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Modified Rosenbrock methods with approximate Jacobian matrices

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1. Introduction

Consider the initial value problem for a stiff system

$$(1.1) \qquad y' = f(y), \quad y(x_0) = y_0,$$

where y is an m-vector and the vector function $f(y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of this problem,

$$(1.2) \qquad x_n = x_0 + nh \qquad (n = 1, 2, \dots, h > 0)$$

and let $J(y)$ be the Jacobian matrix of $f(y)$. We are concerned with the case where the approximations y_j $(j=1, 2, \dots)$ of $y(x_j)$ are obtained by the modified Rosenbrock methods of the form

$$(1.3) \qquad y_{n+1} = y_n + \textstyle\sum_{i=1}^{q} p_i k_i \qquad (n = 0, 1, \dots)$$

which require per step one evaluation of J, k evaluations of f and the solution of a system of m linear equations for q different right hand sides, where

$$(1.4) \quad Mk_i = hf(y_n + \textstyle\sum_{j=1}^{i-1} a_{ij}k_j) + hJ \sum_{j=1}^{i-1} d_{ij}k_j \qquad (i = 1, 2, \dots, q),$$

the matrix $M=I-ahJ$ is nonsingular, $J=J(y_n)$ and a_{ij}, d_{ij} $(j=1, 2, \dots, i-1;$ $i=1, 2, \dots, q)$ and a $(a>0)$ are constants.

Nørsett and Wolfbrandt [3] obtained an A-stable method of order $k+1$ for $k=q=2, 3$. For inexact Jacobian matrices, however, these methods are reduced to methods of lower orders. Steihaug and Wolfbrandt [4] tried to avoid the use of exact Jacobian matrix and considered methods of the form (1.3), called the W-methods, where

$$(1.5) \qquad Wk_i = hf(y_n + \textstyle\sum_{j=1}^{i-1} a_{ij}k_j) + hA \sum_{j=1}^{i-1} d_{ij}k_j \qquad (i = 1, 2, \dots, q),$$

$W=I-ahA$ is nonsingular and A is a matrix approximating J. They have shown that for $q=2^{k-1}$ $(k=2, 3)$ there exists a W-method of order k and that the method of order 2 is $A(0)$-stable under certain conditions.

The first object of this paper is to show that each A-stable modified Rosenbrock method remains A-stable if the Jacobian matrix is approximated with sufficient accuracy. The second object of this paper is to prove that for $q=2^{k-1}(k=1, 2, 3)$ there exists a W-method of order k which is A-stable if A is a sufficiently close approximation to J and that the method of order $p(p=2, 3)$ embeds a method or order $p-1$. Methods of order 4 are also studied.

2. Preliminaries

Let

(2.1) $y_{n+1} = y_n + \Phi(x_n, y_n; h) \qquad (n = 0, 1,\dots),$

(2.2) $\Phi(x_n, y_n; h) = \sum_{i=1}^{k} p_i k_i + \sum_{j=1}^{k-1} q_j l_j + r_1 m_1 \qquad (k = 1, 2, 3),$

(2.3) $t_{n+1} = t(x_n, y_n; h) = \sum_{i=1}^{k} p_i^* k_i + \sum_{j=1}^{k-1} q_j^* l_j + r_1^* m_1,$

(2.4) $T(x; h) = y(x) + \Phi(x, y(x); h) - y(x + h),$

(2.5) $t(x; h) = t(x, y(x); h),$

where

(2.6) $k_i = Cf_i \ (i = 1, 2, 3), \quad l_j = CAk_j \ (j = 1, 2), \quad m_1 = CAl_1,$

(2.7) $f_1 = f(y_n), \quad f_2 = f(y_n + c_2 k_1), \quad f_3 = f(y_n + c_{31}k_1 + c_{32}k_2 + d_3 l_1),$

(2.8) $C = hW^{-1}, \quad W = I - ahA.$

Then in Butcher's notation [1] $T(x_n; h)$ can be expanded into power series in h as follows:

