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Whittaker functions on semisimple Lie groups

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Introduction

Let G be a connected, noncompact, semisimple Lie group with finite center. Let $G=NAK$ be an Iwasawa decomposition of G. That is, K is a maximal compact subgroup, A is a maximal vector subgroup consisting of semisimple elements and N is a maximal simply connected nilpotent subgroup of G.

Our major concern in this article is a so-called (class one) Whittaker function on G, which is closely connected with the Whittaker models of a class one principal series representation of G. Such a function has been studied by many authors (see the reference) in the case when it is associated with a non-degenerate character of N.

In this paper, we do not assume the non-degeneracy of a character of N. We consider the Whittaker function on G from the viewpoint that it appears as a joint eigenfunction of the algebra of all left invariant differential operators on G/K. Our approach is similar to the one employed by Harish-Chandra for his celebrated work concerning the spherical functions on G.

In more detail, let ψ be an arbitrary character of N. We consider the space $C_\psi^\infty(G/K)$ of smooth functions f on G satisfying $f(nxk)=\psi(n)f(x)$ for $n\in N$, $x\in G$ and $k\in K$. The space $C_\psi^\infty(G/K)$ is stable under the action of the algebra of all left invariant differential operators on G/K, or equivalently, under the action of the algebra $U(\mathfrak{g})^{\mathfrak{k}}$ (cf. § 2). So we are allowed to introduce the space $C_\psi^\infty(G/K, \chi_\nu)$ of all joint eigenfunctions of $U(\mathfrak{g})^{\mathfrak{k}}$ in $C_\psi^\infty(G/K)$. Here χ_ν is an algebra homomorphism of $U(\mathfrak{g})^{\mathfrak{k}}$ into $\boldsymbol{C}$ which corresponds to an element ν of the complex dual space $\mathfrak{a}^*$ of the Lie algebra of A (see (2.2)).

We first study the structure of $C_\psi^\infty(G/K, \chi_\nu)$ and obtain the following results.

(I) *Each element of $C_\psi^\infty(G/K, \chi_\nu)$ is a real analytic function on G* (Proposition 3.2).

(II) *The dimension of $C_\psi^\infty(G/K, \chi_\nu)$ is finite and does not exceed the order of the Weyl group W of G relative to A* (Theorem 3.3).

(III) *For those $\nu\in\mathfrak{a}^*$ in general position, we construct the functions $V(x: s\nu, \psi)$ $(s\in W)$ on G explicitly* (cf. (4.1), (4.5) and (4.10)) *and we prove that they form a basis of $C_\psi^\infty(G/K, \chi_\nu)$* (Corollary 4.11 and Theorem 5.4).

Next we define the class one Whittaker function $W(x: \nu, \psi)$ on G associated with $\nu \in \mathfrak{a}^*$ and a character ψ of N by a certain integral formula (see (6.4)). The integral converges for those ν in a certain connected open subset D and is holomorphic there (cf. Proposition 6.1). We have already shown in [4] that for a non-degenerate character ψ, the integral defining $W(x: \nu, \psi)$ can be extended to an entire function of $\nu \in \mathfrak{a}^*$. Here we prove the following.

(IV) *For an arbitrary character ψ of N, the integral defining $W(x: \nu, \psi)$ can be in general meromorphically continued as a function of ν and moreover it belongs to $C_\psi^\infty(G/K, \chi_\nu)$ as a function on G* (Theorem 6.6).

(V) *When we write the Whittaker function $W(x: \nu, \psi)$ as a linear combination of the above constructed basis $V(x: s\nu, \psi)$ $(s \in W)$, the coefficients are explicitly determined in terms of the Harish-Chandra's c-functions and the gamma factors appeared in the functional equations of the Whittaker functions* (Theorem 7.8 and Theorem 7.12).

We describe the main steps of the proofs of the above mentioned results. In view of the fact that each $f \in C_\psi^\infty(G/K)$ can be completely determined by its restriction f_A to A, we construct in § 2 certain differential operator $\delta(z)$ on A for each $z \in U(\mathfrak{g})^{\mathfrak{k}}$ by requiring that $(zf)_A = (e^\rho \circ \delta(z) \circ e^{-\rho}) f_A$ for $f \in C_\psi^\infty(G/K)$. Then if we define $C_\psi^\infty(A, \chi_\nu)$ as the space of all $\Phi \in C^\infty(A)$ satisfying $\delta(z)\Phi = \chi_\nu(z)\Phi$ for $z \in U(\mathfrak{g})^{\mathfrak{k}}$, we can deduce that $C_\psi^\infty(G/K, \chi_\nu)$ is isomorphic to $C_\psi^\infty(A, \chi_\nu)$ under the correspondence $f \mapsto e^{-\rho} f_A$ (see Proposition 3.1). Thus our problem of proving (I), (II) and (III) is reduced to showing the corresponding facts for the space $C_\psi^\infty(A, \chi_\nu)$. In this stage, the operator $\delta(\omega)$ where ω is the Casimir operator on G plays a key role. From the explicit form of $\delta(\omega)$ given in Lemma 2.8, we conclude that it is an elliptic operator on A and hence (I) holds. The statement (II) is based on the fact that any differential operator on A with constant coefficients can be written as the compositions of certain w such operators and the elements of $\delta(U(\mathfrak{g})^{\mathfrak{k}})$ where w is the order of W (cf. Proposition 2.7). To establish (III), we introduce a series $\Phi(h: \nu, \psi) = h^\nu \sum_{\lambda \in L} a_\lambda(\nu) h^\lambda$ on A where the coefficients $a_\lambda(\nu)$ are given by the recursion formula (4.1). Applying the estimate for $a_\lambda(\nu)$ given in Lemma 4.5, we can deduce that $\Phi(h: \nu, \psi)$ is convergent uniformly on every compact subset in A. Moreover we can check directly that $\Phi(h: \nu, \psi)$ is an eigenfunction of $\delta(\omega)$ with eigenvalue $\chi_\nu(\omega)$. This fact plays an essential role in proving that $\Phi(h: \nu, \psi)$ belongs to $C_\psi^\infty(A, \chi_\nu)$ (see Theorem 4.10). Using this function, we can construct an element $V(x: \nu, \psi)$ of $C_\psi^\infty(G/K, \chi_\nu)$ (cf. (4.10)).

The main technique of proving (IV) and (V) is as follows. For each character ψ of N, there corresponds a set of linear forms η_α on the root spaces $\mathfrak{g}_0^\alpha$ where α runs through the set Π of simple roots of G relative to A. We denote by F the set of simple roots α such that $\eta_\alpha \neq 0$. We note that ψ is a non-degenerate character if and only if $F=\Pi$. Put $F_*=-s_0^{-1}F$ where s_0 is the longest element of W. We denote by $P_{F_*}=N_{F_*}A_{F_*}M_{F_*}$ the Langlands decomposition of the parabolic subgroup P_{F_*} of G corresponding to the subset F_* of Π. Then the Whittaker function $W(x\colon \nu, \psi)$ on G can be written as the product of a certain meromorphic function $c^{F_*}(\nu)$ and the Whittaker function $W(m_*\colon \nu_{F_*}, \psi_{F_*})$ on M_{F_*} (see Corollary 6.9). The important fact is that ψ_{F_*} is the non-degenerate character of the maximal nilpotent subgroup $N(F_*)$ of M_{F_*}. In this way, our problem is reduced to that of proving our assertions in the case of non-degenerate characters. As was already mentioned, in this case (IV) follows from Theorem 4.8 in [4]. To establish (V), we need the asymptotic behavior of $W(x\colon \nu, \psi)$ (cf. Lemma 7.1). Applying it, we can determine the coefficient of $V(x\colon s_0\nu, \psi)$. The another coefficients are determined by using the functional equations of the Whittaker functions and the above result (cf. Lemma 7.7).

§ 1. Preliminaries

Let G be a connected, noncompact, semisimple Lie group with finite center. Let $\mathfrak{g}_0$ be the Lie algebra of G. We denote the complexification of $\mathfrak{g}_0$ by $\mathfrak{g}$. Let $B(X, Y)$ $(X, Y\in\mathfrak{g})$ be the Killing form on $\mathfrak{g}$. Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}_0$. We denote by $\mathfrak{p}_0$ the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Killing form. Let θ be the corresponding Cartan involution of $\mathfrak{g}_0$.

Let $\mathfrak{a}_0$ be a maximal abelian subspace in $\mathfrak{p}_0$. For each non-zero element α of the dual space $\mathfrak{a}_0^*$ of $\mathfrak{a}_0$, we set $\mathfrak{g}_0^\alpha=\{X\in\mathfrak{g}_0;\ \mathrm{ad}(H)X=\alpha(H)X \text{ for all } H\in\mathfrak{a}_0\}$. We say that $\alpha\in\mathfrak{a}_0^*-(0)$ is a root of $\mathfrak{g}_0$ relative to $\mathfrak{a}_0$ if $\mathfrak{g}_0^\alpha\neq(0)$. Let Σ be the set of all roots of $\mathfrak{g}_0$ relative to $\mathfrak{a}_0$. We put $m(\alpha)=\dim\mathfrak{g}_0^\alpha$ for every $\alpha\in\Sigma$. Let Σ_+ be a positive system of roots in Σ and let $\Pi=\{\alpha_1,\dots,\alpha_l\}$ be the corresponding set of simple roots. Let W be the Weyl group of the root system Σ, that is, the group generated by the reflections s_α $(\alpha\in\Pi)$. Then W is isomorphic to M^*/M, where M^* (resp. M) denotes the normalizer (resp. centralizer) of $\mathfrak{a}_0$ in K. In what follows, we often write a representative in M^* of an element s of W by the same letter. Since the Killing form is positive definite on $\mathfrak{a}_0$, it induces an inner product $\langle\ ,\ \rangle$ on $\mathfrak{a}_0^*$, which is extended to a non-degenerate symmetric bilinear form on the complex dual $\mathfrak{a}^*$ of $\mathfrak{a}_0$. For each $\nu\in\mathfrak{a}^*$, we define an element H_ν of the complexification $\mathfrak{a}$ of $\mathfrak{a}_0$ by $B(H, H_\nu)=\nu(H)$ for all $H\in\mathfrak{a}_0$. Then it holds that $\langle\mu, \nu\rangle=B(H_\mu, H_\nu)$ for $\mu, \nu\in\mathfrak{a}^*$.

Let $A=\exp\mathfrak{a}_0$ be the analytic subgroup of G with Lie algebra $\mathfrak{a}_0$. For $\nu\in\mathfrak{a}^*$, we set $h^\nu=\exp\nu(H)$ where $h=\exp H\in A$. Let ρ be the element of $\mathfrak{a}_0^*$ such that

$$\rho = 2^{-1}\textstyle\sum_{\alpha\in\Sigma_+} m(\alpha)\alpha.$$

We denote by $\mathfrak{n}_0$ (resp. $\bar{\mathfrak{n}}_0$) the subalgebra of $\mathfrak{g}_0$ given by

$$\mathfrak{n}_0 = \sum_{\alpha\in\Sigma_+} \mathfrak{g}_0^\alpha \quad (\text{resp. } \bar{\mathfrak{n}}_0 = \sum_{\alpha\in\Sigma_+} \mathfrak{g}_0^{-\alpha}).$$

Let $N=\exp\mathfrak{n}_0$ (resp. $\bar{N}=\exp\bar{\mathfrak{n}}_0$) be the analytic subgroup of G corresponding to $\mathfrak{n}_0$ (resp. $\bar{\mathfrak{n}}_0$). Then we know that $\mathfrak{g}_0$ is a direct sum of $\mathfrak{n}_0$, $\mathfrak{a}_0$ and $\mathfrak{k}_0$. Moreover the map $(n, h, k)\mapsto nhk$ is an analytic isomorphism of $N\times A\times K$ onto G and hence $G=NAK$, which is called an Iwasawa decomposition of G.

Let N^* be the set of all characters, namely, all one dimensional unitary respresentations of N. For each $\psi\in N^*$, there exists a unique Lie algebra homomorphism η of $\mathfrak{n}_0$ into $\boldsymbol{R}$ such that $\psi(n)=\exp(i\eta(X))$ where $n=\exp X\in N$. Since η is trivial on $[\mathfrak{n}_0, \mathfrak{n}_0]$, it induces a linear form on $\mathfrak{n}_0/[\mathfrak{n}_0, \mathfrak{n}_0]$. But since

$$\mathfrak{n}_0 = \sum_{\alpha\in\Pi} \mathfrak{g}_0^\alpha \oplus [\mathfrak{n}_0, \mathfrak{n}_0],$$

it can be identified with a linear form on $\sum_{\alpha\in\Pi}\mathfrak{g}_0^\alpha$. Let η_α be the restriction of η to $\mathfrak{g}_0^\alpha$ $(\alpha\in\Pi)$. We say that η is the Lie algebra homomorphism of $\mathfrak{n}_0$ corresponding to ψ and we often write $\psi=\psi_\eta$. If ψ is an element of N^* such that all η_α $(\alpha\in\Pi)$ are nonzero linear forms on $\mathfrak{g}_0^\alpha$, it is called a non-degenerate character of N.

For later use, we shall extend the notion of the non-degenerate character of N to that of certain subgroups of N. Let F be an arbitrary subset of Π. We denote by $\Sigma_+(F)$ the set of roots in Σ_+ which are integral linear combinations of the elements of F. Then $\Sigma_+(F)$ is a positive system of the root system $\Sigma_+(F)\cup -\Sigma_+(F)$ and F is the set of simple roots of $\Sigma_+(F)$. We define a subalgebra of $\mathfrak{n}_0$ by $\mathfrak{n}_0(F)=\sum_{\alpha\in\Sigma_+(F)}\mathfrak{g}_0^\alpha$ and put $N(F)=\exp\mathfrak{n}_0(F)$. Then it is an analytic subgroup of N. We denote by ψ_F the restriction of ψ to $N(F)$. We say that ψ_F is a non-degenerate character of $N(F)$ if $\eta_\alpha\neq 0$ for all $\alpha\in F$.

Now we shall give a normalization of Haar measures of N and $\bar{N}$. Recall that $-B(X, \theta Y)$ $(X, Y\in\mathfrak{g}_0)$ defines an inner product on $\mathfrak{g}_0$. It also induces an inner product on $\mathfrak{g}_0^\alpha$ for all $\alpha\in\Sigma$, with respect to which they are mutually orthogonal. Hence $\bar{\mathfrak{n}}_0$ is an euclidean space with the inner product induced by $-B(X, \theta Y)$. Let dX be the corresponding euclidean measure on $\bar{\mathfrak{n}}_0$. Since the exponential map of $\bar{\mathfrak{n}}_0$ onto $\bar{N}$ is an analytic isomorphism, there exists a unique Haar measure $d\bar{n}$ on $\bar{N}$ that corresponds to dX. Since $N=\theta\bar{N}$, we can normalize a Haar measure dn on N by $dn=\theta(d\bar{n})$.

Finally, for any subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ we write its complexification by $\mathfrak{h}$.

§2. Differential operators on $C_\psi^\infty(G/K)$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, which can be regarded as the algebra of left invariant differential operators on G. We denote the action of

$u \in U(\mathfrak{g})$ on $f \in C^\infty(G)$ at $x \in G$ by $(uf)(x)$, or equivalently by $f(x; u)$.

Let $\{U_d(\mathfrak{g})\}_{d \geq 0}$ be the canonical filtration of $U(\mathfrak{g})$. An element $u \in U(\mathfrak{g})$ is said to be of degree d if $u \in U_d(\mathfrak{g}) - U_{d-1}(\mathfrak{g})$. If $u \in U_d(\mathfrak{g})$ we say that u is of degree $\leq d$. The adjoint action of G on $\mathfrak{g}$ is naturally extended to $U(\mathfrak{g})$, which we denote by u^x with $x \in G$ and $u \in U(\mathfrak{g})$.

Let $U(\mathfrak{k})$, $U(\mathfrak{a})$ and $U(\mathfrak{n})$ be the universal enveloping algebras of $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ respectively, regarded as canonically embedded in $U(\mathfrak{g})$.

LEMMA 2.1 (Harish-Chandra [3]). *The following decomposition of* $U(\mathfrak{g})$ *holds*;

$$U(\mathfrak{g}) = U(\mathfrak{a}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}).$$

Namely, for each $u \in U(\mathfrak{g})$ *there exists a unique element* $\pi(u) \in U(\mathfrak{a})$ *such that* $u - \pi(u) \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$.

Let $p \mapsto p'$ be the unique automorphism of $U(\mathfrak{a})$ which takes $H \in \mathfrak{a}$ to $H + \rho(H)$. We define the map $\gamma\colon U(\mathfrak{g}) \to U(\mathfrak{a})$ by

$$\gamma(u) = \pi(u)' \qquad \text{for} \quad u \in U(\mathfrak{g}). \tag{2.1}$$

Since $\mathfrak{a}$ is abelian, $U(\mathfrak{a})$ can be identified with the symmetric algebra $S(\mathfrak{a})$ and hence with the algebra of polynomial functions on $\mathfrak{a}^*$. Let J be the algebra of W-invariants in $S(\mathfrak{a})$, or equivalently in $U(\mathfrak{a})$. Let $U(\mathfrak{g})^{\mathfrak{k}}$ be the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$. Then the restriction of γ to $U(\mathfrak{g})^{\mathfrak{k}}$ is known to have the following remarkable properties.

THEOREM 2.2 (Harish-Chandra [3]). *The map* γ *induces an algebra homomorphism of* $U(\mathfrak{g})^{\mathfrak{k}}$ *into* $U(\mathfrak{a})$ *with kernel* $U(\mathfrak{g})^{\mathfrak{k}} \cap U(\mathfrak{g})\mathfrak{k}$ *and image* J. *The quotient* $U(\mathfrak{g})^{\mathfrak{k}}/U(\mathfrak{g})^{\mathfrak{k}} \cap U(\mathfrak{g})\mathfrak{k}$ *and hence* J *can be viewed as the algebra of all left invariant differential operators on* G/K.

