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On positive solutions of second order elliptic partial differential equations

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The paper studies necessary and sufficient conditions for the existence of positive solutions for the equation $-\Delta u+pu=0$ on a domain G in terms of the existence of a solution of a related Riccati inequality. Certain results from ordinary differential equations are extended to this setting providing sufficient conditions for positive solutions.

1. Introduction

The purposes of this paper are to extend known results on the existence of positive solutions, on disconjugacy and nonoscillation of the ordinary differential equation (o.d.e.)

$$u''(t) + p(t)u(t) = 0 \tag{1.1}$$

on a nontrivial interval I, which is possibly unbounded, to an analogous p.d.e.

$$-\Delta u(x, y) + p(x, y)u(x, y) = 0 \tag{1.2}$$

on a nontrivial domain, i.e., a connected open set in $\boldsymbol{R}^2$, or possibly on its closure. The coefficient function p in (1.1) is assumed to be continuous, while in (1.2) we assume a local Hölder continuity.

The results readily extend from equations (1.2) in $\boldsymbol{R}^2$ to those in $\boldsymbol{R}^n$, $n>2$; and where the second order Laplacian term $-\Delta u$ is replaced by a more general second term $-\nabla\cdot(r\nabla u)$, where $n>1$ and r is an $n\times n$ symmetric real matrix valued function which is in C^1, and which is uniformly positive definite.

The reason for the minus in (1.2), which is not present in (1.1), is the currently available literature in o.d.e.'s and in p.d.e.'s where these respective forms predominate.

The results to be presented in this paper are obtained by extending the following theorem of M. Bôcher [3] to p.d.e.'s.

THEOREM 1.1 (Bôcher). *Equation* (1.1) *has a positive solution on a non-*

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trivial interval I if and only if threre is a C^1 solution w, on I, of

$$w'(t) + w^2(t) + p(t) \leq 0. \tag{1.3}$$

A. Wintner [15] used

$$w(t) = \int_t^1 p(s)ds + \begin{cases} t^{-1}-4, & 0 < t \leq 1/2, \\ (t-1)^{-1}, & 1/2 < t < 1, \end{cases} \tag{1.4}$$

under the assumption of

$$0 \leq \int_t^1 p(s)ds \leq 4, \qquad 0 \leq t \leq 1, \tag{1.5}$$

to show that w satisfies (1.3) on $I=(0, 1)$, and with a modification he concluded (1.3) has a solution on $[0, 1]$ and, hence, (1.1) has a positive solution on $[0, 1]$. Condition (1.5) relates to a well-known result of Lyapunov, cf. P. Hartman [7], where (1.5) is replaced by

$$(b-a)\int_a^b p^+(s)ds \leq 4, \tag{1.6}$$

when the interval $[0, 1]$ is replaced by a more general interval $[a, b]$ where $p^+(s)=\max\{0, p(s)\}$.

Wintner also showed that if $P(t)=\int_t^\infty p(s)ds$ converges on $I=(a, \infty)$ and if

$$P^2(t) \leq p(t)/4 \tag{1.7}$$

holds on I then $w(t)=2P(t)$ satisfies (1.3) on I. Actually as anticipated, if not in fact known, by Wintner, if $P(t)$ is any antiderivative of $-p(t)$ on I satisfying (1.7) then $w(t)=2P(t)$ still satisfies (1.3) on I and, hence, (1.1) is disconjugate and, hence, nonoscillatory on I.

The concept of nonoscillation on $I=(a, \infty)$ for (1.1) is equivalent to eventual disconjugacy on I in the Calculus of Variations sense which is equivalent to the existence of an eventually strictly positive solution on I, cf. W. T. Reid [12].

The results above as well as others due to A. Wintner [15], P. Hartman [7], and Z. Opial [10] will be extended by first extending the theorem of M. Bôcher to a p.d.e. setting. In our p.d.e. setting, (1.3) will be replaced by

$$-\operatorname{div} W(x, y) - W(x, y)\cdot W(x, y) + p(x, y) \geq 0 \tag{1.8}$$

where W is an exact C^1 vector field, i.e.,

$$\partial W_1/\partial y = \partial W_2/\partial x, \tag{1.9}$$

Finally we mention that one of our first applications will be to establish that the existence of positive solutions, $u(\cdot, y)$, of the family of o.d.e.'s

$$-u_{xx}(x, y) + p(x, y)u(x, y) = 0,$$

indexed by y, on horizontal cross-sections of a domain in $\boldsymbol{R}^2$ proves a sufficient condition for the existence of a positive solution of (1.2) on that domain.

2. Bôcher's theorem extended

The following theorem is a special case of the main theorem of W. Moss and J. Piepenbrink [9] which relates as well to work of W. Allegretto [1]. In what follows,

$$L[\phi] \equiv -\Delta\phi + p\phi,$$

and $\Omega \subseteq \boldsymbol{R}^2$ is a domain, i.e., connected open set, on which p is locally Hölder continuous.

