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# Joins of weak subideals of Lie algebras

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## Introduction

Maruo [4] introduced the concept of weak ideals of Lie algebras generalizing that of subideals and investigated pseudo-coalescency of classes of Lie algebras. Tôgô [6] introduced the concept of weakly ascendant subalgebras of Lie algebras generalizing those of weak ideals and ascendant subalgebras. Weak ideals are called weak subideals in [6]. A class $\mathfrak{X}$ of Lie algebras is pseudo-coalescent [4] if in any Lie algebra the join of any pair of a subideal and a weak subideal belonging to $\mathfrak{X}$ is always a weak subideal belonging to $\mathfrak{X}$. However, it might be meaningless to consider classes $\mathfrak{X}$ such that in any Lie algebra the join of any pair of weak subideals belonging to $\mathfrak{X}$ is always a weak subideal belonging to $\mathfrak{X}$, for there exists a Lie algebra $L$ such that the join of some pair of 1-dimensional weak subideals of $L$ is not a weak subideal of $L$ and is non-abelian simple (cf. [3, Example 5.1]). In this paper we shall investigate several classes of Lie algebras in which the join of any pair of weak subideals (resp. subideals) is always a weak subideal (resp. a subideal).

In Section 2 we shall show that if a Lie algebra $L$ belongs to one of the classes $\overline{\mathfrak{D}}\mathfrak{A}$, $\mathfrak{N}\overline{\mathfrak{M}}_1$ and $\mathfrak{N}\mathfrak{A}_1$ (resp. the classes $\mathfrak{D}\mathfrak{A}$, $\mathfrak{N}\mathfrak{M}_1$ and $\mathfrak{N}\mathfrak{A}_1$), then the set $\mathscr{S}_L(\mathrm{wsi})$ (resp. the set $\mathscr{S}_L(\mathrm{si})$) of all weak subideals (resp. all subideals) of $L$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$ of all subalgebras of $L$ (Theorem 2.11). In Section 3 we shall show that if a Lie algebra $L$ belongs to one of the classes $\mathfrak{A}_1$, $\mathfrak{F}\cap(\mathfrak{N}\overline{\mathfrak{M}}_1)$ and $\mathfrak{F}\cap(\mathfrak{N}\mathfrak{A}_1)$ (resp. the classes $\mathfrak{A}_1$, $\mathfrak{F}\cap(\mathfrak{N}\mathfrak{M}_1)$ and $\mathfrak{F}\cap(\mathfrak{N}\mathfrak{A}_1)$), then $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is a complete sublattice of $\mathscr{S}_L(\leq)$ (Theorem 3.5). In Section 4 we shall construct Lie algebras $L$ such that $\mathscr{S}_L(\mathrm{wsi})$ is a sublattice of $\mathscr{S}_L(\leq)$ but is not a complete sublattice of $\mathscr{S}_L(\leq)$ (Examples 4.1 and 4.2). We shall also construct a Lie algebra $L$ such that $\mathscr{S}_L(\mathrm{wsi})$ is a complete sublattice of $\mathscr{S}_L(\leq)$ (Example 4.3).

Here $\mathfrak{A}$ (resp. $\mathfrak{N}$, $\mathfrak{F}$) is the class of abelian (resp. nilpotent, finite-dimensional) Lie algebras, $\overline{\mathfrak{D}}$ (resp. $\mathfrak{D}$) is the class of Lie algebras in which every subalgebra is a weak subideal (resp. a subideal), and $\overline{\mathfrak{M}}_1$ (resp. $\mathfrak{M}_1$) is the class of Lie algebras in which every weak subideal (resp. every subideal) is an ideal; and $\mathfrak{A}_1$ is the class consisting of either abelian Lie algebras or metabelian Lie algebras $L$ with $\dim(L/L^2)=1$. For classes $\mathfrak{X}$, $\mathfrak{Y}$ of Lie algebras, $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras $L$ having an $\mathfrak{X}$-ideal $I$ such that $L/I\in\mathfrak{Y}$.

## 1.

Throughout the paper we always consider not necessarily finite-dimensional Lie algebras over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified.

Let $L$ be a Lie algebra and let $H$ be a subalgebra of $L$. For an integer $n\geq 0$, $H$ is an $n$-step subideal of $L$, denoted by $H\lhd^n L$, if there exists a finite series $(H_i)_{i\leq n}$ of subalgebras of $L$ such that

(1) $H=H_0\leq H_1\leq\cdots\leq H_n=L$,

(2) $H_i\lhd H_{i+1}\quad(0\leq i<n)$.

$H$ is an $n$-step weak subideal of $L$ [6], denoted by $H\leq^n L$, if there exists a finite chain $(M_i)_{i\leq n}$ of subspaces of $L$ such that

(1) $H=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=L$,

(2) $[M_{i+1},H]\subseteq M_i\quad(0\leq i<n)$.

$H$ is a subideal (resp. a weak subideal) of $L$, denoted by $H$ si $L$ (resp. $H$ wsi $L$), if $H\lhd^n L$ (resp. $H\leq^n L$) for some integer $n\geq 0$.

The ideal closure series of $H$ in $L$ is the descending series $(H^{L,\alpha})_{\alpha\geq 0}$ of subalgebras of $L$ defined inductively by

$$H^{L,0}=L,$$
$$H^{L,\alpha+1}=H^{H^{L,\alpha}}\qquad\text{for any ordinal }\alpha,$$
$$H^{L,\lambda}=\bigcap_{\alpha<\lambda}H^{L,\alpha}\qquad\text{for any limit ordinal }\lambda.$$

It is well known ([2, Lemma 1.3.6]) that for any integer $n\geq 0$

$$H\lhd^n L\quad\text{if and only if}\quad H^{L,n}=H.$$

The weak closure series of $H$ in $L$ is the descending chain $(H_{L,\alpha})_{\alpha\geq 0}$ of subspaces of $L$ defined inductively by

$$H_{L,0}=L,$$
$$H_{L,\alpha+1}=[H_{L,\alpha},H]+H\qquad\text{for any ordinal }\alpha,$$
$$H_{L,\lambda}=\bigcap_{\alpha<\lambda}H_{L,\alpha}\qquad\text{for any limit ordinal }\lambda.$$

Then as a special case of [3, Lemma 1.4 (2)] we have the following

LEMMA 1.1. *For an integer $n\geq 0$, the following conditions are equivalent*:

(1) $H\leq^n L$.

(2) $H_{L,n}=H$.

(3) $[L,{}_n H]\subseteq H$.

In [4] $H$ is called an $n$-step weak ideal of $L$, denoted by $H$ $n$-wi $L$, if $[L,{}_n\,\mathrm{H}]\subseteq H$. $H$ is also called a weak ideal of $L$, denoted by $H$ wi $L$, if $H$ $n$-wi $L$ for some integer $n \geq 0$. It is clear from Lemma 1.1 that $H$ $n$-wi $L$ if and only if $H \leq^n L$ and that $H$ wi $L$ if and only if $H$ wsi $L$.

Now let $H$ wsi $L$ (resp. $H$ si $L$). The weak subideal index (resp. the subideal index) of $H$ in $L$, denoted by wsi$(L: H)$ (resp. si$(L: H)$), is the least integer $n \geq 0$ with respect to $H \leq^n L$ (resp. $H \lhd^n L$).

The following lemma corresponds to [5, Lemma 3.13].

LEMMA 1.2. *Let $\Delta$ be either of the relations* wsi *and* si *and let $\{H_i: i \in I\}$ be a collection such that $H_i \Delta L$ for all $i \in I$. Put $H = \bigcap_{i \in I} H_i$.*

(1) *If $\Delta(L: H_i) \leq n$ for all $i \in I$, then $H \; \Delta \; L$ and $\Delta(L: H) \leq n$.*

(2) *If* card$(I) < \infty$, *then $H \; \Delta \; L$ and $\Delta(L: H) \leq \max\{\Delta(L: H_i): i \in I\}$.*

PROOF. Here we only give a proof for the case $\Delta =$ wsi, since the other case can be similarly proved.

(1) Since $H_{L,n} \subseteq (H_i)_{L,n} = H_i$ for all $i \in I$, $H_{L,n} = H$ and therefore $H \leq^n L$ by using Lemma 1.1.

(2) Put $n = \max\{\text{wsi}(L: H_i): i \in I\}$. Then by (1) we have $H$ wsi $L$ and wsi$(L: H) \leq n$.