Let $\psi \in N^*$ and let $C^\infty_\psi(G/K)$ be the space of smooth functions f on G such that $f(ngk) = \psi(n)f(g)$ for $n \in N$, $g \in G$ and $k \in K$. We shall consider the action of $u \in U(\mathfrak{g})$ on $C^\infty_\psi(G/K)$. We notice that in general uf does not belong to $C^\infty_\psi(G/K)$ even if $f \in C^\infty_\psi(G/K)$, whereas if $u \in U(\mathfrak{g})^{\mathfrak{k}}$ and $f \in C^\infty_\psi(G/K)$ then $uf \in C^\infty_\psi(G/K)$. Because the action of u commutes with the right translation by elements of K. We further remark that since all elements of $C^\infty_\psi(G/K)$ are right K-invariant, each element of $U(\mathfrak{g})\mathfrak{k}$ acts trivially on $C^\infty_\psi(G/K)$.

In the sequel, we often identify $p \in U(\mathfrak{a})$ with a polynomial function on $\mathfrak{a}^*$ and denote the value of p at $\nu \in \mathfrak{a}^*$ by $p(\nu)$. For $\nu \in \mathfrak{a}^*$, we define

$$\chi_\nu(u) = \gamma(u)(\nu) \qquad \text{for} \quad u \in U(\mathfrak{g})^{\mathfrak{k}}. \tag{2.2}$$

Then Theorem 2.2 implies that χ_ν is an algebra homomorphism of $U(\mathfrak{g})^{\mathfrak{k}}$ into

$\boldsymbol{C}$ which is trivial on $U(\mathfrak{g})^{\mathfrak{k}} \cap U(\mathfrak{g})\mathfrak{k}$. Moreover it holds that $\chi_\mu = \chi_\nu$ for $\mu, \nu \in \mathfrak{a}^*$ if and only if there exists $s \in W$ such that $\mu = s\nu$.

Let χ be an algebra homomorphism of $U(\mathfrak{g})^{\mathfrak{k}}$ into $\boldsymbol{C}$. Let $C_\psi^\infty(G/K, \chi)$ be be the space of all joint eigenfunctions in $C_\psi^\infty(G/K)$:

$$C_\psi^\infty(G/K, \chi) = \{f \in C_\psi^\infty(G/K);\ zf = \chi(z)f \text{ for } z \in U(\mathfrak{g})^{\mathfrak{k}}\}.$$

Using the above results on the action of $U(\mathfrak{g})^{\mathfrak{k}}$ on $C_\psi^\infty(G/K)$, we may assume that χ is of the form χ_ν for some $\nu \in \mathfrak{a}^*$. Let f be an arbitrary element of $C_\psi^\infty(G/K)$. Then $f(nhk) = \psi(n) f(h)$ for $n \in N$, $h \in A$, and $k \in K$. Hence f is completely determined by its restriction f_A to A. In fact the map $f \mapsto f_A$ is a linear isomorphism of $C_\psi^\infty(G/K)$ onto $C^\infty(A)$.

For studying the structure of $C_\psi^\infty(G/K, \chi_\nu)$, we shall replace the differential equations on $C_\psi^\infty(G/K)$ by those on $C^\infty(A)$. Let $\mathscr{R}^+$ be the ring of analytic functions of A generated (without 1) by the functions $h^\alpha (\alpha \in \Pi)$ where Π is the set of simple roots in Σ^+.

LEMMA 2.3. *Let $u \in U_d(\mathfrak{g})$. Then we can select a finite set of elements $g_j \in \mathscr{R}^+$, $w_j \in U(\mathfrak{n})$ and $p_j \in U(\mathfrak{a})$ $(1 \leq j \leq r)$ such that*

(i) $\deg(p_j) \leq d-1$ *and* $\deg(w_j) + \deg(p_j) \leq d$,

(ii) *for all* $h \in A$,

$$u \equiv \pi(u) + \textstyle\sum_{1 \leq j \leq r} g_j(h) w_j^{h^{-1}} p_j \mod U(\mathfrak{g})\mathfrak{k}. \tag{2.3}$$

PROOF. We shall proceed the proof by induction on $d = \deg(u)$. The case $d=0$ is trivial. Let $d=1$ and $u = X \in \mathfrak{g}$. If $X \in \mathfrak{a}$ or $\mathfrak{k}$, the lemma is clear. Suppose $X \in \mathfrak{n}$. Since $\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}^\alpha$, we have only to show the lemma when $X \in \mathfrak{g}^\alpha$. But then $X = h^\alpha X^{h^{-1}} (h \in A)$. Since h^α $(\alpha \in \Sigma_+)$ belong to $\mathscr{R}^+$, the lemma holds. Now let $u \in U_d(\mathfrak{g})$. Then by Lemma 2.1, there exists $u_1 \in \mathfrak{n}U(\mathfrak{g})$ such that $u \equiv \pi(u) + u_1 \bmod U(\mathfrak{g})\mathfrak{k}$. By choosing suitable elements $X_\alpha \in \mathfrak{g}^\alpha$ and $u_\alpha \in U_{d-1}(\mathfrak{g})$ $(\alpha \in \Sigma_+)$, we can write

$$u_1 = \textstyle\sum_{\alpha \in \Sigma_+} X_\alpha u_\alpha.$$

Consequently it follows that

$$u \equiv \pi(u) + \textstyle\sum_{\alpha \in \Sigma_+} h^\alpha X_\alpha^{h^{-1}} u_\alpha \mod U(\mathfrak{g})\mathfrak{k}.$$

Applying the induction hypothesis on u_α, we can obtain the lemma.

Using Lemma 2.3, we shall introduce a differential operator $\delta_0(u)$ on A for $u \in U(\mathfrak{g})$ with coefficients in the ring $\mathscr{R}$ of analytic functions on A generated by 1 and $\mathscr{R}^+$. First we note that the differential of ψ induces an algebra homomorphism of $U(\mathfrak{n})$ into $\boldsymbol{C}$, which we denote again by the same letter ψ. Retaining the notations in Lemma 2.3, we define for $u \in U(\mathfrak{g})$, a differential operator on A, by

(2.4) $$\delta_0(u) = \pi(u) + \sum_{1\le j\le r} \psi(w_j)g_j(h)p_j.$$

PROPOSITION 2.4. *For $u \in U(\mathfrak{g})$ and $f \in C^\infty_\psi(G/K)$, we have*

(2.5) $$(uf)(h) = (\delta_0(u)f_A)(h) \qquad (h \in A).$$

Moreover if $z_1, z_2 \in U(\mathfrak{g})^{\mathfrak{k}}$ and $f \in C^\infty_\psi(G/K)$, then

(2.6) $$(z_1 z_2 f)(h) = (\delta_0(z_1)\delta_0(z_2)f_A)(h) \qquad (h \in A).$$

PROOF. Since f is right K-invariant, (2.3) implies that

$$(uf)(h) = f(h;\, \pi(u)) + \sum g_j(h)f(h;\, w_j^{h^{-1}}p_j).$$

But if $X \in \mathfrak{n}_0$, then for $f \in C^\infty_\psi(G/K)$,

$$f(h;\, X^{h^{-1}}) = (d/dt)f(h \exp(tX^{h^{-1}}))\,|_{t=0} = (d/dt)f(\exp(tX)h)\,|_{t=0} = \psi(X)f(h).$$

This implies that

$$f(h;\, w_j^{h^{-1}}p_j) = \psi(w_j)f(h;\, p_j).$$

Thus we obtain

$$(uf)(h) = f(h;\, \pi(u)) + \sum \psi(w_j)g_j(h)f(h;\, p_j).$$

From (2.4) it follows that the right hand side is clearly equal to $\delta_0(u)f_A(h)$. If $z \in U(\mathfrak{g})^{\mathfrak{k}}$ and $f \in C^\infty_\psi(G/K)$, then we know that $zf \in C^\infty_\psi(G/K)$. Thus the assertion (2.6) is a simple consequence of (2.5).

DEFINITION 2.5. The differential operator $\delta_0(u)$ is called *the radial part of* $u \in U(\mathfrak{g})$.

We denote the composition of differential operators D_1, D_2 on A with analytic coefficients by $D_1 \circ D_2$. The multiplication by an analytic function may be regarded as a differential operator on A. Let e^ρ (resp. $e^{-\rho}$) be the analytic function on A defined by $e^\rho(h) = h^\rho$ (resp. $e^{-\rho}(h) = h^{-\rho}$). For each differential operator D on A, we introduce a new differential operator D' by $D' = e^{-\rho} \circ D \circ e^\rho$. Then for $p \in U(\mathfrak{a})$, viewed as a differential operator on A, we see easily that $p' = e^{-\rho} \circ p \circ e^\rho$ is equal to the image of p under the automorphism of $U(\mathfrak{a})$ defined earlier.

We define a differential opeator $\delta(u)$ for $u \in U(\mathfrak{g})$ by $\delta(u) = \delta_0(u)'$. Then $\delta(u)$ is again a differential operator on A with coefficients in $\mathscr{R}$.

LEMMA 2.6. *Let $u \in U_d(\mathfrak{g})$. Then we can choose a finite set of elements $f_j \in \mathscr{R}^+$ and $q_j \in U(\mathfrak{a})$ of degree $\le d-1$ such that*

(2.7) $$\delta(u) = \gamma(u) + \sum f_j q_j.$$

PROOF. If we recall that $\gamma(u)=\pi(u)'$, then the lemma follows immediately from (2.4).

It is well known (cf. Harish-Chandra [3]) that $U(\mathfrak{a})$ is a free J-module of rank w where w is the order of W. Furthermore there exist homogeneous elements $\omega_1=1$, $\omega_2,\dots,\omega_w$ in $U(\mathfrak{a})$ such that $U(\mathfrak{a})=\sum_{1\leq j\leq w}\omega_j J$. Since $\gamma(U(\mathfrak{g})^{\mathfrak{k}})=J$, there exist $z_i\in U(\mathfrak{g})^{\mathfrak{k}}$ $(1\leq i\leq w)$ such that every $p\in U(\mathfrak{a})$ can be written as $p=\sum_{1\leq i\leq w}\omega_i\gamma(z_i)$.

PROPOSITION 2.7. *Let $p\in U(\mathfrak{a})$ and select $z_i\in U(\mathfrak{g})^{\mathfrak{k}}$ $(1\leq i\leq w)$ such that $p=\sum\omega_i\gamma(z_i)$. Then there exist a finite number of elements $g_{ij}\in\mathscr{R}^+$ and $z_{ij}\in U(\mathfrak{g})^{\mathfrak{k}}$ $(1\leq i\leq w,\ 1\leq j\leq r)$ such that*

$$p=\sum\omega_i\circ\delta(z_i)+\sum\sum g_{ij}\omega_i\circ\delta(z_{ij}) \tag{2.8}$$

where the index i (resp. j) runs through $\{1,\dots,w\}$ (resp. $\{1,\dots,r\}$).

PROOF. It follows from Lemma 2.6 that there exist a finite number of elements $f_{ij}\in\mathscr{R}^+$ and $q_{ij}\in U(\mathfrak{a})$ for each i such that $\gamma(z_i)=\delta(z_i)+\sum f_{ij}q_{ij}$. Hence we have

$$p=\sum_{1\leq i\leq w}\omega_i\circ(\delta(z_i)+\sum f_{ij}q_{ij}).$$

Since $\mathscr{R}^+$ is stable under the differentiation by elements of $U(\mathfrak{a})$, we may write

$$p=\sum\omega_i\circ\delta(z_i)+\sum\sum g_{ij}\omega_i\circ p_{ij}$$

for some choice of $g_{ij}\in\mathscr{R}^+$ and $p_{ij}\in U(\mathfrak{a})$. Note that $\deg(\omega_i p_{ij})\leq\deg(p)-1$. Applying the induction hypothesis on $\omega_i p_{ij}\in U(\mathfrak{a})$, we obtain the proposition.

For later use, we shall give the explicit formulas of $\delta_0(\omega)$ and $\delta(\omega)$ for the Casimir operator ω on G. The Casimir operator ω is an element of the center of $U(\mathfrak{g})$ and hence $\omega\in U(\mathfrak{g})^{\mathfrak{k}}$, which is defined as follows. Let $\mathfrak{m}_0$ be the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$. Then it is well known that $\mathfrak{g}_0=\bar{\mathfrak{n}}_0\oplus\mathfrak{m}_0\oplus\mathfrak{a}_0\oplus\mathfrak{n}_0$ where $\bar{\mathfrak{n}}_0=\theta\mathfrak{n}_0$. Let $H_1,\dots,H_l$ be the orthonormal basis of $\mathfrak{a}_0$ with respect to the Killing form and set

$$\omega_{\mathfrak{a}}=\sum_{1\leq i\leq l}H_i^2. \tag{2.9}$$

Let $U_1,\dots,U_r$ be a basis of $\mathfrak{m}_0$ such that $B(U_i,U_j)=-\delta_{ij}$ and set

$$\omega_{\mathfrak{m}}=-\sum_{1\leq i\leq r}U_i^2.$$

For each $\alpha\in\Sigma_+$, let $X_{\alpha,i}$ $(1\leq i\leq m(\alpha))$ be a basis of $\mathfrak{g}_0^{\alpha}$ satisfying $B(X_{\alpha,i},\theta X_{\alpha,j})=-\delta_{ij}$ $(1\leq i,j\leq m(\alpha))$. Using the basis of $\mathfrak{g}_0$ chosen above, we define

$$\omega=\omega_{\mathfrak{a}}+\omega_{\mathfrak{m}}-\sum_{\alpha\in\Sigma_+}\sum_{1\leq i\leq m(\alpha)}(X_{\alpha,i}\theta X_{\alpha,i}+\theta X_{\alpha,i}X_{\alpha,i}).$$

We remark that the definition of ω is independent of the choice of a basis of $\mathfrak{g}_0$.

Let η be the Lie algebra homomorphism of $\mathfrak{n}_0$ into $\boldsymbol{R}$, which corresponds to $\psi \in N^*$. Then we have $\psi(X_{\alpha,j}) = i\eta(X_{\alpha,j})$ for all $\alpha \in \Sigma_+$ and $1 \leq j \leq m(\alpha)$. We remark that $\eta(X_{\alpha,j}) = 0$ unless $\alpha \in \Pi$. For each $\alpha \in \Pi$, we set

$$|\eta_\alpha|^2 = \textstyle\sum_{1 \leq j \leq m(\alpha)} \eta(X_{\alpha,j})^2. \tag{2.10}$$

Then $|\eta_\alpha|$ can be regarded as the length of the restriction η_α of η to $\mathfrak{g}_0^\alpha$.

LEMMA 2.8. *Let ω be the Casimir operator on G. Then the radial part $\delta_0(\omega)$ of ω is given by*

$$\delta_0(\omega) = \pi(\omega) - 2 \textstyle\sum_{\alpha \in \Pi} |\eta_\alpha|^2 h^{2\alpha} \tag{2.11}$$

where $\pi(\omega) = \omega_{\mathfrak{a}} - 2H_\rho$ and hence $\delta(\omega)$ is given by

$$\delta(\omega) = \gamma(\omega) - 2 \textstyle\sum_{\alpha \in \Pi} |\eta_\alpha|^2 h^{2\alpha} \tag{2.12}$$

where $\gamma(\omega) = \omega_{\mathfrak{a}} - \langle \rho, \rho \rangle$.

PROOF. Since $[\theta X_{\alpha,j}, X_{\alpha,j}] = H_\alpha$ for $\alpha \in \Sigma_+$ and $1 \leq j \leq m(\alpha)$, we can deduce from the expression of ω given above,

$$\omega = \omega_{\mathfrak{a}} - 2H_\rho + \omega_{\mathfrak{m}} - 2 \textstyle\sum_{\alpha \in \Sigma_+} \sum_{1 \leq j \leq m(\alpha)} X_{\alpha,j} \theta X_{\alpha,j}.$$

Put $Y_{\alpha,j} = X_{\alpha,j} + \theta X_{\alpha,j}$ for $\alpha \in \Sigma_+$ and $1 \leq j \leq m(\alpha)$. Then $Y_{\alpha,j} \in \mathfrak{k}_0$. Replacing $\theta X_{\alpha,j}$ by $Y_{\alpha,j} - X_{\alpha,j}$ and using the fact that $\omega_{\mathfrak{m}}, X_{\alpha,j} Y_{\alpha,j} \in U(\mathfrak{g})\mathfrak{k}$, we have

$$\omega \equiv \omega_{\mathfrak{a}} - 2H_\rho + 2 \textstyle\sum_{\alpha \in \Sigma_+} \sum_{1 \leq j \leq m(\alpha)} X_{\alpha,j}^2 \mod U(\mathfrak{g})\mathfrak{k}.$$

Hence we obtain

$$\omega \equiv \omega_{\mathfrak{a}} - 2H_\rho + 2 \textstyle\sum_{\alpha \in \Sigma_+} h^{2\alpha} \sum_{1 \leq j \leq m(\alpha)} (X_{\alpha,j}^{h^{-1}})^2 \mod U(\mathfrak{g})\mathfrak{k}.$$

From (2.3), we can deduce that

$$\pi(\omega) = \omega_{\mathfrak{a}} - 2H_\rho$$

and

$$\delta_0(\omega) = \pi(\omega) + 2\textstyle\sum_{\alpha \in \Sigma_+} h^{2\alpha} \sum_{1 \leq j \leq m(\alpha)} \psi(X_{\alpha,j})^2.$$

Since $\psi(X_{\alpha,j}) = i\eta(X_{\alpha,j})$ for $\alpha \in \Sigma_+$ $(1 \leq j \leq m(\alpha))$ and moreover $\eta(X_{\alpha,j}) = 0$ unless $\alpha \in \Pi$, we have

$$\delta_0(\omega) = \pi(\omega) - 2 \textstyle\sum_{\alpha \in \Pi} |\eta_\alpha|^2 h^{2\alpha}.$$

Since $\gamma(\omega) = \pi(\omega)'$ and $\delta(\omega) = \delta_0(\omega)'$, it follows that

$$\gamma(\omega) = \omega_a - \langle \rho, \rho \rangle$$

and

$$\delta(\omega) = \gamma(\omega) - 2 \sum_{\alpha\in\Pi} |\eta_\alpha|^2 h^{2\alpha}.$$

§ 3. Eigenfunctions for $U(\mathfrak{g})^{\mathfrak{k}}$ in $C_\psi^\infty(G/K)$

In this section we shall study the system of differential equations on $C_\psi^\infty(G/K)$:

$$(3.1) \qquad zf = \chi_\nu(z)f \quad \text{for} \quad z \in U(\mathfrak{g})^{\mathfrak{k}}.$$

Here $\chi_\nu(\nu \in \mathfrak{a}^*)$ is an algebra homomorphism of $U(\mathfrak{g})^{\mathfrak{k}}$ into $\boldsymbol{C}$ given by (2.2). As in § 2 we denote the space of all solutions of (3.1) by $C_\psi^\infty(G/K, \chi_\nu)$.

