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On products of the β-elements in the stable homotopy of spheres

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§ 1. Introduction

In his paper [20], H. Toda introduced the elements β_s, $1\leqq s\leqq p-1$, in the p-primary component of the stable homotopy of spheres for an odd prime p, and L. Smith [18] extended them to an infinite family $\{\beta_s\}_{s\geqq 1}$, in case $p\geqq 5$. Later, with the development and plentiful knowledge of the Adams-Novikov spectral sequence based on the Brown-Peterson homology BP such as [5], it is clarified that these β-elements are detected in $\mathrm{Ext}^2_{BP_*BP}(BP_*, BP_*)$, the second line of the E_2-term of the spectral sequence, which consists of an extensive family of elements $\beta_{s/r,i}$ with suitable triple indices including $\beta_s=\beta_{s/1,1}$ (cf. (4.1)). The construction of the homotopy elements β_s is immediate from the one of the 4-cell complex called $V(1)$ and appropriate stable self-maps of $V(1)$ [18], and in this way, L. Smith [19], R. Zahler [23] and the first author [9], [11], [12] constructed homotopy elements which correspond with the generalized β's in Ext^2 including

$$\beta_{sp/r}\ (s\geqq 1,\ 1\leqq r<p),\quad \beta_{sp/p}\ (s\geqq 2),\quad \beta_{sp^2/p,2}\ (s\geqq 2),$$

where $\beta_{sp/r,1}=\beta_{sp/r}$ and some of these were called ε's and ρ's in earlier literatures (see (2.4), (2.5)).

The purpose of this paper is to study the products $\beta_s\beta_{tp/r}$ with $r\leqq p$ and $\beta_s\beta_{tp^2/p,2}$ in π_*^S, the stable homotopy ring of spheres, in case $p\geqq 5$. In particular, we shall study whether they are trivial or not. In this direction, H. Toda [21] obtained a formula of $\beta_s\beta_t$ extending the earlier work of N. Yamamoto [22] and including the relation $\beta_s\beta_{tp}=0$ which is the case $r=1$ of ours.

THEOREM A. *Let p be a prime $\geqq 5$, and r, s, t be positive integers with $r\leqq p$ and $r\leqq p-1$ if $t=1$. Then the element $\beta_s\beta_{tp/r}$ in π_*^S is trivial, if one of the following holds*:

(i) $r\leqq p-2$.
(ii) $r=p-1$ *and* $s\not\equiv -1 \mod p$.
(iii) $r=p-1,\ p$ *and* $t\equiv 0 \mod p$.

The next cases we have to investigate are (iv) $r=p-1$, $s\equiv -1 \mod p$ and $t\not\equiv 0 \mod p$; and (v) $r=p$ and $t\not\equiv 0 \mod p$. For the case (iv), we obtain a weak result that the products are trivial in the E_2-term Ext^4 (§6). In contrast with these cases, the products are shown to be nontrivial for the case (v) with a minor restriction of s, by investigating their images in the cohomology of the Morava stabilizer algebra, in a similar method as in [17].

THEOREM B. *Let p be a prime $\geqq 5$. If $s \not\equiv 0, 1 \bmod p$ and $t \not\equiv 0 \bmod p$ with $t \geqq 2$, then the elements $\beta_s\beta_{tp/p}$ and $\beta_s\beta_{tp^2/p,2}$ in π_*^S are nontrivial.*

In §2, we prepare some lemmas by using the relations in the track groups $[M, M]_*$ and $[V(1), V(1)]_*$, where M is the mod p Moore spectrum and $V(1)$ is the spectrum constructed in [18], and we prove Theorem A in §3. We give in §4 the representation of the β-elements in the E_2-term of the Adams-Novikov spectral sequence, and prove Theorem B in §5 by computing the restriction of them in the cohomology of the Morava stabilizer algebra. Finally in §6, we give some relations concerning the products of two β-elements in the E_2-term of the spectral sequence.

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§2. The β-elements and some lemmas

Throughout this paper, let p be a prime $\geqq 5$ and $q = 2(p-1)$.

Let S be the sphere spectrum, and define the mod p Moore spectrum M and the spectra $X(r)$ for $r \geqq 1$ ($X(1) = V(1)$ in [18]) by the cofiber sequences

(2.1) $$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{\pi} \Sigma S, \text{ where } p \text{ is the map of degree } p,$$

(2.2) $$\Sigma^{rq} M \xrightarrow{\alpha^r} M \xrightarrow{i_r} X(r) \xrightarrow{\pi_r} \Sigma^{rq+1} M \qquad (q = 2(p-1)),$$

where $\alpha: \Sigma^q M \to M$ is the map with $\pi\alpha i = \alpha_1$, the element of Hopf invariant 1, (cf. [9]). Then we have the maps

(2.3) $$\Sigma^q X(r) \xrightarrow{A} X(r+1) \xrightarrow{B} X(r) \qquad (r \geqq 1) \text{ with}$$
$$A i_r = i_{r+1}\alpha, \quad \pi_r = \pi_{r+1}A, \quad i_r = B i_{r+1} \text{ and } \pi_r B = \alpha\pi_{r+1}.$$

Furthermore, consider the maps

$$\beta(=\bar{\psi} \text{ in } [18]): \Sigma^{(p+1)q} X(1) \longrightarrow X(1),$$
(2.4) $$R(r): \Sigma^{p(p+1)q} X(r) \longrightarrow X(r) \qquad (1 \leqq r < p) \text{ with } R(1) = \beta^p,$$
$$R(p)^{(s)}: \Sigma^{sp(p+1)q} X(p) \longrightarrow X(p) \qquad (s \geqq 2),$$

given in [21], [9], [11], respectively. Then the β-elements in the stable homotopy π_*^S are defined as follows:

(2.5) $$\beta_s = \pi\pi_1\beta^s i_1 i \quad (s \geqq 1), \quad \beta_{sp/r} = \pi\pi_r R(r)^s i_r i \quad (1 \leqq r < p,\ s \geqq 1),$$
$$\beta_{sp/p} = \pi\pi_p R(p)^{(s)} i_p i \quad (s \geqq 2),$$

where $\beta_{sp/1} = \beta_{sp}$. We notice that these elements are denoted by ψ_s in [18], $\rho_{s,p-r}$ in [9], $\rho_{s,0}$ in [11], respectively.

