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Weakly serial subgroups

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Introduction

Recently Tôgô [6, 7] introduced the notions of weakly subnormal and weakly ascendant subgroups and investigated their properties. On the other hand, Honda [2] introduced the notion of weakly serial subalgebras in Lie algebras and investigated its property.

In this paper, following the paper [2] we shall introduce the notion of weakly serial subgroups generalizing those of serial subgroups and weakly ascendant subgroups, and develop its property in some types of groups.

Let H be a subgroup of a group G. In Section 2 we shall show that when $G \in \mathrm{L}(\mathrm{Ind(wser)\text{-}Fin})$, H is weakly serial in G if and only if H is serial in G, where Ind(wser)-Fin is the class of all groups in which every non-trivial weakly serial subgroup is of finite index (Theorem 6). We shall also show that if G is locally finite, then $H \operatorname{ser} G$ is equivalent to each of the following conditions: (a) $H \operatorname{ser} K$ for any subgroup K of G containing H; (b) $H \operatorname{ser} \langle H, x\rangle$ for any $x \in G$; (c) $H \operatorname{ser} \langle H, [x, H]\rangle$ for any $x \in G$; (d) for any $x \in G$, there exists an $n = n(x) \in \boldsymbol{N}$ such that $H \operatorname{ser} \langle H, [x, {}_nH]\rangle$ (Theorem 10). In Section 3 we shall show that when $G \in \mathrm{L}(\lhd)\mathfrak{F}$, $G/\zeta_1(G)$ is countable and $H \in \mathfrak{N}$, then H is serial in G if and only if H is $\omega 2$-step descendant in G (Theorem 15). In Section 4 we shall give a characterization of descendant subgroups (Theorem 17).

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1.

Let G be a group. As usual, $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in G$. For non-empty subsets X, Y of G, we write $X^Y = \langle x^y \colon x \in X, y \in Y\rangle$; $[X, Y] = \langle [x, y] \colon x \in X, y \in Y\rangle$; $[X, {}_0Y] = \langle X\rangle$, $[X, {}_{n+1}Y] = [[X, {}_nY], Y]$ for any integer $n \geq 0$.

If H is respectively ascendant, σ-step ascendant, descendant, σ-step descendant and subnormal in G, then we write

$$H \operatorname{asc} G, \quad H \lhd^{\sigma} G, \quad H \operatorname{desc} G, \quad H \lhd_{\sigma} G \quad \text{and} \quad H \operatorname{sn} G,$$

where σ is an ordinal. Let $H \leq G$. Then we write

$$H^{G,0} = G, \quad H^{G,\alpha+1} = H^{H^{G,\alpha}} \quad \text{and} \quad H^{G,\lambda} = \bigcap_{\alpha<\lambda} H^{G,\alpha}$$

for all ordinals α and all limit ordinals λ. We know that for an ordinal σ $H \lhd_\sigma G$ if and only if $H^{G,\sigma} = H$.

Let $H \leq G$ and let Σ be a linearly ordered set. A series from H to G of type Σ is a family $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subgroups of G such that

(a) each Λ_σ and V_σ contains H,

(b) $G - H = \bigcup_{\sigma\in\Sigma} (\Lambda_\sigma - V_\sigma)$,

(c) $\Lambda_\tau \leq V_\sigma$ if $\tau < \sigma$,

(d) $V_\sigma \lhd \Lambda_\sigma$.

H is called serial in G if there exists a series from H to G of type Σ for some Σ. We then write H ser G.

Tôgô [6] introduced the following notion generalizing that of ascendant subgroups: For an ordinal σ, H is a σ-step weakly ascendant subgroup of G, denoted by $H \leq^\sigma G$, if there exists an ascending series $(S_\alpha)_{\alpha\leq\sigma}$ of subsets of G such that

(a) $S_0 = H$ and $S_\sigma = G$;

(b) if α is any ordinal $<\sigma$, then $x^{-1}Hx \subseteq S_\alpha$ for any $x \in S_{\alpha+1}$;

(c) $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is a weakly ascendant subgroup of G, denoted by H wasc G, if $H \leq^\sigma G$ for some ordinal σ. When σ is finite, H is a weakly subnormal subgroup of G, denoted by H wsn G.

