cf. [5, p. 49]) *and ν is the homomorphism defined naturally by sending fibre homotopy classes to their homotopy classes. S. Sasao* [24] *has studied* (7.10) *for sphere bundles over spheres with some conditions, and obtained a generalization of Theorem 7.6 for $n \leqq q-2$.*

Now, we consider the case that $q=3$ or 7.

THEOREM 7.11. *Assume $q=3$ or 7 and $n \geqq 2$, and by* (3.10), *set*

$$k = i_*\tau(k') + t_*(k'') \in \pi_{n-1}(\operatorname{aut} S^q)\,(k' \in \pi_{n+q-1}(S^q),\ k'' \in \pi_{n-1}(S^q)).$$

(i) *Then the short exact sequence*

$$0 \longrightarrow A \longrightarrow \mathscr{L}(E_k) \longrightarrow \mathscr{F}_k(S^q) \longrightarrow 1 \quad (A \cong \mathrm{Coker}\,\partial_k)$$

of (7.2) *is given as follows*:

$$A = (\pi_{n+q}(S^q)+\pi_n(S^q))/A',$$

$$A' = \begin{cases} \{(k'\eta_{n+2}-\eta_3\Sigma k',\ \eta_3\Sigma k'')\} = Z_2 \ \ or\ \ 0 & if\ \ q=3, \\ \{(k'\eta_{n+6}-\eta_7\Sigma k'+(\bar{\nu}_7+\varepsilon_7)\Sigma^8 k'',\ \eta_7\Sigma k'')\} = Z_2 \ \ or\ \ 0 & if\ \ q=7; \end{cases}$$

$$\mathscr{F}_k(S^q) = \begin{cases} Z_2 & if\ \ \langle \iota_q, \iota_q\rangle\Sigma^q k'' = 0 \ \ and\ \ 2k''=0, \\ 1 & otherwise. \end{cases}$$

(ii) *If* E_k *is a* S^q*-bundle over* S^n *with structure group* $SO(q)$, *then the above sequence is split.*

(iii) *For the trivial fibration* $S^q\times S^n$ *over* S^n, *we have the split exact sequence*

$$0 \longrightarrow \pi_{n+q}(S^q)+\pi_n(S^q) \longrightarrow \mathscr{L}(S^q\times S^n) \longrightarrow Z_2 \longrightarrow 1 \ (q=3 \ or\ 7),$$

where Z_2 *acts on* $\pi_{n+q}(S^q)+\pi_n(S^q)$ *as*

$$(-1)\cdot(a, b) = (a+(-1)^n\langle \iota_q, \iota_q\rangle\Sigma^q b,\ -b)\ (a\in\pi_{n+q}(S^q),\ b\in\pi_n(S^q)).$$

PROOF. The results follow immediately from (3.10), Theorems 4.6, 4.8, (5.4), (6.6) and Corollary 2.5. *q.e.d.*

In the rest of this section, we give some examples of this theorem, which are seen by the routine calculations by using the results on the homotopy groups of spheres given in Toda's book [31].

EXAMPLE 7.12 (*the case* $q=3$ *and* $n=4$). *In this case,*

$$k = i_*\tau(k') + t_*(k'')(k'\in\pi_6(S^3) = Z_{12} = \{\omega\},\ k''\in\pi_3(S^3) = Z = \{\iota_3\});$$

and the Hopf bundle $S^7\to S^4$ $(k'=0,\ k''=\iota_3)$ (*cf.* [26]) *and* $SO(5)/SO(3)\to S^4$ $(k'=0,\ k''=2\iota_3)$ (*cf.* [11]) *are typical examples. Then*

$$\mathscr{L}(E_k) = \begin{cases} Z_2 & if\ k''\neq 0\ and\ k'\ is\ odd,\ or\ k''\ is\ odd, \\ Z_2+Z_2 & if\ k''\neq 0\ and\ k'\ are\ even,\ or\ k''=0\ and\ k'\ is\ odd, \\ D_4 & if\ k''=k'=0,\ i.e.,\ k=0, \end{cases}$$

(D_4 *is the dihedral group of order* 8), *and we have a split exact sequence*

$$0 \longrightarrow Z_2+Z_2 \longrightarrow \mathscr{L}(E_k) \longrightarrow Z_2 \longrightarrow 1 \quad otherwise.$$

