HIROSHIMA MATH. J.
13 (1983), 165–202

Some nonlinear degenerate diffusion equations with a nonlocally convective term in ecology

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(Received September 14, 1982)

1. Introduction

In the past several years, from an ecological point of view a number of authors (e.g. Gurney and Nisbet [11], Gurtin and MacCamy [12], Aronson [3], Newman [19] et al) have studied spatial spreading population models in which biological interactions and nonlinear diffusion process called "density-dependent dispersal" are taken into account. This nonlinear diffusion process is described by an equation of degenerate parabolic type.

In this paper, we are concerned with a model for the spatial diffusion of biological population which provides a kind of mechanism of aggregation and which is represented by equation

$$(1.1)\qquad u_t = (u^m)_{xx} - \left[\left\{\int_{-\infty}^{\infty} K(x-y)u(y, t)dy\right\}u\right]_x, \; x \in \boldsymbol{R}^1, \; t > 0$$

subject to an initial condition

$$(1.2)\qquad u(x, 0) = u_0(x), \; x \in \boldsymbol{R}^1,$$

where $u(x, t)$ denotes the population density at point $x \in \boldsymbol{R}^1$ and at time $t>0$ and $1<m<\infty$. We assume the following assumptions on u_0 and K:

(A.1) $u_0 \geq 0$ on $\boldsymbol{R}^1$ and $u_0 \in L^1(\boldsymbol{R}^1) \cap L^\infty(\boldsymbol{R}^1)$;

(A.2) K is differentiable on $\boldsymbol{R}^1$ except for a finite number of discontinuity points of the first kind, $K \in L^\infty(\boldsymbol{R}^1)$ and $K' \in L^1(\boldsymbol{R}^1)$.

Here K' means dK/dx. In what follows we denote the problem (1.1), (1.2) by $P(K, u_0)$.

If the term containing K is absent, the equation (1.1) is reduced to the "porous media equation" occurring in the theory of flow through porous media (see. [5]). The most interesting phenomenon is that, because of the degeneracy of diffusion at $u=0$, an initial smooth disturbance with compact support spreads out at a finite speed (see. [20]) and loses the smoothness (see. [2] and [13]). This contrasts with the property of the heat conduction case ($m=1$). For the second term of the right hand side of (1.1), we give a specific function K defined by

$$(1.3)\qquad K(x)=\begin{cases} 1 & \text{for} \quad -r<x<0, \\ -1 & \text{for} \quad 0<x<r, \\ 0 & \text{otherwise,} \end{cases}$$

where $0<r\leq\infty$. Then the term containing K is rewritten as

$$\left[\left(\int_x^{x+r} u(y, t)dy - \int_{x-r}^{x} u(y, t)dy\right)u\right]_x .$$

This provides the mechanism that moves $u(x, t)$ to the right (resp. left) direction if

$$\int_x^{x+r} u(y, t)dy > \int_{x-r}^{x} u(y, t)dy \quad (\text{resp.} <).$$

Hence, in ecological terms, in the case of K given by (1.3) we would expect that a suitable balance between the diffusion process and the aggregative one gives rise to a pulse-like pattern exhibiting an aggregation of individuals. In the case of K given by (1.3) with $r=\infty$, it was shown by Nagai and Mimura [18] that the phenomenon mentioned just above actually occurs. On the other hand, Satsuma [22] has dealt with the equation (1.1) in the case when $m=1$ and $K(x)=(k/2\delta)\cdot\coth\{\pi x/(2\delta)\}$ not belonging to $L^\infty(\boldsymbol{R}^1)$. He showed two types of exact solutions for (1.1). One is a stationary solution, and the other is a blowing up solution depending on the initial values. The type of equation (1.1) occurs in other fields. Munakata [16] presented it in order to explain liquid instability and freezing, and also Kuramoto [14] in order to explain rhythms and turbulences in populations of chemical oscillators.

From the fact that classical solutions of the Cauchy problem for the porous media equation do not always exist, we have to define solutions of our problem $P(K, u_0)$ in some generalized sense.

