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A boundary value problem for second order functional differential equations*

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1. Introduction

In the present paper will denote by $\boldsymbol{R}$ the real line, by $\boldsymbol{R}^n$ the n-dimensional Euclidean space, and by $|\cdot|$ one of the norms usually considered in $\boldsymbol{R}^n$.

Let J be an interval of the real line $\boldsymbol{R}$ with zero at its right end-point, which belongs to J. The linear space $C^*(J)$ of all continuous and bounded functions on J with values in $\boldsymbol{R}^n$ is considered endowed with the norm $\|\cdot\|_J$ defined by the folmula:

$$\|\varphi\|_J = \sup_{u\in J} |\varphi(u)|$$

which, as it is easily virified, makes it a complete metric space, i.e. a Banach space.

Let $x: I \to \boldsymbol{R}^n$ be a continuous function, where I is an interval of the real line $\boldsymbol{R}$. If $t \in I$ with $J_t = \{\xi \in \boldsymbol{R}: \xi - t \in J\} \subseteq I$, then the function $x_t: J \to \boldsymbol{R}^n$ defined by the formula:

$$x_t(u) = x(t + u),\ u \in J,$$

is called the "past history" of x at t.

This paper is concerned wit the existence of solutions of the two point boundary value problem

$$(\rho(t)x'(t))' = f(t, x_t, x'(t)), \tag{1}$$

$$x_0 = \varphi,\ x(T) = \eta, \tag{2}$$

where the function f is defined on the set

$$[0, T] \times \boldsymbol{D} \times \boldsymbol{R}^n,\ \boldsymbol{D} \subseteq C^*(J),\ T > 0,$$

(φ, η) is a point of $\boldsymbol{D} \times \boldsymbol{R}^n$ and $\rho(t)$ is a positive continuous function defined on $[0, T]$.

Our results generalize previous ones due to Sficas [7] and have their origin from well known results for ordinary differential equations (cf. [1], [4], [5]). In

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the special case when $\rho(t)\equiv 1$ the result of the first section is related to known results due to Grimm and Schmitt [3] for equations of mixed type (see also Bernfeld and Lakshmikantham [2]).

Finally we note that the result concerning the existence of solutions of equation (1), which are defined in an interval $[\alpha, \infty)$ (Theorem 4), is very useful in other branches of the qualititative theory of differential equations, for example in oscillation theory where it is assumed the existence of solutions defined for all large t (see for example Y. Kitamura and T. Kusano [6]).

2. Fixed point method

In this section we deal with the boundary value problem (1)–(2) by using the fixed point technique. In order to applying this technique we need a lemma.

LEMMA 1. *Let the differential equation*

$$(\rho(t)y'(t))' = \omega(t) \tag{3}$$

where ω: $[0, T]\to \boldsymbol{R}^n$ *is a continuous function and* $\rho(t)$ *is as in equation* (1). *If* $\xi, \eta\in \boldsymbol{R}^n$, *then there exists exactly one solution* y *of* (3) *with* $y(0)=\xi$ *and* $y(T)=\eta$, *which is given by*:

$$\begin{aligned} y(t) = &-\left\{(1-A\varphi(t))\int_0^t \varphi(s)\omega(s)ds \right. \\ &\left. +\phi(t)\int_t^T (1-A\varphi(s))\omega(s)ds\right\} + A(\eta-\xi)\varphi(t) + \xi \end{aligned} \tag{4}$$

where $\varphi(t)=\int_0^t \rho(s)^{-1}ds$, $A=\varphi(T)^{-1}$. *Moreover this solution satisfies the relations*:

$$|y(t)| \leqq K_1 \max_{0\leqq s\leqq T} |\omega(s)| + |\eta-\xi| + |\xi|, \tag{5}$$

$$|y'(t)| \leqq K_2 \max_{0\leqq s\leqq T} T|\omega(s)| + Ar^{-1}|\eta-\xi| \tag{6}$$

where $r=\min_{0\leqq t\leqq T}\rho(t)$, $K_1=\int_0^{T-A\Phi(T)} \varphi(s)ds$, $\Phi(T)=\int_0^T \varphi(s)ds$ *and* $K_2=r^{-1}\max\{T-A\,\Phi(T), A\Phi(T)\}$.

PROOF. Differentiating (4) we can easily verify, that $y(t)$ is a solution of (3). Also it is readily seen that $y(0)=\xi$ and $y(T)=\eta$. We will show the relations (5) and (6). The relation (4) can be abbreviated to

$$y(t) = -\int_0^T G(t, s)\omega(s)ds + A(\eta-\xi)\varphi(t) + \xi \tag{7}$$

where $G(t, s)$ is the Green function with respect to the boundary value problem

$$(\rho(t)y'(t))' = 0, \ y(0) = 0, \ y(T) = 0$$

which is given by the formula

$$G(t, s) = \begin{cases} (1-A\varphi(t))\varphi(s), & 0 \leqq s \leqq t \leqq T, \\ \varphi(t)(1-A\varphi(s)), & 0 \leqq t \leqq s \leqq T, \end{cases} \tag{8}$$

(see, e.g. Hartman [4], pp. 325–328 and 418–422).