PROOF. Consider Theorem 7.11 for $q=3$ and $n=4$. Then $\pi_7(S^3)+\pi_4(S^3)=$

$Z_2+Z_2=\{\nu'\eta_6\}+\{\eta_3\}$, and $\eta_3\Sigma k'=\Sigma(\eta_2 k')=0$ and $\eta_6^*\colon \pi_6(S^3)\to\pi_7(S^3)$ is epic ([31, p. 43]). Thus

$$A = Z_2+Z_2 \quad \text{if } k' \text{ and } k'' \text{ are even}, \quad = Z_2 \quad \text{otherwise.}$$

Furthermore $\mathscr{F}_k(S^q)=1$ if $k''\neq 0$, $=Z_2$ otherwise. If $k''=0$ then $k=i_*\tau(k')$ and $J\colon \pi_3(SO(3))\to\pi_6(S^3)(\ni k')$ is epic by Lemma 6.7; hence the sequence $0\to A\to \mathscr{L}(E_k)\to Z_2\to 1$ is split by (4.7) and Theorem 7.11 (ii). If $k=0$, then the splitting action of $0\to Z_2+Z_2\to\mathscr{L}(E_0)\to Z_2\to 1$ in Theorem 7.11 (iii) is given by $(-1)\cdot(a, b)=(a+b, -b)=(a+b, b)$, since $\langle\iota_3, \iota_3\rangle\Sigma^3\eta_3=\omega\eta_6=\nu'\eta_6$ ([31, p. 42]); thus $\mathscr{L}(E_0)=D_4$. *q.e.d.*

EXAMPLE 7.13 (*the case $q=3$ and $n=5$*). *In this case,*

$$k = i_*\tau(k') + t_*(k'')(k'\in\pi_7(S^3) = Z_2 = \{\nu'\eta_6\},\ k''\in\pi_4(S^3) = Z_2 = \{\eta_3\});$$

and $SU(3)\to S^5$ $(k'=0,\ k''=\eta_3)$ (cf. [11]) is a typical example. Then

$$\mathscr{L}(E_k) = \begin{cases} Z_2 & \text{if } k''\neq 0, \\ Z_2+Z_2 & \text{if } k''=0 \text{ and } k'\neq 0, \\ D_4 & \text{if } k''=k'=0, \text{ i.e., } k=0. \end{cases}$$

PROOF. By using the results in [31, pp. 43–45], we have the desired result in the same way as the above proof. *q.e.d.*

EXAMPLE 7.14 (*the case $q=3$ and $n=7$*). *In this case,*

$$k = i_*\tau(k') + t_*(k'')(k'\in\pi_9(S^3) = Z,\ k''\in\pi_6(S^3) = Z_{12} = \{\omega\});$$

and H-spaces of type (3, 7) *are obtained in the case $k'=0$, $k''=n\omega$ $(n=0, 1, 3, 4, 5)$ (cf. [18]), in particular, $Sp(2)\to S^7$ $(k'=0,\ k''=\omega)$ is a typical example. Then*

$$\mathscr{L}(E_k) = Z_{15}+Z_2 \quad \text{if} \quad k''\neq 0,\ 6\omega, \quad = Z_{15}+Z_2+Z_2 \quad \text{if} \quad k=0,$$

and we have the exact sequence

$$0 \longrightarrow Z_{15}+Z_2 \longrightarrow \mathscr{L}(E_k) \longrightarrow Z_2 \longrightarrow 1 \quad \text{if} \quad k''=0 \quad \text{or} \quad 6\omega,$$

which is split if $k'=0$ or $k''=0$.

PROOF. For $k'=0$ and $k''=6\omega$, the sequence is split by the same consideration of [22, Lemma 3.4]. Take $T(x, t, y)=(x, t, -\bar{y})$, $T(z)=-\bar{z}$ $(y, z\in S^3)$ in §6. *q.e.d.*

