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Existence and periodicity of weak solutions of the Navier-Stokes equations in a time dependent domain

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Introduction

This paper deals with the problem of existence and periodicity of weak solutions of the initial-boundary value problem for the Navier-Stokes equations in domains with smoothly moving boundaries. Hopf [5] proved the existence of a global weak solution in a cylindrical domain by using the Faedo-Galerkin approximation. On the other hand, Fujita-Sauer [4] and Lions [8] obtained the same result in the case of time dependent domains with Lipschitz continuous boundaries by a penalty method ([4]) or a singular perturbation method ([8]). Our main purpose in this paper is to show that the method of Hopf [5] can be applied with a slight modification to the case when the domain moves smoothly. An advantage of Hopf's method is that we can show the existence of a periodic solution when the domain moves periodically and the boundary data are small enough.

To show the existence of a weak solution we reduce in Section 1 the given problem to the one in a cylindrical domain, assuming the existence of a diffeomorphism which sends the given time dependent domain to a cylindrical one. In doing so, the velocity and the pressure gradient will be transformed as vector fields. Similar techniques are used in Bock [1] and in Inoue-Wakimoto [6], where the existence of a unique local strong solution is proved by the Faedo-Galerkin method ([1]) or the method of evolution equation in Hilbert space ([6]). However, Bock [1] does not regard the velocity and the pressure gradient as vector fields, and so the calculation given in [1] is complicated. In [6] Inoue and Wakimoto treat the velocity and the pressure gradient as vector fields, but they assume that the Jacobian of the diffeomorphism is equal to 1, which is a strong limitation. In this paper we assume that the Jacobian of the diffeomorphism depends only on time variable. As will be shown in Section 4, this assumption for the Jacobian is of no restriction.

Section 2 deals with the construction and estimate of approxiamte solutions. We first construct approximate solutions for the reduced problem by choosing a suitable Galerkin basis, and then return to the original problem on a time dependent domain to get an energy inequality, which together with a modification of the compactness argument given in [5] enables us to choose a subsequence of the approximate solutions converging in L^2 space to a weak solution. Further, we shall show the uniqueness of our solutions in the case of two-dimensional flow.

Applying the argument in [9, pp. 483–486] to our construction, we show in Section 3 the existence of a periodic solution. In the two-dimensional case, our periodic solution seems to be unique and stable if the data are small enough. This will be discussed in the forthcoming paper.

In solving the initial-boundary value problem for the Navier-Stokes equations under nonhomogeneous boundary conditions one has to extend the given boundary data to the whole of the space-time domain as divergence-free vector fields. So we discuss in Section 4 the extensibility of the data representing the condition that a fluid adheres to the (moving) boundary. We shall show that the boundary data can be extended if and only if the volume of the moving domain is independent of time.

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1. Formulation of the problem

Let $Q_\infty=\bigcup_{t\in R}\Omega(t)\times\{t\}$ be a noncylindrical space-time domain, each $\Omega(t)$ being a bounded domain in R^n $(n=2, 3, 4)$ with smooth boundary $\partial\Omega(t)$. In Q_∞ we consider the initial-boundary value problem for the Navier-Stokes equations:

$$(1.1)\qquad \begin{aligned} \partial v/\partial t - \Delta v + (v, \nabla)v &= f - \nabla p, && x\in\Omega(t), && t>0,\\ \operatorname{div} v &= 0, && x\in\Omega(t), && t\geq 0,\\ v &= \beta, && x\in\partial\Omega(t), && t>0,\\ v(x, 0) &= v_0(x), && x\in\Omega(0). \end{aligned}$$

Here $v=\{v^j(x, t)\}_{j=1}^n$, $p=p(x, t)$ denote respectively the unknown velocity and pressure, while $f=\{f^j(x, t)\}_{j=1}^n$, $v_0=\{v_0^j(x)\}_{j=1}^n$ denote respectively the given external force and initial velocity; $\beta=\{\beta^j(x, t)\}_{j=1}^n$ is given on the boundary $\bigcup_{t\in R}\partial\Omega(t)\times\{t\}$. For $\Omega(t)$ and β we impose the following conditions:

(A.1) There exist a cylindrical domain $\tilde{Q}_\infty=\tilde{\Omega}\times R$ and a level-preserving C^∞ diffeomorphism Φ: $\bar{Q}_\infty\to\bar{\tilde{Q}}_\infty$,

$$(y, s)=\Phi(x, t)=(\phi^1(x, t),\dots, \phi^n(x, t), t)$$

such that

$$(1.2)\qquad \det[\partial\phi^i(x, t)/\partial x^j]\equiv J(t)^{-1}>0, \quad \text{for} \quad (x, t)\in\bar{Q}_\infty.$$

(A.2) β is the restriction to $\cup_{t\in R}\partial\Omega(t)\times\{t\}$ of a C^2 vector field ψ, which is divergence-free on each $\Omega(t)$ and bounded on $\bar{Q}_\infty$ together with its first and second derivatives.

According to (A.2) the problem (1.1) can be reduced to the case of zero boundary values. Setting $v=\psi+u$, we obtain from (1.1),

$$(1.1)' \quad \begin{aligned} &\partial u/\partial t - \Delta u + (u, \nabla)\psi + (\psi, \nabla)u + (u, \nabla)u = F - \nabla p, \; x\in\Omega(t),\; t>0,\\ &\operatorname{div} u = 0, \quad x\in\Omega(t), \quad t\geq 0,\\ &u = 0, \quad x\in\partial\Omega(t), \quad t>0,\\ &u(x, 0) = a(x), \quad x\in\Omega(0), \end{aligned}$$

where $F=f+\Delta\psi-(\psi, \nabla)\psi-\partial\psi/\partial t$; $a(x)=v_0(x)-\psi(x, 0)$.