DEFINITION 1.1. A *solution* $u(x, t)$ of the Cauchy problem $P(K, u_0)$ is defined to be a nonnegative function on $\boldsymbol{R}^1\times(0, \infty)$ which satisfies the following conditions:

(i) $u\in L^\infty(\boldsymbol{R}^1\times(0, T))\cap C(\boldsymbol{R}^1\times(0, \infty))\cap C((0, \infty); L^1(\boldsymbol{R}^1))$ for any $T>0$;

(ii) $\int_{-\infty}^{\infty} K(x-y)u(y, t)dy\in L^\infty(\boldsymbol{R}^1\times(0, T))\cap C(\boldsymbol{R}^1\times(0, \infty))$ for any $T>0$;

(iii) $(u^m)_x\in L^2(\boldsymbol{R}^1\times(0, T))$ for any $T>0$;

(iv) u satisfies the identity

$$\int_0^\infty \int_{-\infty}^\infty \left[uf_t - \left\{ (u^m)_x - \left(\int_{-\infty}^\infty K(x-y)u(y, t)dy \right) u \right\} f_x \right] dxdt$$
$$+ \int_{-\infty}^\infty u_0(x) f(x, 0) dx = 0 \qquad \text{for any} \quad f \in C_0^1(\boldsymbol{R}^1 \times [0, \infty)).$$

The purpose of this paper is to show the uniqueness, existence and regularity results for $P(K, u_0)$ under the assumptions (A.1) and (A.2), and to give some properties of solutions.

In Section 2, notations and preliminaries which will be used later are given. In Section 3, we shall show the uniqueness of solutions for $P(K, u_0)$. Section 4 consists of two parts and gives auxiliary results for approximate solutions of $P(K, u_0)$. As an approximation to $P(K, u_0)$, we consider the Cauchy problem for certain non-degenerate parabolic equations. We deal with the local existence in time in Subsection 4.1 and the global existence in Subsection 4.2. In Section 5, by making use of the results obtained in Section 4, we shall show the global existence of solutions for $P(K, u_0)$ under the assumptions (A.1) and (A.2). In Section 6, we shall give some properties of solutions for $P(K, u_0)$ such that finite propagation of disturbances, the regularity result of solutions and the dependency of solutions for $P(K, u_0)$ on K.

The author would like to express his deepest gratitude to Professor Masayasu Mimura for suggesting this research topic and various discussions. He also thanks Professor Takaŝi Kusano for continued encouragement throughout the course of this work.

2. Notation and preliminaries

In this section we introduce some notations and give some propositions which will play important roles in the proofs of uniqueness and existence of solutions for the problem $P(K, u_0)$ and will be used to derive some properties of the solutions.

Let Q be a set in $\boldsymbol{R}^N$ $(N \geq 1)$. For a nonnegative integer m, $C^m(Q)$ is the set of all continuous functions in Q having continuous derivatives in Q up to order m inclusively. For simplicity we denote $C^0(Q)$ by $C(Q)$. $C^\infty(Q)$ is the set of all functions having continuous derivatives in Q up to arbitrary order. In the case of $Q \subset \boldsymbol{R}^2$ whose points are denoted by (x, t), for nonnegative integers m and n $C^{m,n}(Q)$ is the set of all continuous functions in Q having continuous derivatives in Q up to order m with respect to x and order n with respect to t. $C_0^m(Q)$, $C_0^\infty(Q)$ and $C_0^{m,n}(Q)$ are the sets consisting of all functions in $C^m(Q)$, $C^\infty(Q)$ and $C^{m,n}(Q)$ with compact support in Q, respectively.

$B^m(\boldsymbol{R}^1)$ is the Banach space of functions f in $C^m(\boldsymbol{R}^1)$ having a finite norm

$$|f|_m = \sum_{i=0}^m \sup_{\boldsymbol{R}^1} |(d/dx)^i f(x)|.$$

Let us put $Q_T=\boldsymbol{R}^1\times(0, T)$ for $0<T\leq\infty$. $B^{m,n}(Q_T)$ is the set of functions $f=f(x, t)$ in $C^{m,n}(Q_T)$ having a finite value for the quantity

$$|f|_{m,n,Q_T} = \sum_{i=0}^{m} \sup_{Q_T} |(\partial/\partial x)^i f(x, t)| + \sum_{i=0}^{n} \sup_{Q_T} |(\partial/\partial t)^i f(x, t)|.$$

The norm of the Hölder space on $\boldsymbol{R}^1$ is introduced as follows. For $\alpha\in(0, 1]$ and a positive integer m,

$$[f]_\alpha = |f|_0 + \sup_{x,y\in\boldsymbol{R}^1} |f(x) - f(y)|/|x - y|^\alpha$$

and

$$[f]_{m+\alpha} = \sum_{i=0}^{m} [(d/dx)^i f]_\alpha.$$

The set of all functions for which $[f]_\alpha<\infty$ (resp. $[f]_{m+\alpha}<\infty$) is denoted by $H^\alpha(\boldsymbol{R}^1)$ (resp. $H^{m+\alpha}(\boldsymbol{R}^1)$).