Now, we shall prove that:

$$\max_{0 \leqq t \leqq T} \int_0^T G(t, s)ds = K_1 \tag{9}$$

and

$$\max_{0 \leqq t \leqq T} \int_0^T |\partial G(t, s)/\partial t| ds = K_2. \tag{10}$$

By (8) we obtain

$$h(t) = \int_0^T G(t, s)ds = (1-A\varphi(t)\int_0^t \varphi(s)ds + \varphi(t)\int_t^T (1-A\varphi(s))ds \tag{11}$$

The maximum of $h(t)$ is attained at a point t^* such that:

$$-A\varphi'(t)\int_0^T \varphi(s)ds + (T-t)\varphi'(t) = 0 \quad \text{or} \quad t^* = T - A\int_0^T \varphi(s)ds.$$

Then, by (11), we obtain

$$\begin{aligned} \max_{0 \leqq t \leqq T} \int_0^T G(t, s)ds &= \int_0^T G(t^*, s)ds \\ &= \int_0^{t^*} \varphi(s)ds - A\varphi(t^*)\int_0^{t^*} \varphi(s)ds + \varphi(t^*)(T-t^*) - A\varphi(t^*)\int_{t^*}^T \varphi(s)ds \\ &= \int_0^{t^*} \varphi(s)ds - A\varphi(t^*)\int_0^{t^*} \varphi(s)ds + \varphi(t^*)A\int_0^T \varphi(s)ds - A\varphi(t^*)\int_{t^*}^T \varphi(s)ds \\ &= \int_0^{t^*} \varphi(s)ds = \int_0^{T-A\Phi(T)} \varphi(s)ds = K_1. \end{aligned}$$

In order to prove (10) we observe that

$$h'(t) = \int_0^T (\partial G(t, s)/\partial t)ds = -A\varphi'(t)\int_0^T \varphi(s)ds + (T-t)\varphi'(t)$$

i.e. $\rho(t)h'(t) = -A\int_0^T \varphi(s)ds + T - t$.

Notice that $\rho(t)h'(t)$ is a decreasing function. Moreover since $T-A\int_0^T \varphi(s)ds \geqq T-A\varphi(T)\cdot T=0$, we have

$$\max_{0\leqq t\leqq T}\int_0^T |\partial G(t, s)/\partial t|ds = r^{-1}\max\{T-A\Phi(T), A\Phi(T)\} = K_2.$$

Relations (5) and (6) follow now easily from (7) by using (9) and (10).

REMARK. In the special case where $\rho(t)\equiv 1$ we have $A=T^{-1}$, $\phi(t)=t$, $r=1$, $K_1=T^2/8$ and $K_2=T/2$ and therfore the formulas (5) and (6) lead to some well known formulas for ordinary differential equations (see, e.g. Hartman [4], p. 422, or Jackson [5], p. 101). It is noteworthy that the bounds designated for the solutions y of (3) with $y(0)=\xi$ and $y(T)=\eta$ and its derivative seem to be the smallest which can be found. This fact plays an important role in the conditions stated in Theorem 1.

THEOREM 1. *(Existence and uniqueness) Let f be continuous on the set*

$$E = [0, T] \times C^*(J)\times \boldsymbol{R}^n$$

and satisfy on E a Lipschitz condition of the form

$$|f(t, \varphi, \zeta) - f(t, \tilde{\varphi}, \tilde{\zeta}| \leqq \theta_0\|\varphi-\tilde{\varphi}\|_J + \theta_1|\zeta-\tilde{\zeta}| \tag{12}$$

with Lipschitz constants θ_0 and θ_1 such that:

$$\theta_c K_1 + \theta_1 K_2 < 1 \tag{13}$$

and the numbers K_1 and K_2 are as in Lemma 1. *Then, for every $(\varphi, \eta)\in C^*(J)\times \boldsymbol{R}^n$ there exists exactly one solution of the boundary value problem* (1)–(2).

PROOF. Let $(\varphi, \eta)\in C^*(J)\times\boldsymbol{R}^n$. Consider the set X of all continuous functions $x: J\cup[0, T]\to\boldsymbol{R}^n$ which are continuously differentiable on $[0, T]$ and such that $x_0=\varphi$ for any $x\in X$. The formula

$$\delta(x, y) = \max(\max_{t\in[0,T]}|x(t)-y(t)|, (K_1/K_2)\max_{t\in[0,T]}|x'(t)-y'(t)|) \tag{14}$$

defines in X a metric, which makes it a complete metric space.

Let $x\in X$. Consider the differential equation

$$(\rho(t)y'))' = f(t, x_t, x'(t))$$

which is of the form (3) with $\omega(t)\equiv f(t, x_t, x'(t))$ and hence, by Lemma 1, there exists exactly one solution y with $y(0)=\varphi(0)=\xi$ and $y(T)=\eta$.

Consider now the function $\tilde{x}$, where

$$\tilde{x}(t) = \begin{cases} \varphi(t), & t \in J, \\ y(t), & t \in (0, T]. \end{cases}$$

Evidently $\tilde{x} \in X$. Thus by the formula

$$x \xrightarrow{S} \tilde{x},\ \tilde{x} = S(x),$$

a mapping S of X into itself is defined for which it holds:

$$\tilde{x}_0 = \varphi,\ \tilde{x}(T) = \eta \quad \text{and} \quad (\rho(t)\tilde{x}'(t))' = f(t, x_t, x'(t)) \quad \text{for every} \quad t \in [0, T].$$