THEOREM 2.1 (Moss and Piepenbrink). *If G is a subdomain of Ω and if for every bounded subdomain D of G, with $\bar{D} \subseteq G$, condition*

$$\text{(D)} \qquad \inf_{\phi \in A} \iint_{\bar{D}} \phi L[\phi] dxy \Big/ \iint_{\bar{D}} \phi^2 dxdy > 0$$

$(A = \{\phi \in C_0^\infty(D) : \iint_{\bar{D}} \phi^2 dxdy \neq 0\})$ *holds, then there exists a positive solution $v \in C^2(G)$ of $L[v] = 0$ on G.*

REMARK 2.2. The results of Moss and Piepenbrink also allow G in Theorem 2.1 to be replaced by its closure $\bar{G} \subseteq \Omega$, provided its boundary is smooth. In the case where G is bounded and smooth this is handled by their Lemma 2.3 where the solution $v \in C^2(\bar{G})$. The argument they present for the proof of their Theorem 2.1 shows how this can be extended if G is unbounded and smooth. Results from the literature allow the boundary of G to be piecewise smooth also, in which case $v \in C^2(G) \cap C^0(\bar{G})$. We further remark that in the theorem above, no assumptions are made concerning the regularity of the boundary of G. Also when G is unbounded it need not be an "exterior domain," i.e., contain the complement of some closed ball, such as frequently considered in the literature, cf., e.g., W. Allegretto [1].

From the above the following fundamental result will follow.

THEOREM 2.3. *$L[v] \geq 0$ has a positive $C^2(G)$ solution v on a subdomain G of Ω if and only if $L[u] = 0$ has one. Furthermore, in the preceding sentence if $\bar{G} \subseteq \Omega$ is smooth, we may replace G by $\bar{G}$, while if $\bar{G} \subseteq \Omega$ is piecewise smooth we* *may replace G by $\bar{G}$, but we change $C^2(G)$ to $C^2(G) \cap C^0(\bar{G})$.*

PROOF. Clearly $L[u]=0$ implies $L[u]\geq 0$, so one direction of the equivalence is trivial.

On the other hand, suppose $L[v]\geq 0$ has a positive solution $v\in C^2(G)$ on $G\subseteq\Omega$. The conclusion will follow from Theorem 2.1 provided we establish condition (D).

Assume then that D is a bounded subdomain of G and that F is some bounded smooth subdomain satisfying $D\subseteq F\subseteq\bar{F}\subseteq G$.

Now for $\phi\in C_0^\infty(D)$, let $h=\phi/v\in C^2(\bar{F})$. We apply the "Picone" identity

$$\|\nabla\phi\|^2 + p\phi^2 = \|v\nabla h\|^2 + h^2 vL[v] + \nabla\cdot[\phi(h\nabla v)],$$

and the Divergence Theorem of Gauss to obtain

$$\iint_D \phi L[\phi]dxdy = \iint_F \phi\|^2 + p\phi^2 dxdy = \iint_F v^2\|\nabla h\|^2 dxdy + \iint_F h^2 vL[v]dxdy$$

$$\geq C[\min_F v^2]\cdot[\max_F v^2]^{-1}\iint_F \phi^2 dxdy = K\iint_D \phi^2 dxdy,$$

for some $K>0$, where C is the smallest positive eigenvalue of $-\Delta$ on $\bar{F}$. Thus condition (D) of Theorem 2.1 holds, so there is a positive solution $u\in C^2(G)$ of $L[u]=0$ on G.

Remark 2.2 may be used to establish the rest of the reuslt.

COROLLARY 2.4. *$L[v]\geq 0$ has a positive $C^2(G)$ solution on a domain $G\subseteq\Omega$ if and only if condition* (D) *of Theorem* 2.1 *holds for each bounded subdomain D of G.*

We now indicate how disconjugacy of a family of ordinary differential equations may be used as a sufficient condition for a positive solution of (1.2).

Let $J\subseteq\boldsymbol{R}$ be an open interval and $\bar{J}$ its closure. Suppose $a, b\colon \bar{J}\to\boldsymbol{R}$ are two C^1 functions with $a(y)<b(y)$ for all $y\in\bar{J}$. Let $G=\{(x, y)\colon y\in J \text{ and } a(y)<x<b(y)\}$ and $\bar{G}$ be its closure. Clearly the boundary of G is piecewise smooth.

THEOREM 2.5. *Suppose p is locally Hölder continuous on $\bar{G}$, as defined above. Suppose that for each $y\in\bar{J}$ the ordinary differential equation*

$$-u_{xx}(x, y) + p(x, y)u(x, y) = 0 \tag{2.1}$$

has a positive solution on $I_y=[a(y), b(y)]$.

Then (1.2) *has a positive solution $u\in C^2(G)\cap C^0(\bar{G})$ on $\bar{G}$.*

REMARK 2.6. Protter [11] uses a similar method of cross-sections in establishing lower bounds on the first eigenvalue of an elliptic equation. We note also that we are making no assumptions of the continuity of the family of solutions $u(\cdot, y)$ as y varies in (2.1).

PROOF. By applying results from W. T. Reid [12], in particular Exercise 5 of Chapter V, together with continuity of solutions of o.d.e.'s with respect to initial conditions and parameters, for any $y_0 \in \bar{J}$ there is a $\delta>0$ and a $K_{y_0}>0$ such that for each $y \in \bar{J}$ with $|y-y_0|<\delta$ it follows that

$$\int_{a(y)}^{b(y)} \eta_x^2(x, y) + p(x, y)\eta^2(x, y)dx > K_{y_0} \int_{a(y)}^{b(y)} \eta^2(x, y)dx \tag{2.2}$$

holds for any function $\eta \in C_0^\infty(G)$.