For the closure of $Q_T=\boldsymbol{R}^1\times(0, T)$, say $\bar{Q}_T$, $H^{\alpha,\alpha/2}(\bar{Q}_T)$ is the Hölder space consisting of functions $f=f(x, t)$ on $\bar{Q}_T$ which have a finite norm

$$[f]_{\alpha,Q_T} = \sup_{Q_T} |f(x, t)| + \sup_{(x,s),(y,t)\in\bar{Q}_T} |f(x, s) - f(y, t)|/(|x-y|^\alpha + |s - t|^{\alpha/2}).$$

$H^{1+\alpha,\alpha/2}(\bar{Q}_T)$ (resp. $H^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$) is the set of all functions f satisfying $[f]_{1+\alpha,Q_T}<\infty$ (resp. $[f]_{2+\alpha,Q_T}<\infty$), where

$$[f]_{1+\alpha,Q_T} = [f]_{\alpha,Q_T} + [\partial f/\partial x]_{\alpha,Q_T}$$

and

$$[f]_{2+\alpha,Q_T} = [f]_{1+\alpha,Q_T} + [(\partial/\partial x)^2 f]_{\alpha,Q_T} + [\partial f/\partial t]_{\alpha,Q_T}.$$

For $p\in[1, \infty]$ the norm of the usual $L^p(\boldsymbol{R}^1)$-space is denoted by $\|f\|_p$, and the usual norm of $L^p(Q_T)$-space is denoted by $\|f\|_{p,Q_T}$. We simply denote $\|f\|_{p,Q_\infty}$ by $\|f\|_p$ if there is no confusion.

For a Banach space X and $-\infty<a<b<\infty$, $C((a, b); X)$ is the set of all functions which are continuous from (a, b) into X. $L^p(a, b: X)$ $(1\leq p\leq\infty)$ is the set of all measurable functions f from (a, b) into X such that $t\to\|f(t)\|$ belongs to $L^p(a, b)$, where $\|\cdot\|$ denotes the norm in X.

We next state three propositions which are derived from the definition of solutions for $P(K, u_0)$. Throughout these propositions it is assumed that u_0 and K satisfy the assumptions (A.1) and (A.2), respectively. Let u be a solution of $P(K, u_0)$. Then we have

PROPOSITION 2.1. *For each $\tau\in(0, \infty)$ the solution u satisfies the integral identity*

$$\int_{\tau}^{\infty}\int_{-\infty}^{\infty}\left\{uf_t - \left[(u^m)_x - \left(\int_{-\infty}^{\infty} K(x-y)u(y,t)dy\right)u\right]f_x\right\}dxdt$$

$$+\int_{-\infty}^{\infty} u(x,\tau)f(x,\tau)dx = 0 \qquad \textit{for every} \quad f\in C_0^1(\boldsymbol{R}^1 \times [\tau,\infty)).$$

This proposition implies that the function $u(x, t)$ on $\boldsymbol{R}^1\times[\tau, \infty)$ is a solution of the problem $P(K, u(\cdot, \tau))$ on $\boldsymbol{R}^1\times[\tau, \infty)$, and is proved by using the definition of solutions for $P(K, u_0)$ and a calculation similar to that in the proof of Proposition 1 in [9].

PROPOSITION 2.2. *For any* $t\in(0, \infty)$ u *satisfies*

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx.$$

This proposition means that the total population at each time is fixed. The proof is done by using Proposition 2.1 and the same method as in Theorem 1 in [9].

For solutions of $P(K, u_0)$ we consider a transformation which is a useful tool in giving the uniqueness result and some properties of solutions for $P(K, u_0)$. For a solution u of $P(K, u_0)$ define the functions v and v_0 by

$$v(x,t) = \int_{-\infty}^{x} u(y,t)dy \quad \text{for } x\in\boldsymbol{R}^1 \text{ and } t\in(0,\infty)$$

and

$$v_0(x) = \int_{-\infty}^{x} u_0(y)dy \quad \text{for } x\in\boldsymbol{R}^1,$$

respectively. Then we have the following proposition.

