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Minimally semi-fine limits of Green potentials of general order

Dedicated to Professor Makoto Ohtsuka on the occation of his 60 th birthday

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1. Introduction

In the half space $D=\{x=(x_1,\dots, x_n);\ x_n>0\}$, $n\geqq 2$, let G_α be the Green function of order α, that is,

$$G_\alpha(x, y) = \begin{cases} |x-y|^{\alpha-n} - |\bar{x}-y|^{\alpha-n} & \text{in case} \quad 0 < \alpha < n, \\ \log(|\bar{x}-y|/|x-y|) & \text{in case} \quad \alpha = n, \end{cases}$$

where $\bar{x}=(x_1,\dots, x_{n-1}, -x_n)$ for $x=(x_1,\dots, x_{n-1}, x_n)$. We say that u is a Green potential of order α in D if there is a non-negative measure μ on D such that

$$u(x) = G_\alpha(x, \mu) = \int_D G_\alpha(x, y)d\mu(y) \not\equiv \infty.$$

We say that a function u on D has limit zero in L^p at $\xi\in\partial D$ if

$$\lim_{r\downarrow 0} r^{-n}\int_{B_+(\xi,r)} |u(x)|^p dx=0,$$

where $B_+(\xi, r)=\{x\in D;\ |x-\xi|<r\}$. Letting $\chi_{\Gamma(\xi,a)}$ denote the characteristic function of a cone $\Gamma(\xi, a)=\{x=(x_1,\dots, x_n);\ |x-\xi|<ax_n\}$, we say that u has non-tangential limit zero in L^p at $\xi\in\partial D$ if for any $a>1$, $u\chi_{\Gamma(\xi,a)}$ has limit zero in L^p at ξ. In case $\alpha=2$, it is known (see [5] and [7]) that any Green potential has non-tangential limit zero in L^p, $1\leqq p<n/(n-2)$, at almost every $\xi\in\partial D$.

Next we shall define the minimal α-semi-thinness at $\xi\in\partial D$ of a set $E\subset D$. For this purpose we consider the function

$$k_\alpha(x, y) = \liminf_{(X,Y)\to(x,y),\,(X,Y)\in D\times D} X_n^{-1}Y_n^{-1}G_\alpha(X, Y), \quad x, y\in D\cup\partial D.$$

Note here that $k_\alpha(x, y)=d_\alpha|x-y|^{\alpha-n-2}$ for x and $y\in\partial D$, where $d_\alpha=2(n-\alpha)$ if $\alpha<n$ and $=2$ if $\alpha=n$. Define a capacity C_{k_α} by

$$C_{k_\alpha}(E) = \sup \mu(D), \qquad E\subset D,$$

where the supremum is taken over all non-negative measures μ on D such that S_μ (the support of μ) is included in E and

$$k_\alpha(x, \mu) = \int k_\alpha(x, y)d\mu(y) \leqq 1 \qquad \text{for every} \quad x \in D.$$

A Borel set E in D is called minimally α-semi-thin at $\xi \in \partial D$ if

$$\lim_{r \downarrow 0} r^{\alpha-n-2} C_{k_\alpha}(E \cap B_+(\xi, r)) = 0.$$

We note here that $C_{k_\alpha}(rA) = r^{n-\alpha+2} C_{k_\alpha}(A)$ for $r>0$ and $A \subset D$ and $C_{k_\alpha}(B_+(\xi, 1))>0$, where $rA = \{rx; x \in A\}$. A function u on D is said to have minimally α-semi-fine limit zero at $\xi \in \partial D$ if there exists a Borel set E in D which is minimally α-semi-thin at ξ and for which $\lim_{x \to \xi, x \in D-E} u(x) = 0$.

Finally we shall say that a sequence $\{x^{(j)}\}$ in D is admissible (cf. [1]) if $\lim_{j \to \infty} x^{(j)} = O$ and there exist $a>1$ and $c>0$ such that $x^{(j)} \in \Gamma(O, a)$ and $|x^{(j+1)}| > c|x^{(j)}|$ for every j.

The aim of this note is to prove the following theorem.

THEOREM. *Let $\alpha < 3$ and u be a Green potential of order α in D. Then the following statements are equivalent:*

(i) *For $1 \leqq p < n/(n-\alpha)$, $x_n |x|^{1-\alpha} u(x)$ has limit zero in L^p at O.*

(ii) *There is an admissible sequence $\{x^{(j)}\}$ in D such that*

$$\lim_{j \to \infty} |x^{(j)}|^{2-\alpha} u(x^{(j)}) = 0.$$

(iii) *The function $x_n^{-1} |x|^{3-\alpha} u(x)$ has minimally α-semi-fine limit zero at O.*

In case $\alpha = 2$, this theorem was proved partly by Lelong-Ferrand [3] and Rippon [4].

