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On the space of orderings and the group H

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Let F be a formally real field and P a preordering of F. In his paper [7], M. Marshall introduced an equivalence relation in the space $X(F/P)$ of orderings by making use of fans of index 8, and the notion of connected components of $X(F/P)$ by an equivalence class of the relation.

The main purpose of this paper is to show that the number of connected components of $X(F/P)$ coincides with the dimension of $\boldsymbol{Z}_2$-vector space $H(P)/P$ for a subgroup $H(P)$, which is defined in §2. We also show, in §3, that if $K=F(\sqrt{a})$ is a quadratic extension of F with a an element of Kaplansky's radical, then the number of connected components of $X(K/P')$ equals twice that of $X(F/P)$, where P' is the preordering $\Sigma P\cdot\dot{K}^2$ of K. We should note that the groups $H(P)$ and $H(P')$ are connected by an important relation $N^{-1}(H(P))=F\cdot H(P')$, where N is the norm map of K to F.

For a subset A in a set B, the cardinality of A will be denoted by $|A|$ and the complementary subset of A in B by $B-A$ or A^c.

§1. Preorderings and fans

Throughout this paper, a field F always means a formally real field. We denote by $\dot{F}$ the multiplicative group of F. For a multiplicative subgroup P of $\dot{F}$, P is said to be a preordering of F if P is additively closed and $\dot{F}^2\subseteq P$. We denote by $X(F)$ the space of all orderings σ of F and by $X(F/P)$ the subspace of all orderings σ with $P(\sigma)\supseteq P$, where $P(\sigma)$ is the positive cone of σ. For a subset Y of $X(F)$, we denote by $Y^{\perp}$ the preordering $\cap P(\sigma)$, $\sigma\in Y$. Conversely for any preordering P, there exists a subset $Y\subseteq X(F)$ such that $P=Y^{\perp}$. Thus we have $P=X(F/P)^{\perp}$ and in particular $X(F)^{\perp}=D_F(\infty)=\Sigma\dot{F}^2$. We put $\phi^{\perp}=\dot{F}$ for convenience. The topological structure of $X(F)$ is determined by Harrison sets $H(a)=\{\sigma\in X(F);\ a\in P(\sigma)\}$ as its subbasis, where a ranges over $\dot{F}$. An arbitrary open set in $X(F)$ is thus a union of sets of the form $H(a_1,\dots, a_r)=H(a_1)\cap\cdots\cap H(a_r)$. For a preordering P of F, we write $H(a_1,\dots, a_n/P)=H(a_1,\dots, a_n)\cap X(F/P)$ where $a_i\in\dot{F}$.

For two forms f and g over F, we write $f\sim g(\operatorname{mod} P)$ if for any $\sigma\in X(F/P)$, $sgn_\sigma(f)=sgn_\sigma(g)$ where $sgn_\sigma(f)$ and $sgn_\sigma(g)$ are the signatures at σ of f and g, respectively. If $f\sim g(\operatorname{mod} P)$ and $\dim f=\dim g$, we write $f\cong g(\operatorname{mod} P)$. For a form $f=\langle a_1,\dots, a_n\rangle$ and $b\in F$, if there exist $p_1,\dots, p_n\in P\cup\{0\}$ such that $a_1p_1+\cdots+a_np_n=b$ and $(p_1,\dots, p_n)\neq(0,\dots, 0)$, then we say that the form f represents b over P. We put $D(f/P)=\{b\in\dot{F}; f \text{ represents } b \text{ over } P\}$. We say that f is P-isotropic or f is isotropic over P if f represents 0 and P-anisotropic or antisotropic over P otherwise.

Proofs for the following lemmas can be found in [2].

LEMMA 1.1. ([2], Satz 3, Lemma 4, Satz 7). *Let P be a preordering of a field F and φ, ψ be forms over F. Then the following statements hold.*

(1) *φ is P-isotropic if and only if $D(\varphi/P)=\dot{F}$.*

(2) *If $\varphi\cong\psi(\mathrm{mod}\, P)$, then $D(\varphi/P)=D(\varphi/P)$.*

LEMMA 1.2. ([2], Satz 15). *Let P be a preordering of a field F and a, b be elements of $\dot{F}$. If the form $\langle a, b, ab\rangle$ represents 1 over P, then there exists $c\in\dot{F}$ such that $\langle\langle a, b\rangle\rangle\cong\langle\langle 1, c\rangle\rangle(\mathrm{mod}\, P)$.*

LEMMA 1.3. *Let P be a preordering of a field F and a, b be elements of $\dot{F}$. If $\mathrm{sgn}_\sigma(\langle\langle a, b\rangle\rangle)=0$ for any $\sigma\in X(F/P)$, then the form $\langle 1, a, b\rangle$ is P-isotropic.*

PROOF. We have $\langle 1, a, b\rangle\cong\langle 1, -1, -ab\rangle(\mathrm{mod}\, P)$ by the assumption. Then the assertion follows from Lemma 1.1. Q.E.D.