Our purpose in this paper is to show the existence of a global solution for (1.1)′ As in [6] we regard u, a, and F as vector fields and p a scalar field on $\Omega(t)$. So, if we set

$$\tilde{u}^j(y, s) = \textstyle\sum_{k=1}^n (\partial y^j/\partial x^k)u^k(\Phi^{-1}(y, s))$$

and similarly for a, ψ, F, and

$$\tilde{p}(y, s) = p(\Phi^{-1}(y, s))$$

then (1.1)′ is transformed into the following problem on $\tilde{Q}_\infty$:

$$\begin{aligned} &\partial\tilde{u}/\partial s - L\tilde{u} + M\tilde{u} + N_1\tilde{u} + N_2\tilde{u} = \tilde{F} - \nabla_g\tilde{p}, && y\in\tilde{\Omega}, \quad s>0,\\ &\operatorname{div}\tilde{u} = \textstyle\sum_{j=1}^n \partial\tilde{u}^j/\partial y^j = 0, && y\in\tilde{\Omega}, \quad s\geqq 0,\\ &\tilde{u} = 0, \quad y\in\partial\tilde{\Omega},\; s>0, \quad \tilde{u}(y, 0) = \tilde{a}(y), && y\in\tilde{\Omega}, \end{aligned}$$

where

$$\begin{aligned} &(L\tilde{u})^i = g^{jk}\nabla_j\nabla_k\tilde{u}^i\\ &(M\tilde{u})^i = (\partial y^j/\partial t)\nabla_j\tilde{u}^i + (\partial y^i/\partial x^k)(\partial^2 x^k/\partial s\partial y^j)\tilde{u}^j,\\ &(N_1\tilde{u})^i = \tilde{\psi}^j\nabla_j\tilde{u}^i + \tilde{u}^j\nabla_j\tilde{\psi}^i, \qquad (N_2\tilde{u})^i = \tilde{u}^j\nabla_j\tilde{u}^i,\\ &(\nabla_g\tilde{p})^i = g^{ij}(\partial\tilde{p}/\partial y^j), \end{aligned}$$

and

$$\begin{aligned} &g^{ij} = (\partial y^i/\partial x^k)(\partial y^j/\partial x^k), \quad g_{ij} = (\partial x^k/\partial y^i)(\partial x^k/\partial y^j),\\ &\nabla_j\tilde{u}^i = \partial\tilde{u}^i/\partial y^j + \Gamma^i_{jk}\tilde{u}^k,\\ &\nabla_k\nabla_j\tilde{u}^i = \partial(\nabla_j\tilde{u}^i)/\partial y^k + \Gamma^i_{kl}\nabla_j\tilde{u}^l - \Gamma^l_{kj}\nabla_l\tilde{u}^i,\\ &2\Gamma^k_{ij} = g^{kl}(\partial g_{il}/\partial y^j + \partial g_{jl}/\partial y^i - \partial g_{ij}/\partial y^l)\\ &\qquad = 2(\partial y^k/\partial x^l)(\partial^2 x^l/\partial y^i\partial y^j). \end{aligned}$$

Here and hereafter we use summation convention, i.e. take sum over repeated indices; furthermore for each vector field w on Q_∞, $\tilde{w}$ will always mean a vector field on $\tilde{Q}_\infty$ obtained by the transformation

$$\tilde{w}^j(y, s) = (\partial y^j/\partial x^k)w^k(\Phi^{-1}(y, s)),$$

and conversely.

Note that ∇_j is the covariant differentiation with respect to the Riemannian connection induced from the metric (g_{ij}). From the assumption (A.1) it is easy to see that

$$(1.3) \qquad (g^{ij})^{-1} = (g_{ij}), \quad \det(g_{ij}) = J(t)^2.$$

It is to be noticed that because of (1.2) the divergence operator is left invariant under the coordinate transformation. Finally we note that $\partial\tilde{u}/\partial s+M\tilde{u}$ and $L\tilde{u}$ correspond respectively to $\partial u/\partial t$ and Δu under the transformation Φ; see [6, Th. 2.5] for the details.

2. Existence of weak solutions

Throughout this paper we shall denote by the letter C, with or without indices, various constants; sometimes we shall denote different constants by the same letter, whenever this will not lead to confusion.

We introduce some function spaces. By $C_{0,\sigma}^\infty(\tilde{\Omega})$ we denote the space of all smooth divergence-free vector fields with compact support in $\tilde{\Omega}$. Let $\tilde{H}$ and $\tilde{V}$ be respectively the closures of $C_{0,\sigma}^\infty(\tilde{\Omega})$ in $(L^2(\tilde{\Omega}))^n$ and in $(H^1(\tilde{\Omega}))^n$, and define H_t and V_t similarly on $\Omega(t)$. $\tilde{H}$ is a Hilbert space with respect to any of the inner products defined by

$$(2.1) \qquad \langle \tilde{u}, \tilde{v}\rangle_t \equiv \int_{\tilde{\Omega}} g_{ij}(y, t)\tilde{u}^i(y)\tilde{v}^j(y)J(t)dy;$$

see (1.3). To H_t we give the usual inner product

$$(2.2) \qquad (u, v)_t \equiv \int_{\Omega(t)} u^j(x)v^j(x)dx.$$

For $\tilde{u}, \tilde{v} \in \tilde{V}$ their inner product in $\tilde{V}$ is defined by

$$(2.3) \qquad \langle \nabla_g\tilde{u}, \nabla_g\tilde{v}\rangle_t \equiv \int_{\tilde{\Omega}} g_{ij}(y, t)g^{kl}(y, t)\nabla_k\tilde{u}^i\nabla_l\tilde{v}^j J(t)dy.$$

Note that for any fixed t (2.1) is transformed into (2.2) by the coordinate transformation Φ^{-1} and (2.3) is transformed into

$$(\nabla u, \nabla v)_t \equiv \int_{\Omega(t)} (\partial u^i/\partial x^j)(\partial v^i/\partial x^j)dx. \tag{2.4}$$

For each $\tilde{u} \in \tilde{H}$ we set $|\tilde{u}|_t = \langle \tilde{u}, \tilde{u} \rangle_t^{1/2}$ and for each $\tilde{u} \in \tilde{V}$ $|\nabla_g \tilde{u}|_t = \langle \nabla_g \tilde{u}, \nabla_g \tilde{u} \rangle_t^{1/2}$. The norm corresponding to (2.2) is denoted by $\|\cdot\|_t$. For each t, V_t^* denotes the dual space of V_t. The norm of $f \in V_t^*$ is defined by $\|f\|_t^* = \sup\{\langle f, v \rangle_t;\ v \in V_t, \|\nabla v\|_t \leq 1\}$. Similarly we define the norm on $\tilde{V}^*$ and denote it by $|\cdot|_t^*$. Now we define a weak solution of the problem (1.1)$'$.

DEFINITION 2.1. For $a \in H_0$ and $F \in L^2(0, T; V_t^*)$, $T > 0$ being fixed, we call a function $u \in L^2(0, T; V_t) \cap L^\infty(0, T; H_t)$ a *weak solution of the problem* (1.1)$'$, if and only if the following identity is satisfied:

$$\begin{aligned} -\int_0^T \langle \tilde{u}(t), \tilde{v}'(t) \rangle_t dt - \int_0^T \langle \tilde{u}(t), M\tilde{v}(t) \rangle_t dt + \int_0^T \langle \nabla_g \tilde{u}(t), \nabla_g \tilde{v}(t) \rangle_t dt \\ + \int_0^T \langle N_1 \tilde{u}(t) + N_2 \tilde{u}(t), \tilde{v}(t) \rangle_t\, dt = \langle \tilde{a}, \tilde{v}(0) \rangle_0 + \int_0^T \langle \tilde{F}(t), \tilde{v}(t) \rangle_t dt \end{aligned} \tag{2.5}$$

for any $\tilde{v}(t) = h(t)\tilde{w}$ such that $\tilde{w} \in \tilde{V}$ and $h \in C^1([0, T]; R)$, $h(T) = 0$.

