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Oscillatory solutions of functional differential equations generated by deviation of arguments of mixed type

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1. Introduction

In this paper we consider the higher order functional differential equation of the form

(E) $x^{(n)}(t)+F(t, x(t), x(g_1(t)),\dots, x(g_N(t)),\dots, x^{(n-1)}(t),\dots, x^{(n-1)}(g_N(t)))=0$

and its particular cases

(A) $$x^{(n)}(t) - \sum_{h=1}^{N} p_h(t)f_h(x(g_h(t))) = 0,$$

(B) $$x^{(n)}(t) + \sum_{h=1}^{N} p_h(t)f_h(x(g_h(t))) = 0,$$

where the deviating arguments $g_h(t)$ are of general type.

It is assumed that the function $F(t, u_0,\dots, u_N, u_0^{(1)},\dots, u_N^{(n-1)})$ satisfies either the condition

(1) $$F(t, u_0,\dots, u_N, u_0^{(1)},\dots, u_N^{(1)},\dots, u_N^{(n-1)})u_0 \geq 0$$

or the condition

(2) $$F(t, u_0,\dots, u_N, u_0^{(1)},\dots, u_N^{(1)},\dots, u_N^{(n-1)})u_0 \leq 0$$

in the domain

$$\Omega = \{(t, u_l^{(q)}): t\in[a, \infty), u_0u_l > 0, 0\leq l\leq N, 0\leq q\leq n-1\}.$$

By a *proper solution* of equation (E) we mean a function $x\in C^n[[T_x, \infty), R]$ which satisfies (E) for all sufficiently large t and $\sup\{|x(t)|: t\geq T\}>0$ for any $T\geq T_x$. We make the standing hypothesis that equation (E) does possess proper solutions. A proper solution of (E) is called *oscillatory* if it has arbitrarily large zeros; otherwise the solution is called *nonoscillatory*.

When equation (E) is ordinary ($g_i(t)\equiv t$, $1\leq i\leq N$), the oscillatory properties of (E) satisfying (1) are essentially different from those of (E) satisfying (2). In case (1) holds, the "property A" is typical for equation (E): if n is even, then all proper solutions are oscillatory, and if n is odd, then every proper solution is either oscillatory or monotonically tending to zero as $t\to\infty$. On the other hand, when (2) holds the "property B" is typical: if n is even, then every solution is either oscillatory or else tending monotonically to infinity or to zero as $t\to\infty$, and if n

is odd, then every solution is either oscillatory or tending monotonically to infinity as $t \to \infty$.

In many cases the same situation is typical for retarded equations of the form (E) ($g_i(t) \le t$, $1 \le i \le N$). For standard results in this direction the reader is referred to the book [1].

Oscillatory properties of solutions of functional differential equations with deviating arguments of mixed type are in general essentially different from those of ordinary and retarded differential equations. The first results on oscillation of first order functional differential equations generated by deviation of arguments of general type were independently obtained in the papers [2, 3, 4]. One of the coauthors of the present paper seems to be the first to have discovered that all proper solutions of higher order functional differential equations involving both retarded and advanced arguments may be oscillatory [5]. He showed that all proper solutions of the even order equation

$$x^{(n)}(t) - p^n x(t+n\sigma) - q^n x(t-n\tau) = 0 \tag{3}$$

are oscillatory provided the positive constants p, q, σ and τ satisfy $pe\sigma > 1$ and $qe\tau > 1$, and that the same is true of the equation

$$x^{(n)}(t) - p|x(t+\sigma)|^{\alpha} \operatorname{sgn} x(t+\sigma) - q|x(t-\tau)|^{\beta} \operatorname{sgn} x(t-\tau) = 0$$

for any positive constants p, q, α, β, σ and τ such that $\alpha > 1$ and $0 < \beta < 1$. (Note that the odd order case of (3) has been studied by Ladas and Stavroulakis [6].)