LEMMA 1.4. *Let P be a preordering of a field F and $a_1,\dots, a_n$ be elements of $\dot{F}$. Then $D(\langle\langle a_1,\dots, a_n\rangle\rangle/P)=H(a_1,\dots, a_n/P)^\perp$.*

PROOF. If $D(\langle\langle a_1,\dots, a_n\rangle\rangle/P)\neq\dot{F}$, then $P'=D(\langle\langle a_1,\dots, a_n\rangle\rangle/P)$ is a preordering and it is clear that $P'\subseteq H(a_1,\dots, a_n/P)^\perp$. Conversely the fact $X(F/P')\subseteq H(a_1,\dots, a_n/P)$ implies $P'=X(F/P')^\perp\supseteq H(a_1,\dots, a_n/P)^\perp$. If $D(\langle\langle a_1,\dots, a_n\rangle\rangle/P)=\dot{F}$, then the form $\langle\langle a_1,\dots, a_n\rangle\rangle$ is P-isotropic and $H(a_1,\dots, a_n/P)=\phi$. In this case we have also $D(\langle\langle a_1,\dots, a_n\rangle\rangle/P)=H(a_1,\dots, a_n/P)^\perp$ since we put $\phi^\perp=\dot{F}$. Q.E.D.

If F is not a SAP (Strong Approximation Property) field, then there exist distinct orderings $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ such that $\sigma_4=\sigma_1\sigma_2\sigma_3$ (a fan of index 8) by [4], Satz 3.20.

Let P be a preordering of a field F and $\varphi=\langle 1, a, b, -ab\rangle$ be a quadratic form over F which is P-anisotropic. By Zorn's Lemma, there exists a maximal preordering $P'\supseteq P$ over which φ is anisotropic. In this section, we shall show that P' is a fan of index 8, namely $X(F/P')=\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, $\sigma_4=\sigma_1\sigma_2\sigma_3$.

LEMMA 1.5. *Let P be a preordering of a field F and a, b be elements of $\dot{F}$ such that the form $\langle 1, a, b, -ab\rangle$ is P-isotropic. Then there exists $c\in\dot{F}$ which satisfies the following conditions (1) and (2).*

(1) $D(\langle\langle -a, -b\rangle\rangle/P)=D(\langle\langle -c\rangle\rangle/P)$.

(2) $D(\langle\langle a\rangle\rangle/P)\cap D(\langle\langle b\rangle\rangle/P)=D(\langle\langle c\rangle\rangle/P)$.

PROOF. From the assumption, we can find a non-trivial relation $p_1+ap_2+bp_3-abp_4=0$ with $p_i\in P\cup\{0\}$, $i=1, 2, 3, 4$. If $p_1=0$, then the form $\langle -a, -b, ab\rangle\cong-\langle a, b, -ab\rangle$ is P-isotropic and $D(\langle -a, -b, ab\rangle/P)=\dot{F}$. If $p_1\neq 0$, then we have a relation $1=-ap_2p_1^{-1}-bp_3p_1^{-1}+abp_4p_1^{-1}$ and this shows that the form $\langle -a, -b, ab\rangle$ represents 1 over P. Anyway the form $\langle -a, -b, ab\rangle$, which is the pure part of the 2-fold Pfister form $\langle\langle -a, -b\rangle\rangle$, represents 1 over P. By Lemma 1.2, there exists $c\in\dot{F}$ such that $\langle\langle -a, -b\rangle\rangle\cong\langle\langle 1, -c\rangle\rangle(\mathrm{mod}\, P)$, and we have $D(\langle\langle -a, -b\rangle\rangle/P)=D(\langle\langle 1, -c\rangle\rangle/P)=D(\langle\langle -c\rangle\rangle/P)$.

As for the condition (2), we have $D(\langle\langle a\rangle\rangle/P)\cap D(\langle\langle b\rangle\rangle/P)=H(a/P)^{\perp}\cap H(b/P)^{\perp}=(H(a/P)\cup H(b/P))^{\perp}$ and therefore $D(\langle\langle a\rangle\rangle/P)\cap D(\langle\langle b\rangle\rangle/P)=(H(-a, -b/P)^c)^{\perp}$. It now follows from Lemma 1.4 that $H(-a, -b/P)=H(-c/P)$. Therefore, again by using Lemma 1.4, we have $(H(-a, -b/P)^c)^{\perp}=(H(-c/P)^c)^{\perp}=D(\langle\langle c\rangle\rangle/P)$ from which the condition (2) follows.

LEMMA 1.6. *Let P be a preordering of a field F and $\langle 1, a, b, -ab\rangle$ be a P-an*[illegible]*c form. Then the following statements hold.*

(1) *$H(a, b/P)\neq\phi$.*

(2) *$\tilde{P}=D(\langle\langle a\rangle\rangle/P)\cap D(\langle\langle b\rangle\rangle/P)$ is a preordering and the form $\langle 1, a, b, -ab\rangle$ is $\tilde{P}$-anisotropic.*

PROOF. Suppose, on the contrary, that $H(a, b/P)=\phi$. Then for any $\sigma\in X(F/P)$, $sgn_\sigma(\langle\langle a, b\rangle\rangle)=0$. By Lemma 1.3, the form $\langle 1, a, b\rangle$ is P-isotropic and this contradicts the assumption that the form $\langle 1, a, b, -ab\rangle$ is P-anisotropic. So we have (1). As for the statement (2), since the form $\langle 1, a, b, -ab\rangle$ is P-anisotropic, we have $a\notin -P$ and $b\notin -P$. Then it is clear that $\tilde{P}$ is a preordering. Suppose that the form $\langle 1, a, b, -ab\rangle$ is $\tilde{P}$-isotropic. Then there is a non-trivial relation $p_1+ap_2+bp_3-abp_4=0$ with $p_i\in\tilde{P}\cup\{0\}$, $i=1, 2, 3, 4$. Here $p_4\neq 0$; in fact, by considering p_1 and p_2 as elements of $D(\langle\langle a\rangle\rangle/P)$ and p_3 as an element of $D(\langle\langle b\rangle\rangle/P)$, the relation $p_1+ap_2+bp_3=0$ would imply that the form $\langle 1, a, b\rangle$ is P-isotropic. Thus we may assume that $p_4=1$ without loss of generality, and this implies that the form $\langle 1, a, b\rangle$ represents ab over P. This is a contradiction. Q. E. D.