To study these elements, we use the following elements:

(2.6) $$\delta = i\pi, \quad \beta_{(s)} = \pi_1\beta^s i_1 \quad (s \geqq 1),$$
$$\beta_{(sp/r)} = \pi_r R(r)^s i_r \quad (1 \leqq r < p,\ s \geqq 1) \quad \text{with} \quad \beta_{(sp/1)} = \beta_{(sp)}, \quad \text{and}$$
$$\beta_{(sp/p)} = \pi_p R(p)^{(s)} i_p \quad (s \geqq 2) \quad \text{in} \quad [M, M]_*;$$

(2.7)([21]) $$\alpha' = \alpha_1 \wedge 1_{X(1)}, \quad \beta' = \beta_1 \wedge 1_{X(1)}, \quad \text{and}$$
$$\alpha'' \quad \text{with} \quad \alpha'' i_1 = \alpha' i_1 \delta \quad \text{in} \quad [X(1), X(1)]_*.$$

Then we see immediately that

(2.8) $$\beta_{(tp/r)} = \beta_{(tp/p-1)}\alpha^{p-1-r} \quad (t \geqq 1,\ 1 \leqq r < p),$$
$$\beta_{(tp/r)} = \beta_{(tp/p)}\alpha^{p-r} \quad (t \geqq 2,\ 1 \leqq r \leqq p),$$

by (2.3) and the relations

(2.9)([9; Th. C], [11; Th. CII]) $$AR(r) = R(r+1)A,\ BR(r+1) = R(r)B$$
$$(1 \leqq r < p-1), \quad AR(p-1)^t = R(p)^{(t)}A \quad (t \geqq 2).$$

LEMMA 2.10. *For integers $r \geqq 0$ and $s \geqq 1$, consider the elements*

$$B(r, s) = (\beta_{(1)}\delta)^r \beta_{(s)}, \quad C(r, s) = \alpha\delta B(r, s) \quad \text{in} \quad [M, M]_*.$$

Then

(i) $B(r, s) = \pi_1\beta'^r\beta^s i_1$ if $s \not\equiv -1 \bmod p$. (ii) $C(r, s) = -\pi_1\beta'^r\alpha'\beta^s i_1$.
(iii) $\delta C(r, s) = -\pi_1\beta'^r\alpha''\beta^s i_1$. (iv) $C(r, s)\delta = -\pi_1\beta'^r\beta^s\alpha'' i_1$.

PROOF. The following relations are given in [21; Cor. 2.5, Lemma 3.1, (3.8), (3.9), (3.11), Th. 5.1, and (5.6)]:

(2.11) $$(\beta_{(1)}\delta + \delta\beta_{(1)})\pi_1 = \pi_1\beta', \quad i_1\delta\alpha\delta = -\alpha'' i_1, \quad \alpha\delta\beta_{(s)} = \beta_{(s)}\delta\alpha \quad (s \geqq 1),$$
$$\alpha\delta\pi_1 = -\pi_1\alpha', \quad \delta\alpha\delta\pi_1 = -\pi_1\alpha'', \quad \delta^2 = 0; \quad \beta_{(r)}\beta_{(s)} = 0 \text{ if } r+s \not\equiv 0 \bmod p;$$
$$\text{and } \beta'\xi = \xi\beta' \text{ for any } \xi \in [X(1), X(1)]_*.$$

Then we have

$$(\beta_{(1)}\delta)^r\beta_{(s)}=(\beta_{(1)}\delta+\delta\beta_{(1)})^r\pi_1\beta^s i_1=\pi_1\beta'^r\beta^s i_1 \quad \text{if} \quad 1+s\not\equiv 0 \bmod p, \quad \text{and}$$

$$\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}=\alpha\delta(\beta_{(1)}\delta+\delta\beta_{(1)})^r\pi_1\beta^s i_1=\alpha\delta\pi_1\beta'^r\beta^s i_1=-\pi_1\beta'^r\alpha'\beta^s i_1.$$

Thus (i) and (ii) are proved. (iii) and (iv) follow from (ii) and (2.11). q.e.d.

Let A_* be the subring of $[M, M]_*$ generated by δ and α, and I_* be the two sided ideal of $[M, M]_*$ generated by all indecomposable elements other than δ and α. Then by the structure of A_* which is given in [22; Th. III] (cf. [8; Th. 4.1]), we see that $[M, M]_*=A_*\oplus I_*$ and $\alpha_*: A_*\to A_{*+q}$ is an isomorphism in non-negative dimensions. Hence we obtain the exact sequence

(2.12) $$\cdots \longrightarrow I_{n-q} \xrightarrow{\alpha_*} I_n \xrightarrow{(i_1)_*} [M, X(1)]_n \xrightarrow{(\pi_1)_*} I_{n-q-1} \longrightarrow \cdots \quad (n\geqq q+1)$$

from the exact sequence $\cdots\to[M, M]_{n-q}\xrightarrow{\alpha_*}[M, M]_n\xrightarrow{(i_1)_*}[M, X(1)]_n \xrightarrow{(\pi_1)_*}[M, M]_{n-q-1}\to\cdots$ associated to the cofiber sequence (2.2). The structures of I_n are given by [8; Th. 0.1] for $n<(p^2+3p+1)q-6$. In particular, we have the following (2.14), where

(2.13) $k=(p^2+p+2)q-2$, $l=(p^2+2p)q-2$ (we use these notations in the rest of this section).

(2.14) *Put* $C_i=C(p-i, i+1)(i=1, 2, 3)$ *and* $B_i=B(p-i, 3)(i=1, 2)$. *Then*

$$I_{k-q-1}=\{\delta C_1, C_1\delta\},\ I_{k-q}=\{C_1\},\ I_{k-q+1}=\{\delta B_2\delta\},\ I_{k-q+2}=\{\delta B_2, B_2\delta\};$$
$$I_k=\{\delta C_2\delta\},\ I_{k+1}=\{\delta C_2, C_2\delta\},\ I_{k+2}=\{C_2\};\quad I_{k+q+1}=0,\ I_{k+q+2}=\{\delta C_3\delta\};$$
$$I_{l-q}=0;\quad I_l=\{\delta\beta_{(p+1)}, \beta_{(p+1)}\delta\},\ I_{l+1}=\{\beta_{(p+1)}\};$$
$$I_{l+q}=\{\alpha\delta\beta_{(p+1)}, \delta B_1, B_1\delta\},\ I_{l+q+1}=\{B_1\}.$$

The images of the elements in Lemma 2.10 by α_* in (2.12) are given as follows:

(2.15) $$\alpha_*C(r, s) = 0, \quad \alpha_*\delta C(r, s) = 0, \quad \alpha_*(\delta B(r, s)) = C(r, s).$$

This follows immediately from the definitions and the relations

(2.16) ([22; Th. II]) $$\alpha^2\delta = (2\alpha\delta-\delta\alpha)\alpha, \quad \alpha\beta_{(s)}=0 \quad (s\geqq 1).$$