(2.9) $$T(x_n; h) = hA_1 f + h^2(B_1[f] + B_2 Af) + h^3(C_1[_2 f]_2 + C_2[Af] + C_3 A[f] + C_4 A^2 f + C_5[f^2]) + h^4(D_1[_3 f]_3 + D_2[A[f]] + D_3[_2 Af]_2 + D_4[A^2 f] + D_5 A[_2 f]_2 + D_6 A[Af] + D_7 A^2[f] + D_8 A^3 f + D_9[_2 f^2]_2 + D_{10} A[f^2] + D_{11}[f[f]] + D_{12}[fAf] + D_{13}[f^3]) + O(h^5),$$

where

(2.10) $A_1 = \sum_{i=1}^{k} p_i - 1, \quad B_1 = \sum_{j=2}^{k} c_j p_j - 1/2, \quad B_2 = a + aA_1 + \sum_{i=1}^{k-1} q_i,$

(2.11) $C_1 = c_{32} c_2 p_3 - 1/6, \quad C_2 = a/2 + aB_1 + d_3 p_3, \quad C_3 = a/2 + aB_1 + c_2 q_2,$
$C_4 = 2aB_2 - a^2 A_1 - a^2 + r_1, \quad 2C_5 = \sum_{j=2}^{k} c_j^2 - 1/3, \quad c_3 = c_{31} + c_{32},$

(2.12) $D_1 = -1/24, \quad D_2 = D_3 = D_5 = aC_1 + a/6, \quad D_4 = aC_2 + ad_3 p_3,$
$D_6 = aC_2 + ac_2 q_2, \quad D_7 = aC_3 + ac_2 q_2, \quad 2D_9 = c_2 C_1 + c_2/6 - 1/12,$
$D_8 = 3a^2 B_2 - 2a^3 A_1 + 3ar_1 - 2a^3, \quad 2D_{10} = a/3 + 2aC_5 + c_2^2 q_2,$
$D_{11} = c_3/6 - 1/8 + c_3 C_1, \quad D_{12} = a/3 + 2aC_5 + c_3 d_3 p_3,$
$6D_{13} = \sum_{j=2}^{k} c_j^3 p_j - 1/4.$

Similarly $t(x_n; h)$ is expanded as follows:

$$t(x_n; h) = A_1^* hf + h^2(B_1^*[f] + B_2^* Af) + \cdots, \tag{2.13}$$

where

$$A_1^* = \sum_{i=1}^k p_i^*, \quad B_1^* = \sum_{j=2}^k c_j p_j^*, \quad B_2^* = \sum_{i=1}^k a p_i^* + \sum_{j=1}^{k-1} q_j^*, \cdots . \tag{2.14}$$

To study the stability of (2.1), we apply (2.1) to the scalar test equation $y' = \lambda y$, where λ is a complex number with negative real part. Then hA and (2.1) are reduced to a scalar w and

$$y_{n+1} = R(z, w) y_n \qquad (n = 0, 1, \cdots) \tag{2.15}$$

respectively, where $z = \lambda h$,

$$R(z, w) = 1 + (p_1 + p_2 + p_3) z Y + (c_2 p_2 + c_3 p_3) z^2 Y^2 + (q_1 + q_2) w z Y^2 + c_2 c_{32} p_3 z^3 Y^3 + (d_3 p_3 + c_2 q_2) w z^2 Y^3 + r_1 w^2 z Y^3, \tag{2.16}$$

$$Y = 1/(1 - aw). \tag{2.17}$$

Let

$$R(z, w) = P(z, w)/Q(w), \tag{2.18}$$

where $P(z, w)$ is a polynomial in z and w, $Q(w)$ is a power of $1 - aw$, and $P(z, w)$ and $Q(w)$ have no factor in common. Put

$$z = x + iy \quad (x < 0), \ y = tx, \ r = |z|, \tag{2.19}$$

where x and y are real numbers and i is the imaginary unit. Let α and β be the z- and iz- component of the vector $w - z$ respectively, that is,

$$w - z = (\alpha + i\beta) z, \tag{2.20}$$

where α and β are real numbers. Let

$$\arg(-z) = \theta, \quad \arg(-w) = \phi \qquad (-\pi/2 < \theta, \phi < \pi/2).$$

Then $\beta t > 0$ if and only if $\theta\phi > 0$ and $|\theta| < |\phi|$.