We shall reduce the differential equations (3.1) to a system of differential equations on A by using the results in § 2. Let $C_\psi^\infty(A, \chi_\nu)$ be the space of all solutions of the system of differential equations on A given by

$$(3.2) \qquad \delta(z)\Phi = \chi_\nu(z)\Phi \quad \text{for} \quad z \in U(\mathfrak{g})^{\mathfrak{k}}.$$

PROPOSITION 3.1. *The map $f \mapsto e^{-\rho}f_A$ gives a linear isomorphism of $C_\psi^\infty(G/K, \chi_\nu)$ onto $C_\psi^\infty(A, \chi_\nu)$.*

PROOF. The restriction f_A of $f \in C_\psi^\infty(G/K)$ to A belongs to $C^\infty(A)$. Conversely for $F \in C^\infty(A)$, if we define the function f on G by $f(nhk) = \psi(n)F(h)$ $(n \in N, h \in A, k \in K)$, then $f \in C_\psi^\infty(G/K)$ and $f_A = F$. This implies that the map $f \mapsto f_A$ gives a linear isomorphism of $C_\psi^\infty(G/K)$ onto $C^\infty(A)$. Moreover from Proposition 2.4 we know that $(zf)_A = \delta_0(z)f_A$ $(f \in C_\psi^\infty(G/K), z \in U(\mathfrak{g})^{\mathfrak{k}})$. This means that if $f \in C_\psi^\infty(G/K, \chi_\nu)$ then f_A satisfies

$$(3.3) \qquad \delta_0(z)f_A = \chi_\nu(z)f_A \quad \text{for} \quad z \in U(\mathfrak{g})^{\mathfrak{k}}$$

and conversely. Since $\delta(z) = e^{-\rho}\circ\delta_0(z)\circ e^{\rho}$, the function $\Phi = e^{-\rho}f_A$ $(f \in C_\psi^\infty(G/K, \chi_\nu))$ clearly belongs to $C_\psi^\infty(A, \chi_\nu)$. Conversely if $\Phi \in C_\psi^\infty(A, \chi_\nu)$, then $e^{\rho}\Phi$ satisfies (3.3). But then there exists a unique $f \in C_\psi^\infty(G/K, \chi_\nu)$ such that $f_A = e^{\rho}\Phi$. Thus we obtain the proposition.

PROPOSITION 3.2. *Every element of $C_\psi^\infty(G/K, \chi_\nu)$ is a real analytic function on G.*

PROOF. Since the function e^{ρ} is real analytic on A and the character ψ of N is also real analytic, we have only to show that every $\Phi \in C_\psi^\infty(A, \chi_\nu)$ is real analytic. Whereas Φ satisfies the differential equation $\delta(\omega)\Phi = \chi_\nu(\omega)\Phi$ where ω is the Casimir operator on G. From Lemma 2.8, it follows that

$$(3.4) \qquad (\omega_{\mathfrak{a}} - 2 \sum_{\alpha\in\Pi} |\eta_\alpha|^2 h^{2\alpha})\Phi = \langle \nu, \nu \rangle \Phi.$$

Here we used the fact that $\chi_\nu(\omega)=\langle\nu, \nu\rangle-\langle\rho, \rho\rangle$. Since the Killing form is positive definite on $\mathfrak{a}_0$, the differential operator $\omega_\mathfrak{a}$ defined in (2.9) is an elliptic operator. By the regularity theorem of elliptic operators, we see that the solution of (3.4) is real analytic.

THEOREM 3.3. *The space* $C_\psi^\infty(G/K, \chi_\nu)$ *is finite dimensional and its dimension does not exceed the order* w *of the Weyl group* W.

PROOF. In view of Proposition 3.1, it suffices to show $\dim C_\psi^\infty(A, \chi_\nu)\leq w$. Take an arbitary $h\in A$ and fix it. Define a linear map ε of $C_\psi^\infty(A, \chi_\nu)$ into $\boldsymbol{C}^w$ by $\varepsilon(\Phi)=(\Phi(h; \omega_1),\dots, \Phi(h; \omega_w))$ where $\omega_1=1,\dots, \omega_w$ are homogeneous generators of $U(\mathfrak{a})$ over J introduced in §2. We will show that ε is injective. From Proposition 2.7, it follows that each $p\in U(\mathfrak{a})$ can be written, by taking a finite set of elements z_i, $z_{ij}\in U(\mathfrak{g})^{\mathfrak{k}}$ and $g_{ij}\in\mathscr{R}^+$ $(1\leq i\leq w,\ 1\leq j\leq r)$,

$$p = \sum \omega_i\circ\delta(z_i) + \sum\sum g_{ij}(h)\omega_i\circ\delta(z_{ij}).$$

Consequently if $\Phi\in C_\psi^\infty(A, \chi_\nu)$, then

$$\begin{aligned}\Phi(h; p) &= \sum \chi_\nu(z_i)\Phi(h; \omega_i) + \sum\sum g_{ij}(h)\chi_\nu(z_{ij})\Phi(h; \omega_i)\\ &= \textstyle\sum_{1\leq i\leq w}(\chi_\nu(z_i) + \sum_j g_{ij}(h)\chi_\nu(z_{ij}))\Phi(h; \omega_i).\end{aligned}$$

This implies that if $\Phi(h; \omega_i)=0$ for $1\leq i\leq w$, then $\Phi(h; p)=0$ for all $p\in U(\mathfrak{a})$. Since Φ is real analytic, we can conclude that $\Phi=0$ in a neighborhoood of an arbitrary $h\in A$. But since A is connected, this means that $\Phi=0$ on A. Hence ε is injective and $\dim C_\psi^\infty(A, \chi_\nu)\leq w$.

§4. The functions $\Phi(h: \nu, \psi)$ and $V(x: \nu, \psi)$

Let $\psi\in N^*$ and η be the Lie algebra homomorphism of $\mathfrak{n}_0$ into $\boldsymbol{R}$ corresponding to ψ. Let L denote the set of all linear functions λ on $\mathfrak{a}$ of the form $\lambda=\sum_{\alpha\in\Pi} n_\alpha\alpha$ where n_α $(\alpha\in\Pi)$ are all non-negative integers. For $\lambda=\sum n_\alpha\alpha\in L$, we put $n(\lambda)=\sum n_\alpha$. Let $L'=L-(0)$. Since $\mathfrak{a}$ and $\mathfrak{a}^*$ are identified by means of the Killing form, we can identify the symmetric algebra $S(\mathfrak{a}^*)$ with the algebra of polynomial functions on $\mathfrak{a}^*$, so that $\lambda\in\mathfrak{a}^*$ is a linear function on $\mathfrak{a}^*$ by the rule $\nu\mapsto\langle\lambda, \nu\rangle$ $(\nu\in\mathfrak{a}^*)$. Let $Q(\mathfrak{a}^*)$ be the field of rational functions on $\mathfrak{a}^*$.

For each $\lambda\in L$, we shall define $a_\lambda\in Q(\mathfrak{a}^*)$ by induction on $n(\lambda)$ as follows. Let $a_0=1$ and for $\lambda\in L'$

$$(4.1)\qquad (\langle\lambda, \lambda\rangle+2\lambda)a_\lambda = 2\textstyle\sum_{\alpha\in\Pi}|\eta_\alpha|^2 a_{\lambda-2\alpha}.$$

For the sake of convienience, we put $a_\lambda=0$ if $\lambda\notin L$. Let σ_λ $(\lambda\in L')$ be the hyperplane in $\mathfrak{a}^*$ consisting of ν such that $2\langle\lambda, \nu\rangle+\langle\lambda, \lambda\rangle=0$. We denote by $'\mathfrak{a}^*$ the complement in $\mathfrak{a}^*$ of the union of all hyperplanes σ_λ $(\lambda\in L')$. Then $'\mathfrak{a}^*$ is an open, connected, dense subset in $\mathfrak{a}^*$. It is obvious that the rational functions $a_\lambda\,(\lambda\in L)$ take a well defined value at any point $\nu\in{}'\mathfrak{a}^*$. We remark that any compact subset of $\mathfrak{a}^*$ meets σ_λ for only a finite number of $\lambda\in L'$.

LEMMA 4.1. *If* $\lambda=\sum_{\alpha\in\Pi} n_\alpha\alpha\in L'$ *such that at least one* n_α *is odd, then* $a_\lambda=0$.

PROOF. We shall prove the lemma by induction on $n(\lambda)$. From the recursion formula (4.1), it follows that $a_\alpha=0$ for $\alpha\in\Pi$. Thus the lemma holds when $n(\lambda)=1$. Let $\lambda=\sum n_\alpha\alpha\in L'$ such that n_β is odd for $\beta\in\Pi$. Then all of $\lambda-2\alpha\,(\alpha\in\Pi)$ have an odd integer coefficient. Thus by induction hypothesis $a_{\lambda-2\alpha}=0$ for all $\alpha\in\Pi$. Hence by (4.1), $a_\lambda=0$.

In view of the lemma, we have only to consider those $\lambda\in L$ with even integral coefficients. The following lemma is an improvement of Lemma 4.1. Let F be the subset of Π given by $F=\{\alpha\in\Pi;\ |\eta_\alpha|\neq 0\}$. Then (4.1) can be written as

$$(\langle\lambda,\lambda\rangle+2\lambda)a_\lambda = 2\sum_{\alpha\in F}|\eta_\alpha|^2 a_{\lambda-2\alpha}\quad(\lambda\in L'). \tag{4.2}$$

LEMMA 4.2. *If* $\lambda=2\sum_{\alpha\in\Pi} n_\alpha\alpha\in L'$ *such that* $n_\beta\neq 0$ *for some* $\beta\in\Pi-F$, *then* $a_\lambda=0$.

PROOF. For each non-negative integer n, we set

$$L_{F,n} = \{\lambda = 2\sum_{\alpha\in\Pi} n_\alpha\alpha\in L';\ n_\beta\neq 0 \text{ for some } \beta\in\Pi-F \text{ and } \sum_{\alpha\in F} n_\alpha = n\}.$$

It suffices to show that if $\lambda\in L_{F,n}$ $(n\geq 0)$ then $a_\lambda=0$. We shall prove the lemma by induction on n. Let $n=0$. Then $\lambda\in L_{F,0}$ is of the form $2\sum_{\beta\in\Pi-F} n_\beta\beta$ and hence $\lambda-2\alpha\notin L$ for all $\alpha\in F$. Consequently the right hand side of (4.2) vahishes. But the coefficients $\langle\lambda,\lambda\rangle+2\lambda$ are not identically zero for $\lambda\in L_{F,0}$. Thus $a_\lambda=0$. If we notice that when $\lambda\in L_{F,n}$ then $\lambda-2\alpha\in L_{F,n-1}$ for all $\alpha\in F$, our lemma is an immediate consequence of the induction argument.

REMARK 4.3. If ψ is the trivial character and hence $\eta=0$, then clearly $a_\lambda=0$ for all $\lambda\in L'$.

In what follows we assume that ψ is a fixed non-trivial character unless otherwise stated.

COROLLARY 4.4. *Let* $\psi=\psi_\eta\in N^*$ *such that* $F=\{\alpha\}$, *that is*, $|\eta_\beta|=0$ *for* $\beta\in\Pi-\{\alpha\}$. *Then* $a_\lambda=0$ *unless* $\lambda=2n\alpha$ *and* $a_{2n\alpha}$ *is given by*

$$a_{2n\alpha}(\nu) = \left(\frac{|\eta_\alpha|^2}{2\langle\alpha,\alpha\rangle}\right)^n \frac{\Gamma(\nu_\alpha+1)}{n!\,\Gamma(\nu_\alpha+n+1)} \tag{4.3}$$

where $\nu_\alpha=\langle\nu, \alpha\rangle/\langle\alpha, \alpha\rangle$ *and* $\Gamma(\cdot)$ *is the classical gamma function.*

PROOF. The first assertion is obvious from Lemma 4.2. If $F=\{\alpha\}$ and $\lambda=2n\alpha$, then it follows from (4.2) that for $n\geq 1$

$$(4n^2\langle\alpha, \alpha\rangle+4n\langle\alpha, \nu\rangle)a_{2n\alpha}(\nu) = 2|\eta_\alpha|^2 a_{2(n-1)\alpha}(\nu)$$

and hence

$$a_{2n\alpha}(\nu)=(|\eta_\alpha|^2/2\langle\alpha, \alpha\rangle)\,(1/n(\nu_\alpha+n))\,a_{2(n-1)\alpha}(\nu).$$

This implies (4.3).

For each non-negative integer n, we set $L_n=\{\lambda=2\sum_{\alpha\in\Pi} n_\alpha\alpha\in L;\ \sum_{\alpha\in\Pi} n_\alpha= n\}$. The following estimate on a_λ is important to construct a certain solution of (3.2).

LEMMA 4.5. *Let U be an arbitrary compact subset in $'\mathfrak{a}^*$ and n an arbitrary non-negative integer. Then there exists a positive constant c depending only on U such that for $\nu\in U$ and $\lambda\in L_n$*

$$|a_\lambda(\nu)| \leq c^n/(n!)^2. \tag{4.4}$$

PROOF. The case $n=0$ is obvious. So we may assume $n\geq 1$. It is known (cf. [3]) that we can select a positive constant c_1 depending only on U such that $|\langle\lambda, \lambda\rangle+2\langle\lambda, \nu\rangle|\geq c_1 n^2$ for all $\lambda\in L_n$ and $\nu\in U$. If we put $c_2=\max\{2|\eta_\alpha|^2;\ \alpha\in\Pi\}$, then it follows from (4.1) that

$$|a_\lambda(\nu)|\leq c_2(c_1 n^2)^{-1}\textstyle\sum_{\alpha\in\Pi}|a_{\lambda-2\alpha}(\nu)|\,.$$

For $\nu\in U$, set $A_n(\nu)=\max\{|a_\lambda(\nu)|;\ \lambda\in L_n\}$. Then the above inequality implies that there exists a positive constant c such that $A_n(\nu)\leq cn^{-2}A_{n-1}(\nu)$. We define $B_n(\nu)$ by the recursion formula $B_0(\nu)=1$ and $B_n(\nu)=cn^{-2}B_{n-1}(\nu)$ for $n\geq 1$. Then it is obvious that $B_n(\nu)=c^n/(n!)^2$. On the other hand it holds by induction that $A_n(\nu)\leq B_n(\nu)$ for all n. Hence we obtain $A_n(\nu)\leq c^n/(n!)^2$ for $n\geq 0$. This immediately shows (4.4).

Fix $\psi=\psi_\eta\in N^*$ and consider the series

$$\Phi(h:\nu, \psi) = h^\nu \textstyle\sum_{\lambda\in L} a_\lambda(\nu)h^\lambda \tag{4.5}$$

where $\nu\in{}'\mathfrak{a}^*$, $h\in A$ and a_λ $(\lambda\in L)$ are defined by (4.1). We remark that when $\psi=\psi_0$ (the trivial character) it follows from Remark 4.3 that

$$\Phi(h:\nu, \psi_0) = h^\nu \quad \text{for} \quad h\in A \quad \text{and} \quad \nu\in\mathfrak{a}^*. \tag{4.6}$$

In what follows we again assume that $\psi=\psi_\eta$ is a non-trivial character of N.

LEMMA 4.6. *The series* $\Phi(h: \nu, \psi)$ *converges absolutely and uniformly for* $h \in A$ *and* $\nu \in {}'\mathfrak{a}^*$. *It defines an analytic function of* $(h, \nu) \in A \times {}'\mathfrak{a}^*$.

PROOF. It suffices to show that the series

$$\sum_{n\geq 0} \sum_{\lambda \in L_n} a_\lambda(\nu) h^\lambda$$

converges absolutely and uniformly on $A \times {}'\mathfrak{a}^*$. Let U and V be any relatively compact open subsets in ${}'\mathfrak{a}^*$ and A respectively. From Lemma 4.5, we can deduce that for $\nu \in U$,

$$|\sum_{n\geq 0} \sum_{\lambda \in L_n} a_\lambda(\nu) h^\lambda| \leq \sum_{n\geq 0} c^n/(n!)^2 \sum_{\lambda \in L_n} h^\lambda.$$

Let $\{H_1, H_2, \dots, H_l\}$ be the basis of $\mathfrak{a}_0$ which is dual to $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$. If we write $h = \exp(\sum_{1\leq i\leq l} t_i H_i)$, then $h^\lambda = \exp(2 \sum n_i t_i)$ for $\lambda = 2 \sum_{1\leq i\leq l} n_i \alpha_i \in L_n$. Put

$$r = \sup\{e^{t_i};\ h = \exp(\sum t_i H_i) \in V,\ 1 \leq i \leq l\}.$$

Then $r < +\infty$ and for any $(h, \nu) \in V \times U$

$$|\sum_{n\geq 0} \sum_{\lambda \in L_n} a_\lambda(\nu) h^\lambda| \leq \sum_{n\geq 0} |L_n| (cr^2)^n/(n!)^2 \tag{4.7}$$

where $|L_n|$ denotes the number of elements of L_n. Note that $|L_n| = (n+l-1)!/(l-1)!n!$, which is a polynomial in n of degree l. Hence the right hand side of (4.7) converges. This proves the lemma immediately.