LEMMA 1.7. *Let $\langle 1, a, b, -ab\rangle$ be a form over F and P be a maximal preordering such that $\langle 1, a, b, -ab\rangle$ is P-anisotropic. Then we have*

$$D(\langle\langle a, -b\rangle\rangle/P)\cap D(\langle\langle -a, b\rangle\rangle/P)\cap D(\langle\langle -a, -b\rangle\rangle/P)=P.$$

PROOF. Since $-ab\langle 1, a, b, -ab\rangle\cong\langle 1, -a, -b, -ab\rangle$ is P-anisotropic, Lemma 1.6 says that the form $\langle 1, -a, -b, -ab\rangle$ is anisotropic over $\tilde{P}=D(\langle\langle -a\rangle\rangle/P)\cap D(\langle\langle -b\rangle\rangle/P)$. Thus $\langle 1, a, b, -ab\rangle$ is $\tilde{P}$-anisotropic and we have $P=\tilde{P}$ by the maximality of P. On the other hand, we have $H(a, -b/P)^{\perp}\cap H(-a, -b/P)^{\perp}=(H(a, -b/P)\cup H(-a, -b/P))^{\perp}=H(-b/P)^{\perp}$ and this implies $D(\langle\langle a, -b\rangle\rangle/P)\cap$

$D(\langle\langle -a, -b\rangle\rangle/P)=D(\langle\langle -b\rangle\rangle/P)$ by Lemma 1.4. Similarly $D(\langle\langle -a, b\rangle\rangle/P)\cap D(\langle\langle -a, -b\rangle\rangle/P)=D(\langle\langle -a\rangle\rangle/P)$, and so we have $D(\langle\langle a, -b\rangle\rangle/P)\cap D(\langle\langle -a, b\rangle\rangle/P)\cap D(\langle\langle -a, -b\rangle\rangle/P)=D(\langle\langle -b\rangle\rangle/P)\cap D(\langle\langle -a\rangle\rangle/P)=P$. Q.E.D.

THEOREM 1.8. *Let $\langle 1, a, b, -ab\rangle$ be a form over F and P be a maximal preordering such that $\langle 1, a, b, -ab\rangle$ is P-anisotropic. Then P is a fan of index 8.*

PROOF. By [1], Corollary 3.4, P is a preordering of finite index. In general, let P be a preordering of finite index of a field F and Y be a subset of $X(F/P)$ such that $Y^{\perp}=P$. Then we can find a basis of $X(F/P)$ which is a subset of Y. Lemma 1.7 shows that $(H(a, -b/P)\cup H(-a, b/P)\cup H(-a, -b/P))^{\perp}=P$; thus there exists a basis $B=\{\sigma_{2i}, \sigma_{3j}, \sigma_{4k};\ i\in I, j\in J, k\in K\}$ of $X(F/P)$ where $\sigma_{2i}\in H(a, -b/P)$, $\sigma_{3j}\in H(-a, b/P)$ and $\sigma_{4k}\in H(-a, -b/P)$. There exists an ordering $\sigma_1\in H(a, b/P)$ by Lemma 1.6 (1). Then we can write σ_1 by using the basis B as

$$\sigma_1 = \Pi\sigma_{2i}\cdot\Pi\sigma_{3j}\cdot\Pi\sigma_{4k} \qquad (i\in I', j\in J', k\in K')\cdots\cdots\cdots(A)$$

where $I'\subseteq I$, $J'\subseteq J$ and $K'\subseteq K$. We shall show that each subset I', J' or K' is not empty. Suppose $I'=\phi$. Then by calculating the signature of $-a$ at the both sides of (A), $-a$ is negative at σ_1 and positive at $\Pi\sigma_{2i}\cdot\Pi\sigma_{3j}\cdot\Pi\sigma_{4k}=\Pi\sigma_{3j}\cdot\Pi\sigma_{4k}$. This is a contradiction and we have $I'\neq\phi$. By taking elements $-b$ for J' and $-ab$ for K', we can similarly show that J' and K' are not empty. We now put $\tilde{B}=\{\sigma_{2i}, \sigma_{3j}, \sigma_{4k}, i\in I', j\in J', k\in K'\}$ and $\tilde{P}=\tilde{B}^{\perp}$. Suppose that the form $\langle 1, a, b, -ab\rangle$ is $\tilde{P}$-isotropic. Then by Lemma 1.5, there exists $c\in F$ which satisfies the following conditions (1) and (2):

(1) $D(\langle\langle -a, -b\rangle\rangle/\tilde{P}) = D(\langle\langle -c\rangle\rangle/\tilde{P})$

(2) $D(\langle\langle a\rangle\rangle/\tilde{P}) \cap D(\langle\langle b\rangle\rangle/\tilde{P}) = D(\langle\langle c\rangle\rangle/\tilde{P})$.