Let

$$aw = (u + iv) z, \quad E(x, y, \alpha, \beta) = |Q(w)|^2 - |P(z, w)|^2. \tag{2.21}$$

Then $|R(z, w)| < 1$ if and only if $E(x, y, \alpha, \beta) > 0$. In the sequel $E(x, y, \alpha, \beta)$ is written simply as E. Since E is a polynomial in x, y, α and β, by continuity $E(x, y, \alpha, \beta) > 0$ for sufficiently small $|\alpha|$ and $|\beta|$ if $E(x, y, 0, 0) > 0$. On the other hand $E(x, y, 0, 0) > 0$ for all y and all $x < 0$ if and only if the method (2.1) with

$A=J$ is A-stable. Thus we have the following

THEOREM 1. *The A-stable modified Rosenbrock method remains A-stable if the Jacobian matrix is approximated with sufficient accuracy. The W-method which is A-stable for $A=J$ is A-stable if A is a sufficiently close approximation to J.*

3. Construction of the methods

We shall show the following

THEOREM 2. *For $q=2^{k-1}$ $(k=1, 2, 3)$ there exists a W-method of order k which is A-stable if A is a sufficiently close approximation to J. For $k=2, 3$ there exists also a formula (2.3) such that $t(x; h)=O(h^k)$.*

3.1. Case k=1

The condition $r_1=A_1=0$ yields

(3.1) $y_{n+1} = y_n + k_1,$

(3.2) $T(x_n; h) = h^2(-[f]/2+aAf) + O(h^3),$

(3.3) $R(z, w) = 1 + zY, \quad E = -2x + (2u-1)r^2.$

Hence the method (3.1) is A-stable if and only if $u \geqq 1/2$, that is,

(3.4) $$\alpha \geqq -1 + 1/2a.$$

For instance, when $a=2/3$, it is A-stable if and only if $\alpha \geqq -1/4$.

3.2. Case k=2

The condition $r_1=A_1=B_1=B_2=0$ yields

(3.5) $p_1 = 1-p_2, \quad 2c_2p_2 = 1, \quad q_1 = -a,$

(3.6) $C_1=-1/6, \quad C_2=C_3=a/2, \quad C_4=-a^2, \quad C_5=(3c_2-2)/12,$

(3.7) $R(z, w) = 1 + zY + z^2Y^2/2 - awzY^2,$

(3.8) $E = -2x + b_2x^2 - b_3r^2x + b_4r^4,$

where

(3.9) $b_2=2(4u-1-4vt), \quad b_3=10u^2-6u+1+6v^2-2(2u-1)vt,$

$b_4=(4u-3)v^2+(4u-1)(2u-1)^2/4.$

Hence, for instance, if

(3.10) $$u \geqq 61/100, \quad |v| \leqq 33\sqrt{14}/700, \quad vt \leqq 9/25,$$

the method (2.1) is A-stable because $b_2 \geqq 0$, $b_3 \geqq 4513/5000$, $b_4 \geqq 0$. When $w=z$, that is, $u=a$ and $v=0$, it is A-stable if and only if

(3.11) $$a \geqq 1/4.$$

Choosing $r_1^* = A_1^* = 0$ and $q_1^* = (q_1/p_2)p_2^*$, we have

(3.12) $$p_1^* = -p_2^*, \quad q_1^* = -2ac_2p_2^*,$$

(3.13) $$B_1^* = c_2p_2^*, \quad B_2^* = q_1^*, \quad C_1^* = 0, \quad C_2^* = C_3^* = aB_1^*, \quad C_4^* = 2aq_1^*,$$
$$C_5^* = c_2B_1^*/2.$$

The choice $C_5=0$ yields $c_2=2/3$ and

(3.14) $$y_{n+1} = y_n + (k_1 + 3k_2)/4 - al_1.$$

When $A=J$, $T(x_n, h)$ is reduced to $-(6a^2-6a+1)h^3[_3f]_3/6+O(h^4)$, so that in view of (3.11) we choose

(3.15) $$a = (3 + \sqrt{3})/6.$$

Then the method (3.12) is A-stable if

(3.16) $$\alpha \geqq (83-61\sqrt{3})/100 = -0.2265, \ |\beta| \leqq 33\sqrt{14}(3-\sqrt{3})/700 = 0.2236,$$
$$\beta t \leqq 9(3-\sqrt{3})/25 = 0.4564$$

and it becomes a method of order 3 when $A=J$.