Now we have the following

LEMMA 2.17. *The homotopy group* $[M, X(1)]_n$ *is the* F_p*-vector space generated by*

(i) $\beta'^{p-1}\alpha''\beta^2 i_1$, $\beta'^{p-1}\beta^2\alpha'' i_1$, $\beta'^{p-2}\delta_0\beta^3\alpha'' i_1$ *at* $n=k$,
(ii) $\beta'^{p-1}\alpha'\beta^2 i_1$, $\beta'^{p-2}\delta_1\alpha''\beta^3 i_1$ *at* $n=k+1$,

(iii) *no base* *at* $n=k+2$,
(iv) $\beta'^{p-2}\alpha''\beta^3 i_1\delta$ *at* $n=k+q+1$,
(v) $\beta'^{p-2}\alpha''\beta^3 i_1$, $\beta'^{p-2}\beta^3\alpha'' i_1$, $\beta'^{p-3}\delta_0\beta^4\alpha'' i_1$ *at* $n=k+q+2$,
(vi) $\beta'^{p-1}\delta_1\beta^3 i_1$, $\beta^{p+1} i_1\delta$ *at* $n=k+(p-1)q+1$,

where $\delta_0=i_1\delta\pi_1$, $\delta_1=i_1\pi_1$ *and* $k=(p^2+p+2)q-2$ *is the integer in* (2.13).

PROOF. Consider the sequence (2.12) for $n=k$:

$$[M, X(1)]_{k+1} \xrightarrow{(\pi_1)_*} I_{k-q} \xrightarrow{\alpha_*} I_k \xrightarrow{(i_1)_*} [M, X(1)]_k \xrightarrow{(\pi_1)_*} I_{k-q-1}.$$

By (2.14) and Lemma 2.10, $I_{k-q-1}=(\pi_1)_*\{\beta'^{p-1}\alpha''\beta^2 i_1, \beta'^{p-1}\beta^2\alpha'' i_1\}$ and $I_{k-q}=(\pi_1)_*\{\beta'^r\alpha'\beta^s i_1\}$. Thus both $(\pi_1)_*$'s in the above sequence are epimorphic and hence $(i_1)_*$ is monomorphic. Further $I_k=\{\delta\pi_1\beta'^{p-2}\beta^3\alpha'' i_1\}$ by (2.14) and Lemma 2.10, and $(i_1)_*(\delta\pi_1\beta'^{p-2}\beta^3\alpha'' i_1)=\beta'^{p-2}\delta_0\beta^3\alpha'' i_1$ by the relation $\delta_0\beta'=\beta'\delta_0$ in (2.11) for $\xi=\delta_0$. Therefore (i) follows from the above sequence.

(ii)–(vi) follow similarly from (2.12), (2.14), (2.15) and Lemma 2.10. q.e.d.

We consider the exact sequence

$$\text{(2.18)} \qquad \cdots \longrightarrow [M, X(1)]_{n+1} \xrightarrow{(\alpha^r)^*} [M, X(1)]_{n+1+rq} \xrightarrow{(\pi_r)^*} [X(r), X(1)]_n \xrightarrow{(i_r)^*} [M, X(1)]_n \longrightarrow \cdots$$

associated to the cofiber sequence (2.2).

LEMMA 2.19. $[X(p-1), X(1)]_k$ *is the* F_p*-vector space generated by*

$$\xi_1=\beta'^{p-1}\alpha''\beta^2 B^{p-2},\ \xi_2=\beta'^{p-1}\beta^2\alpha'' B^{p-2},\ \xi_3=\beta'^{p-2}\delta_0\beta^3\alpha'' B^{p-2},$$
$$\xi_4=\beta'^{p-1}\delta_1\beta^3 i_1\pi_{p-1},\ \xi_5=\beta^{p+1} i_1\delta\pi_{p-1}.$$

PROOF. Consider the exact sequence (2.18) for $r=p-1$ and $n=k$:

$$[M, X(1)]_{k+1} \xrightarrow{(\alpha^{p-1})^*} [M, X(1)]_{k+(p-1)q+1} \xrightarrow{(\pi_{p-1})^*} [X(p-1), X(1)]_k \xrightarrow{(i_{p-1})^*} [M, X(1)]_k.$$

By Lemma 2.17(i) and $i_r=Bi_{r+1}$ in (2.3), $[M, X(1)]_k=(i_{p-1})^*\{\xi_1, \xi_2, \xi_3\}$. Furthermore $(\alpha^{p-1})^*=0$ by Lemma 2.17(ii) and $i_1\alpha=0$ in (2.2). Therefore $(\pi_{p-1})^*[M, X(1)]_{k+(p-1)q+1}=\{\xi_4, \xi_5\}$ by Lemma 2.17(vi). Thus the lemma holds. q.e.d.

LEMMA 2.20. *The elements*

$$\beta^{p+1}\delta_0 = \beta\delta_0\beta^p \text{ and } \beta'^{p-1}\delta_1\beta^3\delta_1 \text{ in } [X(1), X(1)]_l \qquad (l=(p^2+2p)q-2)$$

are nontrivial. Furthermore these are linearly independent.

PROOF. We notice that the homomorphisms

$$I_{l+q+1} \xrightarrow{(i_1)_*} [M, X(1)]_{l+q+1} \xrightarrow{(\pi_1)^*} [X(1), X(1)]_l$$

are monomorphic. In fact, $\alpha_*=0: I_{l+1}\to I_{l+q+1}$ in (2.12) for $n=l+q+1$ by (2.14) and (2.16), and hence $(i_1)_*$ is monomorphic. $(i_1)_*: I_{l+1}\to[M, X(1)]_{l+1}$ in (2.12) for $n=l+1$ is epimorphic by (2.14). Thus $\alpha^*=0: [M, X(1)]_{l+1}\to [M, X(1)]_{l+q+1}$ in (2.18) for $r=1$ and $n=l$ by (2.14) and $\beta_{(p+1)}\alpha=0$, and hence $(\pi_1)^*$ is monomorphic.

Since $\beta'^{p-1}\delta_1\beta^3\delta_1=(\pi_1)^*(i_1)_*(B(p-1, 3))$ by Lemma 2.10 and (2.11), it is nontirivial by the above notice and (2.14).