Moreover S is a contraction. In fact let $x_1, x_2 \in X$. We notice that the function

$$v(t) = \tilde{x}_1(t) - \tilde{x}_2(t), \quad t \in J \cup [0, T],$$

is a solution of the equation

$$(\rho(t)v'(t))' = \omega(t)$$

where

$$\omega(t) = f(t, x_{1t}, x_1'(t)) - f(t, x_{2t}, x_2'(t)),\ t \in [0, T],$$

and therefore by (5), (6) and (12) we have:

$$\begin{aligned}
\delta(S(x_1), S(x_2)) &= \delta(\tilde{x}_1, \tilde{x}_2) \\
&= \max(\max_{t\in[0,T]} |\tilde{x}_1(t) - \tilde{x}_2(t)|, (K_1/K_2) \max_{t\in[0,T]} |\tilde{x}_1'(t) - \tilde{x}_2'(t)|) \\
&\leqq \max(K_1 \max_{t\in[0,T]} |f(t, x_{1t}, x_1'(t)) - f(t, x_{2t}, x_2'(t))|, \\
&\quad (K_1/K_2)K_2 \max_{t\in[0,T]} |f(t, x_{1t}, x_1'(t)) - f(t, x_{2t}, x_2'(t))|) \\
&= K_1 \max_{t\in[0,T]} |f(t, x_{1t}, x_1'(t)) - f(t, x_{2t}, x_2'(t))| \\
&\leqq K_1[\theta_0 \max_{t\in[0,T]} \|x_{1t} - x_2\|_J + \theta_1 \max_{t\in[0,T]} |x_1'(t) - x_2'(t)|] \\
&= K_1[\theta_0 \max_{t\in[0,T]} |x_1(t) - x_2(t)| + (\theta_1 K_2/K_1)(K_1/K_2) \max_{t\in[0,T]} |x_1'(t) - x_2'(t)|] \\
&\leqq \delta(x_1, x_2) K_1[\theta_0 + (\theta_1 K_2/K_1)] = \delta(x_1, x_2)\{\theta_0 K_1 + \theta_1 K_2\}.
\end{aligned}$$

Consequently by (13) the mapping S is a contraction and therfore there exists exactly one $x \in X$ with $S(x) = x$. It is obvious that x is a solution of the boundary value problem (1)–(2).

THEOREM 2. (*Existence*) *Let f be continuous and bounded on the set $E = [0, T] \times C^*(J) \times \boldsymbol{R}^n$. Then for every $(\varphi, \eta) \in C^*(J) \times \boldsymbol{R}^n$ there exists at least one solution of the boundary value problem* (1)–(2).

PROOF. Let $(\varphi, \eta) \in C^*(J) \times \boldsymbol{R}^n$ and m a bound of f in E. Consider the set $C^{*,1}(I)$, $I = J \cup [0, T]$ of all continuous and bounded functions $x: I \to \boldsymbol{R}^n$ having continuous first derviatives on $[0, T]$. The formula:

$$\|x\|_I^* = \max(\|x\|_I, (K_1/K_2) \max_{0 \leq t \leq T} |x'(t)|) \tag{15}$$

defines in $C^{*,1}(I)$ a norm, which makes it a Banach space.

Let Y be the set of all $x \in C^{*,1}(I)$ with

(i) $x_0 = \varphi$,
(ii) $|x(t)| \leqq K_1 m + |\eta - \xi| + |\xi|$ for every $t \in [0, T]$,
(iii) $|x'(t)| \leqq K_2 m + Ar^{-1}|\eta - \xi|$ for every $t \in [0, T]$,
(iv) $|\rho(t)x'(t) - \rho(\bar{t})x'(\bar{t})| \leqq m|t - \bar{t}|$ for every $t \in [0, T]$,

where $\xi = \varphi(0)$. This set is not empty, because it contains the function

$$x(t) = \begin{cases} \varphi(t), & t \in J, \\ \varphi(0), & t \in (0, T], \end{cases}$$

and moreover this set is convex. Also the set Y is compact, because by the uniform continuity of ρ, the condition (iv) implies that the derivatives of functions $x \in Y$ are equicontinuous, while by (iii) they are uniformly bounded.

According to Lemma 1 for any $x \in Y$, there exists exactly one solution of the equation:

$$(\rho(t)y'(t)' = f(t, x_t, x'(t))$$

with $y(0) = \varphi(0) = \xi$ and $y(T) = \eta$.

Consider the function $\tilde{x}$ with

$$\tilde{x}(t) = \begin{cases} \varphi(t), & t \in J, \\ y(t), & t \in (0, T]. \end{cases}$$

It is clear that $\tilde{x} \in Y$. Thus by the formula:

$$x \xrightarrow{P} \tilde{x}, \quad \tilde{x} = P(x),$$

is defined a mapping P of Y into itself is defined for which it holds:

$$\tilde{x}_0 = \varphi, \quad \tilde{x}(T) = \eta,$$

$$(\rho(t)\tilde{x}'(t))' = f(t, x_t, x'(t)), \quad \text{for every} \quad t \in [0, T].$$

The mapping P is continuous. In fact as it is easily verified, the set

$$E_Y = \{(t, x_t, x'(t)) \colon x \in Y, t \in [0, T]\}$$

is a compact subset of E. On the other hand by (15) and (5), (6), for every x_1, $x_2 \in Y$ we have:

$$\begin{aligned}(16)\qquad \|P(x_1)-P(x_2)\|_I^* &= \max\left(\max_{t\in[0,T]}|P(x_1(t))-P(x_2(t))|,\right.\\ &\left.(K_1/K_2)\max_{t\in[0,T]}|P'(x_1(t))-P'(x_2(t))|\right)\\ &\leqq K_1\max_{t\in[0,T]}|f(t, x_{1t}, x_1'(t))-f(t, x_{2t}, x_2'(t))|.\end{aligned}$$

Also, by the uniform continuity of f on the set E_Y we have:

$$(\forall\varepsilon>0)(\exists\delta(\varepsilon)>0)(\forall\{(t, z, \zeta) \text{ and } (t, \tilde{z}, \tilde{\zeta})\} \text{ on } E_Y)(\|z-\tilde{z}\|_J<\delta(\varepsilon) \text{ and } |\zeta-\zeta|<\delta(\varepsilon))\Rightarrow|f(t, z, \zeta)-f(t, \tilde{z}, \tilde{\zeta})|<\varepsilon/K_1.$$