Now we introduce the following notion generalizing those of serial subgroups and weakly ascendant subgroups. For a linearly ordered set Σ, we define a weak series from H to G of type Σ to be a family $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subsets of G such that

(a) $H \subseteq V_\sigma \subseteq \Lambda_\sigma \subseteq G$ for all σ,

(b) $G - H = \bigcup_{\sigma\in\Sigma} (\Lambda_\sigma - V_\sigma)$,

(c) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,

(d) $x^{-1}Hx \subseteq V_\sigma$ for any $x \in \Lambda_\sigma$,

(e) $HV_\sigma = V_\sigma$.

We write $H \leq^\Sigma G$ if there exists a weak series from H to G of type Σ. We call H a weakly serial subgroup of G if $H \leq^\Sigma G$ for some linearly ordered set Σ, and write H wser G. Note that if $H \leq^\Sigma G$ and Σ is well-ordered, then H wasc G.

As usual, $\mathfrak{F}$, $\mathfrak{G}$, $\mathrm{E}\mathfrak{A}$ and $\mathfrak{N}$ are the classes of finite, finitely generated, solvable and nilpotent groups respectively, and Min is the class of groups satisfying the minimal condition for subgroups.

For a class $\mathfrak{X}$ of groups and $\Delta = \lhd$, sn or $\leq$, we write $G \in \mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of G there exists an $\mathfrak{X}$-subgroup H of G containing X such that $H \Delta G$. In particular, we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. The class $\mathrm{L}(\lhd)\mathfrak{F}$ is identical with the class $\mathbf{M}\mathfrak{F}$ of locally normal and finite groups in [4, p. 36]. For $H \leq G$ we also write $G \in \mathrm{L}(H)\mathfrak{F}$ if for any finite subset X of G there exists an H-

invariant $\mathfrak{F}$-subgroup of G containing X.

Let S be a non-empty set. A local system $\mathbf{L}$ on S is a collection of subsets of S such that every finite subset of S is contained in some member of $\mathbf{L}$.

Any notation not explained here may be found in [3].

2.

We begin by showing some elementary properties of weakly serial subgroups.

LEMMA 1. *Let G be a group.*

(1) *If $H\leq^{\Sigma} G$ and $K\leq G$, then $H\cap K\leq^{\Sigma}K$.*

(2) *Let f be a homomorphism of G onto a group $\bar{G}$. If $H\leq^{\Sigma}G$ and* $\mathrm{Ker} f\leq H$, *then $f(H)\leq^{\Sigma}\bar{G}$. If $\bar{H}\leq^{\Sigma}\bar{G}$, then $f^{-1}(\bar{H})\leq^{\Sigma}G$.*

PROOF. Assume that $H\leq^{\Sigma}G$ and let $\{\Lambda_\sigma, V_\sigma: \sigma\in\Sigma\}$ be a weak series from H to G. Then $\{\Lambda_\sigma\cap K, V_\sigma\cap K: \sigma\in\Sigma\}$ is a weak series from $H\cap K$ to K and $H\cap K\leq^{\Sigma}K$. If furthermore $\mathrm{Ker} f\leq H$, then $\{f(\Lambda_\sigma), f(V_\sigma): \sigma\in\Sigma\}$ is a weak series from $f(H)$ to $\bar{G}$ and $f(H)\leq^{\Sigma}\bar{G}$.