Next consider the exact sequence

$$I_{l-q} \xrightarrow{\alpha^*} I_l \xrightarrow{(\pi_1)^*} [X(1), M]_{l-q-1}$$

which is obtained in the same way as (2.12) by using the isomorphism $\alpha^*: A_*\to A_{*+q}$ instead of α_*. Then (2.14) implies that $\pi_1\beta^{p+1}\delta_0=(\pi_1)^*(\beta_{(p+1)}\delta)\neq 0$. On the other hand, $\pi_1\beta'^{p-1}\delta_1\beta^3\delta_1=0$ by (2.11) and (2.2). Thus $\beta^{p+1}\delta_0$ and $\beta'^{p-1}\delta_1\beta^3\delta_1$ are linearly independent. The relation $\beta^{p+1}\delta_0=\beta\delta_0\beta^p$ follows from [21; Prop. 4.7(iii)]. q.e.d.

REMARK. We can show that $[X(1), X(1)]_l=\{\beta^{p+1}\delta_0, \beta^p\delta_0\beta, \beta^{p-1}\delta_0\beta^2, \beta'\beta^p, \beta'^{p-1}\delta_1\beta^3\delta_1\}$ by more computations. But we do not use here this stronger form.

LEMMA 2.21. *Put* $\xi=\beta i_1\delta\pi_{p-1}R(p-1)\in[X(p-1), X(1)]_k$. *Then*

$$\xi = \xi_1 + x\xi_3 + \xi_5 \qquad \textit{for some} \quad x\in F_p.$$

PROOF. By Lemma 2.19, we may put $\xi=\sum_{n=1}^5 x_n\xi_n$ $(x_n\in F_p)$. We recall the relation

(2.22)([8; Prop. 6.9]) $\beta_{(1)}\delta\varepsilon = -\delta C(p-1, 2),$ where $\varepsilon = \beta_{(p/p-1)}([9])$.

Then $\pi_1\xi i_{p-1}=-\delta C(p-1, 2)\in I_{k-q-1}$. On the other hand, $\pi_1\xi_1 i_{p-1}= -\delta C(p-1, 2)$ and $\pi_1\xi_2 i_{p-1}=-C(p-1, 2)\delta$ by Lemma 2.10. Also we see that $\pi_1\xi_n i_{p-1}=0$ $(n=3, 4, 5)$ by the relations $\pi_1\delta_0=\pi_{p-1}i_{p-1}=0$ in (2.2). Therefore $x_1=1$, $x_2=0$ by (2.14).

Now $\xi A^{p-2}=\beta\delta_0\beta^p$ by (2.9) and (2.3). On the other hand, $\xi_n A^{p-2}=0$ $(n=1, 2, 3)$ since $B^{p-2}A^{p-2}=0$ by [9; Lemma 1.5]. Furthermore, $\xi_4 A^{p-2}= \beta'^{p-1}\delta_1\beta^3\delta_1$ and $\xi_5 A^{p-2}=\beta^{p+1}\delta_0$ by (2.3). Thus we have $x_4=0$ and $x_5=1$ by Lemma 2.20. q.e.d.

We see easily the following lemma by using Lemma 2.17, the exact sequence (2.18) for $r=1$ and $n=k$ or $k+1$, and the relation $i_1\alpha=0$ in (2.2).

LEMMA 2.23. $[X(1), X(1)]_k=\{\lambda_1=\beta'^{p-1}\alpha''\beta^2,\ \ \lambda_2=\beta'^{p-1}\beta^2\alpha'',$

$$\lambda_3=\beta'^{p-2}\delta_0\beta^3\alpha'',\ \ \lambda_4=\beta'^{p-2}\alpha''\beta^3\delta_0\},$$

$[X(1), X(1)]_{k+1}=\{\mu_1=\beta'^{p-1}\alpha'\beta^2,\ \ \mu_2=\beta'^{p-2}\delta_1\alpha''\beta^3\}\oplus\mathrm{Ker}(i_1)^*$

$\mathrm{Ker}(i_1)^*=\{\mu_3=\beta'^{p-2}\alpha''\beta^3\delta_1,\ \ \mu_4=\beta'^{p-2}\beta^3\alpha''\delta_1,\ \ \mu_5=\beta'^{p-3}\delta_0\beta^4\alpha''\delta_1\}.$

LEMMA 2.24. $(\beta_{p/p-1}\wedge 1_{X(1)})\beta=-2\beta'^{p-1}(\beta\alpha''-\alpha''\beta)\beta$ *in* $[X(1), X(1)]_k$.

PROOF. By Lemma 2.23, we put $(\beta_{p/p-1}\wedge 1_{X(1)})\beta=\sum_{n=1}^4 y_n\lambda_n$. By noting that $\beta_{p/p-1}=\varepsilon_1$ (cf. [8; (5.17)]), we have

(2.25)([8; (6.2)′, (3.3), Prop. 6.9]) $\beta_{p/p-1}\wedge 1_M=\varepsilon\delta+\delta\varepsilon,$

$$\varepsilon\delta\beta_{(1)}=C(p-1, 2)\delta,\ \ \varepsilon\beta_{(s)}=-C(p-1, s+1)\ \ (s\geqq 1).$$

Further we see that $\pi_1(\beta_{p/p-1}\wedge 1_{X(1)})=(\beta_{p/p-1}\wedge 1_M)\pi_1$ by [21; Th. 2.4, Cor. 2.5]. Thus

$$\pi_1(\beta_{p/p-1}\wedge 1_{X(1)})\beta i_1 = C(p-1, 2)\delta - \delta C(p-1, 2).$$

On the other hand, $\pi_1\lambda_1 i_1=-\delta C(p-1, 2)$ and $\pi_1\lambda_2 i_1=-C(p-1, 2)\delta$ by Lemma 2.10, and $\pi_1\lambda_3 i_1=\pi_1\lambda_4 i_1=0$ by (2.2). Therefore $y_1=1$ and $y_2=-1$, i.e. $(\beta_{p/p-1}\wedge 1_{X(1)})\beta=\lambda_1-\lambda_2+y_3\lambda_3+y_4\lambda_4$.

To study y_3 and y_4, recall the homomorphism θ: $[X, Y]_n\to[X, Y]_{n+1}$ defined in [21] and the following

(2.26)([21; Th. 4.1, Th. 2.2]) $\theta(\delta_0)=-\delta_1$, $\theta(\alpha'')=\alpha'$, $\theta(1_{X(1)})=\theta(\beta)=\theta(\beta')=0$; $\theta(\gamma\gamma')=\theta(\gamma)\gamma'+(-1)^{\deg\gamma}\gamma\theta(\gamma')$, $\theta(\gamma\wedge\gamma')=(-1)^{\deg\gamma}\gamma\wedge\theta(\gamma')$ for any γ and γ'.