DEFINITION. We denote by $\mathscr{S}_L(\Delta)$ the set of all subalgebras $H$ of a Lie algebra $L$ such that $H \Delta L$, where $\Delta$ is any one of the relations $\leq$, wsi and si. Let $n$ be any integer $\geq 0$. We furthermore introduce the classes $\overline{\mathfrak{L}}$, $\overline{\mathfrak{L}}^{\infty}$, $\overline{\mathfrak{L}}_{\infty}$, $\overline{\mathfrak{M}}_n$ and $\overline{\mathfrak{M}}$ (resp. the classes $\mathfrak{L}$, $\mathfrak{L}^{\infty}$, $\mathfrak{L}_{\infty}$, $\mathfrak{M}_n$ and $\mathfrak{M}$) as follows. For a Lie algebra $L$, $L \in \overline{\mathfrak{L}}$ (resp. $L \in \mathfrak{L}$) if and only if $\langle H, K \rangle \in \mathscr{S}_L(\text{wsi})$ (resp. $\langle H, K \rangle \in \mathscr{S}_L(\text{si})$) for any $H, K \in \mathscr{S}_L(\text{wsi})$ (resp. $H, K \in \mathscr{S}_L(\text{si})$). $L \in \overline{\mathfrak{L}}^{\infty}$ (resp. $L \in \mathfrak{L}^{\infty}$) if and only if $\langle H_i: i \in I \rangle \in \mathscr{S}_L(\text{wsi})$ (resp. $\langle H_i: i \in I \rangle \in \mathscr{S}_L(\text{si})$) for any subset $\{H_i: i \in I\}$ of $\mathscr{S}_L(\text{wsi})$ (resp. $\mathscr{S}_L(\text{si})$). $L \in \overline{\mathfrak{L}}_{\infty}$ (resp. $L \in \mathfrak{L}_{\infty}$) if and only if $\bigcap_{i \in I} H_i \in \mathscr{S}_L(\text{wsi})$ (resp. $\bigcap_{i \in I} H_i \in \mathscr{S}_L(\text{si})$) for any subset $\{H_i: i \in I\}$ of $\mathscr{S}_L(\text{wsi})$ (resp. $\mathscr{S}_L(\text{si})$). $L \in \overline{\mathfrak{M}}_n$ (resp. $L \in \mathfrak{M}_n$) if and only if wsi$(L: H) \leq n$ (resp. si$(L: H) \leq n$) for any $H \in \mathscr{S}_L(\text{wsi})$ (resp. $H \in \mathscr{S}_L(\text{si})$). $L \in \overline{\mathfrak{M}}$ (resp. $L \in \mathfrak{M}$) if and only if $L \in \overline{\mathfrak{M}}_n$ (resp. $L \in \mathfrak{M}_n$) for some integer $n \geq 0$.

Let $\Delta$ be either of the relations wsi and si. For any Lie algebra $L$, we can consider $\mathscr{S}_L(\leq)$ as a lattice by introducing the usual lattice-structure in it. Then it is clear that $\mathscr{S}_L(\Delta)$ is not necessarily a sublattice of the lattice $\mathscr{S}_L(\leq)$ (cf. [3, Example 5.1] and [2, Lemma 3.1.1]). So it seems to be interesting to investigate the classes of Lie algebras $L$ such that $\mathscr{S}_L(\Delta)$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$. Using Lemma 1.2 we can easily see that $\mathscr{S}_L(\text{wsi})$ (resp. $\mathscr{S}_L(\text{si})$) is a sublattice of the lattice $\mathscr{S}_L(\leq)$ if and only if $L \in \overline{\mathfrak{L}}$ (resp. $L \in \mathfrak{L}$). Moreover, $\mathscr{S}_L(\text{wsi})$ (resp. $\mathscr{S}_L(\text{si})$) is a complete sublattice of the lattice $\mathscr{S}_L(\leq)$ if and only if $L \in \overline{\mathfrak{L}}^{\infty} \cap \overline{\mathfrak{L}}_{\infty}$ (resp. $L \in \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}$).

[6, Theorem 1] states that if $L \in \mathfrak{A}^d$, then every $n$-step weak subideal of $L$

is an $nd$-step subideal of $L$. It follows that if $L \in \mathrm{E}\mathfrak{A}$, then $\mathscr{S}_L(\mathrm{wsi}) = \mathscr{S}_L(\mathrm{si})$. Therefore we obtain

LEMMA 1.3. (1) *For any integers $n$, $d \geq 0$*

$$\overline{\mathfrak{M}}_n \cap \mathfrak{A}^d \leq \mathfrak{M}_{nd} \quad and \quad \mathfrak{M}_n \cap \mathfrak{A}^d \leq \overline{\mathfrak{M}}_n.$$

(2) *If $\mathfrak{X}$ is one of the classes $\mathfrak{L}$, $\mathfrak{L}^\infty$, $\mathfrak{L}_\infty$ and $\mathfrak{M}$, then $\overline{\mathfrak{X}} \cap \mathrm{E}\mathfrak{A} = \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$.*

Any notation not explained in this section may be found in [2].

## 2.

In this section we shall investigate several classes of Lie algebras $L$ such that $\mathscr{S}_L(\Delta)$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$, where $\Delta$ is any one of the relations wsi and si.

We begin with the following lemma due to Maruo [4], which is useful to the argument of this section. We here present its proof simplified by using the notion of weak closure series.

LEMMA 2.1 ([4, Lemma 2.3]). *Let $H$, $K$ wsi $L$ and let $J = \langle H, K \rangle$. If $[H, K] \subseteq H$, then $J$ wsi $L$ and* $\mathrm{wsi}(L: J) \leq \mathrm{wsi}(L: H)\mathrm{wsi}(L: K)$.

PROOF. Put $m = \mathrm{wsi}(L: H)$ and $n = \mathrm{wsi}(L: K)$. Then by Lemma 1.1 $H_{L,m} = H$ and $K_{L,n} = K$. First we show that

$$(*) \qquad [H_{L,i}, K] \subseteq H_{L,i} \qquad \text{for any integer } i \geq 0.$$

To do this we use induction on $i$. It is trivial for $i = 0$. Let $i \geq 0$ and suppose that $[H_{L,i}, K] \subseteq H_{L,i}$. Then

$$\begin{aligned}
[H_{L,i+1}, K] &= [[H_{L,i}, H] + H, K] = [[H_{L,i}, H], K] + [H, K] \\
&\subseteq [[H_{L,i}, K], H] + [H_{L,i}, [H, K]] + [H, K] \\
&\subseteq [H_{L,i}, H] + H = H_{L,i+1}.
\end{aligned}$$

Hence $(*)$ has been proved. Now we define

$$M_{i,j} = H_{L,i+1} + (H_{L,i} \cap K_{L,j}) + K \quad (0 \leq i < m, 0 \leq j \leq n).$$

Then for $0 \leq i < m$

$$H_{L,i+1} + K = M_{i,n} \subseteq M_{i,n-1} \subseteq \cdots \subseteq M_{i,0} = H_{L,i} + K.$$

Furthermore, by using $(*)$ we have

$$\begin{aligned}[M_{i,j}, J] &= [H_{L,i+1} + (H_{L,i} \cap K_{L,j}) + K, H + K] \\ &\subseteq H_{L,i+1} + [H_{L,i+1}, K] + [H_{L,i} \cap K_{L,j}, K] + K \\ &\subseteq H_{L,i+1} + (H_{L,i} \cap K_{L,j+1}) + K = M_{i,j+1}.\end{aligned}$$

Since $M_{m-1,n}=J$ and $M_{0,0}=L$, we have $J \leq^{mn} L$. Thus $J$ wsi $L$ and wsi $(L: J) \leq mn$.

In any Lie algebra $L$, the join of any pair of permutable subideals of $L$ is always a subideal of $L$ (cf. [2, Lemma 2.1.4]). On the other hand, we do not have any reason to say that the join of any pair of permutable weak subideals of $L$ is always a weak subideal of $L$. However, we can show the following

LEMMA 2.2. *Let $H$, $K$ wsi $L$ and let $J=\langle H, K\rangle$. If $[H, K] \subseteq H+K$ and $H$ si $J$, then $J$ wsi $L$.*

PROOF. Let $n=\mathrm{si}(J: H)$. We use induction on $n$ to show that $J$ wsi $L$. By Lemma 2.1 it is trivial for $n \leq 1$. Let $n>1$ and suppose that the result is true for $n-1$. We put $H_1=H^J$. Then clearly $H \lhd^{n-1} H_1 \lhd J$. Since $H_1 \cap K \lhd K$ wsi $L$, $H_1 \cap K$ wsi $L$. By modular law

$$H_1 = H_1 \cap J = H_1 \cap (H+K) = H + (H_1 \cap K).$$

Since $\mathrm{si}(H_1: H)=n-1$, by induction hypothesis we have $H_1$ wsi $L$. Evidently $J=\langle H_1, K\rangle$ and $[H_1, K] \subseteq H_1$. Using Lemma 2.1 we have $J$ wsi $L$.

As a result shown by using Lemma 2.2, we have the following proposition corresponding to [2, Proposition 2.1.6].