But, for $x_1, x_2\in Y$ with $\|x_1-x_2\|_I^*<\min(\delta(\varepsilon), (K_2/K_1)\delta(\varepsilon))\equiv\delta^*(\varepsilon)$ it holds

$$\|x_{1t}-x_{2t}\|_J<\delta(\varepsilon) \quad\text{and}\quad |x_1'(t)-x_2'(t)|<\delta(\varepsilon) \quad\text{for every}\quad t\in[0, T].$$

Hence, we have

$$(\forall\varepsilon>0)(\exists\delta^*(\varepsilon)>0)(\forall x_1, x_2\in \text{ with } \|x_1-x_2\|_I^*<\delta^*(\varepsilon)) \Longrightarrow\|P(x_1)-P(x_2)\|_I^*<\varepsilon,$$

and therefore P is continuous. By Schauder's fixed point theorem we can now derive that there exists at least one $x\in Y$ with $x=P(x)$. It is obvious that x is a solution of the boundary value problem (1)–(2).

3. A priori bounds method

In this section we study again the boundary value problem (1)–(2) by using the method of a priori bounds (cf. Hartman [4]). In what follows we assume that the function ρ appeared in (1) is positive and continuously differentiable on $[0, T]$. Also we use the notations: $\|\cdot\|$, $\|\cdot\|_J$ to denote the Euclidean norm in $\boldsymbol{R}^n$ and respectively the sup-norm defined by it in $C^*(J)$.

We need the following lemmas:

LEMMA 2 (Hartman [4]). *Let $q(s)$, $0\leqq s<+\infty$ be a positive continuous function satisfying*

$$\int^\infty \frac{s\,ds}{q(s)} = +\infty.$$

Let also a^*, k, b^*, T be nonnegative constants. Then there exists a number $M>0$ (depending only on q, a^*, b^*, T, k) with the following property:

If v is a vector function of class C^2 for $0\leqq t\leqq T$ satisfying

a) $\|v(t)\|\leqq b^*$,
b) $\|v''(t)\|\leqq q(\|v'(t)\|)$, $0\leqq t\leqq T$,
c) $\|v''(t)\|\leqq a^*r''(t)+k$,

where $r(t)=\|v(t)\|^2$, then for every $t\in[0, T]$ it holds:

$$\|v'(t)\| \leqq M.$$

LEMMA 3. *Let $q(s)$, $0 \leqq s < +\infty$ be a continuous function as in the Lemma 2. Let also, a, k, b, T be nonegative constants. Then, there exists a $M > 0$ (depending only on q, a, b, k, T and the function ρ) with the property:*

If x is a vector function of class C^2 for $0 \leqq t \leqq T$ satisfying:

a)* $\|x(t)\| \leqq b$,

b)* $\|(\rho(t)x'(t))'\| \leqq q(\rho(t)\|x'(t)\|)$, $0 \leqq t \leqq T$,

c)* $\|\rho(t)x'(t))'\| \leqq a\vartheta''(t) + k$,

where $\vartheta(t) = \|x(t)\rho(t) - \int_0^t x(s)\rho'(s)ds\|^2$, *then*:

$$\rho(t)\|x'(t)\| \leqq M \quad \text{for} \quad \text{every} \quad t \in [0,\ T]. \tag{17}$$

PROOF. It is sufficient to prove that a)*, b)*, c)* lead to the assumptions a), b), c) of Lemma 2 for the function

$$v(t) = \rho(t)x(t) - \int_0^t x(s)\rho'(s)ds,\ t \in [0,\ T].$$

In fact for every $t \in [0,\ T]$ we have:

$$\|v(t)\| \leqq b \max \rho(t) + b \int_0^T |\rho'(s)|ds = b\left\{\max \rho(t) + \int_0^T |\rho'(s)|ds\right\} \equiv b^*,$$

$$\|v''(t)\| = \|(\rho(t)x'(t))'\| \leqq q(\rho(t)\|x'(t)\|) = q(\|v'(t)\|),$$

and hence a) and b) hold. Also,

$$\vartheta'(t) = 2\left\{x(t)\rho(t) - \int_0^t x(s)\rho'(s)ds\right\}(\rho(t)x'(t)),$$

$$\vartheta''(t) = 2\left\{\rho^2(t)\|x'(t)\|^2 + \left[x(t)\rho(t) - \int_0^t x(s)\rho'(s)ds\right](\rho(t)x'(t))'\right\}$$

and consequently

$$\begin{aligned}\|(\rho(t)x'(t))'\| &= \|v''(t)\| \leqq a\vartheta''(t) + k \\ &= 2a\left\{\rho^2(t)\|x'(t)\|^2 + \left[x(t)\rho(t) - \int_0^t x(s)\rho'(s)ds\right](\rho(t)x'(t))'\right\} + k \\ &= 2a\{\|v'(t)\|^2 + v(t)v''(t)\} + k = a^*r''(t) + k\end{aligned}$$

i.e. c) holds with $r(t) = \|v(t)\|^2$ and $a^* = a$.

LEMMA 4. *Let* $f\colon E(T,\ b) \to \boldsymbol{R}^n$, $E(T,\ b) = [0,\ T] \times \bar{U}(0,\ b) \times \boldsymbol{R}^n$, $\bar{U}(0,\ b) = \{\varphi \in C^*(J)\colon \|\varphi\|_J \leqq b\}$ *be a continuous function. If*

$$\beta = \sup_{t \in [0,T]} \rho(t)^{-1} \int_0^t |\rho'(s)|ds$$

and $\hat{a}>0$ *are such that* $1-2\hat{a}b\beta>0$, *then the condition*

$$(18)\qquad \|f(t, z, \zeta)\| \leqq 2\hat{a}\{z(0)f(t, z, \zeta) + \rho(t)\|\zeta\|^2\} + k^*,\ k^* \geqq 0,\ (t, z, \zeta)\in E(T, b),$$

implies that

$$(19)\qquad \|f(t, x_t, x'(t))\| \leqq \frac{2\hat{a}}{1-2\hat{a}b\beta}\left\{f(t, x_t, x'(t))\left[x(t)-\rho(t)^{-1}\int_0^t x(s)\rho'(s)ps\right] + \rho(t)\|x'(t)\|^2\right\} + \frac{k^*}{1-2\hat{a}b\beta}$$

for every function $x\in C(J\cup[0, T])\cap C^1[0, T]$ *with* $\|x_t\|_J\leqq b$, $t\in[0, T]$ *and for every* $t\in[0, T]$.