Some results on oscillation of functional differential equations with general deviating arguments can be found in the papers [7, 8, 9].

Here we deal mainly with equations (A) and (B). Throughout the paper we assume the following conditions to hold:

(A) $\quad p_h, g_h \in C[[a, \infty), R]$, $p_h(t) \ge 0$, $\lim_{t\to\infty} g_h(t) = \infty$, $1 \le h \le N$;

(B) $\quad f_h \in C[R, R]$ and $u f_h(u) > 0$ for $u \ne 0$, $1 \le h \le N$.

Under additional assumptions on the nonlinearity of f_h we give explicit conditions under which all proper solutions of equations (A) and (B) are oscillatory. As a consequence we are able to see that there is a class of functional differential equations with a single deviating argument, all proper solutions of which are oscillatory.

2. Main Results

Let $g_h(t)$, $1 \le h \le N$, be fixed. We define the subsets $\mathscr{A}_h$ and $\mathscr{R}_h$ of $[a, \infty)$ as follows:

$$\mathscr{A}_h = \{t \in [a, \infty): g_h(t) > t\}, \qquad \mathscr{R}_h = \{t \in [a, \infty): g_h(t) < t\}.$$

$\mathscr{A}_h$ and $\mathscr{R}_h$ are the sets on which the deviating argument $g_h(t)$ is advanced and retarded, respectively.

The following four theorems are the main results of the present paper.

THEOREM 1. *Let $n \geq 3$ be odd. Suppose that there are integers i, j and k, $1 \leq i, j, k \leq N$, and a positive number M such that the following conditions are satisfied:*

$$\int_{\mathscr{A}_i} [g_i(t) - t]^{n-1} p_i(t) dt = \infty; \tag{4}$$

$$\int_{\mathscr{A}_j} t^{n-2} [g_j(t) - t] p_j(t) dt = \infty; \tag{5}$$

$$\int_{\mathscr{A}_k} t^{n-1} p_k(t) dt = \infty; \tag{6}$$

$f_h(u)$ is monotonically increasing for $|u| \geq M$ and

$$\int_M^{\infty} \frac{du}{f_h(u)} < \infty, \quad \int_{-M}^{-\infty} \frac{du}{f_h(u)} < \infty, \quad h = i, j, k. \tag{7}$$

Then all proper solutions of (A) *are oscillatory.*

THEOREM 2. *Let n be even. Suppose that there are integers i and j, $1 \leq i$, $j \leq N$, and a positive number M such that the conditions* (4), (5) *and* (7) *for $h = i, j$ of Theorem* 1 *are satisfied. In addition suppose that there are an integer k, $1 \leq k \leq N$, and a positive number m such that*

$$\int_{\mathscr{R}_k} [t - g_k(t)]^{n-1} p_k(t) dt = \infty; \tag{8}$$

$f_k(u)$ is monotonically increasing for $|u| \leq m$ and

$$\int_{+0}^{m} \frac{du}{f_k(u)} < \infty, \quad \int_{-0}^{-m} \frac{du}{f_k(u)} < \infty. \tag{9}$$

Then all proper solutions of (A) *are oscillatory.*

THEOREM 3. *Let n be even. Suppose that there are integers i and j, $1 \leq i$, $j \leq N$, and a positive number M such that the following conditions are satisfied:*

$$\int_{\mathscr{A}_i} t^{n-2} [g_i(t) - t] p_i(t) dt = \infty; \tag{10}$$

$$\int_{\mathscr{A}_j} t^{n-1} p_j(t) dt = \infty; \tag{11}$$

$f_h(u)$ is monotonically increasing for $|u| \geq M$ and

$$\int_M^{\infty} \frac{du}{f_h(u)} < \infty, \quad \int_{-M}^{-\infty} \frac{du}{f_h(u)} < \infty, \quad h = i, j. \tag{12}$$