COROLLARY 4.7. *Under the same assumption as in Corollary* 4.4, *we have*

$$\Phi(h: \nu, \psi) = \Gamma(\nu_\alpha + 1)(|\eta_\alpha|/\sqrt{2\langle\alpha, \alpha\rangle})^{-\nu_\alpha} h^{\nu - \nu_\alpha \alpha} I_{\nu_\alpha}(2|\eta_\alpha| h^\alpha/\sqrt{2\langle\alpha, \alpha\rangle}), \tag{4.8}$$

where $I_{\nu_\alpha}(\cdot)$ *denotes the modified Bessel function of fisrt kind and order* ν_α.

PROOF. In view of Corollary 4.4, we have $\Phi(h: \nu, \psi) = h^\nu \sum_{n\geq 0} a_{2n\alpha}(\nu) h^{2n\alpha}$, and by (4.3)

$$\Phi(h: \nu, \psi) = \Gamma(\nu_\alpha + 1) h^\nu \sum_{n\geq 0} (|\eta_\alpha| h^\alpha/(2\langle\alpha, \alpha\rangle)^{1/2})^{2n}/n!\Gamma(\nu_\alpha + n + 1).$$

Since $I_s(z) = (z/2)^s \sum_{n\geq 0} (z/2)^{2n}/n!\Gamma(s+n+1)$, we can easily obtain the corollary.

Our next aim is to show that as a function of h, $\Phi(h: \nu, \psi)$ belongs to $C_\psi^\infty(A, \chi_\nu)$. We start with the following lemma.

LEMMA 4.8. *Let* ω *be the Casimir operator on* G. *Then for* $h \in A$ *and* $\nu \in {}'\mathfrak{a}^*$,

$$\Phi(h; \delta(\omega): \nu, \psi) = \chi_\nu(\omega) \Phi(h: \nu, \psi).$$

PROOF. If we apply the formula (2.12) of $\delta(\omega)$ to $\Phi(h: \nu, \psi)$, we can obtain

$$\Phi(h;\, \delta(\omega)\colon \nu,\, \psi) = h^\nu \textstyle\sum_{\lambda\in L} (\langle \nu+\lambda,\, \nu+\lambda\rangle - \langle \rho,\, \rho\rangle) a_\lambda(\nu) h^\lambda$$
$$- h^\nu \textstyle\sum_{\lambda\in L} (2 \sum_{\alpha\in\Pi} |\eta_\alpha|^2 a_{\lambda-2\alpha}(\nu)) h^\lambda.$$

Since $\chi_\nu(\omega) = \langle \nu,\, \nu\rangle - \langle \rho,\, \rho\rangle$, it follows that

$$\Phi(h;\, \delta(\omega)\colon \nu,\, \psi) = \chi_\nu(\omega)\Phi(h\colon \nu,\, \psi)$$
$$+ h^\nu \textstyle\sum \{(\langle \lambda,\, \lambda\rangle + 2\langle \lambda, \nu\rangle) a_\lambda(\nu) - 2 \sum_{\alpha\in\Pi} |\eta_\alpha|^2 a_{\lambda-2\alpha}(\nu)\} h^\lambda.$$

However $a_\lambda(\nu)$ is defined by (4.1) and hence the second term vanishes. So we have the lemma.

To show that $\Phi(h\colon \nu,\, \psi) \in C^\infty_\psi(A,\, \chi_\nu)$, we shall need some preparations. Let $\mathscr{B}$ be the set of all mappings $b\colon \lambda \to b_\lambda$ of L into $\boldsymbol{C}$ such that the series $\sum_{\lambda\in L} b_\lambda h^\lambda$ gives an analytic function on A. For $\nu \in \mathfrak{a}^*$ and $b \in \mathscr{B}$, we define an analytic function on A by $\phi_\nu(h) = h^\nu \sum_{\lambda\in L} b_\lambda h^\lambda$. We shall compute $\phi_\nu(h;\, \delta(u))$ where $u \in U(\mathfrak{g})$. From Lemma 2.6 we know that there exist a finite number of elements $f_j \in \mathscr{R}^+$ and $q_j \in U(\mathfrak{a})$ such that $\delta(u) = \gamma(u) + \sum f_j q_j$ for $u \in U(\mathfrak{g})$. We remark that each $f \in \mathscr{R}^+$ can be written as $f(h) = \sum d_\mu h^\mu$ where μ runs through a finite subset of L'. Moreover for each $p \in U(\mathfrak{a})$ it holds that

$$\phi_\nu(h;\, p) = h^\nu \textstyle\sum_{\lambda\in L} p(\nu + \lambda) b_\lambda h^\lambda. \tag{4.9}$$

Combining these facts, we can deduce that $\phi_\nu(h;\, \delta(u))$ is again of the form $\phi_\nu(h;\, \delta(u)) = h^\nu \sum_{\lambda\in L} c_\lambda h^\lambda$ for a suitable choice of $c \in \mathscr{B}$. To make clear the dependence of c on ν, u and b, we will write $c(\nu,\, u,\, b)$ instead of c.

LEMMA 4.9. *Keeping the notations above, we have*

(i) *for fixed u and b, $c_\lambda(\nu,\, u,\, b)$ is a polynomial function of $\nu \in \mathfrak{a}^*$ for all $\lambda \in L$,*

(ii) $c_0(\nu,\, u,\, b) = \gamma(u)(\nu) b_0$,

(iii) $c_\lambda(\nu,\, \omega,\, b) = (\chi_\nu(\omega) + \langle \lambda,\, \lambda\rangle + 2\langle \lambda,\, \nu\rangle) b_\lambda - 2 \sum_{\alpha\in\Pi} |\eta_\alpha|^2 b_{\lambda-2\alpha}$ *for $\lambda \in L'$ where ω is the Casimir operator on G and finally*

(iv) $c_\lambda(\nu,\, z_1 z_2,\, b) = c_\lambda(\nu,\, z_1,\, c(\nu,\, z_2,\, b))$ *for $z_1,\, z_2 \in U(\mathfrak{g})^{\mathfrak{k}}$ and $\lambda \in L$.*

PROOF. The assertion (i) is clear from (4.9). We consider the term $\sum f_j(h)\phi_\nu(h;\, q_j)$ in $\phi_\nu(h;\, \delta(u))$. Since each $f_j \in \mathscr{R}^+$, the term corresponding to $\lambda = 0$ does not appear. This implies (ii). The proof of the assertion (iii) is quite analogous to that of Lemma 4.8. From Proposition 2.4 it follows that $\delta(z_1 z_2) = \delta(z_1)\delta(z_2)$ for $z_1,\, z_2 \in U(\mathfrak{g})^{\mathfrak{k}}$ and hence $\phi_\nu(h;\, \delta(z_1 z_2)) = \phi_\nu(h;\, \delta(z_1)\delta(z_2))$. This implies (iv).

THEOREM 4.10. *Let $\Phi(h\colon \nu,\, \psi)$ be the analytic function on $A \times {}'\mathfrak{a}^*$ defined by (4.5). Then it satisfies for all $z \in U(\mathfrak{g})^{\mathfrak{k}}$, $\Phi(h;\, \delta(z)\colon \nu,\, \psi) = \chi_\nu(z)\Phi(h\colon \nu,\, \psi)$.*

PROOF. For fixed $\nu\in{}'\mathfrak{a}^*$, we denote by $a(\nu)$ the mapping $\lambda\mapsto a_\lambda(\nu)$ of L into $\boldsymbol{C}$ defined by the recursion formula (4.1). Since $\Phi(h\colon \nu, \psi)=h^\nu \sum a_\lambda(\nu)h^\lambda$, Lemma 4.6 implies that $a(\nu)\in\mathscr{B}$. Remembering that the Casimir operator ω lies in the center of $U(\mathfrak{g})$ and hence $\omega z=z\omega$ for all $z\in U(\mathfrak{g})^{\mathfrak{k}}$, we can deduce from (iv) of Lemma 4.9 that

$$c(\nu,\ \omega,\ c(\nu,\ z,\ a(\nu))) = c(\nu,\ z,\ c(\nu,\ \omega,\ a(\nu)))\,.$$

However, we have already seen that $\Phi(h;\ \delta(\omega)\colon \nu,\ \psi)=\chi_\nu(\omega)\Phi(h\colon \nu,\ \psi)$ and hence $c(\nu,\ \omega,\ a(\nu))=\chi_\nu(\omega)a(\nu)$. Thus we get

$$c(\nu,\ \omega,\ c(\nu,\ z,\ a(\nu))) = \chi_\nu(\omega)c(\nu,\ z,\ a(\nu))\,.$$

Applying (iii) in Lemma 4.9, we obtain for $\lambda\in L'$,

$$\begin{aligned}\chi_\nu(\omega)c_\lambda(\nu,\ z,\ a(\nu)) &= (\chi_\nu(\omega)+\langle\lambda,\ \lambda\rangle+2\langle\lambda,\ \nu\rangle)c_\lambda(\nu,\ z,\ a(\nu))\\ &\quad -2\textstyle\sum_{\alpha\in\Pi}|\eta_\alpha|^2 c_{\lambda-2\alpha}(\nu,\ z,\ a(\nu))\end{aligned}$$

and hence

$$(\langle\lambda,\ \lambda\rangle+2\langle\lambda,\ \nu\rangle)c_\lambda(\nu,\ z,\ a(\nu)) = 2\textstyle\sum_{\alpha\in\Pi}|\eta_\alpha|^2 c_{\lambda-2\alpha}(\nu,\ z,\ a(\nu))\,.$$

Therefore $c_\lambda(\nu,\ z,\ a(\nu))$ $(\lambda\in L')$ satisfies the same recursion formula as that of $a_\lambda(\nu)$. The only difference lies in the initial terms. Combining these facts with (ii) in Lemma 4.9, we obtain $c_\lambda(\nu,\ z,\ a(\nu))=\chi_\nu(z)a_\lambda(\nu)$. Since

$$\Phi(h;\ \delta(z)\colon \nu,\ \psi) = h^\nu \textstyle\sum_{\lambda\in L} c_\lambda(\nu,\ z,\ a(\nu))h^\lambda,$$

it follows that $\Phi(h;\ \delta(z)\colon \nu,\ \psi)=\chi_\nu(z)\Phi(h\colon \nu,\ \psi)$.

COROLLARY 4.11. *Let $\psi\in N^*$ and define a function $V(x\colon \nu,\ \psi)$ on $G\times{}'\mathfrak{a}^*$ by*

$$V(x\colon \nu,\ \psi) = \psi(n(x))h(x)^\rho\Phi(h(x)\colon \nu,\ \psi) \tag{4.10}$$

where $x=n(x)h(x)k(x)$ is the Iwasawa decomposition of $x\in G$. Then $V(x\colon \nu,\ \psi)\in C_\psi^\infty(G/K,\ \chi_\nu)$.

PROOF. The corollary is a direct consequence of Proposition 3.1 and the above theorem.

Before ending this section, we will study the dependence of $\Phi(h\colon \nu,\ \psi)$ and hence $V(x\colon \nu,\ \psi)$ on $\psi\in N^*$ more closely. Let $\psi=\psi_\eta\in N^*$ and let $F=F(\psi)$ be the subset of Π such that $F=F(\psi)=\{\alpha\in\Pi;\ |\eta_\alpha|\neq 0\}$ where $|\eta_\alpha|$ is defined in (2.10). We remark that ψ is a non-degenerate character if and only if $F=\Pi$ and ψ is the trivial character if and only if $F=\phi$.

Let $L(F)=\{\lambda\in L;\ \lambda=\sum_{\alpha\in F} n_\alpha\alpha\}$ and $L(F)'=L(F)-(0)$. We denote by ${}'\mathfrak{a}_F^*$

the complement in $\mathfrak{a}^*$ of the union of all hyperplanes σ_λ ($\lambda \in L(F)'$). Clearly $'\mathfrak{a}_F^*$ contains $'\mathfrak{a}^*$. From Lemma 4.2, it follows that a_λ ($\lambda \in L(F)$) are well defined on $'\mathfrak{a}_F^*$ and moreover $\Phi(h: \nu, \psi)$ can be written

$$\Phi(h: \nu, \psi) = h^\nu \sum_{\lambda \in L(F)} a_\lambda(\nu) h^\lambda.$$

Without any essential change of the proof of Lemma 4.5, we can deduce that $\Phi(h: \nu, \psi)$ converges in fact for $(h, \nu) \in A \times {'\mathfrak{a}_F^*}$.

Let P_F be the standard parabolic subgroup of G corresponding to the subset $F = F(\psi)$ of Π. We denote the Langlands decomposition of P_F by $P_F = N_F A_F M_F$. The Lie algebra $\mathfrak{a}_{0,F}$ of A_F is given by $\{H \in \mathfrak{a}_0;\ \alpha(H) = 0 \text{ for all } \alpha \in F\}$. Let $\Sigma_+(F)$ be the subset of Σ_+ consisting of roots which are integral linear combinations of elements of F. Then the Lie algebra $\mathfrak{n}_{0,F}$ of N_F is given by $\mathfrak{n}_{0,F} = \sum_{\alpha \in \Sigma_+ - \Sigma_+(F)} \mathfrak{g}_0^\alpha$. Let $\mathfrak{a}_0(F) = \sum_{\alpha \in F} \boldsymbol{R} H_\alpha$. Then $\mathfrak{a}_0(F)$ is a subalgebra of $\mathfrak{a}_0$ and $\mathfrak{a}_0 = \mathfrak{a}_{0,F} \oplus \mathfrak{a}_0(F)$. If we denote by $A(F)$ the analytic subgroup of A with Lie algebra $\mathfrak{a}_0(F)$, then any $h \in A$ can be written uniquely as $h = h_1 h_2$ where $h_1 \in A_F$ and $h_2 \in A(F)$. Let $\mathfrak{n}_0(F)$ be the subalgebra of $\mathfrak{n}_0$ given by $\mathfrak{n}_0(F) = \sum_{\alpha \in \Sigma_+(F)} \mathfrak{g}_0^\alpha$ and $N(F)$ the corresponding analytic subgroup of N. Then $\mathfrak{n}_0 = \mathfrak{n}_{0,F} \oplus \mathfrak{n}_0(F)$ and the map $(n_1, n_2) \mapsto n_1 n_2$ of $N_F \times N(F)$ into N is an analytic isomorphism of varieties. By definition, $\psi(n_1) = 1$ for all $n_1 \in N_F$ and the restriction ψ_F of ψ to $N(F)$ induces a non-degenerate character of $N(F)$. We further remark that $N(F) = N \cap M_F$, $A(F) = A \cap M_F$ and if we put $K(F) = K \cap M_F$, then $M_F = N(F)A(F)K(F)$ is an Iwasawa decomposition of M_F compatible with that of G.

Using these facts, we proceed the study of $\Phi(h: \nu, \psi)$. Since $h_1^\alpha = 1$ for all $h_1 \in A_F$ and $\alpha \in F$, we can easily obtain

$$\Phi(h_1 h_2: \nu, \psi) = h_1^\nu \Phi(h_2: \nu, \psi) \quad (h_1 \in A_F,\ h_2 \in A(F)). \tag{4.11}$$

Furthermore, we can deduce from the recursion formula (4.2) that $a_\lambda(\nu)$ ($\lambda \in L(F)$) depend only on the restriction ν_F of ν to $\mathfrak{a}_0(F)$ and the restriction ψ_F of ψ to $N(F)$.

In view of the above results, we conclude that the function $\Phi(h_2: \nu, \psi)$ ($h_2 \in A(F)$) is nothing but the one constructed, by replacing the role of G by that of M_F, for the character ψ_F of $N(F)$ and $\nu_F \in \mathfrak{a}(F)^*$. Henceforth we may write $\Phi(h_2: \nu, \psi) = \Phi_F(h_2: \nu_F, \psi_F)$ if we emphasize its dependence on M_F.

Finally, we consider the function $V(x: \nu, \psi)$ introduced in (4.10). Recall that $V(nhk: \nu, \psi) = \psi(n) h^\rho \Phi(h: \nu, \psi)$ where $n \in N$, $h \in A$ and $k \in K$. If we write $n = n_1 n_2$ ($n_1 \in N_F$, $n_2 \in N(F)$) and $h = h_1 h_2$ ($h_1 \in A_F$, $h_2 \in A(F)$), then

$$V(nhk: \nu, \psi) = \psi(n_2) h_1^{\nu+\rho} h_2^\rho \Phi(h_2: \nu, \psi).$$

If we define $\rho(F) = 2^{-1} \sum_{\alpha \in \Sigma_+(F)} m(\alpha)\alpha$ and $\rho_F = \rho - \rho(F)$, then we can easily check that $h_1^\rho = h_1^{\rho_F}$ and $h_2^\rho = h_2^{\rho(F)}$. Hence

$$V(nhk\colon \nu, \psi) = h_1^{\nu+\rho_F}\psi_F(n_2)h_2^{\rho(F)}\Phi_F(h_2\colon \nu_F, \psi_F).$$

At this point, we define a function $V_F(m\colon \nu_F, \psi_F)$ on M_F by

$$(4.12) \qquad V_F(n_2h_2k_2\colon \nu_F, \psi_F) = \psi_F(n_2)h_2^{\rho(F)}\Phi_F(h_2\colon \nu_F, \psi_F),$$

where $n_2 \in N(F)$, $h_2 \in A(F)$ and $k_2 \in K(F)$. Using the decomposition $G = P_FK = N_FA_FM_FK$, we can conclude $V(n_1h_1mk\colon \nu, \psi) = h_1^{\nu+\rho_F}V_F(m\colon \nu_F, \psi_F)$ for $n_1 \in N_F$, $h_1 \in A_F$, $m \in M_F$ and $k \in K$. Thus the essential properties of $V(x\colon \nu, \psi)$ are reduced to those of $V_F(m\colon \nu_F, \psi_F)$, which is defined on the subgroup M_F of lower rank with a non-degenerate character ψ_F.