PROOF. Let x be a continuous function on $J\cup[0, T]$ with continuous first derivative for $0\leqq t\leqq T$ with $\|x_t\|_J\leqq b$, $t\in[0, T]$. Then by (18) we have:

$$\begin{aligned}\|f(t, x_t, x'(t))\| &\leqq 2\hat{a}\{f(t, x'(t))x(t) + \rho(t)\|x'(t)\|^2\} + k^*\\ &= 2\hat{a}\left\{f(t, x_t, x'(t))\left[x(t) - \rho(t)^{-1}\int_0^t x(s)\rho'(s)\,ds + \rho(t)^{-1}\int_0^t x(s)\rho'(s)ds\right] + \rho(t)\|x'(t)\|^2\right\} + k^*\\ &\leqq 2\hat{a}\left\{f(t, x_t, x'(t))\left[x(t) - \rho(t)^{-1}\int_0^t x(s)\rho'(s)ds + \rho(t)\|x(t)\|^2\right\}\right.\\ &\qquad + 2\hat{a}\|f(t, x_t, x'(t))\|b\beta + k^*,\ t\in[0, T].\end{aligned}$$

and consequently

$$(1-2\hat{a}b\beta)\|f(t, x_t, x'(t))\| \leqq 2\hat{a}\left\{f(t, x_t, x'(t))\left[x(t)-\rho(t)^{-1}\int_0^t x(s)\rho'(s)ds\right] + \rho(t)\|x'(t)\|^2\right\} + k^*,\ t\in[0, T],$$

i.e. the relation (19).

LEMMA 5. *Let* $f\colon E(T, b)\to \boldsymbol{R}^n$ *be a continuous function such that*:

(I) $z(0)f(t, z, \zeta)+\rho(t)\|\zeta\|^2>0$ *for every* $(t, z, \zeta)\in E(T, b)$ *with* $z(0)\zeta=0$ *and* $\|z\|_J=b$, $\|z(0)\|>0$.

(II) *For some nonnegative constants* k^* *and* $\hat{a}$ *it holds*:

$$\|f(t, z, \zeta)\| \leqq 2\hat{a}\{z(0)f(t, z, \zeta) + \rho(t)\|\zeta\|^2\} + k^*$$

(III) $\|f(t, z, \zeta)\| \leqq q(\rho(t)\|\zeta\|)$ *for every* $(t, z, \zeta) \in E(T, b)$ *where the function* q *is as in the Lemmas* 2 *and* 3.

Then for any $M>0$ *there exists a continuous bounded function* g: $[0, T] \times C^*(J) \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *such that*:

(I′) $z(0)g(t, z, \zeta)+\rho(t)\|\zeta\|^2>0$ *for every* $(t, z, \zeta) \in [0, T] \times C^*(J) \times \boldsymbol{R}^n$ *with* $z(0)\zeta=0$, $\|z\|_J \geqq b$, $\|z(0)\|>0$.

(II′) $\|g(t, z, \zeta)\| \leqq 2\hat{a}\{z(0)g(t, z, \zeta)+\rho(t)\|\zeta\|^2\}+k^*$ *for every* $(t, z, \zeta) \in [0, T] \times C^*(J) \times \boldsymbol{R}^n$.

(III′) $\|g(t, z, \zeta)\| \leqq q(\rho(t)\|\zeta\|)$ *for every* $(t, z, \zeta) \in [0, T] \times C^*(J) \times \boldsymbol{R}^n$.

(IV′) $f(t, z, \zeta)=g(t, z, \zeta)$ *for every* $(t, z, \zeta) \in E(T, b)$ *with* $\rho(t)\|\zeta \leqq M$, $0 \leqq t \leqq T$, $\|z\|_J \leqq b$.

Proof. We obtain such a function g as follows:

Let $\delta(s)$, where $0 \leqq s < +\infty$, be a continuous real-valued function satisfying

$$\begin{aligned} \delta(s) &= 1, \quad 0 \leqq s \leqq M, \\ 0 < \delta(s) &< 1, \quad M < s \leqq 2M, \\ \delta(s) &= 0, \quad 2M < s < +\infty. \end{aligned}$$

Put

$$g(t, z, \zeta) = \begin{cases} \delta(\rho(t)\|\zeta\|) f(t, z, \zeta) \text{ on } E(T, b), \\ \dfrac{b}{\|z\|_J} g\left(t, b\dfrac{z}{\|z\|_J}, \zeta\right), \text{ for } 0 \leqq t \leqq T,\ \|z\|_J \geqq b,\ \zeta \in \boldsymbol{R}^n. \end{cases} \tag{20}$$

We show that g fulfills the properties (I′)–(III′).