This may be used to establish condition (D) of Theorem 2.1.

Let D be any bounded subdomain of G. Let J_0 be any compact subinterval of J so that $D \subseteq \bar{F} = \{(x, y)\colon y \in J_0 \text{ and } a(y) \leq x \leq b(y)\} \subseteq \bar{G}$.

Now the boundary of $\bar{F}$ is piecewise smooth and $\bar{F}$ is compact. By (2.2) and a compactness argument there is a $K>0$ such that

$$\int_{y \in J_0} \int_{a(y)}^{b(y)} \eta_x^2 + \eta_y^2 + p\eta^2 dx dy \geq K \int_{y \in J_0} \int_{a(y)}^{b(y)} \eta^2 dx dy \tag{2.3}$$

holds for any $\eta \in C_0^\infty(\bar{F})$. Since $C_0^\infty(D) \subseteq C_0^\infty(\bar{F})$, we have established condition (D) Theorem 2.1, as before the left side of (2.3) may be replaced by the same integral of $\eta L[\eta]$. Remark 2.2 is used to reach our conclusion.

A simple corollary now follows from an application of the inequality of Lyapunov, recall (1.6). Here we have $p^-(x, y) = \max\{0, -p(x, y)\}$.

COROLLARY 2.7. *Let $\bar{G}$, a, b, p and $\bar{J}$ be as in the theorem except assume*

$$[b(y)-a(y)] \int_{a(y)}^{b(y)} p^-(x, y)dx \leq 4 \tag{2.4}$$

holds for all $y \in \bar{J}$. Then $L[u]=0$ has a positive solution on $\bar{G}$.

Also, if rather than (2.4) *we assume J is unbounded above and*

$$\limsup\, [b(y)-a(y)] \int_{a(y)}^{b(y)} p^-(x, y)dx < 4 \quad as \quad y \longrightarrow \infty, \tag{2.5}$$

then $L[u]=0$ has an "eventually positive" solution as $y \to \infty$, i.e., a positive solution on $\bar{G} \cap (\boldsymbol{R} \times [y_0, \infty))$ for some $y_0 \in J$.

Since the Laplacian operator is invariant under rotation of axes in the domain space of solutions of (1.2), if G is a piecewise-smooth compact convex set then cross-sections of G parallel to a fixed line are closed line segments. Let

$\{\ell_{t,\alpha}\}_{t=t_1(\alpha)}^{t=t_2(\alpha)}$ denote a one parameter family of such line segments having angle of inclination α relative to the positive X axis, and which cover G as t varies from $t_1(\alpha)$ to $t_2(\alpha)$. By Corollary 2.7, we have a second corollary.

COROLLARY 2.8. *Let G be a piecewise-smooth compact convex set and $\{\ell_{t,\alpha}\}$ be as just described with $L(\ell_{t,\alpha})$ = length of $\ell_{t,\alpha}$. If*

$$\min_{\alpha\in[0,\pi]} \max_{t\in[t_1(\alpha),t_2(\alpha)]} L(\ell_{t,\alpha}) \int_{\ell_{t,\alpha}} p^- \leq 4 \tag{2.6}$$

holds, then $L[u]=0$ has a positive solution on G.

We note that (2.6) implicitly bounds the lengths of linear crossections of G in some direction. We may thus interpret Corollary 2.8 as saying that if a piecewise-smooth compact convex set G is sufficiently narrow then (1.2) has a positive solution in it.

A slight modification in the proof of Theorem 2.5 allows us to expand G from a strip to a sector or half space.

Assume now J is an open interval which is at least unbounded above, and possibly below. Assume $a: J\to \boldsymbol{R}$ is continuous and let $H=\{(x, y): y\in J$ and $a(y)<x\}$.

COROLLARY 2.9. *Suppose that p is locally Hölder continuous on H and that for each $y\in J$ (2.1) has a positive solution on $I_y=(a(y), \infty)$.*
Then (1.2) has a positive solution $u\in C^2(H)$ on H.

PROOF. As in the proof of the theorem, we establish condition (D) of Theorem 2.1. Since the modifications here are slight we omit the proof.

As corollaries of this result conditions such as (1.7) of Wintner or a host of other disconjugacy results may be employed, cf. D. Willett [14] and references therein. For one example, we provide the following, related to Wintner's result.

COROLLARY 2.10. *Let H, a, p, J and I_y be as in Corollary 2.9 except assume*

$$\left(\int_x^\infty p(\tau, y)d\tau\right)^2 \leq -p(x, y)/4, \qquad \textit{for each } x\in I_y, \tag{2.7}$$

holds for each y in J. Then $L[u]=0$ has a positive solution on H.

A change of variable allows us to compare Corollary 2.7 with a result of Glazman [5, p. 158], see also Swanson [13, p. 279], where rather than cross-sectional integrals as in (2.4) and (2.5), cross-sectional infima of p are considered.