PROPOSITION 2.3. v *is continuous on* $\boldsymbol{R}^1\times(0, \infty)$ *and has the following properties*:

(i) $\|v(\cdot, t) - v_0\|_2\to 0$ *as* $t\to 0$;

(ii) $0\le v\le \|u_0\|_1$ *on* $\boldsymbol{R}^1\times(0, \infty)$, *and* $v(-\infty, t)=0$ *and* $v(+\infty, t)=\|u_0\|_1$ *for each* $t\in(0, \infty)$;

(iii) $v_x\ge 0$ *on* $\boldsymbol{R}^1\times(0, \infty)$, v_x *is bounded on* $\boldsymbol{R}^1\times(0, T)$ *for every* $T>0$ *and* $v_x\in C(\boldsymbol{R}^1\times(0, \infty))\cap C((0, \infty): L^1(\boldsymbol{R}^1))$;

(iv) $((v_x)^m)_x$, $v_t\in L^2(\boldsymbol{R}^1\times(0, T))$ *for any* $T>0$;

(v) v *satisfies the integral identity*

$$\int_0^\infty \int_{-\infty}^\infty \left\{ vf_t + \left[((v_x)^m)_x - \left(\int_{-\infty}^\infty K(x-y)u(y, t)dy \right) v_x \right] f \right\} dxdt$$

$$+ \int_{-\infty}^\infty v_0(x) f(x, 0) dx = 0 \qquad \textit{for every} \quad f \in C_0^1(\boldsymbol{R}^1 \times [0, \infty)).$$

PROOF. The properties (ii), (iii) and $((v_x)^m)_x \in L^2(\boldsymbol{R}^1 \times (0, T))$ are easily derived from the definition of solutions for $P(K, u_0)$ and Proposition 2.2. The property (v) is proved by the same method as in the proof of Proposition 3.2 in [17]. $v_t \in L^2(\boldsymbol{R}^1 \times (0, T))$ follows from (v) and $((v_x)^m)_x \in L^2(\boldsymbol{R}^1 \times (0, T))$. The property (i) is shown as follows. Since $v_t \in L^2(\boldsymbol{R}^1 \times (0, T))$, for $0<s<t<T$ we have

$$\int_{-\infty}^\infty |v(x, t) - v(x, s)|^2 dx \leq |t - s| \int_0^T \int_{-\infty}^\infty |v_t|^2 dxdt$$

Combining this relation with the fact

$$\lim_{t \to 0} \int_{-\infty}^\infty v(x, t)\varphi(x) dx = \int_{-\infty}^\infty v_0(x)\varphi(x) dx \qquad \text{for every} \quad \varphi \in C_0^1(\boldsymbol{R}^1),$$

which follows from (v), we obtain

$$\int_{-\infty}^\infty |v(x, t) - v_0(x)|^2 dx \leq t \int_0^T \int_{-\infty}^\infty (v_t)^2 dxdt$$

The relation implies (i). Thus the proof is completed.

The following proposition which is proved in [6] will be used in the proof of Lemma 3.1 in the next section.

PROPOSITION 2.4. *Let f, k_1 and k be nonnegative continuous functions on an interval $[\alpha, \beta]$ and let $0 \leq p < 1$ and $a \geq 0$. If*

$$f(t) \leq a + \int_\alpha^t k_1(s) f(s) ds + \int_\alpha^t k(s) f^p(s) sd \qquad \textit{for any} \quad t \in [\alpha, \beta],$$

then

$$f(t) \leq \left[a^q + q \int_\alpha^t k(s) \exp\left\{ - q \int_\alpha^s k_1(\sigma) d\sigma \right\} ds \right]^{1/q} \exp\left\{ \int_\alpha^t k_1(s) ds \right\}$$

for any $t \in [\alpha, \beta]$, where $q = 1 - p$.

Finally, for the Cauchy problem

$$(2.1) \qquad \begin{cases} \mathscr{L}u \equiv u_t - a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u = f \quad \textit{in} \quad Q_T = \boldsymbol{R}^1 \times (0, T), \\ u(x, 0) = 0 \quad \textit{on} \quad \boldsymbol{R}^1, \end{cases}$$

where $0<T<\infty$, we give an estimate for solutions which will play an important role in Section 4.