Also recall

(2.27)([21; (3.9), (4.3), (4.4)]) $\beta\alpha' = \alpha'\beta$, $\alpha'\delta_0 = \alpha''\delta_1$, $\delta_0\alpha' = \delta_1\alpha''$.

Then $\theta((\beta_{p/p-1}\wedge 1_{X(1)})\beta)=0$ by (2.26). By (2.26), (2.27) and the definitions of λ_n and μ_n, we have $\theta(\lambda_1-\lambda_2)=0$, $\theta(\lambda_3)=\mu_2-\beta'^{p-2}\delta_1\beta^3\alpha''$ and $\theta(\lambda_4)=-\mu_3+\mu_4$. Here $(i_1)^*(\beta'^{p-2}\delta_1\beta^3\alpha'')=0$ by (2.11) and (2.2). Thus $\theta(\lambda_3)$ and $\theta(\lambda_4)$ are linearly independent by Lemma 2.23, and we have $y_3=y_4=0$.

By [21; Th. 4.3], $\alpha''\beta^2-\beta^2\alpha''=-2(\beta\alpha''-\alpha''\beta)\beta$. Therefore $(\beta_{p/p-1}\wedge 1_{X(1)})\beta$ $=\lambda_1-\lambda_2=-2\beta'^{p-1}(\beta\alpha''-\alpha''\beta)\beta$. q.e.d.

§3. On the triviality

In this section we prove Theorem A by the following Propositions 3.1, 3.4 and 3.9.

PROPOSITION 3.1. *$\beta_s\beta_{tp/r}=0$ for $s\geqq 1$, $t\geqq 1$ and $1\leqq r\leqq p-2$.*

PROOF. Assume that $t\geqq 1$ (resp. $t\geqq 2$) if $1\leqq r\leqq p-3$ (resp. $r=p-2$), and put $b=\beta_{(tp/p-1)}$ (resp. $\beta_{(tp/p)}$) and $u=p-1-r$ (resp. $p-r$). Then

(3.2) $\beta_s\beta_{tp/r}=\beta_{tp/r}\beta_s=\pi b\alpha^u\delta\beta_{(s)}i$ (by (2.6) and (2.8)) $=0$ (by (2.16)).

To study the product $\beta_s\beta_{p/p-2}$, we recall the relation

$$\alpha\delta\beta_{(s)} = -(\alpha_1\wedge 1_M)\beta_{(s)} \tag{3.3}$$

which is shown by using [21; Th. 2.4, (3.8)] and (2.16). Now $\beta_s\beta_{p/p-2}=\pi\beta_{(p/p-1)}\alpha\delta\beta_{(s)}i$ by (2.6) and (2.8). Thus, by (3.3), (2.25), Lemma 2.10 and $\alpha_1^2=0$, we see that $\beta_s\beta_{p/p-2}=\alpha_1^2\beta_1^{p-1}\beta_{s+1}=0$. q.e.d.

PROPOSITION 3.4. *For positive integers s and t,*

$$\beta_s\beta_{tp/p-1} = \begin{cases} 0 & \text{if } s\not\equiv -1 \bmod p \text{ or } t\equiv 0 \bmod p, \\ t\beta_{s+(t-1)p}\beta_{p/p-1} & \text{otherwise.} \end{cases}$$

PROOF. Notice that $\beta_s\beta_{tp/p-1}=\pi\pi_1\beta^{s-1}\xi R(p-1)^{t-1}i_{p-1}i$ where ξ is the element in Lemma 2.21. Put $\kappa_n=\pi\pi_1\beta^{s-1}\xi_n R(p-1)^{t-1}i_{p-1}i$. Then $\beta_s\beta_{tp/p-1}=\kappa_1+x\kappa_3+\kappa_5$ by Lemma 2.21.

Now we have the relations

$$\alpha''\beta^{(t-1)p} = \beta^{(t-1)p}\alpha'' \tag{3.5}$$

by using [21; Prop. 4.7(ii)], and

$$\alpha'' i_1 i = 0 \quad \text{and} \quad \pi\pi_1\alpha'' = 0 \tag{3.6}$$

by (2.11) and (2.1). Hence $\kappa_3=\pi\pi_1\beta^{s-1}\beta'^{p-2}\delta_0\beta^3\alpha''\beta^{(t-1)p}i_1 i$ (by (2.3) and $BR(r+1)=R(r)B$ in (2.9)) $=0$ (by (3.5) and (3.6)). By definition, $\kappa_5=\beta_{s+p}\beta_{(t-1)p/p}$ where $\beta_{0/p-1}=0$ when $t=1$. Therefore

$$\beta_s\beta_{tp/p-1} = \kappa_1 + \beta_{s+p}\beta_{(t-1)p/p-1} \quad \text{with} \quad \beta_{0/p-1} = 0. \tag{3.7}$$

To study κ_1 in case $s\not\equiv -1 \bmod p$, we notice that

$$(s+1)\beta^{s-1}\alpha''\beta^{(t-1)p+2} = (s-1)\alpha''\beta^{(t-1)p+s+1}+2\beta^{(t-1)p+s+1}\alpha'', \tag{3.8}$$

which is shown by using [21; Prop. 4.7(ii)]. By (2.3) and (2.9), we have $\kappa_1=\pi\pi_1\beta^{s-1}\beta'^{p-1}\alpha''\beta^{(t-1)p+2}i_1 i$, which is 0 by (3.8) and (3.6). Thus $\kappa_1=0$ in (3.7) and we have $\beta_s\beta_{tp/p-1}=0$ by the repeated use of (3.7).

By Lemma 2.24 and (3.6), we see that $\beta_{p/p-1}\beta_n=-2\pi\pi_1\beta'^{p-1}\beta\alpha''\beta^n i_1 i$ for any positive integer n. Furthermore $\pi\pi_1\beta^{s-1}\alpha''\beta^{(t-1)p+2}=(s-1)\pi\pi_1\beta\alpha''\beta^{(t-1)p+s}$ by [21; Prop. 4.7(ii)] and (3.6). Therefore we obtain $\kappa_1=-((s-1)/2)\beta_{p/p-1}\cdot\beta_{(t-1)p+s}$. Thus by (3.7),

$$\beta_s\beta_{tp/p-1}=\beta_{(t-1)p+s}\beta_{p/p-1}+\beta_{s+p}\beta_{(t-1)p/p-1} \text{ when } s\equiv -1 \mod p.$$

By the repeated use of this equality, we see the equality

$$\beta_s\beta_{tp/p-1}=t\beta_{s+(t-1)p}\beta_{p/p-1} \qquad \text{when} \quad s\equiv -1 \mod p,$$

which is zero if $t\equiv 0 \mod p$. q.e.d.