Consider first the case where $\|z\|_J \leqq b$ i.e. $(t, z, \zeta) \in E(T, b)$. Then, by the obvious identity on $E(T, b)$

$$\begin{aligned} &z(0)g(t, z, \zeta) + \rho(t)\|\zeta\|^2 \\ &\quad = \delta(\rho(t)\|\zeta\|)(z(0)f(t, z, \zeta) + \rho(t)\|\zeta\|^2 + [1-\delta(\rho(t)\|\zeta\|)]\rho(t)\|\zeta\|, \end{aligned}$$

we immediately conclude that (I′) holds. For (II′) we have

$$\begin{aligned} \|g(t, z, \zeta)\| &= \|\delta(\rho(t)\|\zeta\|)f(t, z, \zeta)\| \leqq \delta(\rho(t)\|\zeta\|)\|f(t, z, \zeta)\| \\ &\leqq \delta(\rho(t)\|\zeta\|)[2\hat{a}\{z(0)f(t, z, \zeta)+\rho(t)\|\zeta\|^2\} + k^*] \\ &= 2\hat{a}z(0)\delta(\rho(t)\|\zeta\|)f(t, z, \zeta) + 2\hat{a}\rho(t)\delta(\rho(t)\|\zeta\|)\|\zeta\|^2 + k^*\delta(\rho(t)\|\zeta\|) \\ &\leqq 2\hat{a}\{z(0)g(t, z, \zeta)+\rho(t)\|\zeta\|^2\} + k^* \end{aligned}$$

and therefore (II′) holds.

For (III′) we notice that

$$\|g(t, z, \zeta\| = \|\delta(\rho(t)\|\zeta\|)f(t, z, \zeta)\| = \delta(\rho(t)\|\zeta\|)\|f(t, z, \zeta)\| \leqq q(\rho(t)\|\zeta\|)$$

and hence (III′) is valid.

Consider any $z \in C^*(J)$ with $\|z\|_J > b$. Then, the conditions (I′), (II′), (III′) are obtained correspondingly from (I), (II), (III) if z is replaced by $(b/\|z\|_J)z$.

Finally (IV′) is an immediate consequence of formula (20).

The main result of this section is the following:

THEOREM 3. *Let f be a continuous function on the set $E(T, b)$ satisfying the conditions*:

(i) $z(0)f(t, z, \zeta) + \rho(t)\|\zeta\|^2 \geqq 0$ *for every* $(t, z, \zeta) \in E(T, b)$ *with* $z(0)\zeta = 0$ *and* $\|z\|_J = b$.

(ii) *If* $\hat{a} > 0$ *is such that* $1 - 2\hat{a}b\beta > 0$ (*where* β *is defined in Lemma* 4) *and* $k^* > 0$, *then the following holds*:

$$\|f(t, z, \zeta)\| \leqq 2\hat{a}\{z(0)f(t, z, \zeta) + \rho(t)\|\zeta\|^2\} + k^*$$

for every $(t, z, \zeta) \in E(T, b)$.

(iii) $\|f(t, z, \zeta)\| \leqq q(\rho(t)\|\zeta\|)$ *for every* $(t, z, \zeta) \in E(T, b)$ *where* q *is a function as in Lemma* 2.

Then, for every $(\varphi, \eta) \in \bar{U}(0, b) \times \boldsymbol{R}^n$ *with* $\|\eta\| \leqq b$, *there exists at least one solution of the boundary value problem* (1)–(2).

PROOF. The proof will be given first for the case where f satisfies the condition (I) of Lemma 5 instead of (i). Let M be a constant supplied by Lemma 3 with $a = a'(\min_{t \in [0,T]} \rho(t))^{-1}$, $a' = \hat{a}/(1 - 2\hat{a}b\beta)$, $k = k^*/(1 - 2\hat{a}b\beta)$ and g the associated to M function of Lemma 5. Because g is continuous and bounded on the set $[0, T] \times C^*(J) \times \boldsymbol{R}^n$, by Theorem 2, the equation

$$(\rho(t)x'(t))' = g(t, x_t, x'(t))$$

has at least one solution x with $x_0 = \varphi$ and $x(T) = \eta$. We show that this solution x satisfies the assumptions of Lemma 3. To do this we put

$$r(t) = \|x(t)\|^2$$

when we have:

$$\begin{aligned} r'(t) &= 2x'(t)x(t) = 2x'(t)x_t(0), \\ (\rho(t)r'(t))' &= 2(\rho(t)x'(t))'x_t(0) + 2\rho(t)\|x'(t)\|^2 \\ &= 2(x_t(0)g(t, x_t, x'(t)) + \rho(t)\|x'(t)\|^2) \end{aligned}$$

and consequently, because $\rho(t) > 0$, the conditions $r'(t) = 2x_t(0)x'(t) = 0$ and $r(t) \geqq b^2$ imply, by (*I*′), $r''(t) > 0$. Hence $r(t)$ does not take its maximum at any point $t \in (0, T)$ with $r(t) \geqq b^2$. Since $r(0) = \|\varphi(0)\|^2 \leqq b^2$, $r(T) = \|\eta\|^2 \leqq b^2$ it follows that $r(t) \leqq b^2$ i.e.

(21) $$\|x(t)\| \leqq b, \quad \text{for every} \quad t \in [0, T].$$

The condition b)* of Lemma 3 is fulfilled by (III′) of Lemma 5. We are going now to prove c)*. From the equality

$$(\rho(t)x'(t))' = g(t, x_t, x'(t)), \quad t \in [0, T],$$

and by applying Lemma 4 to function g we have:

$$\begin{aligned}\|(\rho(t)x'(t))'\| &= \|g(t, x_t, x'(t))\| \\ &\leqq 2a' \left\{g(t, x_t, x'(t))'\left[x(t) - \rho(t)^{-1} \int_0^t x(s)\rho'(s)ds\right] + \rho(t)\|x'(t)\|^2\right\} + k \\ &= 2a'\rho(t)^{-1}\left\{(\rho(t)x'(t))'\left[x(t)\rho(t) - \int_0^t x(s)\rho'(s)ds\right] + \rho^2(t)\|x'(t)\|^2\right\} + k \\ &= a'\rho(t)^{-1}\theta''(t) + k \leqq a\theta''(t) + k.\end{aligned}$$

Hence

$$\rho(t)\|x'(t)\| \leqq M, \quad \text{for every} \quad t \in [0, T].$$

By the last relation, (IV′) and (21) we have

$$g(t, x_t, x'(t)) = f(t, x_t, x'(t)), \quad \text{for every} \quad t \in [0, T],$$

and therefore the function x is a solution of the boundary value problem (1)–(2).