The choice

(3.17) $$p_2^* = -3d/4, \quad d = 2-\sqrt{3}$$

yields

(3.18) $$t_{n+1} = 3d(k_1 - k_2)/4 + adl_1,$$

where

(3.19) $$B_1^* = -d/2, \quad B_2^* = ad, \quad C_2^* = C_3^* = -ad/2, \quad C_4^* = 2a^2d, \quad C_5^* = -d/6.$$

Put

(3.20) $$g_2 = 3k_2/4 - al_1.$$

Then (3.13) and (3.14) can be rewritten as follows:

(3.21) $$y_{n+1} = y_n + 3k_1/4 + g_2, \ t_{n+1} = d(3k_1/4 - g_2),$$

where k_1 and g_2 are obtained from the formulas

(3.22) $$Wk_1 = hf_1,\ W(g_2 - k_1) = 3hf_2/4 - k_1.$$

Thus we have $q=2$.

3.3. Case k=3

The conditions $A_1=B_1=B_2=0$ and $C_i=0$ $(i=1, 2, 3, 4, 5)$ yield

(3.23) $$p_1+p_2+p_3=1,\quad q_1+q_2=-a,\quad r_1=a^2,\quad d_3p_3=c_2q_2=-a/2,$$
$$c_2p_2+c_3p_3=1/2,\quad c_3(c_3-c_2)p_3=(2-3c_2)/6,\quad c_{32}c_2p_3=1/6,$$

(3.24) $$D_1=-1/24,\quad D_2=D_3=D_5=a/6,\quad D_4=D_6=D_7=-a^2/2,\ D_8=a^3,$$
$$D_9=(2c_2-1)/24,\quad D_{10}=a(2-3c_2)/12,\quad D_{11}=(4c_3-3)/24,$$
$$D_{12}=a(2-3c_3)/6,\quad D_{13}=[-3+4(c_2+c_3)-6c_2c_3]/72,$$

(3.25) $$R(z, w)=1+zY+z^2Y^2/2-awzY^2+z^3Y^3/6-awz^2Y^3+a^2w^2zY^3,$$

(3.26) $$E=-2x+b_2x^2-b_3x^3+b_4r^2x^2-b_5r^4x+b_6r^6,$$

where

(3.27) $$b_2=2(6u-1-6vt),\quad b_3=6(4u-1-5vt)^2/5+2(9u-1)^2/15+6(v^2+u^2t^2),$$
$$b_4=3(2u-1)v^2t^2-4(12u^2-6u+1)vt+38u^3-27u^2+7u-7/12$$
$$-t^2(2u^3-3u^2+u-1/12)+v^2[3(10u-3)-32vt],$$
$$b_5=12v^4+(36u^2-24u+5)v^2+24u^4-28u^3+13u^2-5u/2+1/6$$
$$-[12u^3-12u^2+4u-1/2+4(3u-2)v^2]vt,$$
$$b_6=6(u-1)v^4+(12u^3-18u^2+8u-5/4)v^2+(3u-1)(6u-1)(12u^3-18u^2$$
$$+9u-1)/36.$$

Hence, for instance, if

(3.28) $$3/8\leqq u\leqq 1,\quad |v|\leqq\sqrt{5g}/120=v_0,\quad -17/56\leqq vt\leqq 9/128,\ g=4\sqrt{489}-57,$$

then the method (2.1) is A-stable because

$$b_2\geqq 53/32,\quad b_3\geqq 0,\quad b_4\geqq 23975/199608,\quad b_5\geqq 119/3072,\quad b_6\geqq 0.$$

For $w=z$ it is A-stable if and only if

(3.29) $$1/3\leqq a\leqq a_1,\quad a_1 = 1.0686\cdots,$$

where a_1 is the largest root of the equation $2a^3-3a^2+a-1/12=0$.