In what follows we write

$$b(u, v, w) = ((u, \nabla)v, w)_t = \int_{\Omega(t)} u^i(\partial v^j/\partial x^i)w^j dx,$$

where u, v and w are elements of V_t. Since $n \leq 4$, it follows from the Sobolev imbedding theorem that the form b can be defined and estimated as follows:

$$|b(u, v, w)| \leq \|u\|_{L_4} \|\nabla v\|_t \|w\|_{L_4} \leq C\|\nabla u\|_t \|\nabla v\|_t \|\nabla w\|_t, \tag{2.6}$$

(see [11]). Note that since $\operatorname{div} u = 0$, integration by parts gives

$$b(u, v, v) = 0. \tag{2.7}$$

THEOREM 2.2. *Fix an arbitrary $T > 0$. Then for each $a \in H_0$ and each $F \in L^2(0, T; V_t^*)$ there exists a weak solution of* (1.1)$'$ *on* $[0, T]$.

We shall prove this in several steps. Let $\{\tilde{\phi}_j\}$ be a sequence of linearly independent vectors in $C_{0,\sigma}^\infty(\tilde{\Omega})$ total in $\tilde{V}$, and $\{\tilde{w}_j(y, t)\}$ be its Schmidt orthogonalization with respect to the inner product (2.1). Note that $\tilde{w}_j(t) = \tilde{w}_j(y, t)$ thus obtained is smooth in (y, t). In fact, it is a finite linear combination of $\{\tilde{\phi}_j\}$ with coefficients in $C^\infty([0, T]; R)$.

We define approximate solutions $\tilde{u}_m(t)$, $m \geq 1$, by the following equations:

$$\begin{aligned} \tilde{u}_m(t) &= \textstyle\sum_{j=1}^m h_{jm}(t)\tilde{w}_j(t), \\ \tilde{u}_m(0) &= \textstyle\sum_{j=1}^m h_{jm}^0 \tilde{w}_j(0),\ h_{jm}^0 = \langle \tilde{a}, \tilde{w}_j(0) \rangle_0, \end{aligned}$$

where $\{h_{jm}(t)\}$ is defined by

$$(2.8)\quad \langle \tilde{u}_m'(t), \tilde{w}_j(t)\rangle_t = \langle L\tilde{u}_m(t), \tilde{w}_j(t)\rangle_t - \langle M\tilde{u}_m(t), \tilde{w}_j(t)\rangle_t - \langle N_1\tilde{u}_m(t) + N_2\tilde{u}_m(t), \tilde{w}_j(t)\rangle_t + \langle \tilde{F}(t), \tilde{w}_j(t)\rangle_t \quad (1\le j\le m).$$

It is easy to see that $\tilde{u}_m(t)$ is determined uniquely by (2.8) in a neighborhood of $t=0$. The proof of the next lemma guarantees that $\tilde{u}_m(t)$ is defined on the whole interval $[0, T]$.

LEMMA 2.3. *$\{u_m(t)\}$ remains bounded in $L^\infty(0, T; H_t)\cap L^2(0, T; V_t)$.*

PROOF. Multiplying (2.8) by $h_{jm}(t)$, taking sum in j and returning to Q_∞, we obtain

$$(2.9)\quad (d/dt)\|u_m(t)\|_t^2+2\|\nabla u_m(t)\|_t^2=-2b(u_m(t), \psi(t), u_m(t))+2(F(t), u_m(t))_t.$$

Here we have used (2.7). By (A.2) there is a constant $C_T>0$ such that

$$(2.10)\qquad |b(u_m(t), \psi(t), u_m(t))|\le \sup_{\Omega(t)}|\nabla\psi(t)|\,\|u_m(t)\|_t^2\le C_T\|u_m(t)\|_t^2.$$

Integrating (2.9) in t, we obtain

$$\begin{aligned}(2.11)\quad \|u_m(t)\|_t^2 + 2\int_0^t \|\nabla u_m(\tau)\|_\tau d\tau &\le \|a\|_0^2 + 2C_T\int_0^t \|u_m(\tau)\|_\tau^2 d\tau + 2\int_0^t \|F(\tau)\|_\tau^*\|\nabla u_m(\tau)\|_\tau d\tau \\ &\le \|a\|_0^2 + 2C_T\int_0^t \|u_m(\tau)\|_\tau^2 d\tau + \int_0^t \|F(\tau)\|_\tau^{*2} d\tau + \int_0^t \|\nabla u_m(\tau)\|_\tau^2 d\tau.\end{aligned}$$

From this we have

$$(2.12)\qquad \|u_m(t)\|_t^2 \le \|a\|_0^2 + 2C_T\int_0^t \|u_m(\tau)\|_\tau^2 d\tau + \int_0^T \|F(\tau)\|_\tau^{*2} d\tau,$$

which brings the boundedness of $\{u_m(t)\}$ in $L^\infty(0, T; H_t)$ by Gronwall's lemma. Then from (2.11)

$$(2.13)\qquad \int_0^T \|\nabla u_m(t)\|_t^2 dt \le \|a\|_0^2 + 2C_T\int_0^T \|u_m(t)\|_t^2 dt + \int_0^T \|F(t)\|_t^{*2} dt.$$

This shows the boundedness in $L^2(0, T; V_t)$, which completes the proof.

Next we shall prove

LEMMA 2.4. *$\{u_m(t)\}$ is precompact in $L^2(0, T; H_t)$.*

To do this we need the following lemma.