PROPOSITION 2.3. *Let $H$, $K$ wsi $L$ with $m=\mathrm{wsi}\,(L: H)>0$ and $n=\mathrm{wsi}\,(L: K)>0$. Assume that $\langle H, K\rangle$ is solvable. Then for any $i \geq m$ and $j \geq n$ we have*

$$J_{i,j} = \langle H^i, K^j\rangle \text{ wsi } L$$

*and $J_{i,j}=(H \cap J_{i,j})+(K \cap J_{i,j})$. Furthermore, for any $i \geq m$ we have $[L, {}_i H] \leq^m L$ and $[L, H^i] \leq^m L$.*

PROOF. Let $i \geq m$ and $j \geq n$. Then it is easy to see that $H^i \leq^m L$ and $K^j \leq^n L$. By Lemma 1.2 (2) we have $H \cap K$ wsi $L$. Put $C=\langle H \cap K, K^j\rangle$. Since $K^j \lhd K$, $C$ wsi $L$ by Lemma 2.1. Clearly $[H^i, K^j] \subseteq H \cap K$. It follows that $[H^i, C] \subseteq H^i+C$. Hence $\langle H^i, C\rangle = H^i+C = H^i+(H \cap K)+K^j$. Since $\langle H, K\rangle$ is solvable, so is $\langle H^i, C\rangle$. Therefore by [6, Theorem 1] $H^i$ si $\langle H^i, C\rangle$. Using Lemma 2.2 we have $\langle H^i, C\rangle$ wsi $L$. We can easily show that $J_{i,j} \lhd H^i+(H \cap K)+K^j$ and $J_{i,j}=(H \cap J_{i,j})+(K \cap J_{i,j})$. Thus we obtain $J_{i,j}$ wsi $L$.

Now $[L, H^i] \subseteq [L, {}_i H] \subseteq H$. By induction on $k$ we have

$$[[L,{}_k H], H_{L,m-1}] \subseteq [L,{}_k H] \quad \text{for any integer } k \geq 0.$$

In particular, $[[L,{}_i H], H_{L,m-1}] \subseteq [L,{}_i H]$ and hence $[L,{}_i H] \leq L$. Furthermore,

$$[L,{}_m [L,{}_i H]] \subseteq [[L,{}_{m-1} H], [L,{}_i H]] \subseteq [H_{L,m-1}, [L,{}_i H]] \subseteq [L,{}_i H].$$

Therefore by Lemma 1.1 we have $[L,{}_i H] \leq^m L$. Since $H_{L,m}=H$, $[H^i, H_{L,m-1}] \subseteq H^i$. Hence we have

$$[[L, H^i], H_{L,m-1}] \subseteq [L, H^i].$$

It follows that $[L, H^i] \leq L$. Moreover,

$$[L,{}_m [L, H^i]] \subseteq [[L,{}_{m-1} H], [L, H^i]] \subseteq [H_{L,m-1}, [L, H^i]] \subseteq [L, H^i].$$

Therefore by Lemma 1.1 we have $[L, H^i] \leq^m L$.

For subalgebras $H$, $K$ of a Lie algebra $L$, the circle product $H \circ K$ of $H$ and $K$ is defined as $H \circ K = [H, K]^{H \cup K}$.

LEMMA 2.4. *Let $\Delta$ be either of the relations* wsi *and* si *and let $H, K \Delta L$. Then the following conditions are equivalent*:

(1) $\langle H, K \rangle \,\Delta\, L$.
(2) $\langle H^K \rangle \,\Delta\, L$.
(3) $H \circ K \,\Delta\, L$.

PROOF. Since $H \circ K \lhd \langle H^K \rangle \lhd \langle H, K \rangle$, (1)$\Rightarrow$(2)$\Rightarrow$(3) is trivial. We have to show that (3) implies (1). Suppose $H \circ K \Delta L$. It is easy to see that $\langle H^K \rangle = H + H \circ K$ and $[H \circ K, H] \subseteq H \circ K$. Hence $\langle H^K \rangle \Delta L$. Since $\langle H, K \rangle = \langle H^K \rangle + K$, we have $\langle H, K \rangle \Delta L$.

Let us recall the class $\mathfrak{D}$ of Lie algebras in which every subalgebra is a subideal. We analogously introduce the class $\bar{\mathfrak{D}}$ of Lie algebras in which every subalgebra is a weak subideal. It is clear that

$$\mathfrak{N} \leq \mathfrak{D} \leq \bar{\mathfrak{D}}.$$

We also have

COROLLARY 2.5. (1) $\bar{\mathfrak{D}}\mathfrak{A} \leq \bar{\mathfrak{L}}$.
(2) $\mathfrak{D}\mathfrak{A} \leq \mathfrak{L}$.

PROOF. Here we only prove (1), since (2) is similarly proved. Let $L \in \bar{\mathfrak{D}}\mathfrak{A}$ and let $H, K \operatorname{wsi} L$. Then $L^2 \in \bar{\mathfrak{D}}$. Since $H \circ K \leq L^2$, $H \circ K \operatorname{wsi} L$. By using Lemma 2.4 we have $\langle H, K \rangle \operatorname{wsi} L$. Therefore $L \in \bar{\mathfrak{L}}$.

Tôgô [7] introduced the closure operation WSI defined as follows: A class $\mathfrak{X}$ of Lie algebras is WSI-closed if every weak subideal of an $\mathfrak{X}$-algebra also belongs to $\mathfrak{X}$. As a relationship between the closure operations I, S and WSI, we have

$$\mathrm{I} < \mathrm{WSI} < \mathrm{S}.$$

On the other hand, Robinson [5] introduced the class $\mathfrak{L}^*$ of groups to investigate classes of groups in which the join of any pair of subnormal subgroups is always subnormal. We analogously introduce the classes $\overline{\mathfrak{L}}^*$ and $\mathfrak{L}^*$ of Lie algebras in the following

DEFINITION. We define the class $\overline{\mathfrak{L}}^*$ of Lie algebras as the largest WSI-closed subclass $\mathfrak{X}$ of $\overline{\mathfrak{L}}$ having the following property:

(A) If $H$, $K$ wsi $L$ and $J=\langle H, K\rangle \in \mathfrak{X}$, then $J$ wsi $L$.

We similarly define the class $\mathfrak{L}^*$ of Lie algebras as the largest I-closed subclass $\mathfrak{X}$ of $\mathfrak{L}$ having the following property:

(B) If $H$, $K$ si $L$ and $J=\langle H, K\rangle \in \mathfrak{X}$, then $J$ si $L$.

We have the following result corresponding to [5, Theorem 3.23].

THEOREM 2.6. (1) $\mathfrak{N}\overline{\mathfrak{L}}^* \leq \overline{\mathfrak{L}}$.

(2) $\mathfrak{N}\mathfrak{L}^* \leq \mathfrak{L}$.

PROOF. Here we only prove (1), since (2) is similarly proved. Let $L \in \mathfrak{N}_c\overline{\mathfrak{L}}^*$. We use induction on $c$ to show that $L \in \overline{\mathfrak{L}}$. It is trivial for $c=0$. Let $c>0$ and suppose that the result is true for $c-1$. $L$ has a nilpotent ideal $I$ of class $\leq c$ such that $L/I \in \overline{\mathfrak{L}}^*$. We put $Z=\zeta_1(I)$. Since $I \lhd L$, $Z \lhd L$. Then clearly $L/Z \in \mathfrak{N}_{c-1}\overline{\mathfrak{L}}^*$ and therefore $L/Z \in \overline{\mathfrak{L}}$ by induction hypothesis. Let $H$, $K$ wsi $L$ and put $J=\langle H, K\rangle$. Since $L/Z \in \overline{\mathfrak{L}}$, we have $J+Z/Z$ wsi $L/Z$, so that $J+Z$ wsi $L$. It is easy to see that $J \cap I \lhd J+Z$. Now $J/J \cap I=\langle H+(J \cap I)/J \cap I,\ K+(J \cap I)/J \cap I\rangle$, where both $H+(J \cap I)/J \cap I$ and $K+(J \cap I)/J \cap I$ are weak subideals of $J+Z/J \cap I$. Since $J+I/I$ wsi $L/I$, $J+I/I \in \mathrm{WSI}\overline{\mathfrak{L}}^* = \overline{\mathfrak{L}}^*$ and so $J/J \cap I \in \overline{\mathfrak{L}}^*$. By the property (A) of $\overline{\mathfrak{L}}^*$ we have $J/J \cap I$ wsi $J+Z/J \cap I$. Hence $J$ wsi $J+Z$ and therefore $J$ wsi $L$. Thus we have $L \in \overline{\mathfrak{L}}$. This completes the proof.