We summarize the above results in the following:

PROPOSITION 4.12. *Let $\psi \in N^*$ and set $F = \{\alpha \in \Pi;\ |\eta_\alpha| \neq 0\}$. If we write $x \in G$ as $x = n_1h_1mk$ according to the decomposition $G = N_FA_FM_FK$, then we have*

$$V(x\colon \nu, \psi) = h_1^{\nu+\rho_F}V_F(m\colon \nu_F, \psi_F)$$

where $V_F(m\colon \nu_F, \psi_F)$ is given by (4.12).

§5. The fundamental solutions

Using the results in the preceding sections, we shall construct w linearly independent elements of $C_\psi^\infty(G/K, \chi_\nu)$ for certain values $\nu \in \mathfrak{a}^*$. Here w is the order of W. The method is quite similar to the one developed by Harish-Chandra in [3].

Let $\nu \in \mathfrak{a}^*$ and define the subgroup W_ν of W by $W_\nu = \{s \in W;\ s\nu = \nu\}$. Let J_ν be the algebra of all W_ν-invariants in $S(\mathfrak{a})$. Then J_ν contains J. For $\mu \in \mathfrak{a}^*$, let $S(\mu)$ be the maximal ideal of $S(\mathfrak{a})$ such that $S(\mu) = \{p \in S(\mathfrak{a});\ p(\mu) = 0\}$ and set $J_\nu(\mu) = J_\nu \cap S(\mu)$.

For any open subset U in $\mathfrak{a}^*$, we denote the algebra of holomorphic functions on U by $\mathcal{O}(U)$. Clearly $S(\mathfrak{a})$ is regarded as a subalgebra of $\mathcal{O}(U)$. For each $\mu \in \mathfrak{a}^*$, let $\partial(\mu)$ be the derivation of $\mathcal{O}(U)$ defined by $f(\nu;\ \partial(\mu)) = (d/dt)f(\nu + t\mu)|_{t=0}$ for $f \in \mathcal{O}(U)$ and $\nu \in \mathfrak{a}^*$. It is obvious that the map $\mu \mapsto \partial(\mu)$ can be uniquely extended to an algebra isomorphism of the symmetric algebra $S(\mathfrak{a}^*)$ into the algebra of holomorphic differential operators on $\mathcal{O}(U)$.

For $\nu \in \mathfrak{a}^*$, let $\mathscr{H}(\nu)$ be the subspace of $S(\mathfrak{a}^*)$ given by

$$\mathscr{H}(\nu) = \{v \in S(\mathfrak{a}^*);\ p(\nu;\ \partial(v)) = 0 \text{ for all } p \in S(\mathfrak{a})J_\nu(\nu)\}.$$

Then it is well known (cf. [3]) that $S(\mathfrak{a}^*) = \mathscr{H}(\nu) \oplus S(\mathfrak{a}^*)J_\nu^+$ where J_ν^+ is an ideal of J_ν of elements of positive degree and moreover $\dim \mathscr{H}(\nu) = w(\nu)$. Here $w(\nu)$ is the order of W_ν.

Now fix $\psi \in N^*$ and let $F = F(\psi)$ be the subset of Π introduced in §4.

LEMMA 5.1. *Let* $\nu \in {}'\mathfrak{a}_F^*$. *For* $v \in \mathscr{H}(\nu)$, *we define a function* A *by* $\Phi_v(h) = \Phi(h: \nu; \partial(v), \psi)$. *Then* $\Phi_v \in C_\psi^\infty(A, \chi_\nu)$.

PROOF. We know from Theorem 4.10 that $\Phi(h; \delta(z): \nu, \psi) = \gamma(z)(\nu)\Phi(h: \nu, \psi)$ for $z \in U(\mathfrak{g})^{\mathfrak{k}}$. Since $\partial(v)$ commutes with $\delta(z)$, we have

$$\Phi_v(h; \delta(z)) = \Phi(h: \nu; \partial(v)\circ\gamma(z), \psi).$$

For each $z \in U(\mathfrak{g})^{\mathfrak{k}}$, let D_z be a differential operator on $\mathfrak{a}^*$ defined by $D_z = \partial(v)\circ \gamma(z) - \gamma(z)(\nu)\partial(v)$. Then for all $z \in U(\mathfrak{g})^{\mathfrak{k}}$, it holds that

$$\Phi_v(h; \delta(z)) - \gamma(z)(\nu)\Phi_v(h) = \Phi(h: \nu; D_z, \psi).$$

Hence it is sufficient to prove $D_z = 0$ for all z. Suppose $D_z \neq 0$ for some $z \in U(\mathfrak{g})^{\mathfrak{k}}$. Then we can select $p_1 \in S(\mathfrak{a})$ such that $p_1(\nu; D_z) \neq 0$. Put $p_2 = (\gamma(z) - \gamma(z)(\nu))p_1$. Then it is clear that $p_1(\nu; D_z) = p_2(\nu; \partial(v))$. On the other hand we know $\gamma(z) \in J$ and hence $\gamma(z) \in J_\nu$. From the definition of p_2, we have $p_2 \in S(\mathfrak{a})J_\nu(\nu)$. But since $v \in \mathscr{H}(\nu)$, it follows that $p_2(\nu; \partial(v)) = 0$ and consequently $p_1(\nu; D_z) = 0$. This contradicts the choice of p_1.

For $\nu \in \mathfrak{a}^*$ we put $r(\nu) = [W: W_\nu]$ and select a set of complete representatives $s_1 = 1, s_2, \dots, s_{r(\nu)}$ of W/W_ν. Then the elements $\nu_i = s_i\nu$ $(1 \leq i \leq r(\nu))$ are all distinct. Moreover each W_{ν_i} is isomorphic to W_ν and hence $w(\nu_i) = w(\nu)$ and $r(\nu_i) = r(\nu)$ for $1 \leq i \leq r(\nu)$.

Let Ω_F be the set of $\nu \in {}'\mathfrak{a}_F^*$ such that

(i) $\nu_i \in {}'\mathfrak{a}_F^*$ for $1 \leq i \leq r(\nu)$ and

(ii) $\nu_i - \nu_j \notin L(F)^\sim$ for any pair of indices $i \neq j$ $(1 \leq i, j \leq r(\nu))$, where $L(F)^\sim = \sum_{\alpha \in F} \boldsymbol{Z}\alpha$.

Then Ω_F is again a connected, open, dense subset of $\mathfrak{a}^*$. For simplicity, put $\mathscr{H}_i = \mathscr{H}(\nu_i)$ $(1 \leq i \leq r(\nu))$. Then $\dim \mathscr{H}_i = w(\nu)$ for all i. Let $\{v_{ij};\ 1 \leq j \leq w(\nu)\}$ be a basis of $\mathscr{H}_i$. We define w functions Φ_{ij} $(1 \leq i \leq r(\nu),\ 1 \leq j \leq w(\nu))$ on A by $\Phi_{ij}(h) = \Phi(h: \nu_i; \partial(v_{ij}), \psi)$.

LEMMA 5.2. *Let* $\nu \in \Omega_F$. *Then the above defined* w *functions* Φ_{ij} *form a basis of* $C_\psi^\infty(A, \chi_\nu)$.

PROOF. From Lemma 5.1, it follows that $\Phi_{ij} \in C_\psi^\infty(A, \chi_\nu)$. So we have only to show the following fact; if we choose non-zero elements $v_i \in \mathscr{H}_i$ $(1 \leq i \leq r(\nu))$, then the functions $\Phi_{v_i}(h) = \Phi(h: \nu_i; \partial(v_i), \psi)$ are linearly independent. For simplicity, we put $\xi_\lambda(h: \nu) = a_\lambda(\nu)h^{\nu+\lambda}$ for $\lambda \in L(F)$. Then we may write $\Phi(h: \nu, \psi) = \sum_{\lambda \in L(F)} \xi_\lambda(h: \nu)$. It can be easily checked that there exists a certain polynomial function $p_{\lambda,v}$ of $\log h \in \mathfrak{a}_0$ for $\lambda \in L(F)$ and $v \in S(\mathfrak{a}^*)$ such that

$$\xi_\lambda(h: \nu; \partial(v)) = p_{\lambda,v}(\log h)\, h^{\nu+\lambda}.$$

Hence we obtain

$$\Phi_{v_i}(h) = \sum_{\lambda\in L(F)} \xi_\lambda(h: v_i; \partial(v_i)) = \sum_{\lambda\in L(F)} p_{\lambda,v_i}(\log h)h^{v_i+\lambda}.$$

Now suppose that c_i $(1\leq i\leq r(v))$ are complex numbers such that $\sum c_i\Phi_{v_i}=0$. Then

$$\sum_{1\leq i\leq r(v)} \sum_{\lambda\in L(F)} c_i p_{\lambda,v_i}(\log h)h^{v_i+\lambda} = 0.$$

Since $v_i - v_j \notin L(F)^\sim (i\neq j)$, the exponents $v_i+\lambda$ $(1\leq i\leq r(v),\ \lambda\in L(F))$ are all distinct. By the above fact and Lemma 4.6, we can apply the corollary to Lemma 57 in [3]. The result is $c_i p_{\lambda,v_i}=0$ for $1\leq i\leq r(v)$ and $\lambda\in L(F)$. On the other hand, it is evident that $p_{0,v_i}(\log h)=v_i(\log h)$ for all i. Since $v_i\neq 0$, it follows that $p_{0,v_i}\neq 0$ and so $c_i=0$.

We say that v is a regular element of $\mathfrak{a}^*$ if $\langle v, \alpha\rangle\neq 0$ for all $\alpha\in\Sigma$. If v is a regular element, then $W_v=(1)$, all sv $(s\in W)$ are distinct and $\mathscr{H}(v)=(0)$.

Let Ω'_F be the set of regular elements $v\in\mathfrak{a}^*$ satisfying

(i) $sv\in {}'\mathfrak{a}_F^*$ for all $s\in W$ and

(ii) $sv-tv\notin L(F)^\sim$ for any pair $(s, t)\in W\times W$ such that $s\neq t$.

COROLLARY 5.3. *Let $v\in\Omega'_F$. Then w functions $\Phi(h: sv, \psi)$ $(s\in W)$ form a basis of $C_\psi^\infty(A, \chi_v)$.*

In view of Proposition 3.1 and the above corollary, we establish the following result.

THEOREM 5.4. *Let $v\in\Omega'_F$. Then the functions $V(x: sv, \psi)$ $(s\in W)$ form a basis of $C_\psi^\infty(G/K, \chi_v)$.*

§6. The Whittaker function $W(x: v, \psi)$

In this section, we introduce a joint eigenfunction $W(x: v, \psi)$ in $C_\psi^\infty(G/K, \chi_v)$, which is closely related to the Whittaker model of a class one principal series representation of G.

Let $v\in\mathfrak{a}^*$. We denote by X_v^∞ the space of all smooth functions φ on G satisfying $\varphi(nhmg)=h^{v+\rho}\varphi(g)$ for $n\in N$, $h\in A$, $m\in M$ and $g\in G$. Let π_v be the representation of G on X_v^∞ defined by $\pi_v(g)\varphi(x)=\varphi(xg)$ for $g, x\in G$ and $\varphi\in X_v^\infty$. The representation π_v is called a class one principal series representation of G. We denote by X_v the subspace of all K-finite elements in X_v^∞.

We define a function 1_v on G by

$$1_v(x) = h(x)^{v+\rho} \quad (x\in G) \tag{6.1}$$

where we write the Iwasawa decomposition of x as $x=n(x)h(x)k(x)$ with $n(x)\in N$,

$h(x) \in A$ and $k(x) \in K$. It can be easily checked that

$$1_\nu(nhmxk) = h^{\nu+\rho} 1_\nu(x) \tag{6.2}$$

for $n \in N$, $h \in A$, $m \in M$, $x \in G$ and $k \in K$. This means that the function 1_ν is a K-fixed element of X_ν. We remark that 1_ν satisfies

$$1_\nu(x; z) = \chi_\nu(z) 1_\nu(x) \quad (x \in G) \tag{6.3}$$

for all $z \in U(\mathfrak{g})^{\mathfrak{k}}$. This follows from Lemma 2.1 and the fact that the space of K-fixed elements in X_ν is one dimensional and stable under $U(\mathfrak{g})^{\mathfrak{k}}$.

Let $\psi = \psi_\eta \in N^*$ and $\nu \in \mathfrak{a}^*$. We introduce an integral $W(x\colon \nu, \psi)$ by

$$W(x\colon \nu, \psi) = \int_N 1_\nu(s_0^{-1} n x) \psi^{-1}(n) dn \quad (x \in G). \tag{6.4}$$

Here dn is the Haar measure on N normalized in § 1 and s_0 is a representative in K of the unique element, denoted by the same letter s_0, in W such that $s_0 \Sigma_+ = -\Sigma_+$. Note that (6.4) does not depend on the choice of the representatives of $s_0 \in W$. When ψ is a non-degenerate character, the above integral was already studied in [2], [4], [6] and [10].

Before considering the convergence of (6.4), we shall examine the formally consistent properties of the integral $W(x\colon \nu, \psi)$. It follows from (6.2) that

$$W(nxk\colon \nu, \psi) = \psi(n) W(x\colon \nu, \psi) \tag{6.5}$$

for $n \in N$, $x \in G$ and $k \in K$. Since A normalizes N and it holds that $d(hnh^{-1}) = h^{2\rho} dn$, we can deduce

$$W(h\colon \nu, \psi) = h^{s_0\nu+\rho} \int_N 1_\nu(s_0^{-1} n) \psi^h(n)^{-1} dn \quad (h \in A)$$

where ψ^h is a character of N given by

$$\psi^h(n) = \psi(hnh^{-1}) \quad (h \in A,\ n \in N).$$

When $x = e$ (the identity element of G), we denote the value $W(e\colon \nu, \psi)$ simply by $W(\nu, \psi)$, that is,

$$W(\nu, \psi) = \int_N 1_\nu(s_0^{-1} n) \psi^{-1}(n) dn. \tag{6.6}$$

Then we can write

$$W(h\colon \nu, \psi) = h^{s_0\nu+\rho} W(\nu, \psi^h) \quad (h \in A). \tag{6.7}$$

Hence we conclude from (6.5) and (6.7) that if $x = nhk$ (the Iwasawa decomposition of x),

$$(6.8) \qquad W(x: \nu, \psi) = \psi(n)h^{s_0\nu+\rho}W(\nu, \psi^h).$$

Thus the study of (6.4) can be reduced to that of $W(\nu, \psi)$. We shall rewrite it in a more convenient form. Recall that the map $\bar{n} \mapsto s_0\bar{n}s_0^{-1}$ is an analytic isomorphism of $\bar{N}$ onto N and it holds that $d(s_0\bar{n}s_0^{-1}) = d\bar{n}$ where $d\bar{n}$ is the Haar measure on $\bar{N}$ introduced in § 1. Since 1_ν is right K-invariant, it follows from (6.6)

$$(6.9) \qquad W(\nu, \psi) = \int_{\bar{N}} 1_\nu(\bar{n})\psi_*(\bar{n})^{-1}d\bar{n}$$

where ψ_* is a character of $\bar{N}$ defined by

$$\psi_*(\bar{n}) = \psi(s_0\bar{n}s_0^{-1}) \quad (\bar{n} \in \bar{N}).$$

Let D be the subset of $\mathfrak{a}^*$ given by

$$D = \{\nu \in \mathfrak{a}^*; \mathrm{Re}(\nu_\alpha) > 0 \text{ for all } \alpha \in \Sigma_+\}$$

where $\nu_\alpha = \langle \nu, \alpha\rangle / \langle \alpha, \alpha\rangle$ and $\mathrm{Re}(\nu_\alpha)$ denotes the real part of $\nu_\alpha \in \boldsymbol{C}$.

PROPOSITION 6.1. *Let $\psi \in N^*$. Then the integral $W(x: \nu, \psi)$ converges absolutely and uniformly for $(x, \nu) \in G \times D$. It gives a smooth function of $x \in G$, which is holomorphic in $\nu \in D$.*

PROOF. First we consider the case when $\psi = \psi_0$ (the trivial character of N). Since $\psi_0^h = \psi_0$ for $h \in A$, it follows from (6.8) and (6.9) that $W(x: \nu, \psi) = h^{s_0\nu+\rho} W(\nu, \psi_0)$ $(x = nhk)$ and

$$W(\nu, \psi_0) = \int_{\bar{N}} 1_\nu(\bar{n})d\bar{n}.$$

But this integral is well known to be uniformly convergent for $\nu \in D$, which is usually called Harish-Chandra's c-function and denoted by $c(\nu)$ (cf. [5]). Thus we obtain the proposition when $\psi = \psi_0$ and moreover

$$(6.10) \qquad W(x: \nu, \psi_0) = c(\nu)h^{s_0\nu+\rho} \quad (x = nhk).$$

Next we consider the general $\psi \in N^*$. Since $|\psi^h{}_*(\bar{n})| = 1$ for $h \in A$ and $\bar{n} \in \bar{N}$, we conclude from (6.9) that

$$|W(\nu, \psi^h)| \leq \int_{\bar{N}} |1_\nu(\bar{n})| d\bar{n}.$$

But since the right hand side is convergent for $\nu \in D$, $W(\nu, \psi^h)$ converges absolutely and uniformly for $(h, \nu) \in A \times D$. From this and (6.8), we get the proposition.