LEMMA 2.5. *For each $\varepsilon>0$ there exists a positive integer $N=N_\varepsilon$ independent of $t\in[0, T]$ such that for any $v\in V_t$ we have*

$$\|v\|_t^2 \le \textstyle\sum_{j=1}^N (v, w_j(t))_t^2 + \varepsilon\|\nabla v\|_t^2. \tag{2.14}$$

PROOF OF LEMMA 2.4. Put $\rho_{mj}(t)=(u_m(t), w_j(t))_t$ for $m\geqq j$. We shall show that $\{\rho_{mj}(t)\}_{m\geq j}$ is uniformly bounded and equicontinuous on $[0, T]$ for each fixed j. In fact, since there exists for each j a constant M_j such that

$$|w_j(x, t)| \le M_j,\ |\nabla w_j(x, t)| \le M_j,\ |(\partial/\partial t)w_j(x, t)| \le M_j, \quad \text{for all} \quad x\in\Omega(t),\ t\in[0, T],$$

it follows from Lemma 2.3 that

$$|\rho_{mj}(t)| \le M_j|\Omega(t)|^{1/2}\|u_m(t)\|_t \le M_j'. \tag{2.15}$$

Furthermore, for $t\in[0, T)$ and $s>0$

$$\begin{aligned}
|\rho_{mj}(t+s) - \rho_{mj}(t)| &= \left|\int_t^{t+s} (d/d\tau)(u_m(\tau), w_j(\tau))_\tau d\tau\right| \\
&\le \left|\int_t^{t+s} (u_m'(\tau), w_j(\tau))_\tau d\tau\right| + \left|\int_t^{t+s} (u_m(\tau), w_j'(\tau))_\tau d\tau\right| \\
&\le \int_t^{t+s} |(\nabla u_m(\tau), \nabla w_j(\tau))_\tau| d\tau + \int_t^{t+s} |b(u_m, u_m, w_j)(\tau)| d\tau \\
&\quad + \int_t^{t+s} |b(\psi, u_m, w_j)(\tau)| d\tau + \int_t^{t+s} |b(u_m, \psi, w_j)(\tau)| d\tau \\
&\quad + \int_t^{t+s} |(F(\tau), w_j(\tau))_\tau| d\tau + \int_t^{t+s} |(u_m(\tau), w_j'(\tau))_\tau| d\tau \\
&\le C_j\Big(\int_t^{t+s} \|\nabla u_m(\tau)\|_\tau d\tau + \int_t^{t+s} \|u_m(\tau)\|_\tau \|\nabla u_m(\tau)\|_\tau d\tau \\
&\quad + \int_t^{t+s} \|u_m(\tau)\|_\tau d\tau + \int_t^{t+s} \|F(\tau)\|_\tau^* d\tau\Big) \\
&\le C_j\Big(s^{1/2}\{1+\sup_t \|u_m(t)\|_t\}\Big\{\int_0^T \|\nabla u_m(\tau)\|_\tau^2 d\tau\Big\}^{1/2} \\
&\quad + s\times\sup_t \|u_m(t)\|_t + s^{1/2}\Big\{\int_0^T \|F(\tau)\|_\tau^{*2} d\tau\Big\}^{1/2}\Big),
\end{aligned} \tag{2.16}$$

where C_j is a constant depending only on n and M_j. So the equicontinuity is obtained. Therefore, applying the diagonal argument we can choose a sequence $\{m_k\}$ of positive integers such that $\{\rho_{m_k,j}(t)\}_{m_k\geq j}$ converges uniformly on $[0, T]$ for each fixed j. Substituting $v=u_{m_k}-u_{m_l}$ into (2.14) and integrating in t, we obtain

$$\int_0^T \|u_{m_k}(t)-u_{m_l}(t)\|_t^2 dt \le \sum_{j=1}^N \int_0^T |\rho_{m_k j}(t)-\rho_{m_l j}(t)|^2 dt + 2\varepsilon \sup_m \int_0^T \|\nabla u_m(t)\|_t^2 dt.$$

Letting k, $l \to \infty$,

$$\limsup \int_0^T \|u_{m_k}(t)-u_{m_l}(t)\|_t^2 dt \le 2\varepsilon \sup_m \int_0^T \|\nabla u_m(t)\|_t^2 dt.$$

Since $\varepsilon > 0$ is arbitrary and $\{u_m\}$ is bounded in $L^2(0, T; V_t)$, the proof is completed.

PROOF OF LEMMA 2.5. Obviously it is enough to show

$$|\tilde{v}|_t^2 \le \sum_{j=1}^N \langle \tilde{v}, \tilde{w}_j(t)\rangle_t^2 + \varepsilon |\nabla_g \tilde{v}|_t^2, \tag{2.17}$$

for all $\tilde{v} \in \tilde{V}$ and all $t \in [0, T]$. Suppose that (2.17) is false. Then there exist a $\delta > 0$ and sequences $\tilde{v}_m \in \tilde{V}$, $t_m \in [0, T]$ such that

$$1 = |\tilde{v}_m|_{t_m}^2 > \sum_{j=1}^m \langle \tilde{v}_m, \tilde{w}_j(t_m)\rangle_{t_m}^2 + \delta |\nabla_g \tilde{v}_m|_{t_m}^2.$$

This shows that $\{\tilde{v}_m\}$ is bounded in $\tilde{V}$. Therefore we may assume, using Rellich's lemma, that $\tilde{v}_m \to \tilde{v}_0 \in \tilde{V}$ weakly in $\tilde{V}$ and strongly in $\tilde{H}$, and further $t_m \to t_0 \in [0, T]$. For each fixed k we have

$$1 = |\tilde{v}_m|_{t_m}^2 > \sum_{j=1}^k \langle \tilde{v}_m, \tilde{w}_j(t_m)\rangle_{t_m}^2 + \delta |\nabla_g \tilde{v}_m|_{t_m}^2 \quad (m \ge k). \tag{2.18}$$

Since $\{g_{ij}(t)\}$ depends smoothly on t, there exists a constant $C > 0$ such that for each $t \in [0, T]$

$$C |\nabla_g \tilde{v}|_{t_0} \le |\nabla_g \tilde{v}|_t, \qquad \text{for all} \quad \tilde{v} \in \tilde{V}.$$

Since the norm $|\nabla_g \cdot|_{t_0}$ is lower semicontinuous with respect to the weak topology of $\tilde{V}$, by letting $m \to \infty$ in (2.18) we obtain

$$1 = |\tilde{v}_0|_{t_0}^2 \ge \sum_{j=1}^k \langle \tilde{v}_0, \tilde{w}_j(t_0)\rangle_{t_0}^2 + \delta C |\nabla_g \tilde{v}_0|_{t_0}^2.$$

Since k is arbitrary and $\{\tilde{w}_j(t_0)\}$ is an orthonormal base in $\tilde{H}$ with respect to the inner product (2.1) at $t = t_0$, we see that $|\nabla_g \tilde{v}_0|_{t_0} = \|\nabla v_0\|_{t_0} = 0$, so that v_0 is constant on $\Omega(t_0)$. But since $v_0 \in V_{t_0}$ it follows that $v_0 = 0$, which is absurd because $|\tilde{v}_0|_{t_0} = 1$. This completes the proof.

PROOF OF THEOREM 2.2. By Lemmas 2.3 and 2.4 we may assume that there exists a u in $L^\infty(0, T; H_t) \cap L^2(0, T; V_t)$ such that

> $u_m \to u$ in the weak topology of $L^2(0, T; V_t)$ and in the weak-star topology of $L^\infty(0, T; H_t)$; $u_m \to u$ in $L^2(0, T; H_t)$.