In this section, we aim to investigate classes of $\overline{\mathfrak{L}}$-algebras (resp. $\mathfrak{L}$-algebras). Theorem 2.6 shows that it is one of the effective measures for this purpose to search classes of $\overline{\mathfrak{L}}^*$-algebras (resp. $\mathfrak{L}^*$-algebras). We can easily see that $\mathfrak{A} \leq \overline{\mathfrak{L}}^* \cap \mathfrak{L}^*$. However, the Hartley example (cf. [2, Lemma 3.1.1]) shows that over any field $\mathfrak{k}$ of characteristic $p>0$

$$\mathfrak{F}_3 \cap \mathfrak{N}_2 \not\leq \overline{\mathfrak{L}}^* \cup \mathfrak{L}^* \quad \text{and} \quad \mathfrak{F} \cap (\mathfrak{A}\mathfrak{N}_2) \cap \mathfrak{G}_3 \not\leq \overline{\mathfrak{L}} \cup \mathfrak{L}.$$

Therefore it seems that both of the classes $\overline{\mathfrak{L}}^*$ and $\overline{\mathfrak{L}}$ (resp. the classes $\mathfrak{L}^*$ and $\mathfrak{L}$) are not so large. But we shall present several interesting subclasses of $\overline{\mathfrak{L}}^*$ (resp. $\mathfrak{L}^*$) in the rest of this section.

First we consider the classes $\overline{\mathfrak{M}}_1$ and $\mathfrak{M}_1$. $\mathfrak{M}_1$ is the class of Lie algebras in which every subideal is an ideal, and is denoted by $\mathfrak{T}$ in [2]. On the other hand, $\overline{\mathfrak{M}}_1$ is the class of Lie algebras in which every weak subideal is an ideal, and is a proper subclass of $\mathfrak{M}_1$. Then we can prove the following

THEOREM 2.7. (1) $\overline{\mathfrak{M}}_1 \leq \overline{\mathfrak{L}}^*$ *and therefore* $\mathfrak{N}\overline{\mathfrak{M}}_1 \leq \overline{\mathfrak{L}}$.
(2) $\mathfrak{M}_1 \leq \mathfrak{L}^*$ *and therefore* $\mathfrak{N}\mathfrak{M}_1 \leq \mathfrak{L}$.

PROOF. Here we only prove (1), since (2) is similarly proved. Clearly we have $\mathrm{wsi}\overline{\mathfrak{M}}_1 = \overline{\mathfrak{M}}_1 \leq \overline{\mathfrak{L}}$. Let $L$ be any Lie algebra and let $H, K$ wsi $L$. Suppose that $J = \langle H, K \rangle \in \overline{\mathfrak{M}}_1$. Since $H$ wsi $J$, $H \lhd J$ and so $[H, K] \subseteq H$. By Lemma 2.1 we have $J$ wsi $L$. Hence the class $\overline{\mathfrak{M}}_1$ has the property (A). Therefore we obtain $\overline{\mathfrak{M}}_1 \leq \overline{\mathfrak{L}}^*$. By using Theorem 2.6 (1) we have $\mathfrak{N}\overline{\mathfrak{M}}_1 \leq \overline{\mathfrak{L}}$.

DEFINITION. (1) We define the class $\overline{\mathfrak{S}}$ of Lie algebras as follows: For a Lie algebra $L$, $L \in \overline{\mathfrak{S}}$ if and only if $L$ has no non-trivial weak subideals. On the other hand, we denote by $\mathfrak{S}$ the class of simple Lie algebras. Then

$$\overline{\mathfrak{S}} < \mathfrak{S}, \ \overline{\mathfrak{S}} < \overline{\mathfrak{M}}_1, \ \mathfrak{S} < \mathfrak{M}_1.$$

(2) We define the subclasses $\mathfrak{A}_1$ and $\mathfrak{A}_0$ of $\mathfrak{A}^2$ as follows: For a Lie algebra $L$, $L \in \mathfrak{A}_1$ if and only if either $L \in \mathfrak{A}$, or $L \in \mathfrak{A}^2$ with $\dim(L/L^2) = 1$. $L \in \mathfrak{A}_0$ if and only if either $L \in \mathfrak{A}$, or $L \in \mathfrak{A}_1 \backslash \mathfrak{A}$ with $L/L^2$ acting on $L^2$ as scalar multiplications. Then

$$\mathfrak{A} < \mathfrak{A}_0 < \mathfrak{A}_1 < \mathfrak{A}^2.$$

Owing to [1, Theorems 3.6 and 3.8], we can easily show that the class $\mathfrak{A}_0$ coincides with the class of Lie algebras in which every subalgebra is a quasi-ideal.

Let us recall the class $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ of Lie algebras which have an ascending ideal series with abelian factors. Stewart proved in his thesis that $\mathfrak{M}_1 \cap \mathrm{E}\mathfrak{A} = \mathfrak{T} \cap \mathrm{E}\mathfrak{A} = \mathfrak{A}_0$ (cf. [2, p. 167]). We can generalize this result in the following

PROPOSITION 2.8. $\mathfrak{A}_0 = \overline{\mathfrak{M}}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} = \mathfrak{M}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

PROOF. Using Lemma 1.3 (1) we have $\overline{\mathfrak{M}}_1 \cap \mathrm{E}\mathfrak{A} = \mathfrak{M}_1 \cap \mathrm{E}\mathfrak{A}$. Hence it follows from the Stewart's result that $\mathfrak{A}_0 \leq \overline{\mathfrak{M}}_1$. Therefore we have

$$\mathfrak{A}_0 \leq \overline{\mathfrak{M}}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{M}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}.$$

Conversely, let $L \in \mathfrak{M}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Then $L$ has an ascending ideal series $(L_\alpha)_{\alpha \leq \sigma}$ with abelian factors. We use transfinite induction on $\alpha$ to show that $L_\alpha \in \mathfrak{A}_0$ for any $\alpha \leq \sigma$. It is trivial for $\alpha = 0$. Let $0 < \alpha \leq \sigma$ and suppose that $L_\beta \in \mathfrak{A}_0$ for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, then $L_\alpha = \bigcup_{\beta < \alpha} L_\beta \in \mathfrak{A}^2$ and hence $L_\alpha \in \mathfrak{M}_1 \cap \mathfrak{A}^2 = \mathfrak{A}_0$. If $\alpha$ is not a limit ordinal, then $L_{\alpha-1} \in \mathfrak{A}_0 \leq \mathfrak{A}^2$. Since $L_\alpha / L_{\alpha-1} \in$

$\mathfrak{A}$, $L_\alpha \in \mathfrak{A}^3$ and therefore $L_\alpha \in \mathfrak{M}_1 \cap \mathfrak{A}^3 = \mathfrak{A}_0$. This completes the induction. In particular, we have $L = L_\sigma \in \mathfrak{A}_0$. Hence $\mathfrak{M}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{A}_0$. Thus we obtain $\mathfrak{A}_0 = \overline{\mathfrak{M}}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} = \mathfrak{M}_1 \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

Next we shall prove that $\mathfrak{A}_1 \leq \overline{\mathfrak{L}}^* \cap \mathfrak{L}^*$. To do this we need the following

LEMMA 2.9. *If* $L \in \mathfrak{A}_1 \backslash \mathfrak{A}$, *then*

$$\mathscr{S}_L(\mathrm{wsi}) = \{H\colon H \leq L^2 \text{ or } H = L\} = \{H\colon H \lhd^2 L\}.$$

PROOF. Since $L^2$ is an abelian ideal of $L$, it is clear that

$$\{H\colon H \leq L^2 \text{ or } H = L\} \subseteq \{H\colon H \lhd^2 L\} \subseteq \mathscr{S}_L(\mathrm{wsi}).$$

Conversely, let $H \in \mathscr{S}_L(\mathrm{wsi})$. Suppose that $H \nleq L^2$. Then $H$ has an element $h \notin L^2$. Since $L \in \mathfrak{A}_1 \backslash \mathfrak{A}$, $L$ has an element $x$ such that $L = L^2 \dotplus \langle x \rangle$. Then clearly $[L^2, x] = L^2$. We can write $h = a + \alpha x$, where $a \in L^2$ and $\alpha \in \mathfrak{k}$. Since $h \notin L^2$, $\alpha \neq 0$. Without loss of generality we may assume $\alpha = 1$, so that $h = a + x$. Then

$$[L^2, h] = [L^2, a+x] = [L^2, x] = L^2.$$

It follows that $[L, h] = L^2$. By induction on $n$ we have $[L,_n h] = L^2$ for any integer $n > 0$. Since $H$ wsi $L$, by Lemma 1.1 $[L,_n H] \subseteq H$ for some integer $n > 0$. Then

$$L^2 = [L,_n h] \subseteq [L,_n H] \subseteq H$$

and hence $L^2 < H$. Since $\dim(L/L^2) = 1$, we have $H = L$. This completes the proof.