Then all proper solutions of (B) *are oscillatory.*

THEOREM 4. *Let $n \geq 3$ be odd. Suppose that there are an integer i, $1 \leq i \leq N$, and a positive number M such that the conditions* (10) *and* (12) *for $h = i$ of Theorem* 3 *are satisfied. In addition suppose that there are an integer k, $1 \leq k \leq N$, and a positive number m such that*

$$\int_{\mathscr{R}_k} [t - g_k(t)]^{n-1} p_k(t) dt = \infty; \tag{13}$$

$f_k(u)$ is monotonically increasing for $|u| \leq m$ and

$$\int_{+0}^{m} \frac{du}{f_k(u)} < \infty, \quad \int_{-0}^{-m} \frac{du}{f_k(u)} < \infty. \tag{14}$$

Then all proper solutions of (B) *are oscillatory.*

Let us now consider the following equations which are particular cases of equations (A) and (B):

$$x^{(n)}(t) - p(t) f(x(g(t))) = 0, \tag{C}$$

$$x^{(n)}(t) + p(t) f(x(g(t))) = 0. \tag{D}$$

We suppose that $p, q \in C[[a, \infty), R]$, $p(t) \geq 0$, $\lim_{t \to \infty} g(t) = \infty$, $f \in C[R, R]$ and $uf(u) > 0$ for $u \neq 0$. Let $\mathscr{A}, \mathscr{R} \subset [a, \infty)$ be the sets on which the deviating argument $g(t)$ is advanced and retarded, respectively, that is,

$$\mathscr{A} = \{t \in [a, \infty): g(t) > t\}, \qquad \mathscr{R} = \{t \in [a, \infty): g(t) < t\}.$$

The following corollaries are immediate consequances of Theorems 1–4. They show that in some cases the presence of a single deviating argument of mixed type is sufficient to force all proper solutions of a functional differential equation to oscillate.

COROLLARY 1. *Let $n \geq 3$ be odd and there exists a positive number M such that $f(u)$ is monotonically increasing for $|u| \geq M$ and*

$$\int_{\mathscr{A}} [g(t) - t]^{n-1} p(t) dt = \infty;$$

$$\int_{\mathscr{A}} t^{n-2} [g(t) - t] p(t) dt = \infty;$$

$$\int_{\mathscr{A}} t^{n-1}p(t)dt = \infty;$$

$$\int_{M}^{\infty}\frac{du}{f(u)} < \infty, \quad \int_{-M}^{-\infty}\frac{du}{f(u)} < \infty.$$

Then all proper solutions of (C) *are oscillatory.*

COROLLARY 2. *Let n be even and there exist positive numbers M, m such that $f(u)$ is monotonically increasing for $|u|\geq M$, $|u|\leq m$ and*

$$\int_{\mathscr{A}} [g(t)-t]^{n-1}p(t)dt = \infty;$$

$$\int_{\mathscr{A}} t^{n-2}[g(t)-t]p(t)dt = \infty;$$

$$\int_{\mathscr{R}} [t-g(t)]^{n-1}p(t)dt = \infty;$$

$$\int_{M}^{\infty}\frac{du}{f(u)} < \infty, \quad \int_{-M}^{-\infty}\frac{du}{f(u)} < \infty;$$

$$\int_{+0}^{m}\frac{du}{f(u)} < \infty, \quad \int_{-0}^{-m}\frac{du}{f(u)} < \infty.$$

Then all proper solutions of (C) *are oscillatory.*

COROLLARY 3. *Let n be even and there exists a positive number M such that $f(u)$ is monotonically increasing for $|u|\geq M$ and*

$$\int_{\mathscr{A}} t^{n-2}[g(t)-t]p(t)dt = \infty;$$

$$\int_{\mathscr{A}} t^{n-1}p(t)dt = \infty;$$

$$\int_{M}^{\infty}\frac{du}{f(u)} < \infty, \quad \int_{-M}^{-\infty}\frac{du}{f(u)} < \infty.$$