PROPOSITION 2.5. *It is assumed that*

(i) *a, b, c and f belong to $B^0(\bar{Q}_T)$,*
(ii) *there exists a positive constant μ satisfying $a(x, t)\geq\mu$ on $\bar{Q}_T$,*
(iii) *$a\in H^{\alpha,\alpha/2}(\bar{Q}_T)$ and for any x, $y\in \boldsymbol{R}^1$ and $t\in[0, T]$*

$$|b(x, t)-b(y, t)|\leq C|x-y|^{\alpha} \quad and \quad |c(x, t)-c(y, t)|\leq C|x-y|^{\alpha},$$

where α is some constant in $(0, 1]$ and C is some positive constant. Let $u\in C^{2,1}(Q_T)$ be a bounded solution of (2.1). Then for any $\delta\in(0, 1)$ we have

$$[u]_{1+\delta,Q_T}\leq MT^{(1-\delta)/2}\|f\|_{\infty,Q_T},$$

where $M=M(\mu, [a]_{\alpha,Q_T}, \|b\|_{\infty,Q_T}, \|c\|_{\infty,Q_T}, T)$ is a positive constant not depending on u such that M increases in all variables but μ and $M\uparrow\infty$ as $\mu\downarrow 0$.

The proof of Proposition 2.5 is done by using a calculation similar to that in the proof of Lemma 2 [7, p. 193], because a solution of (2.1) is represented as the volume potential of f with respect to a fundamental solution of the operator $\mathscr{L}$. In Lemma 2 of [7], only volume potentials on bounded cylindrical domains are dealt with, and the constant corresponding to M in our Proposition 2.5 depends on the lower base of the cylindrical domain. However, by slightly modifying the proof of Lemma 2 of [7], we can remove the dependence on the lower base of the cylindrical domain.

3. Uniqueness of solutions for the problem $P(K, u_0)$

THEOREM 3.1. *Assume the assumptions* (A.1) *and* (A.2) *on u_0 and K. Then there exists at most one solution for the problem $P(K, u_0)$.*

This theorem is an immediate consequence of Lemma 3.1 mentioned below. Before stating Lemma 3.1, we introduce a notation which will be used in the rest of this paper. Let K be a function on $\boldsymbol{R}^1$ satisfying (A.2) and let us put $Q_T=\boldsymbol{R}^1\times(0, T)$ for $0<T<\infty$. For $f\in L^p(Q_T)$ we define

$$K[f](x, t)=\int_{-\infty}^{\infty}K(x-y)f(y, t)dy.$$

If $f\in L^p(Q_T)$ and $f_x\in L^1(Q_T)\cap L^p(Q_T)$ for $p\in(1, \infty]$, then, by making use of an integration by parts and Young's inequality, we obtain

$$\|K[f_x]\|_{p,Q_T}\leq C(K)\|f\|_{p,Q_T}, \tag{3.1}$$

where

$$C(K) = \sum_{i=1}^{n} |K(c_i+0) - K(c_i-0)| + \|K'\|_1 \tag{3.2}$$

and $\{c_i\}$ is the set of discontinuity points of the first kind for K.

LEMMA 3.1. *Let u_1 (resp. u_2) be a solution of the problem $P(K_1, u_{01})$ (resp. $P(K_2, u_{02})$), where K_1 and K_2 satisfy the assumption (A.2) and u_{01} and u_{02} satisfy the assumption (A.1), and for each $i=1, 2$ let us define the functions v_i and v_{0i} by*

$$v_i(x, t) = \int_{-\infty}^{x} u_i(y, t)dy \quad \text{for} \quad x \in R^1 \text{ and } t \in (0, \infty)$$

and

$$v_{0i}(x) = \int_{-\infty}^{x} u_{0i}(y)dy \qquad \text{for} \quad x \in \boldsymbol{R}^1.$$

Assume $v_{01} - v_{02} \in L^2(\boldsymbol{R}^1)$. Then we have the following relation: For any $t \in (0, T)$ with an arbitrarily fixed $T \in (0, \infty)$

$$\begin{aligned} &\|v_1(\cdot, t) - v_2(\cdot, t)\|_2 \\ &\le e^{Mt}\left\{\|v_{01} - v_{02}\|_2 + \int_0^t \|(K_1 - K_2)[u_1](\cdot, s)u_1(\cdot, s)\|_2 e^{-Ms} ds\right\}, \end{aligned}$$

where $M = \max(\|u_1\|_{Q_T}, \|u_2\|_{Q_T})C(K_2) + 1$ and $C(K_2)$ is the constant determined by (3.2) with K replaced by K_2.