THEOREM 2.10. $\mathfrak{A}_1 \leq \overline{\mathfrak{M}}_2 \cap \mathfrak{M}_2 \cap \overline{\mathfrak{L}}^* \cap \mathfrak{L}^*$ *and therefore* $\mathfrak{N}\mathfrak{A}_1 \leq \overline{\mathfrak{L}} \cap \mathfrak{L}$.

PROOF. By Lemma 2.9 we have $\mathfrak{A}_1 \leq \overline{\mathfrak{M}}_2 \cap \mathfrak{M}_2$. On the other hand, by Corollary 2.5 $\mathfrak{A}_1 \leq \mathfrak{A}^2 \leq \overline{\mathfrak{L}} \cap \mathfrak{L}$. Let $H$ be a weak subideal of an $\mathfrak{A}_1$-algebra $L$. If $L \in \mathfrak{A}$, then $H \in \mathfrak{A} \leq \mathfrak{A}_1$. If $L \notin \mathfrak{A}$, then by Lemma 2.9 we have $H \in \mathfrak{A}$ or $H = L$, so that $H \in \mathfrak{A}_1$. Hence $\mathfrak{A}_1$ is WSI-closed and therefore I-closed. Now let $M$ be any Lie algebra and let $H$, $K$ wsi $M$. Suppose that $J = \langle H, K \rangle \in \mathfrak{A}_1$. If $J \in \mathfrak{A}$, then by Lemma 2.1 $J$ wsi $M$. So we may assume $J \notin \mathfrak{A}$. If either $H = J$ or $K = J$, then clearly $J$ wsi $M$. Suppose that $H < J$ and $K < J$. Since $H$, $K$ wsi $J$, by Lemma 2.9 $H, K \leq J^2 \in \mathfrak{A}$ and so $[H, K] = 0$. Using Lemma 2.1 we have $J$ wsi $M$. If both $H$ and $K$ are especially subideals of $M$, then we can similarly show that $J$ si $M$. Therefore $\mathfrak{A}_1$ has the properties (A) and (B). Thus we have $\mathfrak{A}_1 \leq \overline{\mathfrak{L}}^* \cap \mathfrak{L}^*$. By using Theorem 2.6, we obtain $\mathfrak{N}\mathfrak{A}_1 \leq \overline{\mathfrak{L}} \cap \mathfrak{L}$.

REMARK. By the above theorem we have

$$\mathfrak{A}_1 \leq \mathfrak{A}^2 \cap \overline{\mathfrak{M}}_2 \cap \mathfrak{M}_2.$$

However, the Hartley example [2, Lemma 3.1.1] shows that over any field $\mathfrak{f}$ of characteristic $p>0$

$$\mathfrak{A}^2 \cap \overline{\mathfrak{M}}_2 \cap \mathfrak{M}_2 \nleq \overline{\mathfrak{L}}^* \cup \mathfrak{L}^*.$$

Finally we summarize the results in this section as the following main theorem.

THEOREM 2.11. (1) *Let $\mathfrak{X}$ be any one of the classes*

$$\overline{\mathfrak{D}}\mathfrak{A},\ \mathfrak{N}\mathfrak{S},\ \mathfrak{N}\overline{\mathfrak{M}}_1,\ \mathfrak{N}\mathfrak{A}_0,\ \mathfrak{N}\mathfrak{A}_1.$$

*If $L\in\mathfrak{X}$, then $\mathscr{S}_L(\mathrm{wsi})$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$.*

(2) *Let $\mathfrak{X}$ be any one of the classes*

$$\mathfrak{D}\mathfrak{A},\ \mathfrak{N}\mathfrak{S},\ \mathfrak{N}\mathfrak{M}_1,\ \mathfrak{N}\mathfrak{A}_0,\ \mathfrak{N}\mathfrak{A}_1.$$

*If $L\in\mathfrak{X}$, then $\mathscr{S}_L(\mathrm{si})$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$.*

## 3.

In this section we shall investigate several classes of Lie algebras $L$ such that $\mathscr{S}_L(\Delta)$ is a complete sublattice of the lattice $\mathscr{S}_L(\leq)$, where $\Delta$ is any one of the relations wsi and si.

We begin with the following result corresponding to [5, Lemma 3.14].

PROPOSITION 3.1. *Let $L$ be a Lie algebra and let $n$ be any integer $\geq 0$. Then*

(1) *$L\in\overline{\mathfrak{L}}_\infty$ if to any $H\leq L$ there corresponds an integer $m=m(H)\geq 0$ such that $H_{L,m}=H_{L,m+1}$.*

(2) *$L\in\overline{\mathfrak{M}}_n$ if $H_{L,n}=H_{L,n+1}$ for all $H\leq L$.*

(3) *$L\in\mathfrak{L}_\infty$ if and only if to any $H\leq L$ there corresponds an integer $m=m(H)\geq 0$ such that $H^{L,m}=H^{L,m+1}$.*

(4) *$L\in\mathfrak{M}_n$ if and only if $H^{L,n}=H^{L,n+1}$ for all $H\leq L$.*

PROOF. (1) Let $\{H_\lambda\colon \lambda\in\Lambda\}$ be any subset of $\mathscr{S}_L(\mathrm{wsi})$ and let $H=\bigcap_{\lambda\in\Lambda}H_\lambda$. For each $\lambda\in\Lambda$, we put $n(\lambda)=\mathrm{wsi}(L\colon H_\lambda)$. By our assumption we can find an integer $m\geq 0$ such that $H_{L,m}=H_{L,m+1}$. If we put $m(\lambda)=\max\{n(\lambda), m\}$, then

$$H_{L,m}=H_{L,m(\lambda)}\subseteq (H_\lambda)_{L,m(\lambda)}\subseteq (H_\lambda)_{L,n(\lambda)}=H_\lambda.$$

This being true for all $\lambda\in\Lambda$, we have $H_{L,m}=H$. Therefore $H\in\mathscr{S}_L(\mathrm{wsi})$. Thus we obtain $L\in\overline{\mathfrak{L}}_\infty$.

(2) Let $H\in\mathscr{S}_L(\mathrm{wsi})$ with $m=\mathrm{wsi}(L\colon H)$. Then $H=H_{L,m}=H_{L,m+1}=\cdots$. By our assumption we have $H_{L,n}=H_{L,n+1}=\cdots$. It follows that $H_{L,n}=H_{L,m}=H$. Hence $H\leq^n L$ and therefore $\mathrm{wsi}(L\colon H)\leq n$. Thus we obtain $L\in\overline{\mathfrak{M}}_n$.

(3) One implication is similarly proved as in the proof of (1). Suppose $L\in\mathfrak{L}_\infty$ and let $H\leq L$. It is clear that $H^{L,i}\in\mathscr{S}_L(\mathrm{si})$ for any $i<\omega$. Hence $H^{L,\omega}=\bigcap_{i<\omega}H^{L,i}\in\mathscr{S}_L(\mathrm{si})$. If we put $m=\mathrm{si}(L\colon H^{L,\omega})$, then

$$H^{L,m}\leq(H^{L,\omega})^{L,m}=H^{L,\omega}\leq H^{L,m+1}$$

and therefore $H^{L,m}=H^{L,m+1}$.

(4) One implication is similarly proved as in the proof of (2). Suppose $L\in\mathfrak{M}_n$ and let $H\leq L$. Since $H^{L,n+1}\in\mathscr{S}_L(\mathrm{si})$, we have $\mathrm{si}(L\colon H^{L,n+1})\leq n$. Hence

$$H^{L,n}\leq(H^{L,n+1})^{L,n}=H^{L,n+1}\leq H^{L,n}$$

and therefore $H^{L,n}=H^{L,n+1}$.

As a direct consequence of Lemma 1.2 we obtain

LEMMA 3.2. *$\overline{\mathfrak{M}}\leq\overline{\mathfrak{L}}_\infty$ and $\mathfrak{M}\leq\mathfrak{L}_\infty$.*

REMARK. Over any field $\mathfrak{k}$ of characteristic $p>0$, the Unsin example (cf. [2, Chap. 7, §5]) presented a metabelian $\mathfrak{D}$-algebra $L$ having subideals of every non-negative subideal index. Then clearly $L\in\mathfrak{L}_\infty$ and $L\notin\mathfrak{M}$. By using Lemma 1.3 (2), we have $L\in\overline{\mathfrak{L}}_\infty$ and $L\notin\overline{\mathfrak{M}}$. Therefore we have $\overline{\mathfrak{M}}<\overline{\mathfrak{L}}_\infty$ and $\mathfrak{M}<\mathfrak{L}_\infty$.

To prove the main results of this section, we need the following lemma corresponding to [5, Lemma 3.24].