The conditions $A_1^*=B_1^*=B_2^*=0$, $q_2^*=(q_2/p_3)p_3^*$ and $r_1^*=(r_1/p_3)p_3^*$ lead to

(3.30) $$p_1^*+p_2^*+p_3^*=0,\quad c_2p_2^*+c_3p_3^*=0,\quad q_1^*=-q_2^*,\ q_2^*=d_3p_3^*/c_2,$$
$$r_1^*=-2ad_3p_3^*,$$

(3.31) $C_1^* = c_{32}c_2p_3^*,\quad C_2^* = C_3^* = d_3p_3^*,\quad C_4^* = -2ad_3p_3^*,\quad 2C_5^* = c_3(c_3 - c_2)p_3^*,$

(3.32) $D_1^* = 0,\quad D_2^* = D_3^* = D_5^* = aC_1^*,\quad D_4^* = D_6^* = D_7^* = 2ad_3p_3^*,\quad D_8^* = 3ar_1^*,$

$D_9^* = c_2C_1^*/2,\quad D_{10}^* = aC_5^* + c_2d_3p_3^*/2,\quad D_{11}^* = c_3C_1^*,$

$D_{12}^* = 2aC_5^* + c_3d_3p_3^*,\quad 3D_{13}^* = (c_2 + c_2)C_5^*.$

The choice $D_{13} = D_{11} + D_{12} = D_9 + D_{10} = 0$ yields

(3.33) $a = 1/2,\ c_2 = 1,\quad c_{31} = c_{32} = 1/4,\quad d_3 = -3/8,$

(3.34) $y_{n+1} = y_n + (k_1 + k_2 + 4k_3)/6 - (l_1 + l_2)/4 + m_1/4,$

(3.35) $D_2 = D_3 = D_5 = 1/12,\quad D_4 = D_6 = D_7 = -1/8,\quad D_8 = 1/8,$

$D_9 = -D_{10} = -D_{11} = D_{12} = 1/24.$

The method (3.34) is A-stable if

(3.36) $-1/4 \leqq \alpha \leqq 1,\quad |\beta| \leqq 2v_0 = 0.2090,\quad -17/28 \leqq \beta t \leqq 9/64.$

In the case $A = J$, $T(x_n;\, h)$ becomes $-h^4[_3f]_3/24 + O(h^5)$.

For the choice $p_3^* = -1/6$ we have

(3.37) $t_{n+1} = (k_1 + k_2 - 2k_3)/12 - (l_1 - l_2)/16 - m_1/16,$

(3.38) $C_1^* = -1/24,\quad C_2^* = C_3^* = -C_4^* = 1/16,\quad C_5^* = 1/48,$

(3.39) $D_1^* = 0,\quad D_i^* = -1/48\ (i = 2, 3, 5, 9, 11),\quad D_4^* = D_6^* = D_7^* = 1/16,$

$D_8^* = -3/32,\quad D_{10}^* = 1/24,\quad D_{12}^* = 5/96,\quad D_{13}^* = 1/96.$

Let

(3.40) $$g_3 = 4k_3/3 - (l_2 - m_1)/2.$$

Then (3.34) and (3.37) can be rewritten as follows:

(3.41) $$y_{n+1} = y_n + (k_1 + k_2)/6 - l_1/4 + g_3/2,$$

(3.42) $$t_{n+1} = (k_1 + k_2)/12 - l_1/16 - g_3/8,$$

where k_1, k_2, l_1 and g_3 are obtained from the formulas

(3.43) $Wk_1 = hf_1,\quad Wk_2 = hf_2,\quad W(l_1 + 2k_1) = 2k_1,$

$W(g_3 - k_2 + l_1) = 4hf_3/3 - k_2 + l_1,$

(3.44) $f_3 = f(\hat{y}),\quad \hat{y} = y_n + (k_1 + k_2)/4 - 3l_1/8.$

Thus we have $q = 4$.