Before proving Theorem A in case $r=p$ and $t\equiv 0 \mod p$, we have to remark the definition of the homotopy element $\beta_{tp^2/p}$. In general, there would be various β-elements in π_*^S which correspond with a given β-element in Ext^2 of the same name, and, precisely speaking, Theorem A should be stated with appropriate choice of the β-elements in π_*^S although the definition (2.5) would be canonical in the sense that it determines the elements uniquely in case $r\leqq p-1$ [9; Remark on p. 105]. To have Theorem A in case $r=p$ and $t\equiv 0 \mod p$, however, we have to adopt other definitions of $\beta_{tp^2/p}$ already known. The element $\beta_{tp^2/p}$ may be defined from the element $R'(p)$ in [9; Th. C′] in a similar way to (2.5), and this element might be different from the one defined in (2.5). Unfortunately we could not make a discussion on their difference as in [9; Remark on p. 105], because it needs precise information on the stable homotopy of spheres beyond the known limit of computation. In case $t\geqq 2$, there would be one more definition of $\beta_{tp^2/p}$, that is, $\beta_{tp^2/p}=p\beta_{tp^2/p,2}$, where $\beta_{tp^2/p,2}$ is the element in [12]. This might be different from the one in (2.5) as well. Then we have

Proposition 3.9. *For the element $\beta_{tp^2/p}$ defined in either way of above, $\beta_s\beta_{tp^2/p}=0$ for $s\geqq 1$ and $t\geqq 1$ ($t\geqq 2$ in the latter definition).*

Proof. In case of the first new definition, the proof is similar to that of Proposition 3.1, by taking $b=\rho'(tp)$ and $u=p-2$ in (3.2). In case of the second new definition, it is obvious because $p\beta_s=0$. q.e.d.

§4. The β-elements in the E_2-term of the Adams-Novikov spectral sequence

Let BP be the Brown-Peterson spectrum at a prime $p\geqq 5$. Then $BP_*=\mathbf{Z}_{(p)}[v_1, v_2,\cdots]$ and $BP_*BP=BP_*[t_1, t_2,\cdots]$, where $\deg v_i=\deg t_i=2(p^i-1)$ for

$i \geqq 1$. The E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy π_*^S is the cohomology $\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*)$ of the Hopf algebroid (BP_*, BP_*BP), (cf. [1], [2], [4], [7], [13]).

Now we recall the definition of the β-elements in this E_2-term given in [5; §2]:

$$\begin{aligned}(4.1)\qquad & \beta_{sp^n/r,i+1} = \delta\delta'(x_n^s/p^{i+1}v_1^r) \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*),\\ & \beta_{sp^n/r} = \beta_{sp^n/r,1}, \qquad \beta_{sp^n} = \beta_{sp^n/1},\end{aligned}$$

where $n \geqq 0$, $s \geqq 1$, $r \geqq 1$ and $i \geqq 0$ are integers with

$$n \geqq i,\ p \nmid s,\ r \leqq p^n \text{ if } s=1, \text{ and } p^i \mid r \leqq a_{n-i}\ (a_0=1,\ a_k=p^k+p^{k-1}-1\ (k \geqq 1)).$$

Let $x_i \in v_2^{-1}BP_*$ $(i \geqq 0)$ be the elements defined in [5; (2.4)]. Then

$$x_n^s/p^{i+1}v_1^r \in \mathrm{Ext}_\Gamma^{0,*}(A, A/(p^\infty, v_1^\infty)) \qquad (A = BP_*,\ \Gamma = BP_*BP),$$

and we obtain the elements in (4.1) by the boundary homomorphisms

$$\mathrm{Ext}_\Gamma^{0,*}(A, A/(p^\infty, v_1^\infty)) \xrightarrow{\delta'} \mathrm{Ext}_\Gamma^{1,*}(A, A/(p^\infty)) \xrightarrow{\delta} \mathrm{Ext}_\Gamma^{2,*}(A, A).$$

For the BP-homology of the spectra M in (2.1) and $X(r)$ in (2.2), we see the following by definition:

$$\begin{aligned}& BP_*(M) = BP_*/(p), \quad BP_*(X(r)) = BP_*/(p, v_1^r);\\ & \alpha_* = v_1, \quad \beta_* = v_2, \quad R(r)_* = v_2^p, \quad (R(p)^{(s)})_* = v_2^{sp},\end{aligned}$$

where α, β, $R(r)$ and $R(p)^{(s)}$ are the maps in (2.4). Therefore, by using the Geometric Boundary Theorem [3], we see the following

(4.2) *The elements* β_s $(s \geqq 1)$ *and* $\beta_{sp/r}$ $(s \geqq 1$ *for* $1 \leqq r < p$, *and* $s \geqq 2$ *for* $r=p)$ *in* (4.1) *converge to the elements* β_s *and* $\beta_{sp/r}$ *in* (2.5), *respectively*, (cf. [5; §2]).

Furthermore,

(4.3) *The elements* $\beta_{tp^2/p,2}$ $(t \geqq 2)$ *converge to the elements* $\beta_{tp^2/p,2}$ *given in* [12; Def. 5.1].

The E_2-term $\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*)$ is the homology of the cobar complex (Ω^*BP_*, d) (cf. [4]). We can represent the elements of (4.1) in the cobar complex by the following

LEMMA 4.4. *The elements of* (4.1) *can be expressed in the cobar complex* $\Omega^2BP_* = BP_*BP \otimes_{BP_*} BP_*BP$ *as follows*:

(i) $\beta_{sp^2/p,2}=-sv_2^{sp^2-2}t_2\otimes t_1+sv_2^{sp^2-2}t_1\otimes(t_2-t_1^{p+1})-sv_2^{sp^2-1}t_1\otimes\zeta_2+\cdots$

(ii) $$\beta_{sp^k/a_k}=\begin{cases} s(s-1)v_2^{s-2}t_2\otimes t_1^p+\binom{s}{2}v_2^{s-2}t_1\otimes t_1^{2p} \\ \qquad -\sum_{l=1}^{p-1}\frac{s}{p}\binom{p}{l}v_2^{s-1}t_1^l\otimes t_1^{p-l}+\cdots & (k=0), \\ -sv_2^{sp-2}t_2\otimes t_1+sv_2^{sp-2}t_1\otimes(t_2-t_1^{p+1}) \\ \qquad -sv_2^{sp-1}t_1\otimes\zeta_2+\cdots & (k=1), \\ -sv_2^{(sp-1)p^{k-1}}t_1\otimes\zeta_2+\cdots & (k\geqq 2). \end{cases}$$

(iii) $$\beta_{sp^k/a_k-1}=\begin{cases} sv_2^{sp-1}t_1\otimes t_1+\cdots & (k=1), \\ 2sv_2^{(sp-1)p^{k-1}}t_1\otimes t_1+\cdots & (k\geqq 2). \end{cases}$$

(iv) *The other β-elements belong to $(p, v_1)\Omega^2BP_*$.*

Here $\zeta_2=v_2^{-1}t_2+v_2^{-p}(t_2^p-t_1^{p^2+p})-v_2^{-p-1}v_3t_1^p\in v_2^{-1}BP_*BP$, *and* $\cdots$ *denotes an element in* $(p, v_1)\Omega^2BP_*$.