LEMMA 3.3. *Let $L\in\overline{\mathfrak{L}}$ (resp. $L\in\mathfrak{L}$). Then $L\in\overline{\mathfrak{L}}^\infty$ (resp. $L\in\mathfrak{L}^\infty$) if and only if $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is closed under the formation of unions of ascending chains.*

PROOF. One implication is trivial. Suppose that $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is closed under the formation of unions of ascending chains. Let $\{H_\lambda\colon\lambda\in\Lambda\}$ be any subset of $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) and put $J=\langle H_\lambda\colon\lambda\in\Lambda\rangle$. Let the elements of $\Lambda$ be well-ordered as $\Lambda=\{\alpha\colon\alpha<\rho\}$ for some ordinal $\rho$. Then we can define the ascending chain $(J_\alpha)_{\alpha\leq\rho}$ of subalgebras of $L$ as follows:

$$J_0=0,\ J_\alpha=\langle H_\beta\colon\beta<\alpha\rangle\quad(0<\alpha\leq\rho).$$

We use transfinite induction on $\alpha$ to show that $J_\alpha\in\mathscr{S}_L(\mathrm{wsi})$ (resp. $J_\alpha\in\mathscr{S}_L(\mathrm{si})$) for any $\alpha\leq\rho$. It is trivial for $\alpha=0$. Let $0<\alpha\leq\rho$ and suppose that $J_\beta\in\mathscr{S}_L(\mathrm{wsi})$ (resp. $J_\beta\in\mathscr{S}_L(\mathrm{si})$) for all $\beta<\alpha$. If $\alpha$ is not a limit ordinal, then $J_\alpha=\langle J_{\alpha-1}, H_{\alpha-1}\rangle\in\mathscr{S}_L(\mathrm{wsi})$ (resp. $\langle J_{\alpha-1}, H_{\alpha-1}\rangle\in\mathscr{S}_L(\mathrm{si})$) as $L\in\overline{\mathfrak{L}}$ (resp. $L\in\mathfrak{L}$). Assume that $\alpha$ is a limit ordinal. Then it is easy to see that $J_\alpha=\bigcup_{\beta<\alpha}J_\beta$. By our assumption we have $J_\alpha\in\mathscr{S}_L(\mathrm{wsi})$ (resp. $J_\alpha\in\mathscr{S}_L(\mathrm{si})$). This completes the induction. In particular, we have $J=J_\rho\in\mathscr{S}_L(\mathrm{wsi})$ (resp. $J_\rho\in\mathscr{S}_L(\mathrm{si})$). Therefore $L\in\overline{\mathfrak{L}}^\infty$ (resp. $L\in\mathfrak{L}^\infty$).

We now set about showing the main results of this section.

THEOREM 3.4. (1) $\overline{\mathfrak{M}}\cap\overline{\mathfrak{L}}\leq\overline{\mathfrak{L}}^\infty\cap\overline{\mathfrak{L}}_\infty$.
(2) $\mathfrak{M}\cap\mathfrak{L}\leq\mathfrak{L}^\infty\cap\mathfrak{L}_\infty$.

PROOF. Here we only prove (1), since (2) is similarly proved. Let $L\in\overline{\mathfrak{M}}\cap\overline{\mathfrak{L}}$. By Lemma 3.2 we have $L\in\overline{\mathfrak{L}}_\infty$. Let $\{H(\alpha)\colon\alpha<\rho\}$ be any ascending chain in $\mathscr{S}_L(\mathrm{wsi})$, where $\rho$ is an ordinal. We put $J=\bigcup_{\alpha<\rho}H(\alpha)$. Since $L\in\overline{\mathfrak{M}}$, there exists an integer $n\geq 0$ such that $H(\alpha)_{L,n}=H(\alpha)$ for all $\alpha<\rho$. If we put $J_i=\bigcup_{\alpha<\rho}H(\alpha)_{L,i}$ for $0\leq i\leq n$, then we can easily show that $J=J_n\subseteq J_{n-1}\subseteq\cdots\subseteq J_0=L$ is a chain of subspaces of $L$ such that $[J_i, J]\subseteq J_{i+1}$ for $0\leq i<n$. Hence $J\in\mathscr{S}_L(\mathrm{wsi})$ and therefore $L\in\overline{\mathfrak{L}}^\infty$. Thus we have $\overline{\mathfrak{M}}\cap\overline{\mathfrak{L}}\leq\overline{\mathfrak{L}}^\infty\cap\overline{\mathfrak{L}}_\infty$.

Finally we have the following

THEOREM 3.5. (1) *Let $\mathfrak{X}$ be any one of the classes*

$$\overline{\mathfrak{D}},\ \overline{\mathfrak{M}}_1,\ \mathfrak{A}_1,\ \mathfrak{F}\cap(\mathfrak{N}\overline{\mathfrak{M}}_1),\ \mathfrak{F}\cap(\mathfrak{N}\mathfrak{A}_1).$$

*If $L\in\mathfrak{X}$, then $\mathscr{S}_L(\mathrm{wsi})$ is a complete sublattice of the lattice $\mathscr{S}_L(\leq)$.*
(2) *Let $\mathfrak{X}$ be any one of the classes*

$$\mathfrak{D},\ \mathfrak{M}_1,\ \mathfrak{A}_1,\ \mathfrak{F}\cap(\mathfrak{N}\mathfrak{M}_1),\ \mathfrak{F}\cap(\mathfrak{N}\mathfrak{A}_1).$$

*If $L\in\mathfrak{X}$, then $\mathscr{S}_L(\mathrm{si})$ is a complete sublattice of the lattice $\mathscr{S}_L(\leq)$.*

PROOF. If $L\in\overline{\mathfrak{D}}\cup\overline{\mathfrak{M}}_1$ (resp. $L\in\mathfrak{D}\cup\mathfrak{M}_1$), then $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is clearly a complete sublattice of $\mathscr{S}_L(\leq)$. If $L\in\mathfrak{A}_1$, then by Theorems 2.10 and 3.4 $\mathscr{S}_L(\mathrm{wsi})$ $(=\mathscr{S}_L(\mathrm{si}))$ is a complete sublattice of $\mathscr{S}_L(\leq)$. Since $\mathfrak{F}\leq\overline{\mathfrak{M}}\cap\mathfrak{M}$, the results for the other cases are immediately deduced from Theorems 2.11 and 3.4.

## 4.

In this section we shall present several examples in connection with the previous sections.

First we present two examples showing that $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is not necessarily a complete sublattice of the lattice $\mathscr{S}_L(\leq)$, even if $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is a sublattice of $\mathscr{S}_L(\leq)$.

EXAMPLE 4.1. Let $\mathfrak{f}$ be any field of characteristic zero and let $A$ be an abelian

Lie algebra over $\mathfrak{k}$ with basis $\{a_i: i \in \boldsymbol{Z}\}$, where $\boldsymbol{Z}$ denotes the set of integers. For each integer $j \geq 0$, define $x_j \in \mathrm{Der}(A)$ by $a_i x_j = a_{i+j}$ for any $i \in \boldsymbol{Z}$. Let $X$ denote the subalgebra of $\mathrm{Der}(A)$ generated by the elements $x_j$'s. Then $X$ is an abelian Lie algebra with basis $\{x_j: j \geq 0\}$. We construct the split extension $M = A \dotplus X$ of $A$ by $X$. Since $M$ has basis $\{a_i, x_j: i, j \in \boldsymbol{Z}, j \geq 0\}$, we can define $\delta \in \mathrm{End}(M)$ by

$$a_i\delta = a_i + ia_{i-1} \qquad (i \in \boldsymbol{Z}),$$

$$x_j\delta = jx_{j-1} \qquad (j \geq 0),$$

where we regard $x_{-1}$ as the zero map on $A$. By the definition of $\delta$ we can easily see that for any $i, j \in \boldsymbol{Z}$ with $j \geq 0$

$$[a_i, x_j]\delta = [a_i\delta, x_j] + [a_i, x_j\delta].$$

It follows that $\delta$ is a derivation of $M$. We construct the split extension $L = M \dotplus \langle\delta\rangle$ of $M$ by $\langle\delta\rangle$. Then $L = A \dotplus X \dotplus \langle\delta\rangle$. Since $A$ is $\delta$-invariant, $A$ is an ideal of $L$. Define $Y = \langle X, \delta\rangle$. Since $X$ is $\delta$-invariant, $Y$ is the split extension of an abelian ideal $X$ by $\langle\delta\rangle$. Furthermore,

$$Y^2 = [X, \delta] = X\delta = X.$$

Hence $Y^2 \in \mathfrak{A}$ and $\dim(Y/Y^2) = 1$. Therefore we have $Y \in \mathfrak{A}_1 \backslash \mathfrak{A}$. It is clear that $L$ is the split extension of $A$ by $Y$. Thus $L \in \mathfrak{A}\mathfrak{A}_1$. By using Theorem 2.10 we have $L \in \overline{\mathfrak{L}} \cap \mathfrak{L}$. Therefore $\mathscr{S}_L(\mathrm{wsi})$ $(= \mathscr{S}_L(\mathrm{si}))$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$.