2. Proof of the theorem

The following lemma can be proved by elementary calculation.

LEMMA 1. *There exist positive constants c_1, c_2 and $c(\varepsilon)$, $0<\varepsilon<1$, such that*

$$c_1 \frac{x_n y_n}{|x-y|^{n-\alpha} |\bar{x}-y|^2} \leqq G_\alpha(x, y) \leqq c_2 \frac{x_n y_n}{|x-y|^{n-\alpha} |\bar{x}-y|^2} \qquad \text{in case } \alpha < n,$$

$$c_1 \frac{x_n y_n}{|\bar{x}-y|^2} \leqq G_n(x, y) \leqq c(\varepsilon) \frac{x_n y_n}{|x-y|^{\varepsilon} |\bar{x}-y|^{2-\varepsilon}}$$

for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in D.

For a Green potential $u = G_\alpha(\cdot, \mu)$, we set

$$u_1(x) = \int_{\{y \in D;\, |x-y| \geqq |x|/2\}} G_\alpha(x, y) d\mu(y),$$

$$u_2(x) = \int_{\{y \in D;\, |x-y| < |x|/2\}} G_\alpha(x, y) d\mu(y).$$

LEMMA 2. *Let* $\alpha<3$. *If*

$$\text{(A)} \qquad \lim_{r\downarrow 0} r^{1-n}\int_{B_+(O,r)} y_n d\mu(y) = 0,$$

then $\lim_{x\to O} x_n^{-1}|x|^{3-\alpha}u_1(x) = 0$.

PROOF. For $\delta>0$, set $\varepsilon(\delta)=\sup_{0<r\leqq\delta} r^{1-n}\int_{B_+(O,r)} y_n d\mu(y)$. Then by Lemma 1,

$$\limsup_{x\to O} x_n^{-1}|x|^{3-\alpha}u_1(x) \leqq \text{const.} \limsup_{x\to O} |x|^{3-\alpha}\int_{B_+(O,\delta)} \frac{y_n d\mu(y)}{(|x|+|y|)^{n-\alpha+2}}$$

$$\leqq \text{const. } \varepsilon(\delta)$$

for $\delta>0$, which implies that $\lim_{x\to O} x_n^{-1}|x|^{3-\alpha}u_1(x)=0$.

LEMMA 3. *If* (A) *in the above lemma holds, then* $x_n|x|^{1-\alpha}u_2(x)$ *has limit zero in* L^p, $1\leqq p<n/(n-\alpha)$, *at* O.

PROOF. We shall prove only the case $\alpha<n$, because the case $\alpha=n$ can be proved similarly. It sufficies to show that

$$\lim_{r\downarrow 0} r^{-n}\int_{\tilde{B}(r)} [x_n|x|^{1-\alpha}u_2(x)]^p dx=0,$$

where $\tilde{B}(r)=B_+(O, r)-B_+(O, r/2)$. By Lemma 1 and Minkowski's inequality ([6; Appendices A.1]), we obtain

$$\left\{r^{-n}\int_{\tilde{B}(r)} [x_n|x|^{1-\alpha}u_2(x)]^p dx\right\}^{1/p}$$
$$\leqq c_2 2^{|1-\alpha|} r^{1-\alpha-n/p}\left\{\int_{\tilde{B}(r)}\left(\int_{B_+(O,2r)}|x-y|^{\alpha-n}y_n d\mu(y)\right)^p dx\right\}^{1/p}$$
$$\leqq c_2 2^{|1-\alpha|} r^{1-\alpha-n/p}\int_{B_+(O,2r)}\left\{\int_{\tilde{B}(r)}|x-y|^{p(\alpha-n)}dx\right\}^{1/p} y_n d\mu(y)$$
$$\leqq \text{const. } r^{1-n}\int_{B_+(O,2r)} y_n d\mu(y),$$

which tends to zero as $r\downarrow 0$ by (A).

LEMMA 4. *If* (A) *in Lemma* 2 *hods, then* $x_n^{-1}|x|^{3-\alpha}u_2(x)$ *has minimally* α*-semi-fine limit zero at* O.