4. Methods of order 4

Let

(4.1) $y_{n+1} = y_n + \sum_{i=1}^{4} (p_i k_i + q_i l_i) + r_1 m_1 + r_2 m_2 + s_1 n_1,$

(4.2) $t_{n+1} = \sum_{i=1}^{4} p_i^* k_i + p^* h f^* + q_1^* l_1 + q_2^* l_2 + r_1^* m_1,$

where

(4.3) $$k_4 = Cf_4, \quad l_4 = CAk_4, \quad m_2 = CAl_2, \quad n_1 = CAm_1,$$

(4.4) $$f_4 = f(y_n + \sum_{i=1}^{3} c_{4i} k_i + \sum_{j=1}^{2} d_{4j} l_j + e_4 m_1), \quad f^* = f(y_{n+1}).$$

The conditions $A_1 = B_1 = B_2 = 0$, $C_j = 0$ ($j = 1, 2, 3, 4, 5$) and $D_k = 0$ ($k = 1, 2, \ldots, 13$) yield

(4.5) $\sum_{i=1}^{4} p_i = 1, \quad \sum_{j=2}^{4} c_j p_j = 1/2, \quad \sum_{k=3}^{4} g_k p_k = 1/6, \quad 24 u p_4 = 1,$

$q_i = -a p_i \ (i = 1, 2, 3, 4), \quad r_1 + r_2 = a^2, \quad 2 c_2 r_2 = a^2, \quad s_1 = -a^3,$

(4.6) $c_4 = 1, \quad d_3 = -4 a g_3, \quad c_2 d_{43} = c_{43} d_3, \quad e_4 p_4 = a^2/2,$

$c_3(c_3 - c_2) = 2(1 - 2c_2) g_3, \quad (1 - c_3) g_4 = (3 - 4c_3) u, \ (1 - c_3) d_4 = 4a(3c_3 - 2) u,$

$(1 - c_2)(1 - c_3) = 2[3 - 4(c_2 + c_3) + 6 c_2 c_3] u,$

(4.7) $$E(0, y, 0, 0) = y^6 (a^2 b_3 b_2 y^4 + b_1 y^2 + b_0),$$

where

(4.8) $u = c_{43} g_3, \quad c_4 = \sum_{j=1}^{3} c_{4j}, \quad d_4 = \sum_{i=1}^{2} d_{4i}, \quad g_i = \sum_{j=2}^{i-1} c_{ij} c_j \quad (i = 3, 4),$

(4.9) $b_0 = -8a^5 + 12a^4 - 19a^3/3 + 7a^2/4 - a/4 + 1/72,$

$b_1 = 5a^6 - 2a^5 - 19a^4/12 + 4a^3/3 - 13a^2/36 + a/24 - 1/576,$

$b_2 = 2a^4 - 4a^3 + 7a^2/2 - a + 1/12, \quad b_3 = 4a^3 - 7a^2/2 + a - 1/12.$

The method (4.1) with $A = J$ is A-stable if and only if $E(0, y, 0, 0) \geqq 0$ for all y [2], that is,

(4.10) $$a_2 \leqq a \leqq a_3, \quad a_2 = 0.267766, \quad a_3 = 0.788675,$$

where a_2 and a_3 are zeros of b_2 and b_0 respectively.

Choosing $A_1^* = B_1^* = B_2^* = 0$ and $C_i^* = 0$ ($i = 1, 2, 3, 4, 5$), we have

(4.11) $\sum_{i=1}^{4} p_i^* + p^* = 0, \quad \sum_{j=2}^{4} c_j p_j^* + p^* = 0, \quad \sum_{i=3}^{4} g_i p_i^* = -p^*/2,$

$4 u p_4^* = -p^*, \quad q_1^* + q_2^* = a p^*, \quad c_2 q_2^* = a p^*, \quad r_1^* = -a^2 p^*,$

(4.12) $D_1^* = -p^*/12, \quad D_2^* = D_3^* = -D_5^* = a p^*/2, \quad D_4^* = -2a^2 p^*, \quad D_6^* = D_7^* = a^2 p^*,$