Next we shall show that $L \notin \overline{\mathfrak{L}}_\infty \cup \mathfrak{L}_\infty$. To do this it is sufficient to show that $M \notin \overline{\mathfrak{L}}_\infty$. Suppose that it has been shown. Since $M \lhd L$, we have $L \notin \overline{\mathfrak{L}}_\infty$. Clearly $L \in \mathfrak{A}^3$. By Lemma 1.3 (2) we have $L \notin \mathfrak{L}_\infty$. Now for any $i, n \in \boldsymbol{Z}$ with $n > 0$, put

$$a(i, n) = \textstyle\sum_{k=0}^{n} {}_nC_k a_{i+k}$$

and define $A_n = \langle a(i, n): i \in \boldsymbol{Z}\rangle$. Then $A_n$ has basis $\{a(i, n): i \in \boldsymbol{Z}\}$. We put $z = x_0 + x_1$. It is not hard to show that $[A, {}_n z] = A_n$ for any $n > 0$. Since $A \lhd M$, we have $A_1 \geq A_2 \geq \cdots$. For any integer $n > 0$, define $H_n = \langle A_n, z\rangle$. Since $[a(i, n), x_j] = a(i+j, n)$ for all $i, j \in \boldsymbol{Z}$ with $j \geq 0$, we have $[A_n, X] = A_n$. Hence $A_n \lhd M$ and therefore

$$[M, {}_n H_n] = [A + X, {}_n A_n + \langle z\rangle] = [A, {}_n z] = A_n \leq H_n.$$

It follows that $H_n \leq^n M$. This is true for all $n > 0$. Now assume that $M \in \overline{\mathfrak{L}}_\infty$. Then $H = \bigcap_{n>0} H_n \,\mathrm{wsi}\, M$ and so $[M, {}_r H] \subseteq H$ for some $r > 0$. We put $B = \bigcap_{n>0} A_n$, so that $H = B + \langle z\rangle$. Since $[B, X] \subseteq B$, we have

$$[M,{}_rH] = [A+X,{}_rB+\langle z\rangle] = [A,{}_rz] = A_r.$$

Hence $A_r \subseteq H$. By modular law

$$A_r = A_r \cap (B+\langle z\rangle) = B + (A_r \cap \langle z\rangle) = B \leq A_{r+1}$$

and therefore $A_r = A_{r+1}$. Since $a(0, r) \in A_{r+1}$, we can write $a(0, r) = \sum_{k=1}^{s} \alpha_k a(i_k, r+1)$, where $0 \neq \alpha_k \in \mathfrak{k}$ and $i_k \in \boldsymbol{Z}\,(1 \leq k \leq s)$ such that $i_1 < \cdots < i_s$. Then we have $i_1 = 0$ and $\alpha_1 = 1$. Also $i_s + r + 1 = r$ and $\alpha_s = 1$. Hence $i_s = -1 < 0 = i_1$, a contradiction. Therefore we have $M \notin \overline{\mathfrak{L}}_\infty$. Thus $L \notin \overline{\mathfrak{L}}_\infty \cup \mathfrak{L}_\infty$. This implies that $\mathscr{S}_L(\text{wsi})$ $(=\mathscr{S}_L(\text{si}))$ is not a complete sublattice of the lattice $\mathscr{S}_L(\leq)$.

EXAMPLE 4.2. Let $\mathfrak{k}$ be any field of characteristic $p>0$ and let $A$ be the additive abelian group of type $p^\infty$. We consider an abelian Lie algebra $X$ over $\mathfrak{k}$ with basis $\{x_a \colon a \in A\}$. For each $b \in A$, define $\delta(b) \in \mathrm{Der}\,(X)$ by

$$x_a\delta(b) = \textstyle\sum_{i=0}^{p-1} x_{a+ib} \qquad (a \in A).$$

Let $Y$ denote the subalgebra of $\mathrm{Der}\,(X)$ generated by the set $\{\delta(b)\colon b \in A\}$. It is clear that $[\delta(b), \delta(c)] = 0$ for all $b, c \in A$. Hence $Y \in \mathfrak{A}$. We construct the split extension $L = X \dotplus Y$ of $X$ by $Y$. Then $L \in \mathfrak{A}^2$ and therefore by Corollary 2.5 $L \in \overline{\mathfrak{L}} \cap \mathfrak{L}$. Thus $\mathscr{S}_L(\text{wsi})$ $(=\mathscr{S}_L(\text{si}))$ is a sublattice of the lattice $\mathscr{S}_L(\leq)$.

Now let $b \in A$. First we show that in the associative algebra $\mathrm{End}\,(X)$

$$(*) \qquad \delta(b)^{p^n} = \delta(p^n b) \quad \text{for any integer } n \geq 1.$$

To show this we may restrict our attention to the case $n=1$. For any $a \in A$, we have an expression

$$x_a\delta(b)^p = \textstyle\sum_{i_1=0}^{p-1} \cdots \sum_{i_p=0}^{p-1} x_{a+(i_1+\cdots+i_p)b} = \sum_{i=0}^{(p-1)p} n_i x_{a+ib},$$

where each $n_i$ is an integer such that $0 \leq n_i < p$. We introduce an indeterminate $t$ and work in the polynomial algebra $\mathfrak{k}[t]$. Obviously

$$(\textstyle\sum_{i=0}^{p-1} t^i)^p = \sum_{i_1=0}^{p-1} \cdots \sum_{i_p=0}^{p-1} t^{i_1+\cdots+i_p} = \sum_{i=0}^{(p-1)p} n_i t^i.$$

On the other hand, since $\mathrm{char}\,(\mathfrak{k}) = p > 0$ we have

$$(\textstyle\sum_{i=0}^{p-1} t^i)^p = \sum_{i=0}^{p-1} t^{ip}.$$

Hence $\sum_{i=0}^{(p-1)p} n_i t^i = \sum_{i=0}^{p-1} t^{ip}$, so

$$n_i = \begin{cases} 1 & \text{if } p \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$x_a\delta(b)^p = \sum_{i=0}^{(p-1)p} n_i x_{a+ib} = \sum_{i=0}^{p-1} x_{a+ipb} = x_a\delta(pb).$$

This is true for all $a \in A$. Thus $\delta(b)^p = \delta(pb)$ and (*) is proved. Since $A$ is of type $p^\infty$, $|\langle b\rangle| = p^n$ for some integer $n \geq 0$. Then by (*)

$$\delta(b)^{p^n} = \delta(p^n b) = \delta(0) = 0.$$

Hence

$$\langle X, \delta(b)\rangle^{p^n+1} = [X, {}_{p^n}\, \delta(b)] = X\delta(b)^{p^n} = 0.$$

Therefore we have $\langle X, \delta(b)\rangle \in \mathfrak{N}$. Since $L^2 \leq X$, $\langle X, \delta(b)\rangle \lhd L$. Clearly we have $L = \sum_{b\in A}\langle X, \delta(b)\rangle$. It follows that $L$ is a Fitting algebra.

Next we show that $L \notin \overline{\mathfrak{M}} \cup \mathfrak{M} \cup \overline{\mathfrak{L}}^\infty \cup \mathfrak{L}^\infty$. To do this it suffices to show that $L \notin \overline{\mathfrak{L}}^\infty$. Suppose that it has been shown. Since $L \in \overline{\mathfrak{L}}$, by Theorem 3.4 (1) we have $L \notin \overline{\mathfrak{M}}$. Therefore by Lemma 1.3 (2) $L \notin \mathfrak{M} \cup \mathfrak{L}^\infty$. Now assume, to the contrary, that $L \in \overline{\mathfrak{L}}^\infty$. Since $L$ is a Fitting algebra, $\langle\delta(b)\rangle$ si $L$ for all $b \in A$. Hence $Y = \langle\delta(b) \colon b \in A\rangle$ wsi $L$ and so $Y \leq^m L$ for some integer $m \geq 0$. There is an integer $n > 0$ such that $m \leq p^n$. Since $A$ is of type $p^\infty$, we can find an element $b$ of $A$ of order $p^{n+1}$. By (*)

$$[x_{p^n b, p^n}\, \delta(b)] = x_{p^n b}\delta(b)^{p^n} = x_{p^n b}\delta(p^n b) = \sum_{i=0}^{p-1} x_{(i+1)p^n b} \neq 0.$$

Hence $[X, {}_{p^n}\langle\delta(b)\rangle] \neq 0$. However,

$$[X, {}_{p^n}\langle\delta(b)\rangle] \subseteq [X, {}_m Y] \subseteq Y$$

and therefore $[X, {}_{p^n}\langle\delta(b)\rangle] \subseteq X \cap Y = 0$, a contradiction. Thus we have $L \notin \overline{\mathfrak{L}}^\infty$. We consequently obtain $L \notin \overline{\mathfrak{M}} \cup \mathfrak{M} \cup \overline{\mathfrak{L}}^\infty \cup \mathfrak{L}^\infty$. In particular, $\mathscr{S}_L(\mathrm{wsi})$ $(= \mathscr{S}_L(\mathrm{si}))$ is not a complete sublattice of the lattice $\mathscr{S}_L(\leq)$.