Then it follows from (1) and (2) that c is negative at σ_{4k}, $k\in K'$ and positive at $\sigma_1, \sigma_{2i}, \sigma_{3j}$ $(i\in I', j\in J')$. So the equation (A) says that $|K'|$ is even. Therefore $-ab$ is negative at σ_1 and positive at $\Pi\sigma_{2i}\cdot\Pi\sigma_{3j}\cdot\Pi\sigma_{4k}$, $i\in I'$, $j\in J'$, $k\in K'$. This contradiction shows that the form $\langle 1, a, b, -ab\rangle$ is $\tilde{P}$-anisotropic. By the maximality of P, we have $P=\tilde{P}$. Since B is a basis of $X(F/P)$, the fact $P=\tilde{P}$ means $I=I'$, $J=J'$ and $K=K'$. This shows that σ_1 is a unique element of $H(a, b/P)$, namely $H(a, b/P)=\{\sigma_1\}$. Similarly, by considering the forms $\langle 1, a, -b, ab\rangle\cong a\langle 1, a, b, -ab\rangle$, $\langle 1, -a, b, ab\rangle\cong b\langle 1, a, b, -ab\rangle$ and $\langle 1, -a, -b, -ab\rangle\cong -ab\langle 1, a, b, -ab\rangle$ instead of $\langle 1, a, b, -ab\rangle$, we have $|H(a, -b/P)|=1$, $|H(-a, b/P)|=1$ and $|H(-a, -b/P)|=1$. Hence $|X(F/P)|=4$ and the equation (A) shows that P is a fan of index 8. Q.E.D.

§ 2. Connected components and $H(P)$

Let P be a preordering over F. We shall say that two orderings σ, $\tau \in X(F/P)$ are connected in $X(F/P)$ if $\sigma=\tau$ or there exists a fan of index 8 which contains σ and τ, and we denote the relation by $\sigma \sim \tau$. Marshall ([7], Theorem 4.7) showed that the relation $\sim$ is an equivalence relation in $X(F/P)$. An equivalence class of this relation is called a connected component of $X(F/P)$, and a union of some connected components is called full (cf. [3]).

DEFINITION 2.1. Let P be a preordering of a field F. For $x \in \dot{F}$, we denote the multiplicative subgroup $D(\langle\langle x\rangle\rangle/P) \cdot D(\langle\langle -x\rangle\rangle/P)$ by $J(x/P)$, and the set $\{x \in \dot{F};\ J(x/P)=\dot{F}\}$ by $H(P)$.

LEMMA 2.2. *Let P be a preordering of a field F. Then, for elements x and y of $\dot{F}$, the following conditions are equivalent.*

(1) $x \in J(y/P)$.

(2) $\langle 1, y, -x, xy\rangle$ *is P-isotropic.*

(3) $\langle 1, x, -y, xy\rangle$ *is P-isotropic.*

(4) $y \in J(x/P)$.

PROOF. (1)⇒(2). Since $x \in J(y/P)=D(\langle\langle y\rangle\rangle/P) \cdot D(\langle\langle -y\rangle\rangle/P)$, $x=\alpha\beta$ for some $\alpha \in D(\langle\langle y\rangle\rangle/P)$ and $\beta \in D(\langle\langle -y\rangle\rangle/P)$. Thus we have $\alpha\beta^2-x\beta=0$ and it follows from the facts $\alpha\beta^2 \in D(\langle\langle y\rangle\rangle/P)$ and $-x\beta \in D(\langle -x, xy\rangle/P)$ that $\langle 1, y, -x, xy\rangle$ is P-isotropic.

(2)⇒(1). From the assumption, there exists a non-trivial relation $p_1+yp_2-xp_3+xyp_4=0$ with $p_i \in P \cup \{0\}$, $i=1, 2, 3, 4$. If $p_1+yp_2=x(p_3-yp_4)=0$, then at least one of the forms $\langle\langle y\rangle\rangle$ and $\langle\langle -y\rangle\rangle$ is P-isotropic and we have $J(y/P)=D(\langle\langle y\rangle\rangle/P) \cdot D(\langle\langle -y\rangle\rangle/P)=\dot{F}$. If $p_1+yp_2=x(p_3-yp_4)\neq 0$, then $x(p_3-yp_4)^2=(p_1+yp_2)(p_3-yp_4) \in D(\langle\langle y\rangle\rangle/P) \cdot D(\langle\langle -y\rangle\rangle/P)$. Therefore in any case we have $x \in J(y/P)$.

The equivalence of the conditions (2) and (3) is clear from $xy\langle 1, y, -x, xy\rangle \cong \langle 1, x, -y, xy\rangle$. Q.E.D.

REMARK 2.3. (1). $X(F/P)$ satisfies SAP if and only if $\langle 1, x, y, -xy\rangle$ is P-isotropic for any x, $y \in \dot{F}$. By Lemma 2.2, these are equivalent to the condition that $J(y/P)=\dot{F}$ for every $y \in \dot{F}$, namely $H(P)=\dot{F}$.

(2). Since $H(P)=\{x \in \dot{F};\ J(x/P) \ni y$ for any $y \in \dot{F}\}$, it follows from Lemma 2.2 that $H(P)=\{x \in \dot{F};\ x \in J(y/P)$ for every $y \in \dot{F}\}=\cap J(y/P)$, $y \in \dot{F}$. Thus $H(P)$ is a multiplicative subgroup of $\dot{F}$ which contains P, and $H(P)/P$ has a $\boldsymbol{Z}_2$-vector space structure and we denote its dimension by $\dim H(P)/P$.