Under $x=e^\rho \cos\theta$, $y=e^\rho \sin\theta$ it follows that

$$(x^2+y^2)(-u_{xx}-u_{yy}+pu) = -\hat{u}_{\rho\rho} - \hat{u}_{\theta\theta} + e^{2\rho}\hat{p}(\rho, \theta)\hat{u} \tag{2.8}$$

holds, where $\hat{u}(\rho, \theta)=u(e^{\rho}\cos\theta, e^{\rho}\sin\theta)=u(x, y)$ and, likewise, $\hat{p}(\rho, \theta)=p(x, y)$. Thus, (2.8) transforms (1.2) into a similar equation with the new coefficient $e^{2\rho}\hat{p}$.

This transformation associates unbounded circular sectors centered at (0, 0), such as those considered by Allegretto [2], with unbounded horizontal strips in the (ρ, θ) plane having width being the angle measure, less than 2π.

Allegretto considers, after a suitable rotation, that $C_\alpha=\{(x, y)\in \boldsymbol{R}^2 : x>\alpha(x^2+y^2)^{1/2}\}$ for some α, $0<\alpha<1$. With $p^*(\rho)=\inf_{x^2+y^2=e^{2\rho}} p(x, y)$, Theorem 1 of Allegretto [2] states that if

$$\liminf_{\rho\to\infty} e^{2\rho}p^*(\rho)>-1/(1-\alpha) \tag{2.9}$$

then (1.2) is *nonoscillatory on* C_α, i.e., there is a $\rho_0>0$ such that (1.2) has no nodal subdomain on $x^2+y^2\geq e^{2\rho_0}$, or equivalently condition (D) of Theorem 1.2 holds there, which by Corollary 2.4 is equivalent to the existence of a positive solution there.

Condition (2.9) above in turn improves the previously mentioned result of Glazman.

We note that the radian measure, 2β, of the angle of the cone C_α, $0<\alpha<1$ in $\boldsymbol{R}^2$ is

$$2\beta = 2\arctan(1-\alpha^2)^{1/2}/\alpha \tag{2.10}$$

Condition (2.4) of Corollary 2.7 states that if

$$\limsup_{\rho\to\infty} 2\beta\int_{-\beta}^{\beta} e^{2\rho}\hat{p}^-(\rho, \theta)d\theta < 4 \tag{2.11}$$

then (1.2) is nonoscillatory on C_α.

In the special case where $\hat{p}(\rho, \theta)\equiv p^*(\rho)\leq 0$ is independent of θ for $-\beta\leq\theta\leq\beta$, then (2.11) is not as sharp as (2.9), though clearly (2.11) applies to a much broader class of coefficient functions. In this special case (2.11) reduces to

$$\liminf_{\rho\to\infty} e^{2\rho}p^*(\rho) > -\beta^{-2}. \tag{2.12}$$

Now $\beta^2\geq\sin^2\beta=1-\alpha^2\geq 1-\alpha$ holds for all $0<\alpha<1$, so $-\beta^{-2}\geq-(1-\alpha)^{-1}$ and (2.12) is more restrictive than (2.9); however, since $\beta^2\leq\tan^2\beta=(1-\alpha^2)/\alpha^2$, we see $1\leq\beta^2(1-\alpha)^{-1}\leq(1+\alpha)\alpha^{-2}$ holds, so for small β, i.e., α near 1, e.g., for $2/3<\alpha<1$ the quotient is bounded above by 4.

In closing this section we note that a simply connected compact set G spiraling around the origin in $\boldsymbol{R}^2$ may be considered as a vertical type strip $\hat{G}$ in (ρ, θ). An illustration of a spiraling nodal subdomain associated with the n-th gigenvalue of $L[u]\equiv-\Delta u$ under zero boundary conditions is shown on p. 456 of Courant and Hilbert [4]. They point out there how higher eigenvalues may still have as few few as two nodal subdomains for corresponding eigenfunctions.

3. Riccati inequalities

Riccati inequalities and corresponding equations related to the linear operator

$$L[u](x, y) \equiv -\Delta u(x, y) + p(x, y)u(x, y) \tag{3.1}$$

are not new to the literature. Indeed, e.g., P. K. Wong [16, 17] considers both. He also has extensive bibliographies indicating previous occurrences in the literature. While Wong even considers systems of such inequalities, he does not appear to link the concepts of the equivalence as we have in Theorem 2.3, and as we shall in the corresponding Theorem 3.1.

As before we let $\Omega \subseteq \boldsymbol{R}^2$ be a domain on which p is locally Hölder continuous. Let $G \subseteq \Omega$ be a subdomain. Suppose $u \in C^2(G)$ and $u \neq 0$ on G.

Let $w(x, y) = \log|u(x, y)|$, so that $|u(x, y)| = e^{w(x,y)}$ holds. Let $W(x, y) = \nabla w(x, y) = \nabla u(x, y)/u(x, y)$, so that $\nabla u = uW$ holds. We then have $\Delta u = \nabla \cdot (u \nabla u) = u(\nabla \cdot W + W \cdot W)$, so that with

$$K[W] \equiv -\nabla \cdot W - W \cdot W + p \equiv K[W; p], \tag{3.2}$$

it follows that W is a $C^1(G)$ vector field and both

$$L[u](x, y) = u(x, y)K[W](x, y) \tag{3.3}$$

and the exactness condition

$$\partial W_1(x, y)/\partial y = \partial W_2(x, y)/\partial x \tag{3.4}$$

hold on G.