Then all proper solutions of (D) *are oscillatory.*

COROLLARY 4. *Let $n\geq 3$ be odd and there exist positive numbers M, m such that $f(u)$ is monotonically increasing for $|u|\geq M$, $|u|\leq m$ and*

$$\int_{\mathscr{A}} t^{n-2}[g(t)-t]p(t)dt = \infty;$$

$$\int_{\mathscr{R}} [t-g(t)]^{n-1}p(t)dt = \infty;$$

$$\int_{M}^{\infty}\frac{du}{f(u)} < \infty, \quad \int_{-M}^{-\infty}\frac{du}{f(u)} < \infty;$$

$$\int_{+0}^{m} \frac{du}{f(u)} < \infty, \quad \int_{-0}^{-m} \frac{du}{f(u)} < \infty.$$

Then all proper solutions of (D) *are oscillatory.*

EXAMPLE 1. By Theorems 1–4 all proper solutions of the equations
$x^{(n)}(t) - p|x(t+\sin t)|^{\alpha} \operatorname{sgn} x(t+\sin t) - q|x(t-\sin t)|^{\beta} \operatorname{sgn} x(t-\sin t) = 0,$
$x^{(n)}(t) + p|x(t+\sin t)|^{\alpha} \operatorname{sgn} x(t+\sin t) + q|x(t-\sin t)|^{\beta} \operatorname{sgn} x(t-\sin t) = 0,$
are oscillatory provided $p>0$, $q>0$, $\alpha>1$ and $0<\beta<1$.

EXAMPLE 2. Define the function $f(u)$ by

$$f(u) = u^{\alpha} \quad (0 \le u \le 1), \quad f(u) = u^{\beta} \quad (u \ge 1), \quad f(-u) = -f(u),$$

where $0<\alpha<1$ and $\beta>1$. Then from Corollaries 1–4 it follows that all proper solutions of the equations

$$x^{(n)}(t) - f(x(t+\sin t)) = 0,$$

$$x^{(n)}(t) + f(x(t+\sin t)) = 0,$$

are oscillatory.

3. Proofs of Theorems

PROOF OF THEOREMS 1 AND 2. Let $x(t)$ be a nonoscillatory solution of (A). Without loss of generality we may suppose that $x(t)$ is eventually positive. From (A), $x^{(n)}(t)$ is eventually positive, so that there is an integer $l \in \{0, 1, 2, \ldots, n\}$ such that $l \equiv n \pmod 2$ and

$$(15) \qquad x^{(i)}(t) \ge 0, \quad 0 \le i \le l, \quad (-1)^{i-l} x^{(i)}(t) \ge 0, \quad l \le i \le n,$$

for all sufficiently large t, say $t \ge t_0 \ge a$. Let $T \ge t_0$ be so large that $g_h(t) \ge t_0$ for $t \ge T$, $1 \le h \le N$.

First suppose that $l=n$. Then integrating (A) and using (15), we have

$$(16) \qquad x'(t) = \sum_{i=1}^{n-1} \frac{(t-T)^{i-1}}{(i-1)!} x^{(i)}(T) + \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} x^{(n)}(s) ds$$
$$\ge \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} p_i(s) f_i(x(g_i(s))) ds, \quad t \ge T.$$

Take any $T'>T$ and let $T_i = \sup_{T \le t \le T'} \max\{g_i(t), t\}$. Dividing (16) by $f_i(x(t))$ and integrating over $[T, T_i]$, we obtain

$$\int_T^{T_i} \frac{x'(t)}{f_i(x(t))} dt \ge \int_T^{T_i} \frac{1}{f_i(x(t))} \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} p_i(s) f_i(x(g_i(s))) ds dt$$