Take h in $C^1([0, T]; R)$ with $h(T) = 0$ and set $\tilde{v}(t) = h(t)\tilde{w}_j$. Multiplying (2.8) by h, and returning to Q_∞, we obtain by integration by parts

$$
\begin{aligned}
&(u_m'(t),\ v(t))_t \\
&= -(\nabla u_m(t),\ \nabla v(t))_t - b(u_m(t),\ u_m(t),\ v(t)) - b(u_m(t),\ \psi(t),\ v(t)) \\
&\quad - b(\psi(t),\ u_m(t),\ v(t)) + (F(t),\ v(t))_t.
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
&-\int_0^T (u_m(t),\ v'(t))_t dt + \int_0^T (\nabla u_m(t),\ \nabla v(t))_t dt \\
&+\int_0^T b(u_m,\ u_m,\ v)(t)dt + \int_0^T b(u_m,\ \psi,\ v)(t)dt + \int_0^T b(\psi,\ u_m,\ v)(t)dt \\
&= (u_m(0),\ v(0))_0 + \int_0^T (F(t),\ v(t))_t dt.
\end{aligned}
$$

Since $u_m(0)\to a$ in H_0, by letting $m\to\infty$ we obtain

$$
\begin{aligned}
&-\int_0^T (u(t),\ v'(t))_t dt + \int_0^T (\nabla u(t),\ \nabla v(t))_t dt \\
&+\int_0^T b(u,\ u,\ v)(t)dt + \int_0^T b(u,\ \psi,\ v)(t)dt + \int_0^T b(\psi,\ u,\ v)(t)dt \\
&= (a,\ v(0))_0 + \int_0^T (F(t),\ v(t))_t dt.
\end{aligned}
$$

The convergence of the nonlinear term is assured by the fact that

$$
\int_0^T b(u_m,\ u_m,\ v)(t)dt = -\int_0^T b(u_m,\ v,\ u_m)(t)dt
$$

and that v is smooth because of its construction; see [11, Chap. 3]. Expressing the above equality in $\tilde{Q}_\infty$, we have

$$
\begin{aligned}
(2.19)\qquad &-\int_0^T \langle \tilde{u}(t),\ \tilde{v}'(t)\rangle_t dt - \int_0^T \langle \tilde{u}(t),\ M\tilde{v}(t)\rangle_t dt + \int_0^T \langle \nabla_g \tilde{u}(t),\ \nabla_g \tilde{v}(t)\rangle_t dt \\
&+\int_0^T \langle N_1\tilde{u}(t),\ \tilde{v}(t)\rangle_t dt + \int_0^T \langle N_2\tilde{u}(t),\ \tilde{v}(t)\rangle_t dt \\
&= \langle \tilde{a},\ \tilde{v}(0)\rangle_0 + \int_0^T \langle \tilde{F}(t),\ \tilde{v}(t)\rangle_t dt.
\end{aligned}
$$

By linearity this holds for $\tilde{v}=\sum_{j=1}^{l} h_j(t)\tilde{w}_j(t)$, $h_j\in C^1([0,\ T;\ R)$ with $h_j(T)=0$. Now let us recall our construction of $\{\tilde{w}_j(t)\}$. Each $\tilde{\phi}_j$ can be expressed as a finite linear combination of $\{\tilde{w}_j(t)\}$ whose coefficients are functions in $C^1([0,\ T];\ R)$. So, in (2.19) we can take $\tilde{v}=\sum_{j=1}^{l} h_j(t)\tilde{\phi}_j$, h_j as above. Since $\{\tilde{\phi}_j\}$ is total in $\tilde{V}$, we see that (2.5) holds for general $\tilde{\phi}$ in $\tilde{V}$ and $h\in C^1([0,\ T];\ R)$, $h(T)=0$. Thus we have proved Theorem 2.2.

REMARK 2.6. By taking $\tilde{\phi} \in C^{\infty}_{0,\sigma}(\tilde{\Omega})$ and $h \in C^{\infty}_{0}((0, T))$ in (3.5), we see that in the sense of distribution

$$(d/dt)\langle \tilde{u}(t), \tilde{\phi}\rangle_t = \langle \tilde{u}(t), M\tilde{\phi}\rangle_t - \langle \nabla_g \tilde{u}(t), \nabla_g \tilde{\phi}\rangle_t \\ - \langle N_1\tilde{u}(t)+N_2\tilde{u}(t), \tilde{\phi}\rangle_t + \langle \tilde{F}(t), \tilde{\phi}\rangle_t.$$

By the definition of M, N_1 and the estimate (2.6), we see that the right-hand side defines an element of $L^1(0, T; \tilde{V}^*)$. By applying Lemma in [11, Chap. 3, §1], it follows that $\tilde{u}'(t)$ exists as an element of $L^1(0, T; \tilde{V}^*)$ and so $\tilde{u}$ is weakly continuous on $[0, T]$ with values in $\tilde{H}$ since $\tilde{u} \in L^{\infty}(0, T; \tilde{H})$. Hence we have $\tilde{u}(0)=\tilde{a}$ in $\tilde{H}$.

Furthermore, as will be indicated in the proof of Lemma 2.7 below, we have for each $\tilde{\phi} \in \tilde{V}$,

$$(d/dt)\langle \tilde{u}(t), \tilde{\phi}\rangle_t = \langle \tilde{u}'(t)+M\tilde{u}(t), \tilde{\phi}\rangle_t + \langle \tilde{u}(t), M\tilde{\phi}\rangle_t.$$

Hence,

$$\langle \tilde{u}'(t)+M\tilde{u}(t), \tilde{\phi}\rangle_t+\langle \nabla_g\tilde{u}(t), \nabla_g\tilde{\phi}\langle_t+\langle N_1\tilde{u}(t)+N_2\tilde{u}(t), \tilde{\phi}\rangle_t=\langle \tilde{F}(t), \tilde{\phi}\rangle_t. \tag{2.20}$$

Next we shall discuss the uniqueness of our solutions for two dimensional flow. In this case the estimate (2.6) can be replaced by

$$|b(v_1, v_2, w)|=|b(v_1, w, v_2)|\le 2^{1/2}\{\|v_1\|_t\|\nabla v_1\|_t\}^{1/2}\{\|v_2\|_t\|\nabla t_2\|_t\}^{1/2}\|\nabla w\|_t. \tag{2.21}$$

for v_1, v_2, $w \in V_t$; see [11]. From this it follows that the time derivative $\tilde{u}'$ of a weak solution $\tilde{u}$ belongs to $L^2(0, T; \tilde{V}^*)$ since $\tilde{u}$ belongs to $L^{\infty}(0, T; \tilde{H})$.