The process just indicated may also be reversed, where if we start with an exact $C^1(G)$ vector field W then there always is a $C^1(G)$ scalar function w with $\nabla w = W$ and $u = \exp w$ defines a nonzero $C^2(G)$ function for which (3.3) holds.

As a trivial consequence of Theorem 2.3 we have our first result.

THEOREM 3.1. *$K[W] \geq 0$ has an exact $C^1(G)$ vector field solution W on a subdomain $G \subseteq \Omega$ if and only if $K[V] = 0$ has one; and this holds if and only if $L[u] = 0$ has a positive $C^2(G)$ solution on G.*

In what follows we always use capital letters,

$$P \equiv \operatorname{div}^{-1} p \tag{3.5}$$

to signify that P is an exact C^1 vector field and that $\nabla \cdot P = p$. Such P exists on any domain G on which Poisson's equation, $\Delta \tilde{p} = p$, is solvable, and $P = \nabla \tilde{p}$ provides such functions P, which are unique to within the gradient of any harmonic function h, i.e., $P=\nabla(\tilde{p}+h)$. In Section 4 of this paper we indicate a method for computing a specific inverse divergence.

LEMMA 3.2. *$K[W]\geq 0$ has a C^1 exact vector field solution W on a domain if and only if*

$$\nabla\cdot V \geq (V-P)\cdot(V-P) = \|V-P\|^2 \tag{3.6}$$

has one.

PROOF. Let $V=P-W$, $W=P-V$. Then

$$K[W] = \nabla\cdot V-(P-V)\cdot(P-V). \tag{3.7}$$

As corollaries of the preceding we extend several results mentioned in the Introduction; recall (1.7) and (2.7).

COROLLARY 3.3. *If there exists $P=\mathrm{div}^{-1}\, p$ satisfying*

$$4P\cdot P \leq -p \tag{3.8}$$

on a domain G, then $L[u]=0$ has a positive C^2 solution on G.

PROOF. Let $V=-P$, so $W=2P$ is used in (3.7), so that (3.6) and (3.8) are equivalent in this setting.

COROLLARY 3.4. *If for some constant c it follows that*

$$\|P(x, y) + ((4x)^{-1}, c)\|^2 \leq (4x^2)^{-1} \tag{3.9}$$

holds on a domain G containing the half-space $x\geq x_0>0$, and contained in the half-space $x>0$, then $L[u]=0$ has a positive solution on G.

PROOF. Let $V(x, y)=(-(4x)^{-1}, -c)$. Then V is an exact C^1 vector field on the half-space $x>0$. Furthermore, V satisfies (3.6), which completes the proof.

Since rotation and translation of the domain coordinates of solutions does not affect the Laplacian, the previous corollary extends to other half-spaces. Also this result extends the o.d.e. result of A. Wintner [15] showing that

$$-3/4 \leq t\int_t^\infty p(s)ds \leq 1/4$$

or equivalently

$$\left|\int_t^\infty p(s)ds + 1/4t\right| \leq 1/2t$$

holding for all $t \in I=(t_0, \infty)$ implies (1.1) has a positive solution on I. Here it is clear that p may alternate in sign, which is not allowed in condition (1.7).

The following two results are extensions of those found in P. Hartman [6]. These results follow immediately by appealing to Theorem 3.1.

LEMMA 3.5. *If* $W=2^{-1}V+P$, $P=\operatorname{div}^{-1} p$ *then*

$$K[W; p] \geq 2^{-1}K[V; -4P\cdot P]. \tag{3.10}$$

THEOREM 3.6. *If* $K[V; -4P\cdot P]\geq 0$ *has an exact* C^1 *vector field solution on a domain* G *then so does* $K[W; p]\geq 0$ *and hence,* $L[u]=0$ *has a positive* C^2 *solution on* G.

The following generalizes a result of Z. Opial [10]. In what follows, $P=\operatorname{div}^{-1} p$, $\check{p}=P\cdot P$ and $\check{P}=\operatorname{div}^{-1}\check{p}$.

THEOREM 3.7. *If*

$$16\, \check{P}\cdot \check{P} \leq P\cdot P \tag{3.11}$$

holds on a domain G *then* $L[u]=0$ *has a positive* C^2 *solution on* G.

PROOF. Condition (3.11) holding implies, by Corollary 3.3, that $L[u; -4\check{p}]=0$ has a positive C^2 solution on G, so that by (3.3), $K[V; -4\check{p}]=0$ has a C^1 vector field solution. From (3.10) and Theorem 3.1 we now have our result.

In the o.d.e. setting, one advantage of Opial's result over Wintner's, here represented by (3.11) and (3.8), respectively, is that in (3.8), p must be nonpositive while in (3.11), p may be allowed to change sign. Of course, in o.d.e.'s the specific antiderivatives (inverse gradients) are usually expressed in integral form such as those following Corollary 3.4.