$$
\begin{aligned}
&= \int_T^{T_i} p_i(s) \int_s^{T_i} \frac{(t-s)^{n-2}}{(n-2)!} \cdot \frac{f_i(x(g_i(s)))}{f_i(x(t))} dt ds \\
(17) \qquad &\geq \int_T^{T'} p_i(s) \int_s^{T_i} \frac{(t-s)^{n-2}}{(n-2)!} \cdot \frac{f_i(x(g_i(s)))}{f_i(x(t))} dt ds \\
&\geq \int_{\mathscr{A}_i \cap [T,T']} p_i(s) \int_s^{g_i(s)} \frac{(t-s)^{n-2}}{(n-2)!} \cdot \frac{f_i(x(g_i(s)))}{f_i(x(t))} dt ds.
\end{aligned}
$$

Since $l=n\geq 3$, $\lim_{t\to\infty} x(t)=\infty$ and the number T in (17) may be chosen so large that $x(s)\geq M$ for $s\geq T$. Since $f_i(x(t))$ is increasing for $t\geq T$, $f_i(x(g_i(s)))/f_i(x(t))\geq 1$ for $s\leq t\leq g_i(s)$, $s\in\mathscr{A}_i\cap[T, T']$. From (17) it follows that

$$
\begin{aligned}
\int_T^{T_i} \frac{x'(t)}{f_i(x(t))} dt &\geq \int_{\mathscr{A}_i \cap [T,T']} p_i(s) \int_s^{g_i(s)} \frac{(t-s)^{n-2}}{(n-2)!} dt ds \\
&= \frac{1}{(n-1)!} \int_{\mathscr{A}_i \cap [T,T']} p_i(s) [g_i(s)-s]^{n-1} ds.
\end{aligned}
$$

Letting $T'\to\infty$ in the above and using (7), we see that

$$
\int_{\mathscr{A}_i \cap [T,\infty]} [g_i(s)-s]^{n-1} p_i(s) ds \leq (n-1)! \int_{x(T)}^{\infty} \frac{du}{f_i(u)} < \infty,
$$

which contradicts (4).

Next suppose that $2\leq l<n$. In view of (15) we have

$$
\begin{aligned}
(18) \qquad x'(t) &= \textstyle\sum_{i=1}^{l-1} \frac{(t-T)^{i-1}}{(i-1)!} x^{(i)}(T) + \int_T^t \frac{(t-s)^{l-2}}{(l-2)!} x^{(l)}(s) ds \\
&\geq \int_T^t \frac{(t-s)^{l-2}}{(l-2)!} x^{(l)}(s) ds, \quad t\geq T,
\end{aligned}
$$

and

$$
\begin{aligned}
x^{(l)}(t) &= \textstyle\sum_{i=l}^{n-1} (-1)^{i-l} \frac{(s-t)^{i-l}}{(i-l)!} x^{(i)}(s) + (-1)^{n-l} \int_t^s \frac{(r-t)^{n-l-1}}{(n-l-1)!} x^{(n)}(r) dr \\
&\geq \int_t^s \frac{(r-t)^{n-l-1}}{(n-l-1)!} p_j(r) f_j(x(g_j(r))) dr,
\end{aligned}
$$

the latter of which yields in the limit as $s\to\infty$

$$
(19) \qquad x^{(l)}(t) \geq \int_t^{\infty} \frac{(r-t)^{n-l-1}}{(n-l-1)!} p_j(r) f_j(x(g_j(r))) dr, \quad t\geq T.
$$