PROPOSITION 2.4. *Let P be a preordering of F which is of finite index.*

Then for a subset $Y \subseteq X(F/P)$, the following conditions are equivalent.

(1) *Y is full.*

(2) *$Y = H(a/P)$ for some $a \in H(P)$.*

PROOF. First we assume that Y is full. Then for any fan W of index 8, $|W \cap Y| = 0$ or 4. So by [8], Theorem 7.2, we have $Y = H(a/P)$ for some $a \in \dot{F}$. Suppose $a \notin H(P)$. Then $J(a/P) = D(\langle\langle a \rangle\rangle/P) \cdot D(\langle\langle -a \rangle\rangle/P) \subsetneqq \dot{F}$ and so we can take an element $b \in \dot{F} - J(a/P)$. By Lemma 2.2, $\langle 1, a, b, -ab \rangle$ is P-anisotropic and this implies that there exists a preordering $\tilde{P} \supseteq P$ such that $\tilde{P}$ is a fan of index 8 and $\langle 1, a, b, -ab \rangle$ is $\tilde{P}$-anisotropic by Theorem 1.8. Hence we have $|H(a/P) \cap X(F/\tilde{P})| = |Y \cap X(F/\tilde{P})| = 2$, which contradicts the assumption that Y is full.

Conversely suppose that Y is not full. Then there exists a fan $W = \{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ of index 8 such that $Y \cap W \neq \phi$ and $Y^c \cap W \neq \phi$. By [8], Theorem 7.2 $|Y \cap W| = 2$, so we may assume $\sigma_1, \sigma_2 \in Y$ and $\tau_1, \tau_2 \in Y^c$. We let $A_1 = \{b \in \dot{F};\ \mathrm{sgn}_{\sigma_1}(b) \cdot \mathrm{sgn}_{\sigma_2}(b) = 1\}$ and $A_2 = \{b \in \dot{F};\ \mathrm{sgn}_{\tau_1}(b) \cdot \mathrm{sgn}_{\tau_2}(b) = 1\}$. It is clear that A_1 and A_2 are multiplicative subgroups of $\dot{F}$. Moreover, since $\sigma_1, \sigma_2 \in Y$, we have $Y^\perp = D(\langle\langle a \rangle\rangle/P) \subseteq A_1$ and similarly $(Y^c)^\perp = D(\langle\langle -a \rangle\rangle/P) \subseteq A_2$. Now $W = \{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ is a fan of index 8 and $\sigma_1\sigma_2 = \tau_1\tau_2$, and so $A_1 = A_2$. It follows from the assumption $a \in H(P)$, namely $D(\langle\langle a \rangle\rangle/P) \cdot D(\langle\langle -a \rangle\rangle/P) = \dot{F}$, that $A_1 = A_2 = \dot{F}$, which leads to a contradiction $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$. Q. E. D.

THEOREM 2.5. *Let P be a preordering of F which is of finite index. Then the number of connected components of $X(F/P)$ equals* $\dim H(P)/P$.

PROOF. Let S be the set of full sets of $X(F/P)$ and $\varphi: H(P)/P \to S$ be the map defined by $\varphi(\bar{a}) = H(a/P)$ where $\bar{a}$ means the canonical image of $a \in H(P)$. If $ab \in P$, then $H(a/P) = H(b/P)$. From this fact and Proposition 2.4, we can see that φ is well-defined and surjective. We have to show that φ is injective. Suppose $\varphi(\bar{a}) = \varphi(\bar{b})$, namely $H(a/P) = H(b/P)$. Then ab is positive at every $\sigma \in X(F/P)$, and so $ab \in X(F/P)^\perp = P$. This means $\bar{a} = \bar{b}$ and φ is injective. Let n be the number of connected components and m be $\dim H(P)/P$. Since $|H(P)/P| = 2^m$, $|S| = 2^n$ and φ is bijective, we have $2^m = 2^n$ and $m = n$. Q. E. D.

COROLLARY 2.6. *Let P be a preordering of F of index 2^n. Then the number of connected components of $X(F/P)$ is not $n-1$.*

PROOF. It suffices to show that $\dim H(P)/P \neq n-1$. To do this, we have to see that if $H(P) \neq \dot{F}$, then $\dim H(P)/P \leqq n-2$ by Theorem 2.5. Let $b \in \dot{F} - H(P)$. Since $b \in J(b/P) - H(P)$, $J(b/P)$ contains $H(P)$ properly. Moreover the fact $b \notin H(P)$ implies $J(b/P) \neq \dot{F}$. Therefore we see that $\dim \dot{F}/H(P) \geqq 2$ and $\dim H(P)/P \leqq n-2$. Q. E. D.

COROLLARY 2.7. *Let P be a preordering of F of finite index and $Y_1, \ldots, Y_n$*

be connected components of $X(F/P)$. *We write* $P_i=Y_i^{\perp}$ $i=1,\dots,n$. *Then the canonical map* $f\colon \dot{F}/P\to\Pi\dot{F}/P_i$ $(i=1,\dots,n)$ *is isomorphic and the map* $g\colon H(P)/P\to\Pi H(P_i)/P_i$ $(i=1,\dots,n)$ *is isomorphic, where g is the restriction of f to* $H(P)/P$.