REMARK 3.8. Further potential application of the equivalence in Theorem 3.1 is in the area of oscillation of solutions of (1.2), cf. W. Allegretto [1] for the concepts, where oscillation on an unbounded domain is equvialent to the nonexistence of an eventually positive solution exterior to some large ball. The analog of Theorem 3.1 in o.d.e. theory has proved to be very helpful in this regard, cf. D. Willett [14]. The shape and structure of the domain will likely be of much greater significance in these applications than we have presented in this section.

Finally, before we return in the next section to the analog of the result of Wintner, (1.4) and (1.5) of the Introduction, we note that the actual theorem of Moss and Piepenbrink [9], given in part by Theorem 2.1, applies equally well to

$$-\nabla\cdot(r\nabla u) + pu = 0 \tag{3.12}$$

and

$$-\nabla\cdot(r\nabla w)-(r\nabla w)r^{-1}(r\nabla w)+p\geq 0 \tag{3.13}$$

where $W=r\nabla w$ and r is a uniformly positive definite symmetric $n\times n$ matrix valued function whose first partial derivatives are locally Hölder continuous on a domain $\Omega\subseteq \boldsymbol{R}^n$. In this setting $P\equiv \operatorname{div}^{-1}(r, p)$ signifies that $P=r\nabla\tilde{p}$ and $\nabla\cdot P=p$, so that $r^{-1}P$ is exact.

The results of this section extend with little problem to this more general setting.

4. A specific inverse divergence and applications

In this section it becomes simpler to use the notation $x=(x_1, x_2)$ and $\xi=(\xi_1, \xi_2)$ as elements of a piecewise smooth bounded convex domain $G\subseteq\Omega\subseteq\boldsymbol{R}^2$, where as previously, Ω is a domain on which p is locally Hölder continuous.

By a result of K. Kreith [8] when G is smooth and $\tilde{K}(x, \xi)=(2\pi)^{-1}\log\|x-\xi\|$ then

$$\tilde{p}(x)=\int_{\bar{G}}\tilde{K}(x, \xi)p(\xi)d\xi \tag{4.1}$$

provides a solution of Poisson's equation, $\Delta\tilde{p}=p$ on $\bar{G}$. Clearly, by our regularity assumptions of p on Ω, if G is merely piecewise smooth then (4.1) provides a $C^2(G)\cap C^0(\bar{G})$ solution of Poisson's equation on G.

Thus with

$$K(x, \xi)=\nabla_x\tilde{K}(x, \xi)=(2\pi)^{-1}(x-\xi)/\|x-\xi\|^2, \tag{4.2}$$

we have that

$$P(x)\equiv\int_{\bar{G}}K(x, \xi)p(\xi)d\xi=(P_1(x_1, x_2), P_2(x_1, x_2)) \tag{4.3}$$

provides an exact C^1 vector field solution of $\nabla\cdot P=p$, i.e., an "inverse divergence" of p on G (and on $\bar{G}$ if G is smooth). (Recall (3.5) and the following.)

In what follows, capital letters P will always designate the inverse divergence of p given by (4.3). By adding the gradients of harmonic functions to $P(x)$ we obtain the most general inverse divergence of p.

While the inverse divergence (4.3) might appear to be inconsistent with the antiderivatives

$$\int_a^x p(t)dt \quad\text{or}\quad \int_x^b -p(t)dt \tag{4.4}$$

used in o.d.e. theory, by using Green's functions, $g(x, t)=-(b-a)^{-1}(b-x)(t-a)$ for $a\leq t\leq x\leq b$ and $g(x, t)=-(b-a)^{-1}(b-t)(x-a)$ for $a\leq x\leq t\leq b$, then

$$\tilde{p}(x) = \int_a^b g(x, t)p(t)dt, \quad \text{and} \quad P(x) = \int_a^b g_x(x, t)p(t)dt$$

provide solutions of $\tilde{p}''(x) = p(x)$ and $P'(x) = p(x)$, respectively, corresponding to (4.1) and (4.3), respectively. Furthermore, we have

$$(4.5) \qquad P(x) = \int_a^x (b-a)^{-1}(t-a)p(t)dt - \int_x^b (b-a)^{-1}(b-t)p(t)dt.$$

By adding the constants

$$\int_a^b (b-a)^{-1}(b-t)p(t)dt \quad \text{and} \quad \int_a^b -(b-a)^{-1}(t-a)p(t)dt$$

to $P(x)$ in (4.5), respectively, yields the antiderivatives in (4.4). The singularity of $g_x(x, t)$ is the important part of (4.5) for the o.d.e. setting as is the singularity for $K(x, \xi)$ given by (4.2) for the p.d.e. setting.

Now the result (1.4) and (1.5) of Wintner has a direct generalization to our p.d.e. setting; however, the inverse divergence formula (4.3) does not appear to be well suited for it. Instead, it is better suited to a rearrangement of Wintner's result. Indeed, if instead of (1.4) and (1.5) we define

$$(4.6) \qquad w(t) = \int_t^{1/2} p(s)ds + \begin{cases} t^{-1} - 2, & 0 < t \leq 2^{-1}, \\ (t-1)^{-1} + 2, & 2^{-1} < t < 1, \end{cases}$$

and assume

$$(4.7) \qquad 0 \leq \operatorname{sgn}(2^{-1} - t) \int_t^{1/2} p(s)ds \leq 2, \qquad 0 \leq t \leq 1,$$

respectively, then $w(t)$ still satisfies the Riccati inequality (1.3) on (0, 1). This result, as is the case for (1.5), is sharp in the sense that the constants on the right of these inequalities may not be increased without violating the conclusion that a solution of (1.3) exists for some coefficient p. The proof that (4.6) and (4.7) may replace the assumptions (1.4) and (1.5) of Wintner's result is clear from what we shall do here for the p.d.e. case.