PROOF. By the congruences in [5; Lemma 6.8] and the one $\eta_R x_{k+i}^s\equiv\eta_R x_k^{sp^i} \bmod (p^{i+1}, v_1^{2+a_k})$ in [5; p. 499] (η_R is the right unit), and by using the fact that if $dx\equiv y \bmod (p, a)$, then

$$dx^{sp^i}\equiv sp^ix^{sp^i-1}y \bmod (p^{i+1}, p^ia, p^{i-1}a^p,\dots, a^{p^i}),$$

we have easily the following equality in the cobar complex $\Omega^1BP_*/(p^\infty)$:

(4.5) $$\delta'(x_{i+k}^s/p^{i+1}v_1^{p^im})=dx_{i+k}^s/p^{i+1}v_1^{p^im}=dx_k^{sp^i}/p^{i+1}v_1^{p^im}$$
$$=\begin{cases} sv_2^{s-1}t_1^p/p+\binom{s}{2}v_2^{s-2}v_1t_1^{2p}/p+v_1^2X/p & (i=k=0), \\ sv_2^{sp^{i+1}-1}v_1^{p-p^im}(t_1+v_1(v_2^{-1}(t_2-t_1^{p+1})-\zeta_2))/p \\ \qquad +\sum_{l=0}^{i}v_1^{p^l(2+p)-p^im}X_l/p^{l+1} & (0\leqq i\leqq k=1), \\ sv_2^{(sp^{i+1}-1)p^{k-1}}v_1^{a_k-p^im}(2t_1-v_1\zeta_2^{p^{k-1}})/p \\ \qquad +\sum_{l=1}^{i}v_1^{p^l(2+a_k)-p^im}X_l/p^{l+1} & (k\geqq 2, 0\leqq i\leqq k), \end{cases}$$

where $1\leqq p^im\leqq a_k$ and X and X_l are suitable elements in BP_*BP.

Let $V\in BP_*$ and $T\in \mathbf{Z}_{(p)}[t_1, t_2,\dots]$ be any elements. Then by the definition of the differential,

(4.6) $$d(VT)=\eta_R(V)\otimes T-V\Delta T+VT\otimes 1 \quad \text{in } \Omega^2BP_*,$$

where $\Delta: BP_*BP\to BP_*BP\otimes_{BP_*}BP_*BP$ is the diagonal map. Therefore, for any $l, r\geqq 0$ with $p^l|r$, we see that

(4.7) $$\delta(v_1^rVT/p^{l+1})\equiv p^{r-l-1}t_1^r\eta_R(V)\otimes T \bmod (p, v_1)\Omega^2BP_*+\operatorname{Im} d \quad \text{in } \Omega^2BP_*,$$

by using the equality $\eta_Rv_1=v_1+pt_1$. Further we notice that

(4.8) ([5; Lemma 3.19]) ζ_2^p is homologous to ζ_2.

(i)–(iii) are obtained by (4.5−8) and the equalities $\eta_R v_1 = v_1 + pt_1$, $\eta_R v_2 \equiv v_2 + pt_2 \bmod (p^2, v_1)$ and $\Delta t_1 = t_1 \otimes 1 + 1 \otimes t_1$ (cf. [5; §1]), and (iv) follows from (4.5–7). q.e.d.

§5. Reduction to the Morava stabilizer algebra

We make F_p into a BP_*-module by sending v_i ($i \geqq 0$, $i \neq 2$; $v_0 = p$) to 0 and v_2 to 1, and define $S(2)_* = F_p \otimes_{BP_*} BP_*BP \otimes_{BP_*} F_p$ whose dual is called the Morava stabilizer algebra (cf. [15]).

Consider the reduction map

$$r\colon (BP_*, BP_*BP) \longrightarrow (F_p, S(2)_*)$$

of Hopf algebroids. Then we have the ring map

(5.1) $$r^*\colon \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*) \longrightarrow \mathrm{Ext}^*_{S(2)_*}(F_p, F_p),$$

where the second ring is given as follows:

(5.2) ([16; Th. 3.2]) *For* $p \geqq 5$, $\mathrm{Ext}^*_{S(2)_*}(F_p, F_p)$ *is the tensor product of* $E(\zeta_2)$ *with the subalgebra with basis* $\{1, h_{1,0}, h_{1,1}, g_0, g_1, g_0 h_{1,1}\}$ *where* $g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle$ *(the Massey product); and* $h_{1,0}g_1 = g_0 h_{1,1}$, $h_{1,0}g_0 = h_{1,1}g_1 = 0$, $h_{1,0}h_{1,1} = h_{1,0}^2 = h_{1,1}^2 = 0$ *and* $g_i^2 = g_0 g_1 = 0$.

By the definition of the Massey product, we see the following

(5.3) *In the cobar complex for the Hopf algebra* $(F_p, S(2)_*)$, *the generators in* (5.2) *are expressed as follows*: $h_{1,0} = \{t_1\}$, $h_{1,1} = \{t_1^p\}$, $\zeta_2 = \{t_2 + t_2^p - t_1^{p+1}\}$, $g_0 = \{t_1 \otimes t_2^p + t_2 \otimes t_1\}$ *and* $g_1 = \{t_1^p \otimes t_2 + t_2^p \otimes t_1^p\}$.

LEMMA 5.4. *The images of the β-elements in* (4.1) *by the map* r^* *in* (5.1) *are given as follows*:

(i) $r^*\beta_{sp^2/p,2} = -sg_0$.

(ii) $$r^*\beta_{sp^k/a_k} = \begin{cases} \binom{s}{2}\zeta_2 h_{1,1} - \binom{s+1}{2} g_1 & (k=0), \\ -sg_0 & (k=1), \\ s\zeta_2 h_{1,0} & (k \geqq 2). \end{cases}$$

(iii) $r^*\beta_{sp^k/r,i+1} = 0$ *for the other β-elements in* (4.1).