Combining (18) with (19), we have for $t\geq T$

$$
x'(t) \geq \int_T^t \frac{(t-s)^{l-2}}{(l-2)!} \int_s^{\infty} \frac{(r-s)^{n-l-1}}{(n-l-1)!} p_j(r) f_j(x(g_j(r))) dr ds
$$

$$
(20)\quad
\begin{aligned}
&\geq \int_T^t \left(\int_T^r \frac{(t-s)^{l-2}(r-s)^{n-l-1}}{(l-2)!(n-l-1)!} ds \right) p_j(r) f_j(x(g_j(r))) dr \\
&\geq \int_T^t \left(\int_T^r \frac{(r-s)^{n-3}}{(l-2)!(n-l-1)!} ds \right) p_j(r) f_j(x(g_j(r))) dr \\
&= c(n, l) \int_T^t (r-T)^{n-2} p_j(r) f_j(x(g_j(r))) dr,
\end{aligned}
$$

where $c(n, l)$ is a positive constant depending only on n and l. Take any $T'>T$ and let $T_j = \sup_{T \leq t \leq T'} \max\{g_j(t), t\}$. We divide (20) by $f_j(x(t))$ and integrate over $[T, T_j]$, obtaining

$$
(21)\quad
\begin{aligned}
&\int_T^{T_j} \frac{x'(t)}{f_j(x(t))} dt \geq c(n, l) \int_T^{T_j} \frac{1}{f_j(x(t))} \int_T^t (s-T)^{n-2} p_j(s) f_j(x(g_j(s))) ds dt \\
&= c(n, l) \int_T^{T_j} (s-T)^{n-2} p_j(s) \int_s^{T_j} \frac{f_j(x(g_j(s)))}{f_j(x(t))} dt ds \\
&\geq c(n, l) \int_T^{T'} (s-T)^{n-2} p_j(s) \int_s^{T_j} \frac{f_j(x(g_j(s)))}{f_j(x(t))} dt ds \\
&\geq c(n, l) \int_{\mathscr{A}_j \cap [T, T']} (s-T)^{n-2} p_j(s) \int_s^{g_j(s)} \frac{f_j(x(g_j(s)))}{f_j(x(t))} dt ds.
\end{aligned}
$$

Note that $\lim_{t \to \infty} x(t) = \infty$, since $l \geq 2$. We can take T so large that $x(s) \geq M$ for $s \geq M$. Since $f_j(x(g_j(s)))/f_j(x(t)) \geq 1$ for $s \leq t \leq g_j(s)$, $s \in \mathscr{A}_j \cap [T, T']$, we conclude from (21) that

$$
\int_{\mathscr{A}_j \cap [T, \infty)} (s-T)^{n-2} [g_j(s) - s] p_j(s) ds \leq c(n, l)^{-1} \int_{x(T)}^{\infty} \frac{du}{f_j(u)} < \infty,
$$

which contradicts (5).

Now suppose that $l=1$. This is possible only if n is odd. Then it follows from (19) that

$$
(22)\quad x'(t) \geq \int_t^\infty \frac{(r-t)^{n-2}}{(n-2)!} p_k(r) f_k(x(g_k(r))) dr, \quad t \geq T.
$$

Take any $T'>T$ and let $T_k = \sup_{T \leq t \leq T'} \max\{g_k(t), t\}$. Dividing (22) by $f_k(x(t))$ and integrating over $[T, T_k]$, we obtain

$$
(23)\quad
\begin{aligned}
&\int_T^{T_k} \frac{x'(t)}{f_k(x(t))} dt \geq \int_T^{T_k} \frac{1}{f_k(x(t))} \int_t^\infty \frac{(r-t)^{n-2}}{(n-2)!} p_k(r) f_k(x(g_k(r))) dr dt \\
&\geq \int_T^{T_k} p_k(r) \int_T^r \frac{(r-t)^{n-2}}{(n-2)!} \cdot \frac{f_k(x(g_k(r)))}{f_k(x(t))} dt dr \\
&\geq \int_{\mathscr{A}_k \cap [T, T']} p_k(r) \int_T^r \frac{(r-t)^{n-2}}{(n-2)!} \cdot \frac{f_k(x(g_k(r)))}{f_k(x(t))} dt dr.
\end{aligned}
$$