REMARK. Let $L$ be the Lie algebra constructed in Example 4.2. Then $L$ is a Fitting algebra, but $L$ does not satisfy the idealizer condition. In fact, we can prove that $I_L(Y) = Y < L$. Let $x \in C_X(Y)$. Since $A$ is of type $p^\infty$, we can write

$$x = \sum_{i=0}^{p^n-1} \alpha_i x_{ia}$$

where $a \in A$ with $|\langle a\rangle| = p^n$ and $\alpha_i \in \mathfrak{k}$ $(0 \leq i < p^n)$. Since $A$ is a radicable group, $A$ has an element $b$ such that $pb = a$. Then

$$x\delta(b) = \sum_{i=0}^{p^n-1} \alpha_i x_{ipb}\, \delta(b) = \sum_{i=0}^{p^n-1} \sum_{j=0}^{p-1} \alpha_i x_{(ip+j)b}.$$

On the other hand, $x\delta(b) = [x, \delta(b)] = 0$ as $x \in C_X(Y)$. Hence we have $\sum_{i=0}^{p^n-1} \sum_{j=0}^{p-1} \alpha_i x_{(ip+j)b} = 0$. Since $b$ is of order $p^{n+1}$, we have $\alpha_i = 0$ $(0 \leq i < p^n)$. Therefore $x = 0$. Thus we have $C_X(Y) = 0$. It follows that $I_L(Y) \cap X = 0$. By modular law

$$I_L(Y) = I_L(Y) \cap (X+Y) = (I_L(Y) \cap X) + Y = Y.$$

Therefore we obtain $I_L(Y) = Y < L$.

Finally we shall construct an example of $(\overline{\mathfrak{L}}^* \cap \mathfrak{L}^*)$-algebras $L$ such that both $\mathscr{S}_L(\mathrm{wsi})$ and $\mathscr{S}_L(\mathrm{si})$ are complete sublattices of the lattice $\mathscr{S}_L(\leq)$.

EXAMPLE 4.3. Let $\mathfrak{k}$ be a field in which $\alpha^2+\beta^2+\gamma^2=0$ always implies $\alpha=\beta=\gamma=0$ (for example, take as $\mathfrak{k}$ the field of real numbers). Let $A$ be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{a_0, a_1, \dots\}$ and let $x$, $y$, $z$ be respectively the derivations of $A$ defined by

$$\begin{aligned}
&a_{3i}x = a_{3i+1},\ a_{3i+1}x = -a_{3i},\ a_{3i+2}x = 0 && (i \geq 0),\\
&a_{3i}y = 0, \quad a_{3i+1}y = a_{3i+2},\ a_{3i+2}y = -a_{3i+1} && (i \geq 0),\\
&a_{3i}z = a_{3i+2},\ a_{3i+1}z = 0, \quad a_{3i+2}z = -a_{3i} && (i \geq 0).
\end{aligned}$$

We denote by $S$ the subalgebra of $\mathrm{Der}(A)$ generated by the elements $x$, $y$ and $z$. It is easy to see that

$$[x, y] = z,\ [y, z] = x,\ [z, x] = y.$$

Hence $S$ is a 3-dimensional simple Lie algebra with basis $\{x, y, z\}$. First we show that $S \in \overline{\mathfrak{S}}$. To do this it is sufficient to show that $S$ has no 1-dimensional weak subideals. Assume, to the contrary, that $S$ has a 1-dimensional weak subideal, say $\langle w \rangle$. Since $\langle w \rangle$ has codimension 2 in $S$, $\langle w \rangle \leq^2 S$ and hence $[S, {}_2\langle w \rangle] \subseteq \langle w \rangle$. Write $w = \alpha x + \beta y + \gamma z$, where $\alpha, \beta, \gamma \in \mathfrak{k}$. Then we have $\beta \neq 0$ or $\gamma \neq 0$. In fact, suppose that $\beta = \gamma = 0$. Then $\alpha \neq 0$, whence $\langle w \rangle = \langle x \rangle$. Therefore $-z = [z, {}_2 x] \in \langle x \rangle$, a contradiction. Now we can find a $\mu \in \mathfrak{k}$ such that $[x, {}_2 w] = \mu w$. By calculation we have

$$[x, {}_2 w] = -(\beta^2+\gamma^2)x + \alpha\beta y + \alpha\gamma z.$$

Hence $-(\beta^2+\gamma^2) = \mu\alpha$, $\alpha\beta = \mu\beta$ and $\alpha\gamma = \mu\gamma$. Since $\beta \neq 0$ or $\gamma \neq 0$, we have $\alpha = \mu$. Therefore $\alpha^2+\beta^2+\gamma^2=0$. This contradicts our assumption on $\mathfrak{k}$. Thus we obtain $S \in \overline{\mathfrak{S}}$.

We now construct the split extension $L = A \dotplus S$ of $A$ by $S$. Then by Theorem 2.7 we have

$$L \in \mathfrak{A}\overline{\mathfrak{S}} \leq \mathfrak{A}\overline{\mathfrak{M}}_1 \leq \overline{\mathfrak{L}} \cap \mathfrak{L}.$$

Let $H$ wsi $L$ and suppose that $H \not\leq A$. Since $0 \neq H+A/A$ wsi $L/A \in \overline{\mathfrak{S}}$, $H+A/A = L/A$ and so $H+A=L$. To each element $t$ of $\{x, y, z\}$, there correspond an $h_t \in H$ and an $a_t \in A$ such that $t = h_t + a_t$. Then for any integer $i \geq 0$,

$$[a_{3i}, h_x] = [a_{3i}, x-a_x] = a_{3i+1},$$
$$[a_{3i+1}, h_y] = [a_{3i+1}, y-a_y] = a_{3i+2},$$
$$[a_{3i+2}, -h_z] = [a_{3i+2}, -z+a_z] = a_{3i}.$$

Since $H$ wsi $L$, we can easily see that $\{a_{3i}, a_{3i+1}, a_{3i+2}\} \subseteq H$. This being true for all $i \geq 0$, we have $A \leq H$. Hence $L = H + A = H$. Therefore we have proved that

$$\mathscr{S}_L(\text{wsi}) = \{H\colon H \leq A \text{ or } H = L\} = \{H\colon H \lhd^2 L\}.$$

It follows that $L \in \overline{\mathfrak{M}}_2 \cap \mathfrak{M}_2$. Using Theorem 3.4 we have $L \in \overline{\mathfrak{L}}^\infty \cap \overline{\mathfrak{L}}_\infty \cap \mathfrak{L}^\infty \cap \mathfrak{L}_\infty$. Therefore both $\mathscr{S}_L(\text{wsi})$ and $\mathscr{S}_L(\text{si})$ are complete sublattices of the lattice $\mathscr{S}_L(\leq)$.

Finally we show that $L \in \overline{\mathfrak{L}}^* \cap \mathfrak{L}^*$. Let $\mathfrak{X}$ denote the class $\mathfrak{A} \cup (L)$ of Lie algebras over $\mathfrak{k}$, where $(L)$ is the smallest class containing $L$. Evidently $\mathfrak{X} \leq \overline{\mathfrak{L}} \cap \mathfrak{L}$. Since $\mathscr{S}_L(\text{wsi}) = \{H\colon H \leq A \text{ or } H = L\}$, $\mathfrak{X}$ is wsi-closed and hence i-closed. Now let $M$ be any Lie algebra over $\mathfrak{k}$ and let $H, K$ wsi $M$ (resp. $H, K$ si $M$). Suppose that $J = \langle H, K\rangle \in \mathfrak{X}$. We must show that $J$ wsi $M$ (resp. $J$ si $M$). To do this we may assume that $J \notin \mathfrak{A}$, $H < J$ and $K < J$. Then $J \cong L$. Hence by the above argument we have $[H, K] = 0$. By using Lemma 2.1 (resp. [2, Lemma 2.1.4]), we obtain $J$ wsi $M$ (resp. $J$ si $M$). Therefore $\mathfrak{X}$ has the property (A) (resp. the property (B)). It follows that $\mathfrak{X} \leq \overline{\mathfrak{L}}^*$ (resp. $\mathfrak{X} \leq \mathfrak{L}^*$). Thus we have $L \in \overline{\mathfrak{L}}^* \cap \mathfrak{L}^*$.

REMARK. Let $L$ be the Lie algebra over any field $\mathfrak{k}$ constructed as in Example 4.3. Then we can prove as above that $L \in \mathfrak{M}_2 \cap \mathfrak{L}^\infty \cap \mathfrak{L}_\infty \cap \mathfrak{L}^*$.

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