PROOF. By Proposition 2.4, for any $i=1,\dots,n$, there exists $a_i\in H(P)$ such that $Y_i=H(a_i/P)$. Then we have $P_i=D(\langle\langle a_i\rangle\rangle/P)$ and $D(\langle\langle -a_i\rangle\rangle/P)=(Y_i^c)^{\perp}=\cap P_j$, $j\neq i$. Since $D(\langle\langle a_i\rangle\rangle/P)\cdot D(\langle\langle -a_i\rangle\rangle/P)=\dot{F}$, $P_i\cdot(\cap P_j)=\dot{F}$, $j\neq i$. Then we can readily see that the canonical injection f is surjective. As for g, it is clear that $H(P)\subseteq H(P_i)$ for any $i=1,\dots,n$, and therefore g is well-defined. Clearly g is injective and it follows from Theorem 2.5 that $\dim H(P)/P=n$ and $\dim H(P_i)/P_i=1$ for any $i=1,\dots,n$. Hence $\dim H(P)/P=\dim \Pi H(P_i)/P_i$ and this implies that g is an isomorphism. Q. E. D.

§3. Quadratic extensions

Let P be a preordering of F and $K=F(\sqrt{a})$ be a quadratic extension of F with $a\in F-(-P\cup F^2)$. Since $a\notin -P$, $H(a/P)$ is not an empty set and every ordering $\sigma\in H(a/P)$ can be extended to an ordering of K. Let τ be an extension of $\sigma\in H(a/P)$ such that $\sqrt{a}$ is positive at τ. Then the positive cone $P(\tau)$ of τ is the set of $x+y\sqrt{a}\in\dot{K}$, where (x,y) satisfies one of the following conditions (1), (2), (3):

(1) $x,\ y\in P(\sigma)$.

(2) $x,\ -y\in P(\sigma)$ and $x^2-ay^2\in P(\sigma)$.

(3) $-x,\ y\in P(\sigma)$ and $-(x^2-ay^2)\in P(\sigma)$.

This is easily shown by using $x^2-ay^2=(x-y\sqrt{a})(x+y\sqrt{a})$. This observation implies the uniqueness of τ. Thus for any $\sigma\in H(a/P)$, there exist exactly two extensions $\sigma_1,\sigma_2\in X(K)$ of σ such that $\sqrt{a}$ is positive at σ_1 and $\sqrt{a}$ is negative at σ_2. Put $P'=\Sigma P\dot{K}^2$ and $X'=\{\tau\in X(K)$; the restriction of τ to F belongs to $H(a/P)\}$. It is clear that P' is a preordering of K which is contained in $P(\tau)$ for any $\tau\in X'$.

LEMMA 3.1. *The following statements hold.*

(1) $P'=(X')^{\perp}$ (2) $P'\cap F=D_F(\langle\langle a\rangle\rangle/P)$.

PROOF. (1). Since $P'\subseteq P(\tau)$ for any $\tau\in X'$, we have $P'\subseteq(X')^{\perp}$. Conversely, $D_F(\langle\langle a\rangle\rangle/P)\subseteq P'$ by the definition of P', which implies $X(K/P')\subseteq X'$. Thus $P'=X(K/P')^{\perp}\supseteq(X')^{\perp}$.

(2). In (1), we have shown that $D_F(\langle\langle a\rangle\rangle/P)\subseteq P'$. For the reverse inclusion, we take $b\in\dot{F}-D_F(\langle\langle a\rangle\rangle/P)$; then there exists $\sigma\in H(a/P)$ such that b is negative at σ. Let τ be an extension of σ in K. The fact that b is negative at τ implies $b\notin(X')^{\perp}=P'$. This shows $P'\cap F\subseteq D_F(\langle\langle a\rangle\rangle/P)$. Q. E. D.

PROPOSITION 3.2. *Let* $N: K \to F$ *be the norm map. Then* $N^{-1}(P)=\dot{F}\cdot P'$.

PROOF. Let S be the set of all Pfister forms $\langle\langle p_1,\dots, p_n\rangle\rangle$ where $p_i \in P$. By [5], Norm Principle 2.13, $N^{-1}(D_F(\rho))=\dot{F}\cdot D_K(\rho_K)$ for any $\rho \in S$, where $\rho_K=\rho\otimes K$. Hence $N^{-1}(\cup D_F(\rho))=\cup(N^{-1}(D_F(\rho)))=\cup(\dot{F}\cdot D_K(\rho_K))$, $\rho\in S$. Then the facts $P=\cup D_F(\rho)$ and $P'=\cup D_K(\rho_K))$ imply the assertion. Q. E. D.

COROLLARY 3.3. *Let* $\varepsilon: \dot{F}\to\dot{K}$ *be the canonical injection. Then the sequence*

$$1 \longrightarrow \dot{F}/D_F(\langle\langle a\rangle\rangle/P) \xrightarrow{\bar{\varepsilon}} \dot{K}/P' \xrightarrow{\bar{N}} \dot{F}/P$$

is exact, where $\bar{\varepsilon}$ *and* $\bar{N}$ *are induced maps of* ε *and* N *respectively.*

PROOF. Lemma 3.1 (2) shows that $\bar{\varepsilon}$ is well-defined and injective. Proposition 3.2 shows that $\bar{N}$ is well-defined and $Ker\ \bar{N}=Im\ \bar{\varepsilon}$. Q. E. D.

In [6], we called a quadratic extension $K=F(\sqrt{a})$ a radical extension if $a\in R(F)-\dot{F}^2$, where $R(F)$ is Kaplansky's radical of F.

LEMMA 3.4. *Let* $K=F(\sqrt{a})$ *be a radical extension of* F. *Let* σ *and* τ *be arbitrary orderings of* F *and* σ_i, τ_i $(i=1, 2)$ *be the extensions in* K *of* σ, τ *respectively. Then* $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ *is not a fan of index* 8.