EXAMPLE 7.15 (*the case $q=7$ and $n=8$*). *In this case,*

$$k = i_*\tau(k') + t_*(k'')(k' \in \pi_{14}(S^7) = Z_{120} = \{\lambda_7\},\ k'' \in \pi_7(S^7) = Z = \{\iota_7\});$$

and the Hopf bundle $S^{15} \to S^8$ $(k'=0,\ k''=\iota_7)$ *(cf.* [26]) *and* $SO(9)/SO(7) \to S^8$ $(k'=0,\ k''=2\iota_7)$ *(cf.* [11]) *are typical examples. Then*

$$\mathcal{L}(E_k) = \begin{cases} Z_2+Z_2+Z_2+Z_2 & \text{if } k'' \neq 0 \text{ and } k' \text{ are even,} \\ Z_2+Z_2+Z_2 & \text{if } k'' \neq 0 \text{ is even and } k' \text{ is odd, or } k'' \text{ is odd,} \\ Z_2+Z_2+D_4 & \text{if } k = 0, \end{cases}$$

and we have the split exact sequences

$$0 \to Z_2+Z_2+Z_2+Z_2 \to \mathcal{L}(E_k) \to Z_2 \to 1 \quad \text{if } k'' = 0 \text{ and } k' \text{ is even,}$$
$$0 \to Z_2+Z_2+Z_2 \to \mathcal{L}(E_k) \to Z_2 \to 1 \quad \text{if } k'' = 0 \text{ and } k' \text{ is odd.}$$

PROOF. If $k=0$, then the splitting action of $0 \to Z_2+Z_2+Z_2+Z_2 \to \mathcal{L}(E_0) \to Z_2 \to 1$ in Theorem 7.11 (iii) is given by $(-1)\cdot(a, b, c, d) = (a, b, c, a+d)$, since $\langle \iota_7, \iota_7 \rangle \Sigma^7 \eta_7 = \sigma' \eta_{14}$; thus $\mathcal{L}(E_0) = Z_2+Z_2+D_4$. *q.e.d.*

EXAMPLE 7.16 (*the case* $q=7$ *and* $n=9$). *In this case*

$$k = i_*\tau(k') + t_*(k'')(k' \in \pi_{15}(S^7) = Z_2+Z_2+Z_2 = \{\sigma'\eta_{14}\} + \{\bar{\nu}_7\} + \{\varepsilon_7\},$$
$$k'' \in \pi_8(S^7) = Z_2 = \{\eta_7\});$$

and $U(5)/U(3) \to S^9$ $(k'=0,\ k''=\eta_7)$ *(cf.* [11]) *is a typical example. Then*

$$\mathcal{L}(E_k) = \begin{cases} Z_2+Z_2+Z_2+Z_2 & \text{if } k'' \neq 0, \\ Z_2+Z_2+Z_2+D_4 & \text{if } k = 0, \end{cases}$$

and we have the exact sequences

$$0 \to Z_2+Z_2+Z_2+Z_2+Z_2 \to \mathcal{L}(E_k) \to Z_2 \to 1 \quad \text{if } k'' = 0 \text{ and } k' = \bar{\nu}_7 \text{ or } \varepsilon_7,$$
$$0 \to Z_2+Z_2+Z_2+Z_2 \to \mathcal{L}(E_k) \to Z_2 \to 1 \quad \text{if } k'' = 0 \text{ and } k' = \sigma'\eta_{14},$$

which is split if $k''=0$ *and* $k'=\sigma'\eta_{14}$.

PROOF. The result for the trivial case is seen similarly to the proof of Example 7.15. *q.e.d.*

EXAMPLE 7.17 (*the case* $q=7$ *and* $n=11$). *In this case,*

$$k = i_*\tau(k') + t_*(k'')(k' \in \pi_{17}(S^7) = Z_{24}+Z_2,\ k'' \in \pi_{10}(S^7) = Z_{24});$$

and $Sp(3)/Sp(1) \to S^{11}$ *is a typical example* $(k'=0,\ k''=\nu_7)$ *(cf.* [11]). *Then*

$$\mathscr{L}(E_k) = \begin{cases} Z_{504} & \text{if } k'' \not\equiv 0 \mod 8, \\ Z_{504}+Z_2 & \text{if } k'' \equiv 0 \mod 8 \text{ and } k'' \neq 0, \\ Z_{504}+Z_2+Z_2 & \text{if } k = 0, \end{cases}$$

and we have the exact sequence

$$0 \longrightarrow Z_{504}+Z_2 \longrightarrow \mathscr{L}(E_k) \longrightarrow Z_2 \longrightarrow 1 \quad \textit{otherwise},$$

which is split if $k''=0$ *and* $k' \equiv 0 \mod 3$.

PROOF. The sequence is split by Theorem 7.11 (ii) if $k''=0$ and $k' \equiv 0 \mod 3$. For the trivial case, the action is trivial by Theorem 7.11 (iii), since $\pi_{11}(S^7)=0$. *q.e.d.*

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