COROLLARY 6.2. *Let $\psi \in N^*$ and $\nu \in D$. Then $W(x: \nu, \psi) \in C_\psi^\infty(G/K, \chi_\nu)$.*

PROOF. The corollary is a direct consequence of (6.3) and the above proposition.

REMARK 6.3. We have already shown in [4] that if ψ is a non-degenerate character then $W(x: \nu, \psi)$ can be extended to an entire function of $\nu \in \mathfrak{a}^*$.

Our next aim is to prove that for general $\psi \in N^*$, the integral $W(x: \nu, \psi)$ can be continued to a meromorphic function of $\nu \in \mathfrak{a}^*$. For that purpose, we first write down the explicit formula of $c(\nu)$. Let Σ_+° be the set of $\alpha \in \Sigma_+$ such that $\alpha/2$ is not a root. For each $\alpha \in \Sigma_+^\circ$, we set

$$(6.11) \qquad c_\alpha(\nu) = d_\alpha \frac{\Gamma(\nu_\alpha)\Gamma(2^{-1}(\nu_\alpha + m(\alpha)/2))}{\Gamma(\nu_\alpha + m(\alpha)/2)\Gamma(2^{-1}(\nu_\alpha + m(\alpha)/2 + m(2\alpha)))}$$

where d_α is the constant given by

$$d_\alpha = 2^{(m(\alpha)-m(2\alpha))/2}(\pi/\langle\alpha, \alpha\rangle)^{(m(\alpha)+m(2\alpha))/2}.$$

Then it is well known (cf. [10]) that under the normalization of a Haar measure on $\bar{N}$ introduced in § 1, the c-function is given by

$$(6.12) \qquad c(\nu) = \prod_{\alpha \in \Sigma_+^\circ} c_\alpha(\nu).$$

This implies that $c(\nu)$ and hence $W(x: \nu, \psi_0)$ are in fact meromorphic functions of ν.

To proceed further, we shall need some preparations. Let $F = F(\psi)$ be the subset of Π such that $F = \{\alpha \in \Pi; |\eta_\alpha| \neq 0\}$. To begin with, we shall consider the map $\alpha \mapsto -s_0^{-1}\alpha$ of Σ into itself. Since $s_0^{-1} = s_0$ in W and $s_0\Sigma_+ = -\Sigma_+$, we have $-s_0^{-1}\Sigma_+ = \Sigma_+$ and hence $-s_0^{-1}\Pi \subset \Sigma_+$. But $-s_0^{-1}\Pi$ is a simple root system and consequently $-s_0^{-1}\Pi = \Pi$. If we set $F_* = -s_0^{-1}F = \{-s_0^{-1}\alpha; \alpha \in F\}$, then F_* is again a subset of Π and it holds that $-s_0F_* = F$.

Let P_{F_*} be the standard parabolic subgroup of G corresponding to the subset F_* of Π. We denote the Langlands decomposition of P_{F_*} by $P_{F_*} = N_{F_*}A_{F_*}M_{F_*}$. Let $\Sigma_+(F_*)$ be the subset of Σ_+ of integral linear combinations of the roots of F_*. Then the Lie algebra $\mathfrak{a}_{0,F_*}$ of A_{F_*} is given by $\{H \in \mathfrak{a}_0; \alpha(H) = 0 \text{ for all } \alpha \in F_*\}$ and the Lie algebra $\mathfrak{n}_{0,F_*}$ of N_{F_*} is of the form $\sum_{\alpha \in \Sigma_+ - \Sigma_+(F_*)} \mathfrak{g}_0^\alpha$. Put $\mathfrak{a}_0(F_*) = \sum_{\alpha \in \Sigma_+(F_*)} \boldsymbol{R}H_\alpha$ and let $A(F_*)$ be the analytic subgroup of A with Lie algebra $\mathfrak{a}_0(F_*)$. Moreover set $\mathfrak{n}_0(F_*) = \sum_{\alpha \in \Sigma_+(F_*)} \mathfrak{g}^\alpha$ and denote by $N(F_*)$ the analytic subgroup of N with Lie algebra $\mathfrak{n}_0(F_*)$. Then $A(F_*) = A \cap M_{F_*}$ and $N(F_*) = N \cap M_{F_*}$. Furthermore if we put $K(F_*) = K \cap M_{F_*}$, then it holds that $M_{F_*} = N(F_*)A(F_*)K(F_*)$ and it is an Iwasawa decomposition of M_{F_*} compatible with that of G. Finally we define subalgebras $\bar{\mathfrak{n}}_0(F_*)$ and $\bar{\mathfrak{n}}_{0,F_*}$ of $\bar{\mathfrak{n}}_0$ respectively by

$$\bar{\mathfrak{n}}_0(F_*) = \sum_{\alpha \in \Sigma_+(F_*)} \mathfrak{g}_0^{-\alpha}, \quad \bar{\mathfrak{n}}_{0,F_*} = \sum_{\alpha \in \Sigma_+ - \Sigma_+(F_*)} \mathfrak{g}_0^{-\alpha}.$$

Let $\bar{N}(F_*)$ and $\bar{N}_{F_*}$ be the analytic subgroup of $\bar{N}$ with Lie algebras $\bar{\mathfrak{n}}_0(F_*)$ and $\bar{\mathfrak{n}}_{0,F_*}$ respectively. Then the map $(\bar{n}_1, \bar{n}_2) \mapsto \bar{n}_1\bar{n}_2$ is an analytic isomorphism of $\bar{N}_{F_*} \times \bar{N}(F_*)$ onto $\bar{N}$.

LEMMA 6.4. *For $\nu \in D$, the integral $W(\nu, \psi)$ can be reduced to*

$$W(\nu, \psi) = c^{F_*}(\nu) \int_{\bar{N}(F_*)} 1_\nu(\bar{n}_2)\psi_*(\bar{n}_2)^{-1} d\bar{n}_2 \tag{6.13}$$

where $c^{F_*}(\nu)$ is given by

$$c^{F_*}(\nu) = \prod_{\alpha \in \Sigma_+^\circ - \Sigma_+(F_*)} c_\alpha(\nu). \tag{6.14}$$

PROOF. From (6.9) it follows that

$$W(\nu, \psi) = \int_{\bar{N}_{F_*} \times \bar{N}(F_*)} 1_\nu(\bar{n}_1\bar{n}_2)\psi_*(\bar{n}_1)^{-1}\psi_*(\bar{n}_2)^{-1} d\bar{n}_1 d\bar{n}_2.$$

We remark that since $-s_0F_* = F$ and consequently $s_0\bar{N}(F_*)s_0^{-1} = N(F)$, it follows that $\psi_*(\bar{n}_1) = 1$ for all $\bar{n}_1 \in \bar{N}_{F_*}$ and the restriction of ψ_* to $\bar{N}(F_*)$ is a non-degenerate character of $\bar{N}(F_*)$. Hence we have

$$W(\nu, \psi) = \int_{\bar{N}_{F_*} \times \bar{N}(F_*)} 1_\nu(\bar{n}_1\bar{n}_2)\psi_*(\bar{n}_2)^{-1} d\bar{n}_1 d\bar{n}_2.$$

Let $\bar{n}_2 = n_2h_2k_2$ be the Iwasawa decompsition of $\bar{n}_2$. Then $n_2 \in N(F_*)$, $h_2 \in A(F_*)$ and $k_2 \in K(F_*)$. Since the function 1_ν is right K-invariant, it holds that $1_\nu(\bar{n}_1\bar{n}_2) = 1_\nu(\bar{n}_1 n_2h_2)$. Moreover since $n_2h_2 \in M_{F_*}$, it follows that $v_1 = (n_2h_2)\bar{n}_1(n_2h_2)^{-1} \in \bar{N}_{F_*}$ and $dv_1 = d\bar{n}_1$. Using these facts, we obtain

$$W(\nu, \psi) = \int_{\bar{N}_{F_*} \times \bar{N}(F_*)} 1_\nu(n_2h_2\bar{n}_1)\psi_*(\bar{n}_2)^{-1} d\bar{n}_1 d\bar{n}_2.$$

But by (6.2), we know that $1_\nu(n_2h_2\bar{n}_1) = h_2^{\nu+\rho}1_\nu(\bar{n}_1)$ and hence $1_\nu(n_2h_2\bar{n}_1) = 1_\nu(\bar{n}_1)1_\nu(\bar{n}_2)$. Therefore the above integral can be decomposed into

$$W(\nu, \psi) = \int_{\bar{N}_{F_*}} 1_\nu(\bar{n}_1) d\bar{n}_1 \int_{\bar{N}(F_*)} 1_\nu(\bar{n}_2)\psi_*(\bar{n}_2)^{-1} d\bar{n}_2. \tag{6.15}$$

The first integral is evaluated as follows. We note that $c(\nu) = W(\nu, \psi_0)$ can be written, as in the same manner,

$$c(\nu) = \int_{\bar{N}_{F_*}} 1_\nu(\bar{n}_1) d\bar{n}_1 \int_{\bar{N}(F_*)} 1_\nu(\bar{n}_2) d\bar{n}_2.$$

The second integral can be viewed as the c-function for M_{F_*} and hence its value is given by $\prod_{\Sigma_+^\circ(F_*)} c_\alpha(\nu)$ where $\Sigma_+^\circ(F_*) = \Sigma_+^\circ \cap \Sigma_+(F_*)$. Consequently we can deduce from (6.12) that

$$\int_{\bar{N}_{F_*}} 1_\nu(\bar{n}_1)\, d\bar{n}_1 = \prod_{\alpha\in\Sigma_+^\circ-\Sigma_+(F_*)} c_\alpha(\nu).$$

Let W_{F_*} be the subgroup of W generated by the reflections $s_\alpha\,(\alpha\in F_*)$. We denote the longest element in W_{F_*} by s_1. Then $s_1^{-1}=s_1$ and $s_1\,\Sigma_+(F_*)=-\Sigma_+(F_*)$. Let s_* be the element of W such that $s_*=s_0 s_1^{-1}$. Then $F=-s_0(F_*)=s_*(F_*)$. Recall that we denote by P_F the standard parabolic subgroup of G corresponding to $F\subset\Pi$ and we write the Langlands decomposition of P_F as $P_F=N_F A_F M_F$. Furthermore we remember that $M_F=N(F)A(F)K(F)$ is an Iwasawa docomposition of M_F, which was constructed in §4. Since $s_*(F_*)=F$, it holds that $s_* P_{F_*} s_*^{-1}=P_F$, $s_* M_{F_*} s_*^{-1}=M_F$, $s_* A(F_*)s_*^{-1}=A(F)$ and $s_* N(F_*)s_*^{-1}=N(F)$.

Let ψ_{F_*} be a character of $N(F_*)$ defined by $\psi_{F_*}(n_2)=\psi(s_* n_2 s_*^{-1})$ for $n_2\in N(F_*)$. Since the restriction ψ_F of ψ to $N(F)$ is a non-degenerate character, the character ψ_{F_*} of $N(F_*)$ is also non-degenerate. In what follows, we denote the restriction of ν to $\mathfrak{a}_0(F_*)$ by ν_{F_*} if necessary.

We now introduce an integral $W_{F_*}(m_*\colon \nu_{F_*}, \psi_{F_*})$ with $m_*\in M_{F_*}$ by

$$W_{F_*}(m_*\colon \nu_{F_*}, \psi_{F_*}) = \int_{N(F_*)} 1_\nu(s_1^{-1} n_2 m_*)\psi_{F_*}(n_2)^{-1} dn_2. \tag{6.16}$$

Then the value $W_{F_*}(\nu_{F_*}, \psi_{F_*})$ at e of (6.16) can be written, by using the facts that $s_1^{-1}N(F_*)s_1=\bar{N}(F_*)$ and $\psi_{F_*}(s_1\bar{n}_2 s_1^{-1})=\psi_*(\bar{n}_2)$ for $\bar{n}_2\in\bar{N}(F_*)$,

$$W_{F_*}(\nu_{F_*}, \psi_{F_*}) = \int_{\bar{N}(F_*)} 1_\nu(\bar{n}_2)\psi_*(\bar{n}_2)^{-1} d\bar{n}_2. \tag{6.17}$$

COROLLARY 6.5. *For $\nu\in D$, the integral $W(\nu, \psi)$ can be written as*

$$W(\nu, \psi) = c^{F_*}(\nu) W_{F_*}(\nu_{F_*}, \psi_{F_*}). \tag{6.18}$$

Moreover it can be continued to a meromorphic function of $\nu\in\mathfrak{a}^*$.

PROOF. The first assertion follows from Lemma 6.4 and (6.17). We can deduce from (6.14) that $c^{F_*}(\nu)$ is in fact a meromorphic function of ν. On the other hand, the integral (6.16) is exactly the same as the Whittaker integral for M_{F_*} with $\nu_{F_*}\in\mathfrak{a}(F_*)^*$ and the non-degenerate character ψ_{F_*} of $N(F_*)$. Hence it follows from Theorem 4.8 in [4] that the integral (6.16) can be extended to an entire function on $\mathfrak{a}(F_*)^*$. Consequently we obtain the corollary.

We summarize the above results in the following;

THEOREM 6.6. *For any $\psi\in N^*$, the integral $W(x\colon \nu, \psi)$ $(x\in G)$ can be continued to a meromorphic function of $\nu\in\mathfrak{a}^*$, which remains to be an element of $C_\psi^\infty(G/K, \chi_\nu)$.*

DEFINITION 6.7. We say that $W(x\colon \nu, \psi)$ is the *class one Whittaker function* *on G of type* (ν, ψ), or simply the Whittaker function on G.

In what follows, we shall relate the Whittaker function $W(x: \nu, \psi)$ on G with the Whittaker function $W_{F_*}(m_*: \nu_{F_*}, \psi_{F_*})$ on M_{F_*}. We recall that $s_*^{-1}P_F s_* = P_{F_*}$ and $s_*^{-1}M_F s_* = M_{F_*}$.

LEMMA 6.8. *Keeping the above notations, we have*

$$W(m: \nu, \psi) = c^{F_*}(\nu)\, W_{F_*}(m_*: \nu_{F_*}, \psi_{F_*}) \tag{6.19}$$

where $m \in M_F$ *and* $m_* = s_*^{-1} m s_* \in M_{F_*}$.

PROOF. To begin with, we shall show the lemma when $h \in A(F)$. Remember that $W(h: \nu, \psi) = h^{s_0\nu+\rho} W(\nu, \psi^h)$ and moreover it holds from Corollary 6.5 that $W(\nu, \psi^h) = c^{F_*}(\nu) W_{F_*}(\nu_{F_*}, (\psi^h)_{F_*})$. By definition, we have

$$(\psi^h)_{F_*}(n_2) = \psi^h(s_* n_2 s_*^{-1}) = \psi(s_* h_* n_2 h_*^{-1} s_*^{-1})$$

where $n_2 \in N(F_*)$ and $h_* = s_*^{-1} h s_*$. Since $h \in A(F)$ and hence $h_* \in A(F_*)$, we can conclude that $(\psi^h)_{F_*} = (\psi_{F_*})^{h_*}$. Consequently,

$$W(\nu, \psi^h) = c^{F_*}(\nu) W_{F_*}(\nu_{F_*}, (\psi_{F_*})^{h_*}) \quad (h \in A(F)).$$

On the other hand, we can easily obtain, as in (6.7),

$$W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*}) = h_*^{s_1\nu+\rho(F_*)} W_{F_*}(\nu_{F_*}, (\psi_{F_*})^{h_*})$$

where $\rho(F_*) = 2^{-1} \sum_{\alpha \in \Sigma_+(F_*)} m(\alpha)\alpha$. Since

$$h_*^{s_1\nu+\rho(F_*)} = h^{s_*(s_1\nu+\rho(F_*))} = h^{s_0\nu+\rho(F)}$$

where $\rho(F) = 2^{-1} \sum_{\alpha \in \Sigma_+(F)} m(\alpha)\alpha$ and moverover $h^{\rho(F)} = h^{\rho}$ for $h \in A(F)$, we have

$$W(h: \nu, \psi) = h^{s_0\nu+\rho} W(\nu, \psi^h) = c^{F_*}(\nu) W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*}) \tag{6.20}$$

where $h \in A(F)$ and $h_* = s_*^{-1} h s_*$. This proves the lemma when $m = h \in A(F)$. Let $m = nhk$ be the Iwasawa decomposition of $m \in M_F$. Then $n \in N(F)$, $h \in A(F)$ and $k \in K(F)$. Correspondingly, the Iwasawa decomposition of $m_* = s_*^{-1} m s_* \in M_{F_*}$ is given by $m_* = n_* h_* k_*$ where $n_* = s_*^{-1} n s_* \in N(F_*)$, $h_* = s_*^{-1} h s_* \in A(F_*)$ and $k_* = s_*^{-1} k s_* \in K(F_*)$. From (6.8), we know that $W(m: \nu, \psi) = \psi(n) W(h: \nu, \psi)$. On the other hand, we can easily obtain $W_{F_*}(m_*: \nu_{F_*}, \psi_{F_*}) = \psi_{F_*}(n_*) W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*})$. Since $n_* = s_*^{-1} n s_*$, we have $\psi_{F_*}(n_*) = \psi(n)$. Combining these facts with (6.20), we obtain the lemma.