PROOF. Consider the sets

$$E_j = \{x\in D;\ 2^{-j}\leqq |x| < 2^{-j+1},\ x_n^{-1}u_2(x)\geqq a_j^{-1}2^{j(3-\alpha)}\}$$

for $j=1, 2,\ldots$, where $\{a_j\}$ is a sequence of positive numbers such that $\lim_{j\to\infty} a_j=$

∞ but $\lim_{j\to\infty} a_j 2^{j(n-1)} \int_{B_+(O,2^{-j+2})} y_n d\mu(y)=0$. If ν is a non-negative measure on D such that $S_\nu \subset E_j$ and $k_\alpha(x, \nu) \leqq 1$ for $x \in D$, then we have

$$\int_D d\nu(x) \leqq a_j 2^{-j(3-\alpha)} \int x_n^{-1} u_2(x) d\nu(x)$$
$$\leqq a_j 2^{-j(3-\alpha)} \int_{B_+(O,2^{-j+2})} k_\alpha(y, \nu) y_n d\mu(y)$$
$$\leqq a_j 2^{-j(3-\alpha)} \int_{B_+(O,2^{-j+2})} y_n d\mu(y).$$

Hence by the definition of C_{k_α},

$$C_{k_\alpha}(E_j) \leqq a_j 2^{-j(3-\alpha)} \int_{B_+(O,2^{-j+2})} y_n d\mu(y),$$

so that

$$\lim_{j\to\infty} 2^{j(n-\alpha+2)} C_{k_\alpha}(E_j) = 0.$$

Setting $E = \cup_{j=1}^{\infty} E_j$, we see that E is minimally α-semi-thin at O on account of the countable subadditivity of C_{k_α}, and

$$\limsup_{x\to O, x\in D-E} x_n^{-1} |x|^{3-\alpha} u_2(x) \leqq \limsup_{j\to\infty} 2^{|3-\alpha|} a_j^{-1} = 0.$$

The proof of our lemma is thus completed.

We are now ready to prove the theorem.

PROOF OF THE THEOREM. If the statement (i) holds, then for any $a>1$, we can find a sequence $\{x^{(j)}\}$ such that $x^{(j)} \in \Gamma(O, a) \cap B_+(O, 2^{-j+1}) - B_+(O, 2^{-j})$ and $\lim_{j\to\infty} |x^{(j)}|^{2-\alpha} u(x^{(j)}) = 0$. Since the sequence $\{x^{(j)}\}$ is admissible, (ii) follows.

If the statement (ii) holds, then Lemma 1 gives

$$|x^{(j)}|^{1-n} \int_{B_+(O,|x^{(j)}|)} y_n d\mu(y) \leqq \text{const.}\ |x^{(j)}|^{2-\alpha} u(x^{(j)}),$$

so that

$$\lim_{j\to\infty} |x^{(j)}|^{1-n} \int_{B_+(O,|x^{(j)}|)} y_n d\mu(y) = 0,$$

which implies (A) since $\{x^{(j)}\}$ is admissible. Thus (ii) implies (iii) by Lemmas 2 and 4, and (i) by Lemmas 2 and 3.

To prove that (iii) implies (ii), it sufficies to note that $C_{k_\alpha}(\Delta(a, r) - \Delta(a, r/2)) = r^{n-\alpha+2} C_{k_\alpha}(\Delta(a, 1) - \Delta(a, 1/2))$ and $0 < C_{k_\alpha}(\Delta(a, 1) - \Delta(a, 1/2)) < \infty$ for any $a>1$ and $r>0$, where $\Delta(a, r) = \Gamma(O, a) \cap B_+(O, r)$.

3. Remarks

(a) Each of (i), (ii) and (iii) in the theorem is equivalent to condition (A).

(b) Let $0 \leqq \beta \leqq 2$ and $\beta < \alpha < 3$. Then each of (A), (i), (ii) and (iii) is equivalent to the following:

(i)′ *For* p, $1 \leqq p < n/(n-\alpha+\beta)$, $x_n^{1-\beta}|x|^{1+\beta-\alpha}u(x)$ *has limit zero in* L^p *at* O.

For this it suffices to show the next lemma.

LEMMA 3′. *If* (A) *holds, then* $x_n^{1-\beta}|x|^{1+\beta-\alpha}u_2(x)$ *has limit zero in* L^p, $1 \leqq p < n/(n-\alpha+\beta)$, *at* O.

Lemma 3′ can be proved in the same way as Lemma 3, if one notes

$$\left\{\int_{B_+(O,r)} \frac{x_n^{p(2-\beta)}}{|x-y|^{p(n-\alpha)}|\bar{x}-y|^{2p}}\, dx\right\}^{1/p} \leqq \text{const.} \left\{\int_{B_+(O,r)} |x|^{p(\alpha-\beta-n)} dx\right\}^{1/p} = \text{const.}\ r^{\alpha-\beta-n+n/p}.$$

(c) If $\alpha < 3$ and $G_\alpha(\cdot, \mu) \not\equiv \infty$, then $\lim_{r\downarrow 0} r^{1-n}\int_{B_+(\xi,r)} y_n d\mu(y) = 0$ for almost every $\xi \in \partial D$, so that $x_n|x-\xi|^{1-\alpha}G_\alpha(x, \mu)$ has limit zero in L^p, $1 \leqq p < n/(n-\alpha)$, at almost every $\xi \in \partial D$.