THEOREM 4.1. *In* (4.3) *let* $\bar{G} = [0, 1] \times [0, 1]$. *Let G denote its interior. If*

$$(4.8) \qquad 0 \leq [\operatorname{sgn}(2^{-1} - x_i)]P_i(x) \leq 2, \qquad \text{for} \quad i = 1, 2,$$

holds for $x \in \bar{G}$ *then* (1.2) *has a positive solution on* $\bar{G}$.

PROOF. Let Z be the $C^1(G)$ vector field whose components are given by

$$(4.9) \qquad Z_i(x) = \begin{cases} x_i^{-1} - 2, & 0 < x_i \leq 2^{-1}, \\ (x_i - 1)^{-1} + 2, & 2^{-1} < x_i < 1, \end{cases} \qquad 0 < x_j < 1,$$

$$\text{for } i = 1, 2, j = 1, 2 \quad \text{and} \quad i \neq j.$$

Let $W(x) = P(x) + Z(x)$ on G. Then W is exact and we shall show

$$(4.10) \qquad -\nabla \cdot W(x) - W(x) \cdot W(x) + p(x) \geq 0$$

holds on G. This will be accomplished by establishing

$$(4.11) \qquad W_i^2(x) \leq -\partial Z_i/\partial x_i = \begin{cases} x_i^{-2}, & 0 < x_i \leq 2^{-1}, \\ (x_i - 1)^{-2}, & 2^{-1} < x_i < 1, \end{cases} \qquad 0 < x_j < 1,$$

$$\text{for} \quad i = 1, 2, \quad j = 1, 2 \quad \text{and} \quad i \neq j.$$

We see now that

$$(4.12) \qquad W_i^2(x) = P_i(x)\,[P_i(x) + 2Z_i(x)] + Z_i^2(x).$$

For $0 < x_i < 2^{-1}$, by (4.8) and (4.9) we have

$$P_i(x) + 2Z_i(x) \leq 2 + 2[x_i^{-1} - 2] = 2x_i^{-1} - 2,$$

where the expression on the right is positive.

We multiply the extremes of the last display by $P_i(x) \geq 0$ and then add $Z_i^2(x)$ to obtain

$$W_i^2(x) \leq 2[2x_i^{-1} - 2] + (x_i^{-1} - 2)^2 = x_i^{-2}$$

which establishes (4.11) in the case $0 < x_i \leq 2^{-1}$.

An analogous argument holds if $2^{-1} < x_i < 1$, except here $0 \geq P_i(x) \geq -2$, so that by going back to (4.12) we have

$$P_i(x) + 2Z_i(x) \geq -2 + 2(x_i - 1)^{-1} + 4 = 2(x_i - 1)^{-1} + 2,$$

where the expression on the right is negative. By multiplying by $P_i(x) \leq 0$ and adding $Z_i^2(x)$ we obtain

$$W_i^2(x) \leq -2[2(x_i - 1)^{-1} + 2] + [(x_i - 1)^{-1} + 2]^2 = (x_i - 1)^{-2}.$$

Thus (4.11) holds and, consequently, so does (4.10).

Finally, since p is continuous on $\bar{G}$ and $Z_i(x)$ tends to $+\infty$, $-\infty$ (in a uniform fashion for $0 < x_j < 1$, $j \neq i$) as x_i tends to 0 and 1, respectively (from the right and left, respectively), the inequality (4.10) must be strict in G in a neighborhood of the boundary. As concluded by Wintner in the o.d.e. case we may here claim that by redefining $W(x)$ appropriately in that neighborhood of the boundary, (4.10) will hold on $\bar{G}$. Hence, by applying Theorem 3.1 to (4.10) we have proved the theorem.

By an appropriate change of variable we have the following result.

CCOROLLARY 4.2. *Let* $\bar{G}=[a_1, a_1+c]\times[a_2, a_2+c]$ *for some* $c>0$ *be used in* (4.3). *If*

$$0 \leq [\operatorname{sgn}(a_i+2^{-1}c-x_i)]cP_i(x) \leq 2, \quad i = 1, 2, \tag{4.13}$$

holds for $x \in \bar{G}$ *then* (1.2) *has a positive solution on* $\bar{G}$.

PROOF. With $a=(a_1, a_2)$ the corollary follows from the theorem. Under the change of variables

$$u = c^{-1}(x-a), \quad \eta = c^{-1}(\xi-a),$$
$$y(x) \equiv \bar{y}(u), \quad p(x) \equiv \bar{p}(u),$$

we have

$$c^2[-\Delta y(x)+p(x)y(x)] \equiv -\Delta\bar{y}(u) + c^2\bar{p}(u)\bar{y}(u)$$

and

$$c\int_{\bar{G}} K(x, \xi)p(\xi)d\xi = \int_{[0,1]\times[0,1]} K(u, \eta)c^2 p(\eta)d\eta$$

holding where u varies over $[0, 1]\times[0, 1]$.