COROLLARY 6.9. *Retain the above notations. If we write* $x \in G$ *as* $x = n_1 h_1 m k$ *according to the decomposition* $G = N_F A_F M_F K$, *we obtain*

$$W(x: \nu, \psi) = c^{F_*}(\nu) h_1^{s_0\nu+\rho_F} W_{F_*}(m_*: \nu_{F_*}, \psi_{F_*})$$

where $m_* = s_*^{-1} m s_*$.

§7. The connection between $W(x: \nu, \psi)$ and $V(x: \nu, \psi)$

We have already seen in Theorem 5.4 that for $\nu \in \Omega'_F$, the functions $V(x: s\nu, \psi)$ $(s \in W)$ form a basis of $C_\psi^\infty(G/K, \chi_\nu)$. On the other hand, we have shown in Theorem 6.6 that $W(x: \nu, \psi) \in C_\psi^\infty(G/K, \chi_\nu)$. Hence there exist complex numbers $b_s(\nu, \psi)$ $(s \in W)$ depending on ν and ψ such that for $\nu \in \Omega'_F$, $\psi \in N^*$ and $x \in G$,

$$W(x: \nu, \psi) = \sum_{s \in W} b_s(\nu, \psi) V(x: s\nu, \psi). \tag{7.1}$$

Our aim is to decide $b_s(\nu, \psi)$ for $s \in W$. We start with the following lemmas. Let $\mathfrak{a}_0^+$ (resp. $\mathfrak{a}_0^-$) be the set of $H \in \mathfrak{a}_0$ such that $\alpha(H) > 0$ (resp. $\alpha(H) < 0$) for all all $\alpha \in \Sigma_+$.

LEMMA 7.1. *Put* $h_t = \exp(tH)$ *where* $t > 0$ *and* $H \in \mathfrak{a}_0^-$. *Then for* $\nu \in D$ *and* $\psi \in N^*$, *we have*

$$\lim_{t \to \infty} h_t^{-s_0\nu - \rho} W(h_t: \nu, \psi) = c(\nu) \tag{7.2}$$

where $c(\nu)$ *denotes Harish-Chandra's c-function.*

PROOF. It follows from (6.7) that for $\nu \in D$,

$$h_t^{-(s_0\nu + \rho)} W(h_t: \nu, \psi) = \int_N 1_\nu(s_0^{-1} n) \psi^{h_t}(n)^{-1} dn.$$

If we assume that $\psi = \psi_\eta$ and $n = \exp(\sum X_\alpha)$ where $X_\alpha \in \mathfrak{g}_0^\alpha$ $(\alpha \in \Sigma_+)$, then $\psi^{h_t}(n) = \exp(i\eta(\sum h_t^\alpha X_\alpha))$. Since $h_t \in \exp(\mathfrak{a}_0^-)$, it follows that $\lim_{t \to +\infty} h_t^\alpha = 0$ for all $\alpha \in \Sigma_+$ and hence $\lim_{t \to +\infty} \psi^{h_t}(n) = 1$ for all $n \in N$. Thus we conclude from Proposition 6.1 that for $\nu \in D$,

$$\lim_{t \to +\infty} h_t^{-(s_0\nu + \rho)} W(h_t: \nu, \psi) = \int_N 1_\nu(s_0^{-1} n) dn.$$

But the right hand side is clearly equal to $c(\nu)$.

LEMMA 7.2. *Let* h_t *be as in Lemma* 7.1. *Then for* $\nu \in D \cap \Omega'_F$, $\psi \in N^*$ *and* $s \in W$, *we have*

$$\lim_{t \to +\infty} h_t^{-(s_0\nu + \rho)} V(h_t: s\nu, \psi) = \begin{cases} 1 & \text{if } s = s_0, \\ 0 & \text{if } s \neq s_0. \end{cases}$$

PROOF. We note that

$$h^{-(s_0\nu + \rho)} V(h: s\nu, \psi) = h^{s\nu - s_0\nu} \sum_{\lambda \in L(F)} a_\lambda(s\nu) h^\lambda$$

where the right hand side is convergent absolutely and uniformly for $(h, \nu) \in A \times \Omega'_F$. Since $\lim_{t \to +\infty} h_t^{\lambda} = 0$ for $\lambda \in L(F)'$, to prove the lemma we have only to show that $\lim_{t \to +\infty} h_t^{s\nu - s_0\nu} = 0$ if $s \neq s_0$. Note that $(s\nu - s_0\nu)(H) = (s_0^{-1}s\nu - \nu)(s_0^{-1}H)$ for $H \in \mathfrak{a}_0$ and if $H \in \mathfrak{a}_0^-$ then $s_0^{-1}H \in \mathfrak{a}_0^+$. Since $\nu \in D$, that is, $\mathrm{Re}(\langle \nu, \alpha \rangle) > 0$ for $\alpha \in \Sigma_+$, we can deduce from Lemma 3.3.2.1 in [14] that $\mathrm{Re}(\nu(s_0^{-1}H)) > \mathrm{Re}(s_0^{-1}s\nu(s_0^{-1}H))$ for $H \in \mathfrak{a}_0^-$ and $s \neq s_0$. This means that $\mathrm{Re}((s\nu - s_0\nu)(H)) < 0$ for $H \in \mathfrak{a}_0^-$ and $s \neq s_0$. Hence $\lim_{t \to +\infty} h_t^{s\nu - s_0\nu} = 0$.

Applying Lemma 7.1 and Lemma 7.2 to (7.1), we obtain the following lemma.

LEMMA 7.3. *For $\nu \in D \cap \Omega'_F$ and $\psi \in N^*$ we have*

$$b_{s_0}(\nu, \psi) = c(\nu). \tag{7.3}$$

To proceed further, we first assume that ψ is a non-degenerate character and hence $F = \Pi$. In this case we simply write $\Omega' = \Omega'_\Pi$. If we set

$$\Psi(h: \nu, \psi) = h^{-\rho} W(h: \nu, \psi) \qquad \text{for} \quad h \in A,$$

then it follows from (7.1) that

$$\Psi(h: \nu, \psi) = \textstyle\sum_{s \in W} b_s(\nu, \psi) \Phi(h: s\nu, \psi). \tag{7.4}$$

LEMMA 7.4. *Let $\omega_1, \omega_2, \dots, \omega_w$ be the homogeneous generators of $S(\mathfrak{a})$ over J introduced in § 2. Then $w \times w$ matrix*

$$(\Phi(h_0;\, \omega_i\colon s\nu, \psi))_{1 \leq i \leq w, s \in W}$$

is non-singular for any $h_0 \in A$ and $\nu \in \Omega'$.

PROOF. For otherwise, we can choose complex numbers a_s ($s \in W$), not all zero, such that $\sum_{s \in W} a_s \Phi(h_0;\, \omega_i\colon s\nu, \psi) = 0$ $(1 \leq i \leq w)$. Put $f(h) = \sum_{s \in W} a_s \Phi(h: s\nu, \psi)$ for $h \in A$. Then $f \in C_\psi^\infty(A, \chi_\nu)$. Since $f(h_0;\, \omega_i) = 0$ $(1 \leq i \leq w)$, we conlcude from the proof of Theorem 3.3 that $f(h_0;\, p) = 0$ for all $p \in U(\mathfrak{a})$. But since f is analytic and A is connected, this implies $f = 0$ on A. On the other hand, $\Phi(h: s\nu, \psi)$ ($s \in W$) are linearly independent and hence $a_s = 0$ for all $s \in W$. This contradicts our choice of a_s.

LEMMA 7.5. *The coefficients $b_s(\nu, \psi)$ ($s \in W$) are holomorphic functions on Ω'.*

PROOF. Fix $h \in A$. From the above lemma, there exist holomorphic functions $a_{si}(\nu)$ on Ω' ($s \in W$, $1 \leq i \leq w$) such that $\sum_{1 \leq i \leq w} a_{si}(\nu) \Phi(h;\, \omega_i\colon t\nu, \psi) = 1$ or 0 according as $t = s$ or not. Hence from (7.4) we conclude

$$b_s(\nu, \psi) = \textstyle\sum_{1 \leq i \leq w} a_{si}(\nu) \Psi(h;\, \omega_i\colon \nu, \psi).$$

Since ψ is a non-degenerate character, $W(h: \nu, \psi)$ is an entire function of ν and hence $\Psi(h;\omega_i\colon\nu,\psi)$ are also entire functions of ν. Thus we establish the lemma.

We have shown in [4] that for a non-degenerate character ψ, the Whittaker function $W(x\colon\nu,\psi)$ satisfies the functional equations

$$W(x\colon\nu,\psi)=M(s,\nu,\psi)W(x\colon s\nu,\psi) \tag{7.5}$$

for each $s\in W$. Here $M(s,\nu,\psi)$ $(s\in W)$ are meromorphic functions of ν, which are determined recursively as follows. If $s=s_\alpha$ $(\alpha\in\Pi)$, then

$$M(s_\alpha,\nu,\psi)=e_\alpha(\nu)e_\alpha(-\nu)^{-1}(|\eta_\alpha|/2(2\langle\alpha,\alpha\rangle)^{1/2})^{2\nu_\alpha} \tag{7.6}$$

where $e_\alpha(\nu)$ is given by

$$e_\alpha(\nu)^{-1}=\Gamma(2^{-1}(\nu_\alpha+m(\alpha)/2+1))\Gamma(2^{-1}(\nu_\alpha+m(\alpha)/2+m(2\alpha))).$$

If $s\in W$ and $\alpha\in\Pi$ such that $l(s_\alpha s)=l(s)+1$, then

$$M(s_\alpha s,\nu,\psi)=M(s,\nu,\psi)M(s_\alpha,s\nu,\psi). \tag{7.7}$$

Here $l(s)$ denotes the length of $s\in W$.

LEMMA 7.7. *For $s\in W$, we have*

$$b_s(\nu,\psi)=M(s_0s,\nu,\psi)b_{s_0}(s_0s\nu,\psi).$$

PROOF. Combining (7.5) with (7.1), we can easily obtain that

$$b_s(\nu,\psi)=b_{st^{-1}}(t\nu,\psi)M(t,\nu,\psi)$$

for $s,t\in W$. In particular, if we take $t=s_0^{-1}s=s_0s$, we have the lemma.

THEOREM 7.8. *Let ψ be a non-degenerate character of N. Then $b_s(\nu,\psi)$ $(s\in W)$ are holomorphic functions on Ω' and they are given by*

$$b_s(\nu,\psi)=M(s_0s,\nu,\psi)c(s_0s\nu) \tag{7.8}$$

and consequently it holds that

$$W(x\colon\nu,\psi)=\textstyle\sum_{s\in W}M(s_0s,\nu,\psi)c(s_0s\nu)V(x\colon s\nu,\psi). \tag{7.9}$$

PROOF. In view of Lemma 7.7, it is enough to show that $b_{s_0}(\nu,\psi)=c(\nu)$ for $\nu\in\Omega'$. But from Lemma 7.3, it follows that $b_{s_0}(\nu,\psi)=c(\nu)$ for $\nu\in D\cap\Omega'$. Since Ω' is connected and both $b_{s_0}(\nu,\psi)$ and $c(\nu)$ are holomorphic on Ω', we conclude that $b_{s_0}(\nu,\psi)=c(\nu)$ on Ω'.

Now we shall consider the case when ψ is not necessarily a non-degenerate character. We set $F=\{\alpha\in\Pi;\,|\eta_\alpha|\neq0\}$ and $F_*=-s_0^{-1}F$. Let $\mathfrak{m}_*$ (resp. $\mathfrak{k}_*$) be the complexification of the Lie algebra of M_{F_*} (resp. $K(F_*)$) and let $U(\mathfrak{m}_*)^{\mathfrak{k}_*}$ be the centralizer of $\mathfrak{k}_*$ in the universal enveloping algebra $U(\mathfrak{m}_*)$ of $\mathfrak{m}_*$. For $\nu_* \in \mathfrak{a}(F_*)^*$ (the complex dual space of $\mathfrak{a}_0(F_*)$), we define, as in (2.2), an algebra homomorphism χ_{ν_*} of $U(\mathfrak{m}_*)^{\mathfrak{k}_*}$ into $\boldsymbol{C}$. Let ψ_* be a character of $N(F_*)$. We denote by $C^\infty_{\psi_*}(M_{F_*}/K(F_*), \chi_{\nu_*})$ the space of $f \in C^\infty(M_{F_*})$ such that

(I) $f(n_*m_*k_*) = \psi_*(n_*)f(m_*)$ $(n_* \in N(F_*),\ m_* \in M_{F_*},\ k_* \in K(F_*))$,

(II) $zf = \chi_{\nu_*}(z)f$ for all $z \in U(\mathfrak{m}_*)^{\mathfrak{k}_*}$.

As in §4, we shall construct a basis of $C^\infty_{\psi_*}(M_{F_*}/K(F_*), \chi_{\nu_*})$. Let $L(F_*)$ be the set of all linear forms on $\mathfrak{a}_0(F_*)$ which are linear combinations of elements of F_* with nonnegative integer coefficients. We consider a series

$$\Phi_{F_*}(h_*: \nu_*, \psi_*) = h_*^{\nu_*} \textstyle\sum_{\lambda \in L(F_*)} a_\lambda(\nu_*) h_*^\lambda \tag{7.10}$$

where $h_* \in A(F_*)$ and a_λ $(\lambda \in L(F_*))$ are defined by the recursion formula: $a_0 = 1$ and

$$(\langle \lambda, \lambda \rangle + 2\langle \lambda, \nu_* \rangle) a_\lambda(\nu_*) = 2 \textstyle\sum_{\alpha \in F_*} |\eta^*_\alpha|^2 a_{\lambda - 2\alpha}(\nu_*) \tag{7.11}$$

for $\lambda \in L(F_*) - (0)$. Here η^* denotes the Lie algebra homomorphism of $\mathfrak{n}_0(F_*)$ into $\boldsymbol{R}$ that corresponds to ψ_*. Then, as in Lemma 4.6, it defines a smooth function on $A(F_*)$, which is holomorphic in $\nu_* \in {}'\mathfrak{a}(F_*)^*$. Here ${}'\mathfrak{a}(F_*)^*$ denotes the complement in $\mathfrak{a}(F_*)^*$ of all hyperplanes σ_λ $(\lambda \in L(F_*) - (0))$. Moreover if we set

$$V_{F_*}(m_*: \nu_*, \psi_*) = \psi_*(n_*) h_*^{\rho(F_*)} \Phi_{F_*}(h_*: \nu_*, \psi_*) \tag{7.12}$$

where $m_* = n_* h_* k_*$ is the Iwasawa decomposition of $m_* \in M_{F_*}$, then we can deduce from Corollary 4.11 that $V_{F_*}(m_*: \nu_*, \psi_*)$ belongs to $C^\infty_{\psi_*}(M_{F_*}/K(F_*), \chi_{\nu_*})$. Let $\Omega(F_*)'$ be the set of regular elements ν_* in $\mathfrak{a}(F_*)^*$ such that $s\nu_* \in {}'\mathfrak{a}(F_*)^*$ for all $s \in W_{F_*}$ and $s\nu_* - t\nu_* \notin L(F_*)^\sim$ for any pair $(s, t) \in W_{F_*} \times W_{F_*}$ with $s \neq t$. Then as in Theorem 5.4, we see that for $\nu_* \in \Omega(F_*)'$ the functions $V_{F_*}(m_*: s\nu_*, \psi_*)$ $(s \in W_{F_*})$ form a basis of $C^\infty_{\psi_*}(M_{F_*}/K(F_*), \chi_{\nu_*})$.

In the following, we assume that $\nu_* = \nu_{F_*}$ and $\psi_* = \psi_{F_*}$. We remark that since ν_* is the restriction of $\nu \in \mathfrak{a}^*$ to $\mathfrak{a}_0(F_*)$ it holds that $(\nu_*)_\alpha = \nu_\alpha$ where $(\nu_*)_\alpha = \langle \nu_*, \alpha \rangle / \langle \alpha, \alpha \rangle$ for $\alpha \in F_*$. Moreover we remark that ψ_* is a non-degenerate character of $N(F_*)$ and it follows from the definition of ψ_{F_*} that $\eta^*_\alpha = \eta_{s_*\alpha}$ for $\alpha \in F_*$.

LEMMA 7.9. *Let $\nu_* = \nu_{F_*}$ and $\psi_* = \psi_{F_*}$. Then it holds that*

$$\Phi_{F_*}(h_*: \nu_*, \psi_*) = \Phi(h: s_*\nu, \psi)$$

where $h \in A(F)$, $h_* = s_*^{-1} h s_* \in A(F_*)$.

PROOF. We recall that $\Phi(h: s_*\nu, \psi)$ is defined by

$$\Phi(h: s_*\nu, \psi) = h^{s_*\nu} \textstyle\sum_{\mu \in L(F)} a_\mu(s_*\nu) h^\mu$$

where a_μ ($\mu \in L(F)$) are given by $a_0 = 1$ and

$$(7.13) \qquad (\langle \mu, \mu \rangle + 2\langle \mu, s_* \nu \rangle) a_\mu(s_* \nu) = 2 \textstyle\sum_{\beta \in F} |\eta_\beta|^2 a_{\mu - 2\beta}(s_* \nu)$$

for $\mu \in L(F)'$. Since $s_* F_* = F$ and the map $\lambda \mapsto s_* \lambda$ is a bijection of $L(F_*)$ onto $L(F)$, we can rewrite (7.13) as

$$(\langle s_*\lambda, s_*\lambda \rangle + 2\langle s_*\lambda, s_*\nu \rangle) a_{s_*\lambda}(s_*\nu) = 2 \textstyle\sum_{\alpha \in F_*} |\eta_{s_*\alpha}|^2 a_{s_*(\lambda - 2\alpha)}(s_*\nu)$$

where $\lambda \in L(F_*)'$. Since s_* preserves $\langle\ ,\ \rangle$, we have

$$(7.14) \qquad (\langle \lambda, \lambda \rangle + 2\langle \lambda, \nu \rangle) a_{s_*\lambda}(s_*\nu) = 2 \textstyle\sum_{\alpha \in F_*} |\eta_{s_*\alpha}|^2 a_{s_*(\lambda - 2\alpha)}(s_*\nu).$$

On the other hand, the recursion formula of $a_\lambda(\nu_*)$ in (7.11) can be written as

$$(7.15) \qquad (\langle \lambda, \lambda \rangle + 2\langle \lambda, \nu \rangle) a_\lambda(\nu) = 2 \textstyle\sum_{\alpha \in F_*} |\eta_{s_*\alpha}|^2 a_{\lambda - 2\alpha}(\nu),$$

since $\nu_* = \nu_{F_*}$ and $\psi_* = \psi_{F_*}$. Comparing (7.14) with (7.15), we can conclude that $a_{s_*\lambda}(s_*\nu) = a_\lambda(\nu)$ for all $\lambda \in L(F_*)$. Hence

$$\Phi(h : s_*\nu, \psi) = h^{s_*\nu} \textstyle\sum_{\alpha \in L(F_*)} a_\lambda(\nu) h^{s_*\lambda} = h_*^\nu \textstyle\sum_{\alpha \in L(F_*)} a_\lambda(\nu) h_*^\lambda,$$

which implies the lemma.