Assume that $\bar{H}\leq^{\Sigma}\bar{G}$ and let $\{\bar{\Lambda}_\sigma, \bar{V}_\sigma: \sigma\in\Sigma\}$ be a weak series from $\bar{H}$ to $\bar{G}$. Then $\{f^{-1}(\bar{\Lambda}_\sigma), f^{-1}(\bar{V}_\sigma): \sigma\in\Sigma\}$ is a weak series from $f^{-1}(\bar{H})$ to G and $f^{-1}(\bar{H})\leq^{\Sigma}G$.

Next, following the line of the paper [1], we shall give some characterizations of weakly serial subgroups.

LEMMA 2. *Let G be a group and let $H\leq G$. Then H wser G if and only if there exists a binary relation $\subset$ on G satisfying the following conditions, where $x, y, z\in G$:*

(i) *$x\subset y$ and $y\subset z$ imply $x\subset z$.*

(ii) *Either $x\subset y$ or $y\subset x$.*

(iii) *If $h\in H$, then $h\subset x$.*

(iv) *If $h\in H$ and $x\notin H$, then $x\not\subset h^x$.*

(v) *If $h\in H$ and $y\not\subset x$, then $y\not\subset hx$.*

PROOF. Let $\{\Lambda_\sigma, V_\sigma: \sigma\in\Sigma\}$ be a weak series from H to G. For any $x\in G-H$ there exists the unique element $\sigma(x)$ of Σ such that $x\in\Lambda_{\sigma(x)}-V_{\sigma(x)}$. For any $h\in H$ we write $\sigma(h)=\sigma(1)$ and put $\sigma(h)<\sigma(y)$ if $y\in G-H$. We then define a binary relation $\subset$ on G as follows: $x\subset y$ if and only if $\sigma(x)\leq\sigma(y)$. This binary relation $\subset$ satisfies (i)–(v) above. Here we only check that $\subset$ satisfies (iv) and (v).

(iv) Let $h\in H$ and $x\notin H$. We may assume that $h^x\notin H$. Since $x\in\Lambda_{\sigma(x)}$, we have $h^x\in V_{\sigma(x)}$ by (d) in the definition of weakly serial subgroups. If $x\subset h^x$, then $\sigma(x) \leq \sigma(h^x)$, which contradicts $h^x \notin V_{\sigma(h^x)}$. Hence $x \not\subset h^x$.

(v) Let $h \in H$ and $y \not\subset x$. Then $\sigma(x) < \sigma(y)$. We may assume that $x \notin H$. Hence we have $x \in \Lambda_{\sigma(x)} \subseteq V_{\sigma(y)}$ by (c), and so $hx \in V_{\sigma(y)}$ by (e). If $y \subset hx$, then $\sigma(y) \leq \sigma(hx)$, which contradicts $hx \notin V_{\sigma(hx)}$. Hence $y \not\subset hx$.

Conversely, let $\subset$ be a binary relation on G satisfying (i)–(v). Let $x \sim y$ mean that $x \subset y$ and $y \subset x$. Then $\sim$ is an equivalence relation on G. By (iii) and (iv) we have $H = \{x \in G: x \sim 1\}$. Let Σ be the set of all $\sim$-equivalence classes except H. For $\sigma, \tau \in \Sigma$, we write $\sigma < \tau$ if $\sigma \neq \tau$ and there exist $x \in \sigma$ and $y \in \tau$ such that $x \subset y$. It is a simple matter to check that $\leq$ is a well-defined linear order on Σ. Now we define the terms of a weak series determined by $\subset$:

$$\Lambda_\sigma = \{x \in G: x \subset y \quad \text{for some } y \in \sigma\},$$

$$V_\sigma = \begin{cases} H & \text{if } \sigma \text{ is the first element of } \Sigma, \\ \bigcup_{\tau < \sigma} \Lambda_\tau & \text{otherwise.} \end{cases}$$

It is easy to show that $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ is a weak series from H to G. Here we verify only (d) and (e) in the definition of weakly serial subgroups. Assume that we have verified (a), (b) and (c).