REMARK 4.3. Theorem 4.1 remains valid if (4.8) is changed to

$$0 \leq P_i(x) \leq 4 \quad \text{for} \quad i = 1, 2 \tag{4.14}$$

for all x in $\bar{G}$.

At this time we return to the discussion preceding Theorem 4.1.

In the case when p is a negative constant it follows that $P_i(x)>0$ holds for $0<x_i<2^{-1}$ and $P_i(x)<0$ holds for $2^{-1}<x_i<1$, so that condition (4.14) is not realistic, whereas when this constant is sufficiently close to zero then (4.8) will hold. Indeed, if p is negative and "pyramid" in shape, symmetric about the lines $x_1=1/2$ and $x_2=1/2$ in $\bar{G}$ and monotone decreasing in each of the variables x_1 and x_2 on $[0, 1/2]\times[0, 1/2]$ then the first inequality in (4.8) will hold. Again if p is not too negative the right inequality in (4.8) will also hold. This provides a large class of coefficient functions to which Theorem 4.1 applies, but of course there is a larger class, as described by (4.8) itself.

Finally in closing, with $\bar{G}=[0, 1]\times[0, 1]$ and with $p(x_1, x_2)=p_0$ we note that for $p_0>-2\pi^2$, by a comparison, (1.2) has a positive solution on $\bar{G}$. Indeed,

$u(x_1, x_2) = \sin \pi x_1 \sin \pi x_2$ satisfies $-\Delta u - 2\pi^2 u = 0$ on $\bar{G}$ with $u(x_1, x_2) = 0$ on the boundary. As a comparison, (4.8) shows that if

$$p_0 \geq -8\pi(4 \arctan 2^{-1} + \log 5)^{-1} \cong -7.255 \tag{4.15}$$

then (1.2) has a positive solution.

The ratio of the constant on the right in (4.15) to the "best" constant $2\pi^2$ here in the p.d.e. setting is essentially the same as corresponding ratio of 4, as provided by (1.5) to the "best" constant of π^2 in the o.d.e. setting but, of course, (4.8) and (1.5) apply to much broader classes of functions.

The results of Section 2 applied to this example yield

$$p_0 \geq -4 \tag{4.16}$$

which, of course, is not as good as (4.15).

This illustrates the difference in the general nature of the results of this section over those in Section 2. There we show that if a domain is inside a sufficiently narrow strip then equation (1.2) has a positive solution on it. Here we show that if it is inside a sufficiently small square, which need not be as narrow as the strip, then (1.2) has a positive solution on it.

When we consider solutions having domains in $\boldsymbol{R}^n$, $n>2$, we may consider various subdimensional cross-sections with results analogous to those of both Sections 2 and 4 being used.

References

[1] W. Allegretto, On the equivalence of two types of oscillation for elliptic operators, *Pacific J. Math.* **55** (1974), 319–328.

[2] ———, Nonoscillation criteria for elliptic equations in conical domains, *Proc. Amer. Math. Soc.* **63** (1977), 245–250.

[3] M. Bôcher, Nonoscillatory linear differential equations of second order, *Bull. Amer. Math. Soc.* **7** (1900–01), 333–340.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. I,* Interscience Publishers, New York, 1953.

[5] I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators.* (Israel Program for Scientific Translations, Daniel Davey and Co., New York, 1965.)

[6] P. Hartman, On linear second order differential equations with small coefficients, *Amer. J. Math.* **73** (1951), 955–962.

[7] ———, *Ordinary Differential Equations,* J. Wiley and Sons, New York, 1964.

[8] K. Kreith, Analogy in solving Poisson's equation, *Canad. Math. Bull.* **18** (4) (1975), 517–521.

[9] W. Moss and J. Piepenbrink, Positive solutions of elliptic equations, *Pacific J. Math.* **75** (1978), 219–226.

[10] Z. Opial, Sur les integrales oscillantes de l'equation differentielle $u''+f(t)u=0$, *Ann.*

Polon. Math. **4** (1958), 308–313.

[11] M. Protter, Lower bounds for the first eigenvalue of elliptic equations, *Annlas of Math.* **71** (1960), 423–444.

[12] W. T. Reid, *Ordinary Differential Equations,* J. Wiley and Sons, Inc., New York, 1971.

[13] C. A. Swanson, Nonoscillation criteria for elliptic equations, *Canad. Math. Bull.* **12** (3) (1969), 275–280.

[14] D. Willett, Classification of second order linear differential equations with respect to oscillation, *Advances in Math.* **3** (1969), 594–623.

[15] A. Wintner, On the non-existence of conjugate points, *Amer. J. Math.* **73** (1951), 368–380.

[16] P. K. Wong, Wirtinger type inequalities and elliptic differential inequalities, *Tôhuku Math. J.* **23** (1971), 117–127.

[17] ———, A Sturmian theorem for first order partial differential equations, *Trans. Amer. Math. Soc.* **166** (1972), 125–131.

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