If we suppose that $\lim_{t\to\infty} x(t)=x_0<\infty$, then it follows from (23) that

$$\int_{\mathscr{A}_k\cap[T,\infty)} p_k(r)(r-T)^{n-1}\,dr \le c(n-2)!\int_{x(T)}^{x_0}\frac{du}{f_k(u)} < \infty,$$

where c is a positive constant. But this contradicts (6), and so we must have $\lim_{t\to\infty} x(t)=\infty$. In (23) T may be chosen so large that $x(t)\ge M$ for $t\ge T$. Then (23) yields

$$\int_T^{T_k}\frac{x'(t)}{f_k(x(t))}dt \ge \int_{\mathscr{A}_k\cap[T,T']} p_k(r)\int_T^r \frac{(r-t)^{n-2}}{(n-2)!}\cdot\frac{f_k(x(g_k(r)))}{f_k(x(t))}dt\,dr$$
$$\ge \int_{\mathscr{A}_k\cap[T,T']} p_k(r)\int_T^r\frac{(r-t)^{n-2}}{(n-2)!}dt\,dr = \frac{1}{(n-1)!}\int_{\mathscr{A}_k\cap[T,T']} p_k(r)(r-T)^{n-1}dr,$$

and letting $T'\to\infty$ and using (7), we are again led to a contradiction to (6).

Finally suppose that $l=0$, which is possible only if n is even. In this case we have

$$-x'(t) = \sum_{i=1}^{n-1}(-1)^i x^{(i)}(s)\frac{(s-t)^{i-1}}{(i-1)!} + \int_t^s\frac{(r-t)^{n-2}}{(n-2)!}x^{(n)}(r)\,dr$$

for $s\ge t\ge T$. Choose $T'>T$ so that $g_h(t)\ge T$ for $t\ge T'$, $1\le h\le N$, and let $T^*>T'$ be fixed. Using (15) and (A), we see from the above that

$$(24)\qquad -x'(t)\ge\int_t^{T^*}\frac{(s-t)^{n-2}}{(n-2)!}p_k(s)f_k(x(g_k(s)))\,ds,\quad T\le t\le T^*.$$

Dividing (24) by $f_k(x(t))$ and integrating over $[T, T^*]$, we obtain

$$-\int_T^{T^*}\frac{x'(t)}{f_k(x(t))}dt \ge \int_T^{T^*}\frac{1}{f_k(x(t))}\int_t^{T^*}\frac{(s-t)^{n-2}}{(n-2)!}p_k(s)f_k(x(g_k(s)))\,ds\,dt$$
$$= \int_T^{T^*}p_k(s)\int_T^s\frac{(s-t)^{n-2}}{(n-2)!}\cdot\frac{f_k(x(g_k(s)))}{f_k(x(t))}dt\,ds$$
$$\ge \int_{T'}^{T^*}p_k(s)\int_T^s\frac{(s-t)^{n-2}}{(n-2)!}\cdot\frac{f_k(x(g_k(s)))}{f_k(x(t))}dt\,ds.$$

If $\lim_{t\to\infty} x(t)=x_0>0$, then from (25) we see that

$$\int_{T'}^{\infty} p_k(s)(s-T)^{n-1}ds < \infty,$$

which contradicts (8). If follows that $\lim_{t\to\infty} x(t)=0$. In (25) T may be taken so large that $x(s)\le m$ and $x(g_k(s))\le m$ for $s\ge T$. Noting that $f_k(x(g_k(s)))/f_k(x(t))\ge 1$ for $g_k(s)\le t\le s$, $s\in\mathscr{R}_k\cap[T', T^*]$, we conclude from (25) that

$$\int_{\mathscr{R}_k\cap[T',\infty)}[s-g_k(s)]^{n-1}p_k(s)\,ds \le (n-1)!\int_{+0}^{x(T)}\frac{du}{f_k(u)} < \infty,$$

which also contradicts (8). Thus the proof of Theorems 1 and 2 is complete.