In fact, define a measure λ_δ on R^{n-1} by

$$\lambda_\delta(e) = \int_{e\times(0,\delta)} y_n d\mu(y) \qquad \text{for a Borel set} \quad e \subset R^{n-1}$$

and note, by a well-known theorem from the theory of integration, that

$$\lim_{r\downarrow 0} r^{1-n}\lambda_\delta(\{x' \in R^{n-1};\ |x'-\xi'| < r\})$$

exists and is finite for almost every $\xi' \in R^{n-1}$. From Fatou's theorem it follows that for almost every $\xi' \in R^{n-1}$,

$$\lim_{\delta\downarrow 0} \lim_{r\downarrow 0} r^{1-n}\lambda_\delta(\{x' \in R^{n-1};\ |x'-\xi'| < r\}) = 0,$$

which implies that $\lim_{r\downarrow 0} r^{1-n}\int_{B_+(\xi,r)} y_n d\mu(y) = 0$, $\xi = (\xi', 0)$.

(d) If a Borel set E in D is minimally 2-semi-thin at O, then we can find a positive superharmonic function u in D such that $\lim_{x\to O, x\in E} x_n^{-1}|x|u(x) = \infty$ and $\liminf_{x\to O, x\in\Gamma(O,a)} u(x) = 0$ for any $a > 1$.

To show this, we need the following lemma, which can be proved by using [2: Théorème 7.8].

LEMMA 5. *For a Borel set A in D, we have*

$$C_{k_\alpha}(A) = \inf \lambda(\bar{D}),$$

where the infimum is taken over all non-negative measures λ *on* $\bar{D}$ *such that* $k_\alpha(\lambda, y) = \int k_\alpha(x, y)d\lambda(x) \geqq 1$ *for every* $y \in A$.

Let E be a Borel set in D which is minimally 2-semi-thin at O, and take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\lim_{j\to\infty} a_j 2^{nj} C_{k_2}(E_j) = 0,$$

where $E_j = E \cap B_+(O, 2^{-j+1}) - B_+(O, 2^{-j})$. By Lemma 5, for each j we can find a non-negative measure λ_j on $\bar{D}$ such that $\lambda_j(\bar{D}) < C_{k_2}(E_j) + a_j^{-2}2^{-nj}$ and $k_2(\lambda_j, z) \geqq 1$ for $z \in E_j$. Denoting by λ'_j the restriction of λ_j to the set $\{x \in \bar{D};\ 2^{-j-1} < |x| < 2^{-j+2}\}$, we have for $z \in E_j$,

$$\begin{aligned} k_2(\lambda'_j, z) &\geqq 1 - c_2 \int_{\{|x| \leqq 2^{-j-1}\} \cup \{|x| \geqq 2^{-j+2}\}} |x-z|^{-n} d\lambda_j(x) \\ &\geqq 1 - c_2 4^n \{2^{nj} C_{k_2}(E_j) + a_j^{-2}\}. \end{aligned}$$

Set $\lambda = \sum_{j=1}^{\infty} a_j 2^j \lambda'_j$, and define $u(z) = z_n k_2(\lambda, z)$, $z \in D$. Then

$$\liminf_{z\to O, z\in E} z_n^{-1}|z|u(z) \geqq \liminf_{j\to\infty} a_j\{1 - c_2 4^n(2^{nj} C_{k_2}(E_j) + a_j^{-2})\} = \infty.$$

Further we note the following properties:

(d_1) The function u satisfies that $u \not\equiv \infty$, and is of the form

$$u(z) = G_2(z, \mu) + \text{a Poisson integral},$$

so that u is superharmonic in D.

(d_2)
$$\lim_{r\downarrow 0} r^{1-n} \int_{|x| \leqq r} d\lambda(x) = 0.$$

From (d_2) and the proof of the theorem, it follows that for any $a > 1$, $\liminf_{z\to O, z\in\Gamma(O,a)} u(z) = 0$. Thus u has the required properties.

If E is a Borel subset of a cone $\Gamma(O, a)$ which is minimally 2-semi-thin at O, then there exists a Green potential $G_2(\cdot, \mu)$ such that $\lim_{x\to O, x\in E} G_2(x, \mu) = \infty$.

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