(d) Let $x \in \Lambda_\sigma$ and $h \in H$. We may assume that $x \notin H$ and $h^x \notin H$. Since $x \not\subset h^x$ by (iv) and $x \in \Lambda_\sigma$, we have $h^x \subset x \subset y$ for some $y \in \sigma$ by (ii). Let $h^x \in \tau$. Then $\tau \leq \sigma$. If $\tau = \sigma$, then we have $x \subset y \subset h^x$, which contradicts $x \not\subset h^x$. Hence $\tau < \sigma$. It follows that $h^x \in \Lambda_\tau \subseteq V_\sigma$.

(e) Let $h \in H$ and $x \in V_\sigma$. Note that $z \in \sigma$ if and only if $z \in \Lambda_\sigma - V_\sigma$. We may assume that $x \notin H$ and $hx \notin H$. Let $x \in \tau$. Then $x \in \Lambda_\tau - V_\tau$ and therefore $\tau < \sigma$. Since $x \in V_\sigma \subseteq \Lambda_\sigma$, we have $x \subset y$ for some $y \in \sigma$ and $y \not\subset x$. By (v) $y \not\subset hx$. Let $hx \in \delta$. Then we see that $\delta < \sigma$. It follows that $hx \in \Lambda_\delta \subseteq V_\sigma$.

PROPOSITION 3. *Let G be a group and let $H \leq G$. Then H* wser *G if and only if there is a local system* $\mathbf{L}$ *of subgroups of G such that $H \cap L$* wser *L for each $L \in \mathbf{L}$.*

PROOF. If H wser G, then we may take $\mathbf{L} = \{L \leq G: L \in \mathfrak{G}\}$ by Lemma 1(1). Conversely, suppose that $H \cap L$ wser L for each $L \in \mathbf{L}$. By making use of Lemma 2, for each $L \in \mathbf{L}$ there is a binary relation $\subset_L$ on L satisfying (i)–(v) with G replaced by L and H by $H \cap L$. It follows from the theory of Mal'cev systems [3, Lemma 5.13 and pp. 136–139] that there exists a binary relation $\subset$ on G with the property that if $x_1, \dots, x_n, y_1, \dots, y_n$ are finitely many elements of G then there exists a member L of $\mathbf{L}$ such that $x_i, y_i \in L (1 \leq i \leq n)$ and

$$x_i \subset y_i \Longleftrightarrow x_i \subset_L y_i \qquad (1 \leq i \leq n). \tag{$*$}$$

By ($*$) we easily see that $\subset$ satisfies (i)–(v). Hence by using of Lemma 2 again, we conclude that H wser G.

As a special case of Proposition 3 we have the following

COROLLARY 4. *Let $\mathfrak{X}$ be a class of groups and let $G \in L\mathfrak{X}$. Then for a subgroup H of G, H* wser *G if and only if $H \cap F$* wser *F for any $\mathfrak{X}$-subgroup F of G.*

Now we introduce new classes of groups as follows. Let Δ be any of the relations $\leq$, wser, ser, wasc, asc, desc, wsn, sn, $\lhd$. We define a class Ind(Δ)-Fin of groups: $G \in \mathrm{Ind}(\Delta)$-Fin if and only if either $G=1$, or for any non-trivial subgroup H of G such that $H \Delta G$ H is of finite index in G. We then have

$$\mathfrak{F} \lneqq \mathrm{Ind}(\leq)\text{-Fin} \leqq \mathrm{Ind}(\mathrm{wser})\text{-Fin},$$

$$L\mathfrak{F} \lneqq L(\mathrm{Ind}(\leq)\text{-Fin}) \leqq L(\mathrm{Ind}(\mathrm{wser})\text{-Fin}).$$

In fact, by [4, Theorem 4.33] $G \in \mathrm{Ind}(\leq)$-Fin if and only if either $G \in \mathfrak{F}$ or G is infinite cyclic.

The following proposition is a generalization of [6, Corollary(a) to Theorem 4] (cf. [8, Theorem 3]).