LEMMA 2.7. *If $\tilde{w} \in L^2(0, T; \tilde{V})$ and $\tilde{w}' \in L^2(0, T; \tilde{V}^*)$, then $\tilde{w}$ is continuous on $[0, T]$ with values in $\tilde{H}$. Further we have*

$$(d/dt)|\tilde{w}|_t^2 = 2\langle \tilde{w}'+M\tilde{w}, \tilde{w}\rangle_t. \tag{2.22}$$

PROOF. We calculate directly $(d/dt)|\tilde{w}|_t^2$, assuming $\tilde{w} \in C^1([0, T]; \tilde{V})$. For general $\tilde{w}$ one has only to regularize it in t after defining $\tilde{w}=0$ outside $(0, T)$; see [11, Chap. 3].

$$\begin{aligned}(d/dt)|\tilde{w}(t)|_t^2 &= \int_{\tilde{\Omega}} g_{jk}'(t)\tilde{w}^j(t)\tilde{w}^k(t)J(t)dy \\ &\quad + 2\int_{\tilde{\Omega}} g_{jk}(t)\tilde{w}^{j\prime}(t)\tilde{w}^k(t)J(t)dy \\ &\quad + \int_{\tilde{\Omega}} g_{jk}(t)\tilde{w}^j(t)\tilde{w}^k(t)J'(t)dy \\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

$J'(t)$ in I_3 will be calculated as follows: Consider on Q_∞ the divergence-free vector field "$\partial/\partial t$" which is transformed by Φ_* into $(\partial y^j/\partial t)(\partial/\partial y^j)+(\partial/\partial t)$ on $\tilde{Q}_\infty$. By claculating the divergence of the transformed vector field, we obtain

$$\begin{aligned}0 &= J(t)^{-1}\{(\partial/\partial y^j)(J(t)(\partial y^j/\partial t)) + (\partial/\partial t)J(t)\}\\ &= \nabla_j(\partial y^j/\partial t) + J(t)^{-1}J'(t),\end{aligned}$$

so that

$$J'(t) = -J(t)\nabla_j(\partial y^j/\partial t).$$

Since $\tilde{w} \in \tilde{V}$, substitution of this into I_3 gives

$$\begin{aligned}I_3 &= -\int_{\tilde{\Omega}} g_{jk}(t)\tilde{w}^j(t)\tilde{w}^k(t)J(t)\nabla_l(\partial y^l/\partial t)dy\\ &= \int_{\tilde{\Omega}} \nabla_l\{g_{jk}(t)\tilde{w}^j(t)\tilde{w}^k(t)J(t)\}(\partial y^l/\partial t)dy.\end{aligned}$$

Note that ∇_l is the covariant differentiation with respect to $(g_{ji}(t))$, so $\nabla_l g_{jk}(t)=0$, $\nabla_l J(t)=0$. Therefore,

$$I_3 = 2\int_{\tilde{\Omega}} g_{jk}(t)(\partial y^l/\partial t)(\nabla_l\tilde{w}^j(t))\tilde{w}^k(t)J(t)dy.$$

On the other hand, since

$$g_{jk}'(t) = (\partial^2 x^l/\partial t\partial y^j)(\partial x^l/\partial y^k) + (\partial^2 x^l/\partial t\partial y^k)(\partial x^l/\partial y^j),$$

the integrand in I_1 is written as follows:

$$\begin{aligned}&\{(\partial^2 x^l/\partial t\partial y^j)(\partial x^l/\partial y^k)\tilde{w}^j(t)\tilde{w}^k(t) + (\partial^2 x^l/\partial t\partial y^k)(\partial x^l/\partial y^j)\tilde{w}^j(t)\tilde{w}^k(t)\}J(t)\\ &= 2(\partial^2 x^l/\partial t\partial y^m)(\partial x^l/\partial y^k)\tilde{w}^m(t)\tilde{w}^k(t)J(t)\\ &= 2g_{jk}(t)(\partial y^j/\partial x^l)(\partial^2 x^l/\partial t\partial y^m)\tilde{w}^m(t)\tilde{w}^k(t)J(t).\end{aligned}$$

So we have

$$I_1 = 2\int_{\tilde{\Omega}} g_{jk}(t)(\partial y^j/\partial x^l)(\partial^2 x^l/\partial t\partial y^m)\tilde{w}^m(t)\tilde{w}^k(t)J(t)dy.$$

Adding these we have the desired equality; refer to the notations in Section 1. We have also proved the absolute continuity of $|\tilde{w}|_t$. From this and the weak continuity of $\tilde{w}$ stated in Remark 2.6 it follows that $\tilde{w}$ is continuous on $[0, T]$ with values in $\tilde{H}$.

THEOREM 2.8. *When $n=2$, the solution given in Theorem 2.2 is unique.*

PROOF. Let u, v be solutions for the same a and F. Setting $w=u-v$ we obtain from (2.20)

$$\begin{aligned}(2.23)\qquad &\langle \tilde{w}'(t)+M\tilde{w}(t),\ \tilde{\phi}\rangle_t+\langle \nabla_g\tilde{w}(t),\ \nabla_g\tilde{\phi}\rangle_t+\langle N_1\tilde{w}(t),\ \tilde{\phi}\rangle_t\\ &+\langle N_2\tilde{u}(t)-N_2\tilde{v}(t),\ \tilde{\phi}\rangle_t=0 \qquad \text{for each} \quad \tilde{\Phi}\in\tilde{V}\end{aligned}$$

and $\tilde{w}(0)=0$. On the other hand,

$$(N_2\tilde{u})^i-(N_2\tilde{v})^i=\tilde{u}^j\nabla_j\tilde{u}^i-\tilde{v}^j\nabla_j\tilde{v}^i=\tilde{u}^j\nabla_j\tilde{w}^i-\tilde{w}^j\nabla_j\tilde{v}^i,$$

so that, by (2.21)

$$\begin{aligned}|\langle N_2\tilde{u}(t)-N_2\tilde{v}(t),\ \tilde{w}(t)\rangle_t|&=|b(u,\ w,\ w)(t)-b(w,\ v,\ w)(t)|\\ &=|b(w,\ v,\ w)(t)|\le C|\tilde{w}(t)|_t|\nabla_g\tilde{w}(t)|_t|\nabla_g\tilde{v}(t)|_t.\end{aligned}$$

Since $\tilde{w}'\in L^2(0,\ T;\ \tilde{V}^*)$ it follows from (2.23) and Lemma 2.7 that

$$\begin{aligned}&(1/2)(d/dt)|\tilde{w}(t)|_t^2+|\nabla_g\tilde{w}(t)|_t^2\\ &=-\langle N_1\tilde{w}(t),\ \tilde{w}(t)\rangle_t-\langle N_2\tilde{u}(t)-N_2\tilde{v}(t),\ \tilde{w}(t)\rangle_t\\ &\le C_1|\tilde{w}(t)|_t|\nabla_g\tilde{w}(t)|_t+C_2|\tilde{w}(t)|_t|\nabla_g\tilde{w}(t)|_t|\nabla_g\tilde{v}(t)|_t.\end{aligned}$$

By Schwarz's inequality,

$$(d/dt)|\tilde{w}(t)|_t^2\le(C_3+C_4|\nabla_g\tilde{v}(t)|_t^2)|\tilde{w}(t)|_t^2.$$

Hence

$$|\tilde{w}(t)|_t^2\le\int_0^t(C_3+C_4|\nabla_g\tilde{v}(\tau)|_\tau^2)|\tilde{w}(\tau)|_\tau^2d\tau.$$

Applying Gronwall's lemma we obtain $\tilde{w}=0$, which completes the proof.