PROOF. Put $P=P(\sigma)\cap P(\tau)$. The norm map $N: K\to F$ is surjective since $a\in R(F)$ and by Corollary 3.3, we have the exact sequence

$$1 \longrightarrow \dot{F}/P \xrightarrow{\bar{\varepsilon}} K/P' \xrightarrow{\bar{N}} \dot{F}/P \longrightarrow 1$$

where $P'=\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$. Since $\dim \dot{F}/P=2$, we have $\dim \dot{K}/P'=4$, which implies that $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ is linearly independent. Q. E. D.

Let P be a preordering of a field F, $K=F(\sqrt{a})$ be a quadratic extension of F with $a\in\dot{F}-(-P\cup F^2)$. Let P', X' be the preordering of K and the set of orderings defined in Lemma 3.1. We denote by bar the Galois map of K over F and for a subset A of K, we put $\bar{A}=\{\bar{x}; x\in A\}$. For a ordering τ of K, we denote by $\bar{\tau}$ the ordering of K with the positive cone $P(\tau)^{-}$. For a subset $B\subseteq X'$, we also write $\bar{B}=\{\bar{\tau}; \tau\in B\}$. It is clear that $\bar{P}'=P'$, $\bar{X}'=X'$ and $\bar{\sigma}_1=\sigma_2$ where σ_1 and σ_2 are the extensions of $\sigma\in H(a/P)$.

COROLLARY 3.5. *Let* P *be a preordering of* F *and* $K=F(\sqrt{a})$ *be a radical extension of* F. *Then for any connected component* Y *of* $X'=X(K/P')$, $Y\cap\bar{Y}=\phi$.

PROOF. Suppose $Y\cap\bar{Y}\neq\phi$. Then there exists $\sigma\in X(F/P)$ such that $\sigma_1\sim\sigma_2$ where σ_1 and σ_2 are the extensions of σ. Let $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ be a fan of index 8 and τ_1', τ_2' be the restriction of τ_1, τ_2 to F respectively. Since $\sigma_1\sigma_2\tau_1\tau_2=1$, we have $\tau'_1=\tau'_2$, which is a contradiction by Lemma 3.4. Q.E.D.

LEMMA 3.6. *Let* $K=F(\sqrt{a})$ *be a radical extension of* F. *Then for any* $x\in\dot{F}$, $N^{-1}(J_F(x/P))=\dot{F}\cdot J_K(x/P')$.

PROOF. If $x\in P$ or $x\in -P$, then $J_F(x/P)=\dot{F}$, $J_K(x/P')=\dot{K}$ and the assertion follows immediately in this case. We now proceed to the case when $x\notin P$ and $-x\notin P$. Then $D_F(\langle\langle x\rangle\rangle/P)$ is a preordering of F and $N^{-1}(D_F(\langle\langle x\rangle\rangle/P)=\dot{F}\cdot D_K(\langle\langle x\rangle\rangle/P')$ by Proposition 3.2. Similarly $N^{-1}(D_F(\langle\langle -x\rangle\rangle/P)=\dot{F}\cdot D_K(\langle\langle -x\rangle\rangle/P')$. Therefore we see that $N^{-1}(J_F(x/P))\supseteq\dot{F}\cdot D_K(\langle\langle x\rangle\rangle/P)\cdot D_K(\langle\langle -x\rangle\rangle/P)=\dot{F}\cdot J_K(x/P')$. We note that $N(\dot{F}\cdot D_K(\langle\langle x\rangle\rangle/P'))=D_F(\langle\langle x\rangle\rangle/P)$, $N(\dot{F}\cdot D_K(\langle\langle -x\rangle\rangle/P'))=D_F(\langle\langle -x\rangle\rangle/P)$ since $K=F(\sqrt{a})$ is a radical extension of F.

To show the reverse inclusion, we take $\alpha\beta\in J_F(x/P)$, where $\alpha\in D_F(\langle\langle x\rangle\rangle/P)$ and $\beta\in D_F(\langle\langle -x\rangle\rangle/P)$. There exist $f_1, f_2\in\dot{F}$, $b_1\in D_K(\langle\langle x\rangle\rangle/P')$ and $b_2\in D_K(\langle\langle -x\rangle\rangle/P')$ such that $N(f_1b_1)=\alpha$ and $N(f_2b_2)=\beta$. Then for any $z\in N^{-1}(\alpha\beta)$, $N(f_1b_1f_2b_2z)=(\alpha\beta)^2\in\dot{F}^2$ and this implies $f_1b_1f_2b_2z\in\dot{F}\cdot\dot{K}^2$ by Hilbert Theorem 90. Hence $z\in f_1f_2b_1b_2\dot{F}\cdot\dot{K}^2\subseteq\dot{F}\cdot D_K(\langle\langle x\rangle\rangle/P')\cdot D_K(\langle\langle -x\rangle\rangle/P')$ and we see that $N^{-1}(J_F(x/P))\subseteq\dot{F}\cdot J_K(x/P')$. Q.E.D.

LEMMA 3.7. *Let* $K=F(\sqrt{a})$ *be a radical extension of* F. *Then for any* $b\in\dot{F}$, $J_K(b/P')\cap\dot{F}=J_F(b/P)$.