For the proof of the Theorem in the general case when (i) holds, we note that for every $\varepsilon: 0 < \varepsilon \leqq 1$ the function $f(t, z, \zeta) + \varepsilon z(0)$ satisfies (I) of Lemma 5 (ii) and (iii) if k^* and q are replaced by $k^* + \varepsilon b$, $q + \varepsilon b$ respectively.

Hence the equation

$(21)_\varepsilon$ $$(\rho(t)x'(t))' = f(t, x_t, x'(t)) + \varepsilon x(t)$$

has a solution x_ε with $x_{\varepsilon,0} = \varphi$ and $x_\varepsilon(T) = \eta$. By Lemma 3 there exists a constant M^* (independent of ε) such that:

$$\rho(t)\|x_\varepsilon'(t)\| \leqq M^*, \quad \text{for every} \quad t \in [0, T].$$

Consequently if $N = \max_{0 \leqq s \leqq M^*} q(s) + b$ then

$$\|\rho(t)x_\varepsilon'(t)'\| \leqq N, \quad \text{for every} \quad t \in [0, T].$$

Thus, the functions $\rho x_\varepsilon'$, $0 < \varepsilon \leqq 1$ are uniformly bounded and equicontinuous on the interval $[0, T]$. Hence, because $\rho(t)$ is a positive bounded function on $[0, T]$, there exists a sequence ε_n: $n = 1, 2, \ldots$, $0 < \varepsilon_n \leqq 1$ with $\lim_{n \to +\infty} \varepsilon_n = 0$ and a function $y \in C^1[0, T]$ such that $\lim_{n \to \infty} x'_{\varepsilon_n} = y'$ and $\lim_{n \to \infty} x_{\varepsilon_n} = y$ uniformly on $[0, T]$.

Consider now the function

$$x(t) = \begin{cases} \varphi(t), & t \in J, \\ y(t), & t \in (0, T]. \end{cases}$$

Evidently $\|\cdot\|_J - \lim_{n\to\infty} x_{\varepsilon_n,t} = x_t$ uniformly on $[0, T]$. Furthermore, by $(21)_\varepsilon$ we have:

$$\rho(t)x'_{\varepsilon_n}(t) = c_n + \int_0^t \{f(s, x_{\varepsilon_n,s}, x'_{\varepsilon_n}(s)) + \varepsilon_n x_{\varepsilon_n}(s)\} ds$$

where

$$c_n = \left[\eta - \xi - \int_0^T \rho(s_1)^{-1} \int_0^{s_1} \{f(s, x_{\varepsilon_n,s}, x'_{\varepsilon_n}(s)) + \varepsilon_n x_{\varepsilon_n}(s)\} ds\, ds_1\right] \Big/ \int_0^T \rho(s)^{-1} ds$$

and $\xi = \varphi(0)$.

By Lebesgue convergence Theorem we conclude that $\lim_{n\to\infty} x_{\varepsilon_n}(t) = x(t)$, $t \in J \cup [0, T]$ and hence x is a solution of the boundary value problem (1)–(2).

EXAMPLE 1. Consider the differential equation

$$((1+t)x'(t))' = u(t)x(t)\|x_t\|_J + \mu(t)x'(t)\|x'(t)\| \tag{22}$$

where $u(t) \geqq 0$, $\mu(t) > 0$ are continuous on $[0, \infty)$ and moreover the function μ is such that

$$\vartheta = \inf_{t\in[0,\infty} (1+t)/\mu(t) > 1 .$$

For any $T > 0$ we consider a b:

$$0 < b < \min\{(T+1)T^{-1}, \vartheta - 1\}$$

and furthermore the corresponding set $E(T, b)$.

We shall show that for equation (22) the conditions (i), (ii), (iii) of Theorem 3 are fulfilled on $E(T, b)$. Here it is $\rho(t) = 1 + t$ and

$$f(t, z, \zeta) = u(t)z(0)\|z\|_J + \mu(t)\|\zeta\|\zeta.$$

In fact for (i) we notice that:

$$\begin{aligned} & z(0)f(t, z, \zeta) + \rho(t)\|\zeta\|^2 \\ & = u(t)\|z(0)\|^2\|z\|_J + \mu(t)z(0)\zeta\|\zeta\| + (1+t)\|\zeta\|^2 \\ & = u(t)\|z(0)^2\|z\|_J + (1+t)\|\zeta\|^2 \geqq 0 \end{aligned}$$

since $z(0)\zeta = 0$.

We shall show that (ii) holds for $\hat{a} = 1/2$ and a proper choice of k^*. First of all notice that $1-2\hat{a}b\beta>0$, because:

$$\beta = \sup_{t\in[0,T]}(1+T)^{-1}\int_0^t ds = \sup_{t\in[0,T]} t(1+t)^{-1} = T(T+1)^{-1}$$

and by the assumption we have $b<(T+1)T^{-1}$. It is enough

$$u(t)\|z(0)\|z\|_J + \mu(t)\|\zeta\|^2 \leqq u(t)\|z(0)\|^2\|z\|_J + \mu(t)z(0)\|\zeta\|\zeta + (1+t)\|\zeta\|^2 + k^*$$

or

$$k^* + u(t)\|z(0)\|\ \|z\|_J(\|z(0)\|-1) + \mu(t)\|\zeta\|(\zeta z(0)-\|\zeta\|) + (1+t)\|\zeta\|^2 \geqq 0$$

for every $(t, z, \zeta)\in E(T, b)$.