PROPOSITION 5. *Let Δ be any of the relations* wser, ser, wasc, asc, desc, wsn, *and let $G \in \mathrm{Ind}(\Delta)$-Fin. If $H \Delta G$, then H* sn *G.*

PROOF. We shall only give the proof for the case that Δ is wser, since the others can be similarly proved. Let H wser G and $H \neq 1$. Then we see that H has finite index in G. Therefore there exists a normal subgroup N of G contained in H such that N has finite index in G. By Lemma 1(2) we have

$$H/N \text{ wser } G/N \in \mathfrak{F},$$

and so

$$H/N \text{ wsn } G/N \in \mathfrak{F}.$$

It follows from [6, Corollary(a) to Theorem 4] that

$$H/N \text{ sn } G/N.$$

Thus we obtain

$$H \text{ sn } G.$$

From the proof above we can show that if Δ is any one of wser, ser, wasc, asc, desc, wsn and if $H \Delta G$ and H has finite index in G, then H sn G.

In Lie algebras the notion of weakly serial subalgebras is identical with that of serial subalgebras in locally solvable and finite Lie algebras [2, Theorem 2.7]. But in group case we have the corresponding result in a wider class.

THEOREM 6. *Let* $G \in \mathrm{L}(\mathrm{Ind}\,(\mathrm{wser})\text{-}\mathrm{Fin})$. *If* H wser G, *then* H ser G.

PROOF. Let F be any subgroup of G belonging to Ind (wser)-Fin. By Lemma 1(1) we have

$$H \cap F \text{ wser } F \in \mathrm{Ind}\,(\mathrm{wser})\text{-}\mathrm{Fin}.$$

By using Proposition 5 we obtain

$$H \cap F \text{ sn } F.$$

Therefore by [1, Lemma 1] we conclude that

$$H \text{ ser } G.$$

COROLLARY 7. *Let* $G \in \mathrm{L}\mathfrak{F}$. *If* H wser G, *then* H ser G.

In connection with the second statement of [1, Theorem A] we shall show the following

PROPOSITION 8. *Let* $G \in \mathrm{L}(\mathrm{Ind}\,(\mathrm{ser})\text{-}\mathrm{Fin})$ *and let* θ *be a homomorphism of* G *such that* $\mathrm{Ker}\,\theta$ *is finitely generated. If* H ser G, *then* $\theta(H)$ ser $\theta(G)$.

PROOF. We put $N = \mathrm{Ker}\,\theta$. It suffices to show that HN/N ser G/N. For any finite subset X of G there exists a subgroup F_X of G such that

$$\langle X, N\rangle \leq F_X \in \mathrm{Ind}\,(\mathrm{ser})\text{-}\mathrm{Fin}.$$

We then define

$$\mathbf{L} = \{F_X/N : X \text{ is a finite subset of } G\},$$

which is a local system on G/N. Now for any finite subset X of G we have

$$H \cap F_X \text{ ser } F_X \in \mathrm{Ind}\,(\mathrm{ser})\text{-}\mathrm{Fin}.$$

By using Proposition 5 we obtain

$$H \cap F_X \text{ sn } F_X,$$

and so

$$(H \cap F_X)N/N \text{ sn } F_X/N.$$

Since

$$(H \cap F_X)N/N = (HN \cap F_X)/N = HN/N \cap F_X/N,$$

we have

$$HN/N \cap F_X/N \text{ ser } F_X/N \qquad \text{for any } F_X/N \in \mathbf{L}.$$

Thus by making use of [1, Lemma 1] we see that

$$HN/N \text{ ser } G/N.$$

COROLLARY 9. *Let* $G \in \mathrm{L}(\mathrm{Ind}\,(\mathrm{wser})\text{-}\mathrm{Fin})$ *and let* θ *be a homomorphism of* G *such that* $\mathrm{Ker}\,\theta$ *is finitely generated. If* H wser G, *then* $\theta(H)$ wser $\theta(G)$.