PROOF. The definition of solutions for $P(K, u_0)$ and Proposition 2.3 give us the relations such that $K_i[u_i](v_i)_x \in L^p(Q_T)$ $(1 \le p \le \infty)$, $(((v_i)_x)^m)_x$, $(v_i)_t \in L^2(Q_T)$ for any $T > 0$ and

$$(v_1 - v_2)_t = [(u_1)^m - (u_2)^m]_x - K_1[u_1](v_1)_x + K_2[u_2](v_2)_x \text{ a.e. in } Q_\infty. \tag{3.3}$$

For each positive integer N let χ_N be a smooth function on $\boldsymbol{R}^1$ such that $0 \le \chi_N \le 1$ on $\boldsymbol{R}^1$, $\chi_N(x) = 1$ for $|x| < N$, $\chi_N(x) = 0$ for $|x| \ge N+1$, $\|\chi_N'\|_\infty \le C$, where C is a positive constant independent of N. Multiply (3.3) by $(v_1 - v_2)\chi_N$ and integrate over Q_s $(0 < s < \infty)$. Then the use of an integration by parts in the resulting relation yields that

$$\begin{aligned} &\int_{-\infty}^{\infty} |v_1(x, s) - v_2(x, s)|^2 \chi_N(x) dx \\ &\le \int_{-\infty}^{\infty} |v_{01}(x) - v_{02}(x)|^2 dx - 2\int_0^s \int_{-\infty}^{\infty} \{(u_1)^m - (u_2)^m\}(v_1 - v_2)\chi_N' dx dt \\ &\quad - 2\int_0^s \int_{-\infty}^{\infty} \chi_N(v_1 - v_2)(K_1[u_1]u_1 - K_2[u_2]u_2) dx dt \\ &= \|v_{01} - v_{02}\|_2^2 + I - II. \end{aligned}$$

Note that $\{(u_1)^m-(u_2)^m\}(v_1-v_2)$, $(v_1-v_2)(K_1[u_1]u_1-K_2[u_2]u_2)\in L^1(Q_s)$. We let N pass to infinity to obtain

$$|I| \leq 2C\int_0^s\int_{N\leq|x|\leq N+1} |(u_1)^m - (u_2)^m|\,|v_1 - v_2|dxdt \longrightarrow 0$$

and

$$II \longrightarrow 2\int_0^s\int_{-\infty}^{\infty} (v_1-v_2)(K_1[u_1]u_1 - K_2[u_2]u_2)dxdt.$$

Therefore, for any $s\in(0, \infty)$ we get

$$\begin{aligned}&\|v_1(\cdot, s) - v_2(\cdot, s)\|_2^2\\ &\leq \|v_{01} - v_{02}\|_2^2 - 2\int_0^s\int_{-\infty}^{\infty} (v_1-v_2)(K_1[u_1]u_1 - K_2[u_2]u_2)dxdt\\ &= \|v_{01} - v_{02}\|_2^2 + III.\end{aligned}$$

The term III is rewritten as

$$\begin{aligned}&-2\int_0^s\int_{-\infty}^{\infty} (v_1-v_2)(u_1-u_2)K_2[u_2]dxdt - 2\int_0^s\int_{-\infty}^{\infty} (v_1-v_2)u_1K_2[u_1-u_2]dxdt\\ &-2\int_0^s\int_{-\infty}^{\infty} (v_1-v_2)u_1(K_1-K_2)[u_1]dxdt\\ &= III_1 + III_2 + III_3.\end{aligned}$$

From the fact that $u_1-u_2=(v_1-v_2)_x$ and $v_1(x, t)-v_2(x, t)\to 0$ as $|x|\to\infty$ for any $t\in(0, \infty)$ it follows that

$$\begin{aligned}III_1 &= \int_0^s\int_{-\infty}^{\infty} (v_1-v_2)^2(K_2[u_2])_x dxdt\\ &\leq \|(K_2[u_2])_x\|_\infty \int_0^s \|v_1(\cdot, t) - v_2(\cdot, t)\|_2^2 dt.\end{aligned}$$