$D_8^* = -a^3 p^*, \quad D_9^* = (3c_2 - 1)p^*/12, \quad D_{10}^* = a(c_2 - 1)p^*/2,$

$D_{11}^* = (2c_3 - 1)p^*/4, \quad D_{12}^* = a(1 - 2c_3)p^*, \quad D_{13}^* = (2c_2 - 1)(1 - 2c_3)p^*/2.$

The choice $c_2 = 2/5$ and $c_3 = 3/5$ yields

(4.13) $p_1 = p_4 = 11/72, \quad p_2 = p_3 = 25/72, \quad r_1 = -a^2/4, \quad r_2 = 5a^2/4,$

(4.14) $c_{31} = -3/20, \quad c_{32} = 3/4, \quad d_3 = -6a/11, \quad c_{41} = 19/44, \quad c_{42} = -15/44,$

$c_{43} = 10/11, \quad d_{41} = 24a/11, \quad d_{42} = -30a/11, \quad e_4 = 36a^2/11,$

(4.15) $p_1^* = p_2^* = p^*/6, \quad p_3^* = -5p^*/12, \quad p_4^* = -11p^*/12, \quad q_1^* = -3ap^*/2,$

$q_2^* = 5ap^*/2, \quad r_1^* = -a^2 p^*.$

When $A = J$, $T(x_n; h)$ becomes

$$(h^5/5!)\{(-120a^4 + 180a^3 - 80a^2 + 15a - 1)[_4 f]_4 + a[_3 f^2]_3 - 2a[_2[f]f]_2 - 2[_2 f^3]_2/15 + (20a^2 - 5a + 1)[[_2 f]_2 f] + (2800a^2 + 200a + 9)[[f^2]]/220 + a[[f]f^2] + [f^4]/30\} + O(h^6)$$

and $t(x; h)$ is reduced to

$$(h^4/4!)\{(-24a^2 + 12a - 2)[_3 f]_3 + 2(1 - 18a)[_2 f^2]_2/5 + 6(1 - 4a)[[f]f]/5 + 12[f^3]/25\}p^* + O(h^5).$$

Let

(4.16) $v_3 = C(f_3 - 3ak_2 + 18a^2 l_1/5), \; v_4 = C(f_4 - 15ak_2/11 - 18a^2 l_1/11),$

$v = CA(q_3 v_3 + q_4 v_4 + 65ak_2/72 - 5a^2 l_1/4).$

Then (4.1), (4.2) and (4.4) can be rewritten as follows:

(4.17) $y_{n+1} = y_n + p_1 k_1 + p_2 k_2 + p_3 v_3 + p_4 v_4 + q_1 l_1,$

(4.18) $t_{n+1} = p_1^* k_1 + p_2^* k_2 + p_3^* v_3 + p_4^* v_4 + p^* h f^* + q_1^* l_1 - a^2 p^* m_1,$

(4.19) $f_4 = f(y_n + c_{41} k_1 + c_{42} k_2 + c_{43} v_3 + d_{41} l_1).$

Hence we have $q = 7$ and we have shown the following

THEOREM 3. *For $k = 4$ and $q = 7$ there exist a formula* (4.2) *such that $t_{n+1} = O(h^4)$ and a W-method* (4.1) *of order* 4 *which is* A-*stable if A is a sufficiently close approximation to J.*

References

[1] J. C. Butcher, *Coefficients for the study of Runge-Kutta integration processes,* J. Austral. Math. Soc. **3** (1963), 185–201.

[2] S. P. Nørsett, *C-polynomials for rational approximation to the exponential function,* Numer. Math. **25** (1975), 39–56.

[3] S. P. Nørsett and A. Wolfbrandt, *Order conditions for Rosenbrock-type methods,* Numer. Math. **32** (1979), 1–16.

[4] T. Seihaug and A. Wolfbrandt, *An attempt to avoid exact Jacobian and nonlinear equations in the numerical solution of stiff differential equations,* Math. Comp. **33** (1979), 521–534.

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