In [5] Stewart showed the equivalence of (1) and (3) in the following theorem.

THEOREM 10. *Let* $G \in \mathrm{L}\mathfrak{F}$ *and* $H \leq G$. *Then the following conditions are equivalent*:

(1) H ser G.
(2) H ser K *for any subgroup* K *of* G *containing* H.
(3) H ser $\langle H, x\rangle$ *for any* $x \in G$.
(4) H ser $\langle H, [x, H]\rangle$ *for any* $x \in G$.
(5) *For any* $x \in G$, *there exists an* $n = n(x) \in \boldsymbol{N}$ *such that* H ser $\langle H, [x, {}_nH]\rangle$.

PROOF. It is clear that (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5). We have to show that (5)$\Rightarrow$(1). Let F be any finite subgroup of G and let $x \in F$. By the condition (5) there exists an $n = n(x) \in \boldsymbol{N}$ such that

$$H \text{ ser } \langle H, [x, {}_nH]\rangle .$$

Hence

$$H \cap F \text{ sn } \langle H, [x, {}_nH]\rangle \cap F.$$

Since $\langle H \cap F, [x, {}_nH \cap F]\rangle \leq \langle H, [x, {}_nH]\rangle \cap F$, we have

$$H \cap F \text{ sn } \langle H \cap F, [x, {}_nH \cap F]\rangle .$$

Owing to [7, Theorem 2] we see that

$$H \cap F \text{ sn } F.$$

Therefore [1, Lemma 2] shows that

$$H \text{ ser } G.$$

3.

In this section we shall mainly state the relationship of weakly serial subgroups, serial subgroups, weakly ascendant subgroups, ascendant subgroups, etc. in some types of groups. We shall first give the following result which contains [7, Corollary to Theorem 1].

PROPOSITION 11. *Let G be a group and $H \leq G$. Assume that $G \in \mathrm{L}(H)\mathfrak{F} \cap$* Min. *Then the following conditions are equivalent:*

(1) H ser G.

(2) H wser G.

(3) H asc G.

(4) H wasc G.

PROOF. (1)$\Leftrightarrow$(2): Since $G \in \mathrm{L}\mathfrak{F}$, the assertion follows from Corollary 7.

(1)$\Rightarrow$(3): There exists a series from H to G which may be well-ordered since G satisfies Min. Namely there exists an ascending series from H to G. Hence H asc G.

(3)$\Rightarrow$(1) is trivial.

(3)$\Leftrightarrow$(4): See [7, Corollary to Theorem 1].

An infinite group all of whose proper subgroups are finite is called quasifinite. Here we define a class $\mathrm{q}\mathfrak{F}$ of groups as follows: $G \in \mathrm{q}\mathfrak{F}$ if and only if G is either finite or quasifinite. Now we have

PROPOSITION 12. *Let $G \in \mathrm{q}\mathfrak{F}$ and $H \leq G$. Then the following conditions are equivalent:*

(1) H ser G.

(2) $H \lhd^{\omega} G$.

(3) H asc G.

(4) H wasc G.

PROOF. (1)$\Rightarrow$(3): If H ser G, then H asc G since $G \in$ Min.

(4)$\Rightarrow$(3): Assume (4) and let $H \neq G$. Since $H \in \mathfrak{F}$, [6, Theorem 5] asserts that $H \leq^{\omega} G$. Since $G \in$ Min, we can use [6, Corollary (b) to Theorem 4] to see that H asc G.

(3)$\Rightarrow$(2): Let $(H_{\alpha})_{\alpha \leq \sigma}$ be an ascending series from H to G. If σ is a limit ordinal, then $H_{\alpha} \in \mathfrak{F}$ for any $\alpha < \sigma$. Hence $\alpha < \omega$ for any $\alpha < \sigma$. Therefore $\sigma = \omega$. If σ is a non-limit ordinal, then $H_{\sigma-1} \in \mathfrak{F}$. It follows that $\sigma < \omega$. Thus we have $H \lhd^{\omega} G$.