3. Existence of periodic solutions

In this section we assume that the movement of $\Omega(t)$, the boundary data $\psi(x,\ t)$ and the diffeomorphism $\Phi(x,\ t)$ are periodic with period $T>0$. This implies that the tensor $(g_{ij}(y,\ t))$ is also periodic with period T. So, by our construction, $\{\tilde{w}_j(y,\ t)\}$ and $\{w_j(x,\ t)\}$ are periodic with the same period.

THEOREM 3.1. *Let $\Omega(t)$, $\psi(x,\ t)$ and Φ be as above. If $\max_{0\le t\le T}\|\nabla\psi(t)\|_t$ is sufficiently small, then for each $F\in L^2(0,\ T;\ V_t^*)$ there exists a function u in $L^\infty(0,\ T;\ H_t)\cap L^2(0,\ T;\ V_t)$ satisfying (2.5) with some $a=u(0)=u(T)$ in H_0.*

PROOF. Take for each $m\ge 1$ a function $\tilde{u}_m^0$ arbitrarily from the subspace of $\tilde{V}$ spanned by $\{\tilde{w}_j(y,\ 0)\}_{1\le j\le m}$, and then determine $\tilde{u}_m(t)$ with $\tilde{u}_m(0)=\tilde{u}_m^0$ by the equation (2.8). Using (2.6), we obtain from (2.9)

$$\begin{aligned}&(d/dt)\|u_m(t)\|_t^2 + 2\|\nabla u_m(t)\|_t^2\\&\leq 2\|F(t)\|_t^*\|\nabla u_m(t)\|_t - 2b(u_m, \psi, u_m)(t)\\&\leq \|F(t)\|_t^{*2} + \|\nabla u_m(t)\|_t^2 + CK\|\nabla u_m(t)\|_t^2,\end{aligned}$$

where $K=\max_{0\leq t\leq T}\|\nabla\psi(t)\|_t$ and C is the constant in (2.6) which depends on the size of $\Omega(t)$. Hence,

$$(3.1)\qquad (d/dt)\|u_m(t)\|_t^2 + \|\nabla u_m(t)\|_t^2 \leq \|F(t)\|_t^{*2} + CK\|\nabla u_m(t)\|_t^2.$$

Now assume that

$$(3.2)\qquad CK < 1.$$

Then, by the Poincaré inequality we obtain

$$(d/dt)\|u_m(t)\|_t^2 + C'\|u_m(t)\|_t^2 \leq \|F(t)\|_t^{*2},$$

so that

$$(3.3)\qquad e^{C'T}\|u_m(T)\|_T^2 \leq \|u_m(0)\|_0^2 + \int_0^T e^{C't}\|F(t)\|_t^{*2}dt.$$

If we choose $r>0$ so that

$$r^2(1-e^{-C'T}) \geq \int_0^T e^{-C'(T-t)}\|F(t)\|_t^{*2}dt,$$

it follows from (3.3) that $\|u_m(T)\|_T\leq r$ if $\|u_m(0)\|_0\leq r$. On the other hand, it is easily checked that the map: $u_m(0)\mapsto u_m(T)$ is continuous. Since by periodicity both $u_m(0)$ and $u_m(T)$ are in the finite dimensional linear span $[w_1(x, 0), \ldots, w_m(x, 0)]=[w_1(x, T),\ldots, w_m(x, T)]$, the Brouwer fixed point theorem ensures the existence of a u_m such that $u_m(0)=u_m(T)$ and $\|u_m(0)\|_0=\|u_m(T)\|_T\leq r$. From now on we shall fix such $r>0$ and u_m for each m. Since r is independent of m, the arguments in the proofs of Lemmas 2.3 and 2.4 are applicable. So we may assume that there exists a u in $L^\infty(0, T; H_t)\cap L^2(0, T; V_t)$ such that

$u_m\to u$ in the weak topology of $L^2(0, T; V_t)$ and in the weak-star topology of $L^\infty(0, T; H_t)$; $u_m\to u$ in $L^2(0, T; H_t)$.

$u_m(0)=u_m(T)$ converges in the weak topology of $H_0=H_T$ to an element a.

Obviously u satisfies (2.5). Further, multiplying (2.8) by $h\in C^1([0, T]; R)$ with $h(0)=0$, integrating in t and letting $m\to\infty$, we see that $\tilde{u}$ satisfies (2.19) with $\langle\tilde{a}, \tilde{v}(0)\rangle_0$ replaced by $\langle\tilde{a}, \tilde{v}(T)\rangle_T$. Since $\tilde{u}(t)$ is weakly continuous on $[0, T]$ with values in $\tilde{H}$, it follows that $\tilde{u}(0)=\tilde{u}(T)=\tilde{a}$. This completes the proof.

REMARK 3.2. By applying the method in [6] we can show the existence of a unique strong solution on $[0, T]$ when $n=2$, assuming $a \in H_0$ and $\tilde{F} \in C([0, T]; \tilde{H})$. This seems to enable us to apply the result in Serrin [10], concerning the uniqueness and stability of periodic solutions, to the two-dimensional flow. This problem will be discussed in the forthcoming paper.

4. A remark on the assumptions (A.1)–(A.2)

In this paper we have discussed the solvability of the initial-boundary value problem (1.1) assuming (A.1)–(A.2). In this section we first show that the condition (1.2) in (A.1) is of no restriction. The techniques of the proof of this can be applied to the discussion on the extensibility of the boundary data representing the condition that a fluid adheres to the (moving) boundary, i.e., the so-called no-slip condition. In fact, it will be shown that (A.2) holds for β corresponding to the no-slip condition if and only if the volume $|\Omega(t)|$ of $\Omega(t)$ is independent of t. All of our arguments are based on the following result due to Ebin-Marsden ([2, Th. 8.6]).