COROLLARY 7.10. *Under the same assumption as in Lemma* 7.9, *we have*

$$V_{F_*}(m_* : \nu_*, \psi_*) = V(m : s_*\nu, \psi)$$

where $m \in M_F$ *and* $m_* = s_*^{-1} m s_* \in M_{F_*}$.

PROOF. Let $m = nhk$ be the Iwasawa decomposition of m. Then the Iwasawa decomposition of m_* is given by $m_* = n_* h_* k_*$ where $n_* = s_*^{-1} n s_*$, $h_* = s_*^{-1} h s_*$ and $k_* = s_*^{-1} k s_*$. By definition, we have

$$V_{F_*}(m_* : \nu_*, \psi_*) = \psi_*(n_*) h_*^{\rho(F_*)} \Phi_{F_*}(h_* : \nu_*, \psi_*).$$

Since $\psi_*(n_*) = \psi(n)$, $h_*^{\rho(F_*)} = h^{\rho(F)}$ and $\Phi_{F_*}(h_* : \nu_*, \psi_*)$ is equal to $\Phi(h : s_*\nu, \psi)$, we get

$$V_{F_*}(m_* : \nu_*, \psi_*) = \psi(n) h^{\rho(F)} \Phi(h : s_*\nu, \psi).$$

But the right hand side is clearly equal to $V(m : s_*\nu, \psi)$.

Keeping the assumption $\nu_* = \nu_{F_*}$ and $\psi_* = \psi_{F_*}$, we shall consider the Whittaker function $W_{F_*}(m_* : \nu_*, \psi_*)$ on M_{F_*} introduced in (6.16). Following the same line of the proof of Theorem 6.6, we can conclude that $W_{F_*}(m_* : \nu_*, \psi_*) \in C^\infty_{\psi_*}(M_{F_*}/K(F_*), \chi_{\nu_*})$. Hence it can be written as

$$(7.16) \qquad W_{F_*}(m_* : \nu_*, \psi_*) = \textstyle\sum_{s \in W_{F_*}} b_s(\nu_*, \psi_*) V_{F_*}(m_* : s\nu_*, \psi_*)$$

for suitable constants $b_s(\nu_*, \psi_*)$ $(s \in W_{F_*})$. Since ψ_* is a non-degenerate character of $N(F_*)$, $W_{F_*}(m_*: \nu_*, \psi_*)$ is an entire function of $\nu_* \in \mathfrak{a}(F_*)^*$ and satisfies the functional equations

$$W_{F_*}(m_*: \nu_*, \psi_*) = M(s, \nu_*, \psi_*)W_{F_*}(m_*: s\nu, \psi_*)$$

for all $s \in W_{F_*}$. Here $M(s, \nu_*, \psi_*)$ $(s \in W_{F_*})$ are defined recursively, by replacing ν and η by ν_* and η_* respectively in (7.6) and (7.7). We remark that since $(\nu_*)_\alpha = \nu_\alpha$ and $\eta^*_\alpha = \eta_{s_*\alpha}$ for $\alpha \in F_*$ we may write $e_\alpha(\nu_*) = e_\alpha(\nu)$ and

$$M(s_\alpha, \nu_*, \psi_*) = e_\alpha(\nu)e_\alpha(-\nu)^{-1}(\eta_{s_*\alpha}/2(2\langle\alpha, \alpha\rangle)^{1/2})^{2\nu_\alpha} \tag{7.17}$$

for $\alpha \in F_*$. Furthermore we can deduce, as in Theorem 7.8, that the coefficients $b_s(\nu_*, \psi_*)$ are holomorphic in $\Omega(F_*)'$ and they are given by

$$b_s(\nu_*, \psi_*) = M(s_1 s, \nu_*, \psi_*)c_{F_*}(s_1 s\nu) \tag{7.18}$$

where s_1 is the longest element of W_{F_*} and c_{F_*} is the c-function of M_{F_*}, which is given by

$$c_{F_*}(\nu) = \prod_{\alpha \in \Sigma^\circ_+(F_*)} c_\alpha(\nu). \tag{7.19}$$

LEMMA 7.11. *Let $\nu_* = \nu_{F_*}$ and $\psi_* = \psi_{F_*}$. Then we have*

$$W(m: \nu, \psi) = c^{F_*}(\nu)\sum_{s \in W_{F_*}} c_{F_*}(s_1 s\nu)M(s_1 s, \nu_*, \psi_*)V(m: s_* s\nu, \psi) \tag{7.20}$$

for $m \in M_F$ and $\nu_* \in \Omega(F_*)'$.

PROOF. We have already seen in Lemma 6.8 that $W(m: \nu, \psi) = c^{F_*}(\nu)W_{F_*}\cdot(m_*: \nu_*, \psi_*)$ where $m_* = s_*^{-1} m s_*$. On the other hand, from Corollary 7.10 it follows that $V_{F_*}(m_*; \nu_*, \psi_*) = V(m: s_*\nu, \psi)$. Hence by (7.16), we have

$$W(m: \nu, \psi) = c^{F_*}(\nu)\sum_{s \in W_{F_*}} b_s(\nu_*, \psi_*)V(m: s_* s\nu, \psi).$$

Since $b_s(\nu_*, \psi_*)$ $(s \in W_{F_*})$ are given by (7.18), the lemma follows.

THEOREM 7.12. *Let ψ be a character of N and define F and F_* by $F = \{\alpha \in \Pi; |\eta_\alpha| \neq 0\}$ and $F_* = -s_0^{-1}F$. Let ν_* be the restriction of $\nu \in \mathfrak{a}^*$ to $\mathfrak{a}_0(F_*)$ and let ψ_* be the character of $N(F_*)$ defined by $\psi_*(n_*) = \psi(s_* n_* s_*^{-1})$ for $n_* \in N(F_*)$. Then the Whittaker function $W(x: \nu, \psi)$ on G can be expressed for $\nu_* \in \Omega(F_*)'$ as follows;*

$$W(x: \nu, \psi) = c^{F_*}(\nu)\sum_{s \in W_{F_*}} c_{F_*}(s_1 s\nu)M(s_1 s, \nu_*, \psi_*)V(x: s_* s\nu, \psi).$$

Here the functions $c^{F_*}(\nu)$ and $c_{F_*}(\nu)$ are meromorphic functions on $\mathfrak{a}^*$ given by (6.14) and (7.19) respectively. Moreover $M(s, \nu_*. \psi_*)$ $(s \in W_{F_*})$ are meromorphic functions of ν_*, which are determined recursively as follows; if $s = s_\alpha$ $(\alpha \in F_*)$, *then* $M(s_\alpha, \nu_*, \psi_*)$ *is given by* (7.17) *and if* $s \in W_{F_*}$ *and* $\alpha \in F_*$ *such that* $l(s_\alpha s) = l(s)+1$, *then* $M(s_\alpha s, \nu_*, \psi_*) = M(s, \nu_*, \psi_*) M(s_\alpha, s\nu_*, \psi_*)$. *Finally the function* $V(x: \nu, \psi)$ *on* G *is already introduced in* (4.10).

PROOF. If we write $x = n_1 h_1 m k$ following the decomposition $G = N_F A_F M_F K$, we can easily obtain

$$W(x: \nu, \psi) = h_1^{s_0\nu+\rho_F} W(m: \nu, \psi)$$

and

$$V(x: s_* s\nu, \psi) = h_1^{s_* s\nu+\rho_F} V(m: s_* s\nu, \psi).$$

But since $h_1^{s_* s\nu} = (h_1)_*^{s\nu}$ and $(h_1)_* = s_*^{-1} h_1 s_* \in A_{F_*}$, it holds that $h_1^{s_* s\nu+\rho_F} = h_1^{s_*\nu+\rho_F}$ for all $s \in W_{F_*}$. Similarly since $s_0 = s_* s_1$, we have $h_1^{s_0\nu+\rho_F} = h_1^{s_*\nu+\rho_F}$. Consequently, by Lemma 7.11 we can obtain the theorem immediately.

§8. An example

In this section we consider the case when $\psi \in N^*$ such that the corresponding subset $F(\psi)$ of Π consists of only one element α. We will show that in this case the Whittaker function $W(x: \nu, \psi)$ can be written in terms of the modified Bessel function of second kind. In what follows, we set $F = F(\psi) = \{\alpha\}$, $\beta = -s_0^{-1}\alpha$ and hence $F_* = \{\beta\}$.

THEOREM 8.1. *Let* $\psi \in N^*$ *satisfying the above condition. If we write* $h \in A$ *as* $h = h_1 h_2$ *where* $h_1 \in A_F$ *and* $h_2 \in A(F)$, *we have*

$$(8.1) \qquad W(h: \nu, \psi) = c(\nu)\Gamma(-(s_0\nu)_\alpha)^{-1} h_1^{s_0\nu+\rho_\alpha} K(h_2: \nu, \psi)$$

and

$$(8.2) \qquad K(h_2: \nu, \psi) = 2(|\eta_\alpha|/(2\langle\alpha, \alpha\rangle)^{1/2})^{-(s_0\nu)_\alpha} h_2^{\rho(\alpha)} K_{(s_0\nu)_\alpha}(2|\eta_\alpha| h_2^\alpha/(2\langle\alpha, \alpha\rangle)^{1/2})$$

where $\rho(\alpha) = (m(\alpha)/2 + m(2\alpha))\alpha$, $\rho_\alpha = \rho - \rho(\alpha)$ *and* $K_{(s_0\nu)_\alpha}$ *is the modified Bessel function of second kind and order* $(s_0\nu)_\alpha$.

In particular when G *is of real rank one and* $F(\psi) = \Pi = \{\alpha\}$, *then for* $h \in A$ *we get*

$$(8.3) \quad W(h: \nu, \psi) = 2c_\alpha(\nu)\Gamma(\nu_\alpha)^{-1}(|\eta_\alpha|/(2\langle\alpha, \alpha\rangle)^{1/2})^{\nu_\alpha} h^\rho K_{\nu_\alpha}(2|\eta_\alpha| h^\alpha/(2\langle\alpha, \alpha\rangle)^{1/2}).$$

PROOF. Put $s_* = s_0 s_\beta^{-1}$. Then we have already seen in Corollary 6.9 that for $h_1 \in A_F$ and $h_2 \in A(F)$,

$$(8.4) \qquad W(h_1 h_2: \nu, \psi) = c(\nu) c_\beta(\nu)^{-1} h_1^{s_0\nu+\rho_\alpha} W_{F_*}((h_2)_*: \nu_{F_*}, \psi_{F_*})$$

where $(h_2)_* = s_*^{-1} h_2 s_*$. Here we used the facts that $\rho_F = \rho_\alpha$ and $c^{F_*}(\nu) =$

$c(\nu)c_\beta(\nu)^{-1}$. In the following we compute $W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*})$ for $h\in A(F)$ explicitly. In the proof of Lemma 6.8 we have shown that

$$W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*}) = h^{s_0\nu+\rho(\alpha)}W_{F_*}(\nu_{F_*}, (\psi^h)_{F_*}),$$

which can be given by the integral

$$h^{s_0\nu+\rho(\alpha)}\int_{\bar N(F_*)} 1_\nu(\bar n)\psi^h(s_0\bar n s_0^{-1})^{-1}d\bar n$$

(cf. (6.17)). Since $F_*=\{\beta\}$, we have $\bar{\mathfrak n}_0(F_*)=\mathfrak g_0^{-\beta}\oplus\mathfrak g_0^{-2\beta}$ and hence each $\bar n\in\bar N(F_*)$ can be written uniquely as $\bar n=\exp(Y+Z)$ where $Y\in\mathfrak g_0^{-\beta}$ and $Z\in\mathfrak g_0^{-2\beta}$. But since $-s_0\beta=\alpha$ and hence $\mathrm{Ad}(s_0)\mathfrak g_0^{-\beta}=\mathfrak g_0^{\alpha}$, we conclude that if $\bar n=\exp(Y+Z)$,

$$\psi^h(s_0\bar n s_0^{-1}) = \exp\{ih^\alpha\eta_\alpha(\mathrm{Ad}(s_0)Y)\}.$$

For simplicity, we introduce a linear form ζ_β on $\mathfrak g_0^{-\beta}$ by $\zeta_\beta(Y)=\eta_\alpha(\mathrm{Ad}(s_0)Y)$. Then it is clear that $|\zeta_\beta|=|\eta_\alpha|$. On the other hand, G. Schiffmann showed in [10] that if $\bar n=\exp(Y+Z)$,

$$1_\nu(\bar n) = \{(1+2^{-1}\langle\beta, \beta\rangle|Y|^2)^2 + 2\langle\beta, \beta\rangle|Z|^2\}^{-\mu}$$

where $|Y|^2=-B(Y, \theta Y)$, $|Z|^2=-B(Z, \theta Z)$ and $\mu=(\nu_\beta+m(\beta)/2+m(2\beta))/2$. Consequently $W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*})$ is given by

$$h^{s_0\nu+\rho(\alpha)}\int_{\mathfrak g_0^{-\beta}\times\mathfrak g_0^{-2\beta}}\{(1+2^{-1}\langle\beta, \beta\rangle|Y|^2)^2 + 2\langle\beta, \beta\rangle|Z|^2\}^{-\mu}\exp\{-ih^\alpha\zeta_\beta(Y)\}\,dY\,dZ.$$

The above integral can be explicitly calculated (cf. [4]) and the result is

$$2c_\beta(\nu)\Gamma(\nu_\beta)^{-1}(|\zeta_\beta|/(2\langle\beta, \beta\rangle)^{1/2})^{\nu_\beta}h^{s_0\nu+\rho(\alpha)+\nu_\beta\alpha}K_{-\nu_\beta}(2|\zeta_\beta|h^\alpha/(2\langle\beta, \beta\rangle)^{1/2}).$$

Since $\beta=-s_0^{-1}\alpha$, it follows that $\langle\beta, \beta\rangle=\langle\alpha, \alpha\rangle$, $\nu_\beta=-(s_0\nu)_\alpha$ and hence $h^{s_0\nu+\nu_\beta\alpha}=1$ for $h\in A(F)$. In view of the fact that $|\zeta_\beta|=|\eta_\alpha|$, we can deduce that $W_{F_*}(h_*: \nu_{F_*}, \psi_{F_*})$ is equal to

$$c_\beta(\nu)\Gamma(-(s_0\nu)_\alpha)^{-1}K(h: \nu, \psi).$$

Combining this with (8.4), we obtain (8.1). If G is of real rank one, then $s_0=s_\alpha$ and $-s_0\nu=\nu$. Moreover since $F=\Pi$, it holds that $A(F)=A$. If we note that $c(\nu)=c_\alpha(\nu)$ and the modified Bessel function satisfies $K_{-\nu_\alpha}=K_{\nu_\alpha}$, we conclude that (8.3) is a direct consequence of (8.1).

References

[1] W. Casselman and J. Shalika, The unramified principal series of p-adic groups II; The Whittaker function, Compositio Math., **41** (1980), 387–406.

[2] R. Goodman and N. Wallach, Whittaker vectors and conical vectors, J. Functional Analysis, **39** (1980), 199–279.

[3] Harish-Chandra, Spherical functions on a semi-simple Lie group I, Amer. J. Math., **80** (1958), 241–310, II, ibid., 553–613.

[4] M. Hashizume, Whittaker models for real reductive groups, Japan. J. Math., **5** (1979), 349–401.

[5] S. Helgason, A duality for symmetric spaces with applications to group representations, Advances in Math., **5** (1970), 1–154.

[6] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France, **95** (1967), 243–309.

[7] H. Jacquet and R. Langlands, Automorphic forms on $GL(2)$, Lecture Notes in Math. No. 114, Springer-Verlag, 1970.

[8] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, Automorphic forms on $GL(3)$ I, Ann. of Math., **109** (1979), 169–212, II, ibid., 213–258.

[9] B. Kostant, On Whittaker vectors and representation theory, Inventiones Math., **48** (1978), 101–184.

[10] G. Schiffmann, Intégrales d'entrelacement et fonctions de Whittaker, Bull. Soc. Math. France, **99** (1971), 3–72.

[11] F. Shahidi, Whittaker models for real groups, Duke Math. J., **47** (1980), 99–125.

[12] J. Shalika, The multiplicity one theorem for $GL(n)$, Ann. of Math., **100** (1974), 171-193.

[13] T. Shintani, On a explicit formula for class 1 Whittaker functions on GL_n over p-adic fields, Proc. Japan Acad., **52** (1976), 180–182.

[14] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups Vols. I, II, Springer-Verlag, Berlin-New York, 1972.

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