(2)$\Rightarrow$(1) and (2)$\Rightarrow$(4) are trivial.

In connection with Proposition 5 we have the following proposition, which is stronger than [7, Proposition 2].

PROPOSITION 13. *Let Δ be any of the relations* wser, ser, wasc, asc, desc, wsn, *and let $G \in \mathrm{L}(\mathrm{sn})(\mathrm{Ind}(\Delta)\text{-Fin})$. If $H \Delta G$ and $H \in \mathfrak{G}$, then H* sn *G.*

PROOF. By assumption, there exists a subgroup K of G such that

$$H \leq K \operatorname{sn} G \quad \text{and} \quad K \in \operatorname{Ind}(\Delta)\text{-Fin}.$$

Since $H \Delta K$, we have $H \operatorname{sn} K$ by Proposition 5. Thus $H \operatorname{sn} G$.

Next we present a condition for a subgroup to be a descendant subgroup in a locally normal and finite group. To do this we need the following lemma due to Hall.

LEMMA 14 ([4, Theorem 4.34]). *Let G be a group such that $G \in \mathrm{L}(\lhd)\mathfrak{F}$ and $G/\zeta_1(G)$ is countable. Then $G/\zeta_1(G)$ is embedded in a direct product of countably many finite groups.*

PROOF. Put $\bar{G}=G/\zeta_1(G)$. Since $\bar{G} \in \mathrm{L}(\lhd)\mathfrak{F}$, we can write $\bar{G}$ as the union of a countable ascending chain $(\bar{G}_i)_{i<\omega}$ of finite normal subgroups with $\bar{G}_0=1$. Since $\bar{G}$ is residually finite, for any $i<\omega$ there exists a normal subgroup $\bar{K}_i$ of finite index in $\bar{G}$ such that $\bar{K}_i \cap \bar{G}_i=1$. Put $\bar{N}_i=\bar{G}_{i-1}\bar{K}_i$ for any $i<\omega$. Then $\bar{N}_i$ is of finite index and normal in $\bar{G}$. Further $\bar{G}_i \cap \bar{N}_i=\bar{G}_{i-1}$, from which it follows that $\bigcap_{i<\omega} \bar{N}_i=1$. Let θ be the natural homomorphism of $\bar{G}$ into the Cartesian product of the $\bar{G}/\bar{N}_i$. Then θ is injective and $\operatorname{Im}\theta$ is contained in the direct product of the $\bar{G}/\bar{N}_i$.

As a group analogue of [2, Theorem 4.5] we have

THEOREM 15. *Let G be a group such that $G \in \mathrm{L}(\lhd)\mathfrak{F}$ and $G/\zeta_1(G)$ is countable. Assume that H is a subgroup of G such that $H\zeta_1(G)/\zeta_1(G)$ is nilpotent of class c. If H ser G, then $H \lhd_{\omega+c+1} G$.*

PROOF. Put $\bar{G}=G/\zeta_1(G)$ and $\bar{H}=H\zeta_1(G)/\zeta_1(G)$. By the proof of Lemma 14, $\bar{G}$ has a collection $(\bar{N}_i)_{i<\omega}$ of normal subgroups of finite index in $\bar{G}$ such that $\bigcap_{i<\omega}\bar{N}_i=1$ and such that the image of $\bar{G}$ under the natural monomorphism θ: $\bar{G} \to \mathrm{Cr}_{i<\omega}(\bar{G}/\bar{N}_i)$ is contained in $\mathrm{Dr}_{i<\omega}(\bar{G}/\bar{N}_i)$. Let π_i denote the projection mapping $\mathrm{Dr}_{i<\omega}(\bar{G}/\bar{N}_i) \to \bar{G}/\bar{N}_i$ and put $G_i=\pi_i\theta(\bar{G})$ and $H_i=\pi_i\theta(\bar{H})$. By [1, Theorem A] H_i ser G_i. Since $G_i \in \mathfrak{F}$, H_i sn G_i. Hence there exists a $k=k(i) \in \boldsymbol{N}$ such that $H_i^{G_i,k}=H_i$. It follows that

$$\bigcap_{k<\omega}(\mathrm{Dr}_{i<\omega} H_i^{G_i,k}) = \mathrm{Dr}_{i<\omega}(\bigcap_{k<\omega} H_i^{G_i,k}) = \mathrm{Dr}_{i<\omega} H_i.$$