PROOF OF THEOREMS 3 AND 4. Let $x(t)$ be a nonoscillatory solution of (B). We may suppose without loss of generality that $x(t)$ is eventually positive. From (B), $x^{(n)}(t)$ is eventually negative, and so there is an integer $l \in \{0, 1, \dots, n-1\}$ such that $l \equiv n-1 \pmod 2$ and

$$x^{(i)}(t) \geq 0, \quad 0 \leq i \leq l, \quad (-1)^{i-l} x^{(i)}(t) \geq 0, \quad l \leq i \leq n, \tag{26}$$

for all sufficiently large t, say $t \geq t_0 \geq a$. Let $T \geq t_0$ be so large that $g_h(t) \geq t_0$ for $t \geq T$, $1 \leq h \leq N$.

First suppose that $2 \leq l \leq n-1$. Then we have

$$\begin{aligned} x'(t) &= \sum_{i=1}^{l-1} \frac{(t-T)^{i-1}}{(i-1)!} x^{(i)}(T) + \int_T^t \frac{(t-s)^{l-2}}{(l-2)!} x^{(l)}(s)\, ds \\ &\geq \int_T^t \frac{(t-s)^{l-2}}{(l-2)!} x^{(l)}(s)\, ds, \quad t \geq T, \end{aligned} \tag{27}$$

and

$$\begin{aligned} x^{(l)}(t) &= \sum_{i=l}^{n-1} (-1)^{i-l} \frac{(s-t)^{i-l}}{(i-l)!} x^{(i)}(s) + (-1)^{n-l} \int_t^s \frac{(r-t)^{n-l-1}}{(n-l-1)!} x^{(n)}(r)\, dr \\ &\geq \int_t^s \frac{(r-t)^{n-l-1}}{(n-l-1)!} p_i(r) f_i(x(g_i(r)))\, dr, \quad s \geq t \geq T. \end{aligned} \tag{28}$$

Combining (27) with (28) yields inequality (20) and the subsequent proof proceeds exactly as in the corresponding part of the proof of Theorems 1 and 2.

Next suppose that $l=1$, which is possible only if n is even. From (28) we then have

$$x'(t) \geq \int_t^\infty \frac{(r-t)^{n-2}}{(n-2)!} p_j(r) f_j(x(g_j(r)))\, dr, \quad t \geq T,$$

wherefrom the proof is the same as in the corresponding part of the proof of Theorems 1 and 2.

Finally we consider the case where $l=0$. Note that this is possible only if n is odd. From the equation

$$-x'(t) = \sum_{i=1}^{n-1} (-1)^i x^{(i)}(s) \frac{(s-t)^{i-1}}{(i-1)!} - \int_t^s \frac{(r-t)^{n-2}}{(n-2)!} x^{(n)}(r)\, dr$$

and in view of (26) and (B) we obtain

$$-x'(t) \geq \int_t^{T^*} \frac{(s-t)^{n-2}}{(n-2)!} p_k(s) f_k(x(g_k(s)))\, ds$$

for $T \leq t \leq T^*$, where T^* is sufficiently large. Thus we have obtained inequality

(24) and from this point on the proof is entirely the same as in the corresponding proof of Theorems 1 and 2. This completes the proof of Theorems 3 and 4.

4. We conclude with the remark that the results obtained for equations (A) and (B) may easily be extended to more general equations of the form (E). For instance, in case n is even, if we assume that the function $F(t, u_0,\dots, u_N, u_0^{(1)},\dots, u_N^{(n-1)})$ satisfies the inequality

$$F(t, u_0,\dots, u_N, u_0^{(1)},\dots, u_N^{(n-1)}) \operatorname{sgn} u_h \geq p_h(t) f_h(u_h) \operatorname{sgn} u_h, \quad h = i, j,$$

in the domain Ω and that the functions $p_h(t)$ and $f_h(u)$ satisfy the conditions of Theorem 3, then we conclude that all proper solutions of (E) are oscillatory. Theorems 1, 2 and 4 allow similar generalizations.

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