We choose $k^*>0$ such that

$$k^* - u(t)b^2(b+1) \geqq 0 \quad \text{for every} \quad t\in[0, T]. \tag{23}$$

Thus, it is sufficient to show that

$$\mu(t)\|\zeta\|\zeta z(0) - \mu(t)\|\zeta\|^2 + (1+t)\|\zeta\|^2 \geqq 0 \quad \text{for every} \quad (t, z, \zeta)\in E(T, b)$$

or $(1+t)\|\zeta\|^2 - \mu(t)\|\zeta\|^2(1+\|z(0)\|)\leqq 0$

or $\|\zeta\|^2\{1+t-\mu(t)(b+1)\} \geqq 0$

or $1 + t - \mu(t)(b+1) \geqq 0 \Longrightarrow b \leqq (t+1)/\mu(t) - 1$

which hold by the assumption that $b<\vartheta-1$.

Finally, for (iii) we notice that

$$\|f(t, z, \zeta)\| \leqq u(t)b^2 + \mu(t)\|\zeta\|^2 = u(t)b^2 + (\mu(t)/\rho^2(t))\rho^2(t)\|\zeta\|^2$$

and if we put

$$\lambda_1 = \max_{0\leqq t\leqq T}(1+t)^{-2}\mu(t), \quad \lambda_2 = b^2\max_{0\leqq t\leqq T} u(t),$$

we have

$$\|f(t, z, \zeta)\| \leqq \lambda_1\rho^2(t)\|\zeta\|^2 + \lambda_2.$$

It is clear that, for the function q

$$q(s) = \lambda_1 s^2 + \lambda_2, \quad 0 \leqq s < \infty,$$

it holds

$$\int^\infty (s/q(s)ds = \int^\infty (s/\lambda_1 s^2+\lambda_2))ds = \infty.$$

Hence by Theorem 3 we conclude that for every $(\varphi, \eta)\in \overline{U}(0, b)\times \boldsymbol{R}^n$ with

$\|\eta\| \leqq b$ the boundary value problem

$$((1+t)x'(t))' = u(t)x(t)\|x_t\|_J + \mu(t)x'(t)\|x'(t)\|, \; x_0 = \varphi, \quad x(T) = \eta,$$

has at least one solution.

The following theorem is a criterion for the existence of a solution of (1) with $+\infty$ as the right and end-point of its interval of definition.

THEOREM 4. *Suppose that f is continuous on the set*

$$E(b) = [0, \infty) \times \bar{U}(0, b) \times \boldsymbol{R}^n$$

and moreover that for any $T>0$, f satisfies the conditions of Theorem 3, *where the constants* $\hat{a}$, k^* *and the function q can depend on T. Then, for every $\varphi \in \bar{U}(0, b)$ the equation*

$$(\rho(t)x'(t))' = f(t, x_t, x'(t))$$

has at least one solution $x\colon J \cup [0, \infty) \to \boldsymbol{R}^n$ with $x_0 = \varphi$.

PROOF. According to Theorem 3 for every $m=1, 2,\dots$, there exists a solution x_m of (1) with $x_{m,0}=\varphi$ and $x_m(m)=0$. We notice that, if T is any positive number, in view of Lemma 3 there exists a $M>0$ (depending only on T) such that:

$$\rho(t)\|x'_m(t)\| \leqq M, \quad \text{for every} \quad t \in [0, T] \quad \text{and for every} \quad m \geqq T.$$

Also,

$$\|(\rho(t)x'_m(t))'\| \leqq \max_{0 \leqq s \leqq M} q(s), \quad \text{for every} \quad t \in [0, T] \quad \text{and for every } m \geqq T$$

where q is the function corresponding to T.

From these relations it is clear that the sequence $\rho x'_m$, $m \geqq T$ is equicontinuous and uniformly bounded on $[0, T]$. Hence, there exists a subsequence ${}_Tx_n$, $n = 1, 2,\dots$, of x_m, $m=1, 2,\dots$, and a function ${}_Ty\colon [0, T] \to \boldsymbol{R}^n$ such that:

$$\lim_{n\to\infty} {}_Tx_n = {}_Ty, \quad \lim_{n\to\infty} {}_Tx'_n = y'$$

uniformly on $[0, T]$.

As in the proof of Theorem 3, we can derive that the function

$${}_Tx(t) = \begin{cases} \varphi(t), & t \in J, \\ {}_Ty(t), & t \in (0, T], \end{cases}$$

is a solution of (1) on $[0, T]$ with $x_0=\varphi$. Evidently $\lim_{n\to\infty} {}_Tx_n = {}_Tx$ and $\lim_{n\to\infty} {}_Tx'_n = {}_Tx'$ on $J \cup [0, T]$.

By Arzela-Ascoli's Theorem and the diagonization Theorem we conclude that we conclude that there exists a subsequence of functions x_m, $m=1, 2,\dots$ which converges for every $t \in [0, \infty)$ and defines a limit-function x, which is a solution of (1) with $x_0=\varphi$.

EXAMPLE 2. Consider the same equation as in Example 1, where we now choose b so that

$$b < \min\{1, \vartheta - 1\}.$$

Furthermore suppose that the function $u(t)$, $t \geqq 0$ is bounded (then always there exists a $k^* > 0$ so that (23) holds).

It is now clear that the conditions of Theorem 4 are satisfied and therfore the equation (22) has a solution x defined on $J \cup [0, \infty)$ and such that $x_0 = \varphi$, $\varphi \in \bar{U}(0, b)$.

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