PROOF. It is clear that $J_K(b/P')\cap\dot{F}\supseteq J_F(b/P)$. Conversely, we take an element $x\in J_K(b/P')\cap\dot{F}$. By Lemma 2.2, the form $\langle 1, b, -x, bx\rangle$ over K is P'-isotropic. So by the definition of P', a form $\langle p_1,\dots, p_n\rangle\otimes\langle 1, b, -x, bx\rangle$ over K is isotropic for some $p_1,\dots, p_n\in P$. If the form $\langle p_1,\dots, p_n\rangle\otimes\langle 1, b, -x, bx\rangle$ over F is anisotropic, then there is a subform which is similar to the universal binary form $\langle 1, -a\rangle$, a contradiction. Therefore the form $\langle p_1,\dots, p_n\rangle\otimes\langle 1, b, -x, bx\rangle$ over F is isotropic. So the form $\langle 1, b, -x, bx\rangle$ over F is P-isotropic and $x\in J_F(b/P)$ by Lemma 2.2. Thus we have $J_K(b/P')\cap F\subseteq J_F(b/P)$. Q.E.D.

PROPOSITION 3.8. *Let* $K=F(\sqrt{a})$ *be a radical extention of* F. *Then* $H_F(P)=H_K(P')\cap\dot{F}$.

PROOF. For any $b\in H_F(P)$, we have $J_F(b/P)=\dot{F}$ and this implies $\dot{K}=N^{-1}(\dot{F})=N^{-1}(J_F(b/P))=\dot{F}\cdot J_K(b/P')$ by Lemma 3.6. Since $\dot{F}\subseteq J_K(b/P')$, we have $\dot{K}=J_K(b/P')$ and so $b\in H_K(P')$. Hence $H_F(P)\subseteq H_K(P')\cap F$.

Conversely we take an element $b\in H_K(P')\cap\dot{F}$. Then $J_K(b/P')=\dot{K}\supsetneqq\dot{F}$ and we have $b\in H(P)$ since $J_F(b/P)=\dot{F}$ by Lemma 3.7. Q.E.D.

PROPOSITION 3.9. *Let* $K=F(\sqrt{a})$ *be a radical extension of* F. *Then* $N(H_K(P'))\subseteq H_F(P)$.

PROOF. It is clear that if $J_K(x/P')=\dot{K}$, then $J_K(\bar{x}/P')=\dot{K}$ by the fact $\bar{P}'=P'$. So we have $H_K(P')^-=H_K(P')$. It follows from Proposition 3.8 that, for $\alpha\in H_K(P')$, $N(\alpha)\in H_K(P')\cap\dot{F}=H_F(P)$. Thus we have $N(H_K(P'))\subseteq H_F(P)$. Q.E.D.

THEOREM 3.10. *Let P be a preordering of F of finite index, and $K=F(\sqrt{a})$ be a radical extension of F. Then the sequence*

$$1 \longrightarrow \dot{F}/H_F(P) \xrightarrow{\bar{\varepsilon}} K/H_K(P) \xrightarrow{\bar{N}} \dot{F}/H_F(P) \longrightarrow 1$$

is exact. In particular $N^{-1}(H_F(P))=\dot{F}\cdot H_K(P')$ and the number of connected components of $X(K/P')$ is $2n$, where n is the number of connected components of $X(F/P)$.

PROOF. The map $\bar{\varepsilon}$ is well-defined and injective by Proposition 3.8 and $\bar{N}$ is well-defined by Proposition 3.9. Since $K=F(\sqrt{a})$ is a radical extension of F, $\bar{N}$ is surjective and it is clear that $Im\,\bar{\varepsilon}\subseteq Ker\,\bar{N}$. We need to show that $\dim\dot{K}/H(P')=2\dim\dot{F}/H(P)$. Since $\dim\dot{K}/H(P')\geqq 2\dim\dot{F}/H(P)$, we have only to show that $\dim\dot{K}/H(P')\leqq 2\dim\dot{F}/H(P)$. By Corollary 3.3, the sequence

$$1 \longrightarrow \dot{F}/P \xrightarrow{\bar{\varepsilon}} \dot{K}/P' \xrightarrow{\bar{N}} \dot{F}/P \longrightarrow 1$$

is exact, and so $\dim\dot{K}/P'=2\dim\dot{F}/P$. Thus it suffices to show that $\dim H(P')/P'\geqq 2\dim H(P)/P$ by the facts $\dim\dot{K}/P'=\dim\dot{K}/H(P')+\dim H(P')/P'$ and $\dim\dot{F}/P=\dim\dot{F}/H(P)+\dim H(P)/P$.

The number n of connected components of $X(F/P)$ equals $\dim H(P)/P$ by Theorem 2.5. Let $X_1,\dots,X_n$ be the connected components of $X(F/P)$. By Proposition 2.4, there exist $a_i\in H(P)$, $i=1,\dots,n$, such that $X_i=H(a_i/P)$. Let $Y_i=H(a_i/P')\subseteq X'$, $i=1,\dots,n$; then each Y_i is full since $a_i\in H_K(P')$ by Proposition 3.8. Since the restriction of Y_i to F is X_i for every i, the sets Y_i, $i=1,\dots,n$, are disjoint to each other. It is clear that $\bar{Y}_i=Y_i$ from the definition of Y_i. So Corollary 3.5 implies that Y_i is not connected for any i. Hence the number of connected components of X' is at least $2n$. Thus, it follows from Theorem 2.5 that $\dim H(P')/P'\geqq 2n=2\dim H(P)/P$. Q.E.D.

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