LEMMA 4.1. *Let $\omega_t (t \in R)$ be a family of volume elements on $\overline{\Omega(0)}$ depending smoothly on t such that ω_0 is the canonical volume element in R^n. Assume that*

$$\int \omega_t = \int \omega_0 \quad \textit{for all} \quad t \in R.$$

Then there exist diffeomorphisms Φ_t: $\overline{\Omega(0)} \to \overline{\Omega(0)}$ with $\Phi_0 =$ the identity such that $\Phi_t^* \omega_t = \omega_0$ for all t.

Using this we can prove

LEMMA 4.2. *In Lemma 4.1 we can choose Φ_t which, moreover, satisfy $\Phi_t =$ the identity on $\partial\Omega(0)$.*

PROOF. Let $\tilde{\Phi}_t$ be the diffeomorphisms given in Lemma 4.1. If the diffeomorphisms μ_t: $\overline{\Omega(0)} \to \overline{\Omega(0)}$ satisfy

$$\mu_t^* \omega_0 = \omega_0; \; \mu_t = \tilde{\Phi}_t^{-1} \quad \text{on} \quad \partial\Omega(0), \tag{4.1}$$

then $\Phi_t = \tilde{\Phi}_t \circ \mu_t$ are the desired ones. We shall show that such μ_t exist. Let $Y_t = (d\tilde{\Phi}_t^{-1}/dt) \circ \tilde{\Phi}_t$ be the vector fields which are tangential to $\partial\Omega(0)$. It is known (see [7]) that there exist vector fields X_t on $\overline{\Omega(0)}$ such that $\operatorname{div} X_t = 0$ in $\Omega(0)$ and $X_t = Y_t$ on $\partial\Omega(0)$. If we define $\mu_t(x)$, $x \in \overline{\Omega(0)}$, as the solution of the initial value problem:

$$dy/dt = X_t(y), \quad y(0) = x,$$

it is easy to see that μ_t actually satisfies (4.1). This completes the proof.

We are now ready to prove the following which shows that condition (1.2) is of no restriction.

THEOREM 4.3. *Suppose we are given diffeomorphisms* ϕ_t: $\overline{\Omega(0)} \to \overline{\Omega(t)}$, $t \in R$, *such that* $\phi_0 =$ *the identity. Let* $K(t) = |\Omega(t)|$ *be the volume of* $\Omega(t)$. *Then there exist diffeomorphisms* ψ_t: $\overline{\Omega(0)} \to \overline{\Omega(t)}$ *with* $\psi_0 =$ *the identity such that*

$$\text{(i)} \quad \psi_t = \phi_t \quad on \quad \partial\Omega(0), \qquad \text{(ii)} \quad \psi_t^* \omega_0 = K(t)K(0)^{-1}\omega_0,$$

where ω_0 *is the canonical volume element in* R^n.

PROOF. Choose linear transformations ζ_t on R^n depending smoothly on t such that $\zeta_0 =$ the identity and $\zeta_t^* \omega_0 = K(0)K(t)^{-1}\omega_0$, and set $\Omega'(t) \equiv \zeta_t(\Omega(t))$. Then $|\Omega'(t)| = |\Omega(0)|$ for all t. So the volume elements $\omega_t \equiv \tilde{\phi}_t^* \omega_0$ ($\tilde{\phi}_t = \zeta_t \circ \phi_t$) on $\overline{\Omega(0)}$ satisfy the assumption of Lemma 4.1, and we can choose, by Lemma 4.2, diffeomorphisms Φ_t: $\overline{\Omega(0)} \to \overline{\Omega(0)}$ such that $\Phi_t^* \omega_t = \omega_0$; $\Phi_0 =$ the identity; $\Phi_t =$ the identity on $\partial\Omega(0)$. It is easy to see that the diffeomorphisms $\psi_t = \phi_t \circ \Phi_t$ satisfy the desired properties. This completes the proof.

Let us now discuss the extensibility of the boundary data representing the no-slip condition, which is expressed as

$$\beta^i(x, t) = (\partial x^i/\partial t)(y(x, t), t), \quad x \in \partial\Omega(t) \tag{4.2}$$

where $(y, t) = \Phi(x, t) = (\phi_t^{-1}(x), t) \in \overline{\Omega(0)} \times R$.

THEOREM 4.4. *For* β *defined by* (4.2) *we can choose* ψ *satisfying assumption* (A.2) *if and only if* $|\Omega(t)| = |\Omega(0)|$ *for all* $t \in R$.

PROOF. Suppose that $|\Omega(t)| = |\Omega(0)|$ for all t. By Theorem 4.3 there exist diffeomorphisms ξ_t: $\overline{\Omega(0)} \to \overline{\Omega(t)}$ such that $\xi_t^* \omega_0 = \omega_0$ and $\xi_t = \phi_t$ on $\partial\Omega(0)$. Thus, the vector field

$$\psi^i(x, t) = (d\xi_t^i/dt)(\xi_t^{-1}(x)), \quad x \in \Omega(t)$$

satisfies $\operatorname{div} \psi = 0$ in each $\Omega(t)$ and $\psi = \beta$ on each $\partial\Omega(t)$.

Conversely suppose that there exists ψ such that $\operatorname{div} \psi = 0$ in each $\Omega(t)$ and $\psi = \beta$ on each $\partial\Omega(t)$. Define $\eta_t(y)$, $y \in \overline{\Omega(0)}$, as the solution of the initial value problem:

$$dz/dt = \psi(z, t), \quad z(0) = y.$$

It is easy to see that $\Psi_t \equiv \eta_t$: $\overline{\Omega(0)} \to \overline{\Omega(t)}$ satisfy $\Psi_t^* \omega_0 = \omega_0$ and $\Psi_t = \phi_t$ on $\partial\Omega(0)$. Hence $|\Omega(t)| \equiv |\Omega(0)|$, which completes the proof.

References

[1] D. N. Bock, On the Navier-Stokes equations in noncylindrical domains, J. Differential Equations **25** (1977), 151–162.

[2] D. G. Ebin and J. E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math. **92** (1970), 102–163.

[3] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[4] H. Fujita and N. Sauer, On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries, J. Fac. Sci. Univ. Tokyo Sec. IA **17** (1970), 403–420.

[5] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. **4** (1951), 213–231.

[6] A. Inoue and M. Wakimoto, On existence of solutions of the Navier-Stokes equation in a time dependent domain, J. Fac. Sci. Univ. Tokyo Sec. IA **24** (1977), 303–319.

[7] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1969.

[8] J. L. Lions, Singular perturbations and some non linear boundary value problems, MRC Tech. Summary Rep. no. 421, University of Wisconsin Press, Madison, 1963.

[9] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod et Gauthier-Villars, Paris, 1969.

[10] J. Serrin, A note on the existence of periodic solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. **3** (1959), 120–122.

[11] R. Temam, Navier-Stokes Equations, North-Holland Publ. Co., Amsterdam, 1977.

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