PROOF. By Lemma 4.4(i), (ii) and (5.3).

$$r^*\beta_{sp^2/p,2} = \{-st_2\otimes t_1+st_1\otimes(t_2-t_1^{p+1})-st_1\otimes\zeta_2\} = -sg_0,$$

$$r^*\beta_{sp^k/a_k} = \begin{cases} \{-st_2\otimes t_1+st_1\otimes(t_2-t_1^{p+1})-st_1\otimes\zeta_2\} = -sg_0 & (k=1), \\ \{-st_1\otimes\zeta_2\} = -sh_{1,0}\zeta_2 & (k\geqq 2). \end{cases}$$

By Lemma 4.4(iii) and (iv),

$r^*\beta_{sp^k/a_k-1} = \{s't_1\otimes t_1\} = \{d(s't_1^2/2)\} = 0$ ($s'=s$ if $k=1$, and $s'=2s$ if $k\geqq 2$),
$r^*\beta_{sp^k/r,i+1} = 0$ for the elements in (iv).

We turn now to $r^*\beta_s$. Using the equality in [16; Th. 1.2], we have

$$(5.5) \qquad d(t_1^pt_2) = -t_1\otimes t_1^{2p} - 2t_2\otimes t_1^p + \zeta_2 h_{1,1} - g_1,$$

$$d(t_3^p) = -g_1 + \textstyle\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k} t_1^{p-k}\otimes t_1^k.$$

These and Lemma 4.4(ii) imply

$$r^*\beta_s = \left\{s(s-1)t_2\otimes t_1^p + \binom{s}{2} t_1\otimes t_1^{2p} - \textstyle\sum_{l=1}^{p-1}\frac{s}{p}\binom{p}{l} t_1^l\otimes t_1^{p-l}\right\}$$

$$= \binom{s}{2}\zeta_2 h_{1,1} - \binom{s+1}{2} g_1.$$

q.e.d.

The next theorem follows immediately from (5.2) and Lemma 5.4.

THEOREM 5.6. *Let p be a prime$\geqq 5$ and s, t be positive integers. Then the following* (i) *and* (ii) *hold in the E_2-term* $\mathrm{Ext}^4_{BP_*BP}(BP_*, BP_*)$ *of the Adams-Novikov spectral sequence*:

(i) $\beta_s\beta_{tp/p}\neq 0$ *and* $\beta_s\beta_{tp^2/p,2}\neq 0$ *if* $s\not\equiv 0, 1 \bmod p$ *and* $t\not\equiv 0 \bmod p$.
(ii) $\beta_s\beta_{tp^k/a_k}\neq 0$ *if* $k\geqq 2$, $s\not\equiv 0, -1 \bmod p$ *and* $t\not\equiv 0 \bmod p$.

Now we are ready to prove Theorem B.

PROOF OF THEOREM B. By (4.2), (4.3) and Theorem 5.6(i), $\beta_s\beta_{tp/p}$ $\beta_s\beta_{tp^2/p,2}$ for s, $t\geqq 2$ in the E_2-term are the nontrivial permanent cycles. Furthermore they are not bounded because of the sparseness of the Adams-Novikov spectral sequence. q.e.d.

§6. Concluding remarks

In the first place, we give more relations in Ext^4. We notice that the β-elements in (4.1) can be defined also for $p=3$ and Lemma 4.4 holds.

PROPOSITION 6.1. *Let p be an odd prime and s, t be positive integers. Then the following* (i)–(iii) *hold in the E_2-term* $\mathrm{Ext}^4_{BP_*BP}(BP_*, BP_*)$ *of the Adams-Novikov spectral sequence*:

(i) $\beta_s\beta_{tp^k/r} = 0$ *for* $k\geqq 1$ *and* $1\leqq r\leqq a_k-1$, *and especially* $\beta_s\beta_{tp/p-1} = 0$.

(ii) $\beta_s\beta_{tp^2/p,2} = \beta_{s+t(p^2-p)}\beta_{tp/p}$.

(iii) $\beta_s\beta_{tp^k/a_k} = \beta_{s+(tp-1)(p^{k-1}-p)}\beta_{tp^2/a_2}$
$= (t/2)\beta_{s+(tp-1)p^{k-1}-(2p-1)p}\beta_{2p^2/a_2}$, *for* t, $k\geqq 2$.

PROOF. Recall the Greek letter map η [5]. Then by Lemma 4.4(iii), $\beta_s\beta_{tp^k/a_k-1} = s'\eta(v_2^{s+(tp-1)p^{k-1}}t_1\otimes t_1/pv_1)$, where $s'=s$ if $k=1$ and $s'=2s$ if $k\geqq 2$. On the other hand, $v_2^{s+(tp-1)p^{k-1}}t_1\otimes t_1/pv_1=0$ in $\mathrm{Ext}^2_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty))$ since this is bounded by $v_2^{s+(tp-1)p^{k-1}}t_1^2/2pv_1$. Therefore $\beta_s\beta_{tp^k/a_k-1}=0$.

The other relations follow similarly from Lemma 4.4. q.e.d.

REMARK. By using Lemma 4.4(ii) for $k=0$, we can also prove Toda's relation ([21; Th. 5.3])

$$uv\beta_s\beta_t = st\beta_u\beta_v \quad (s+t=u+v) \qquad \text{in } \mathrm{Ext}^4.$$

If the β-elements in the relations of the above proposition exist in π_*^S, then the same relations hold in π_*^S modulo F^{2p+2}, where F^n is the filtration which defines the spectral sequence. In particular, since $F^{2p+2}=0$ in dimension $(2p^2-1)(2p-2)-4$ [10], we have

COROLLARY 6.2. $\beta_{p-1}\beta_{p/p-1}=0$ *in* π_*^S *for* $p\geqq 5$.

Next we notice the following

LEMMA 6.3. $\beta_2\beta_{2p/p}=x\{k_{1,0}b_{11}a_2\}$ *for some* $x\not\equiv 0 \bmod p$, *where* $p\geqq 5$ *and* $\{k_{1,0}b_{11}a_2\}$ *is the element in* [6; p. 324, (20)].

PROOF. By [6; Th. 4.1], we see that the $((2p^2+3p+1)q-4)$stem of the stable homotopy π_*^S is generated by one element $\{k_{1,0}b_{11}a_2\}$. $\beta_2\beta_{2p/p}$ is also nontrivial by Theorem B and belongs to this stem. q.e.d.

By this lemma we can restate the problem in [6; p. 324] as follows:
Is $\beta_1\beta_2\beta_{2p/p}$ trivial?

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