As $\mathrm{Dr}_{i<\omega} H_i^{G_i,k+1} \lhd \mathrm{Dr}_{i<\omega} H_i^{G_i,k}$, we have $\mathrm{Dr}_{i<\omega} H_i \lhd_\omega \mathrm{Dr}_{i<\omega} G_i$. Put $\bar{K}=\theta^{-1}($ $\mathrm{Dr}_{i<\omega} H_i \cap \theta(\bar{G}))$. Then $\bar{K} \lhd_\omega \bar{G}$. On the other hand, it is easy to see that $\bar{K}=\bigcap_{i<\omega} \bar{N}_i\bar{H}$. Hence

$$[\bar{K}, {}_c\bar{H}] \subseteq \bigcap_{i<\omega}[\bar{N}_i\bar{H}, {}_c\bar{H}] \subseteq \bigcap_{i<\omega}\bar{N}_i\gamma_{c+1}(\bar{H}) = \bigcap_{i<\omega}\bar{N}_i = 1.$$

By [3, Corollary to Lemma 3.12] we have $\bar{H} \lhd^c \bar{K}$. Therefore $\bar{H} \lhd_{\omega+c} \bar{G}$, whence

$H\zeta_1(G) \lhd_{\omega+c} G$. Thus we have $H \lhd_{\omega+c+1} G$.

COROLLARY 16. *Let G be a group such that* $G \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \cap$ Min *and* $G/\zeta_1(G)$ *is countable. Assume that H is a nilpotent subgroup of G. Then the following conditions are equivalent*:

(1) H ser G.
(2) H wser G.
(3) H asc G.
(4) $H \lhd^{\omega} G$.
(5) H desc G.
(6) $H \lhd_{\omega 2} G$.

PROOF. The assertion follows from Proposition 11, Theorem 15 and [7, Theorem 3].

REMARK. We may define a weakly serial subgroup as a subgroup having a weak series satisfying the conditions (a)–(d). That is, we may exclude the condition (e) in the definition of a weakly serial subgroup in Section 1. The statements of Lemma 1 (except the first part of (2)), Lemma 2 (excluding (v)), Proposition 3, Corollary 4, Corollary 7, Proposition 11 and Corollary 16 hold for such a weakly serial subgroup.

4.

The following result is due to Tôgô: For a subgroup H of G, H asc G if and only if H wasc G with a weakly ascending series $(S_\alpha)_{\alpha \leq \sigma}$ consisting of subgroups.

A similar characterization holds for descendant subgroups. Namely we have the following

THEOREM 17. *Let G be a group and* $H \leq G$. *Then* H desc G *if and only if there exists a descending series* $(H_\alpha)_{\alpha \leq \sigma}$ *of subgroups of G such that*

(a) $H_0 = G$ *and* $H_\sigma = H$;
(b) *if* α *is any ordinal* $< \sigma$, *then* $x^{-1}Hx \subseteq H_{\alpha+1}$ *for any* $x \in H_\alpha$;
(c) $H_\lambda = \bigcap_{\alpha < \lambda} H_\alpha$ *for any limit ordinal* $\lambda \leq \sigma$.

PROOF. If there exists such a series, then we can easily show that $H^{G,\alpha} \leq H_\alpha$ for any $\alpha \leq \sigma$. Hence $H^{G,\sigma} = H$. Therefore $H \lhd_\sigma G$.

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