Noting that $(K_2[u_2])_x=K_2[(u_2)_x]$ and using (3.1) for $K=K_2$, we obtain

$$III_1 \leq C(K_2)\|u_2\|_\infty \int_0^s \|v_1(\cdot, t) - v_2(\cdot, t)\|_2^2 dt.$$

Since $u_1-u_2=(v_1-v_2)_x$, by making use of Hölder's inequality and (3.1) in III_2 we have

$$III_2 \leq \int_0^s \|v_1(\cdot, t) - v_2(\cdot, t)\|_2^2 dt + \|u_1\|_\infty C(K_2)\int_0^s \|v_1(\cdot, t) - v_2(\cdot, t)\|_2^2 dt.$$

Next the term III_3 is dominated by

$$2\int_0^s \|(K_1-K_2)[u_1](\cdot, t)u_1(\cdot, t)\|_2\|v_1(\cdot, t) - v_2(\cdot, t)\|_2 dt.$$

Hence we obtain

$$\|v_1(\cdot, s) - v_2(\cdot, s)\|_2^2$$
$$\leq \|v_{01} - v_{02}\|_2^2 + 2M\int_0^s \|v_1(\cdot, t) - v_2(\cdot, t)\|_2^2 dt$$
$$+ 2\int_0^s \|(K_1 - K_2)[u_1](\cdot, t)u_1(\cdot, t)\|_2 \|v_1(\cdot, t) - v_2(\cdot, t)\|_2 dt,$$

where $M = \max(\|u_1\|_{Q_T}, \|u_2\|_{Q_T})C(K_2) + 1$. Using Lemma 2.4 for the inequality obtained just above, we get

$$\|v_1(\cdot, s) - v_2(\cdot, s)\|_2^2$$
$$\leq e^{2Ms}\left\{\|v_{01} - v_{02}\|_2 + \int_0^t \|(K_1 - K_2)[u_1](\cdot, t)u_1(\cdot, t)\|_2 e^{-Mt} dt\right\}^2$$

for any $s \in (0, T)$, which implies the desired inequality. Thus the proof is completed.

4. Auxiliary results for approximate solutions of $P(K, u_0)$

As will be shown in Section 5, a soluton of $P(K, u_0)$ will be constructed as a limit of a sequence of nonnegative classical solutions for the Cauchy problem of nonlinear (non-degenerate) parabolic equations

$$u_t = (m(u+\varepsilon)^{m-1}u_x)_x - (K[u]u)_x \quad \text{in} \quad Q_T \tag{4.1}$$

subject to an initial condition

$$u(x, 0) = u_0(x) \quad \text{on} \quad \boldsymbol{R}^1, \tag{4.2}$$

where ε is a positive constant, $0 < T < \infty$ and

$$K[u](x, t) = \int_{-\infty}^{\infty} K(x-y)u(y, t)dy.$$

Throughout this section we impose the following assumptions on u_0 and K:

(A.3) $u_0 \in H^{2+\alpha}(\boldsymbol{R}^1) \cap L^1(\boldsymbol{R}^1)$ for some $\alpha \in (0, 1]$ and $u_0 \geq 0$ on $\boldsymbol{R}^1$;

(A.4) K satisfies the assumption (A.2) and $K \in L^1(\boldsymbol{R}^1)$.

4.1. Local existence of nonnegative classical solutions for the problem (4.1), (4.2)

In this subsection, for this non-degeneracy of parabolicity we are concerned with nonnegative classical solutions of (4.1), (4.2). We shall show the local existence of nonnegative classical solutions in time for the problem (4.1), (4.2) under the assumptions (A.3) and (A.4). The result is as follows.

THEOREM 4.1. *There is a time $T(>0)$ such that the problem* (4.1), (4.2) *has a nonnegative classical solution belonging to $H^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$.*

The proof of this theorem will be done by making use of the fixed point theorem for a contraction mapping. Let f be a function belonging to $H^{1+\alpha,\alpha/2}(\bar{Q}_T)$. Consider the Cauchy problem for the equation

$$(4.3) \qquad u_t = (m(u+\varepsilon)^{m-1}u_x)_x - (K[f]u)_x \quad \text{in} \quad Q_T.$$

We denote the Cauchy problem (4.3), (4.2) by $P_\varepsilon(K, u_0, f)$. If there exists a unique nonnegative classical solution u of $P_\varepsilon(K, u_0, f)$, then we can consider the mapping F assigning u to f. A fixed point of F is a nonnegative classical solution of (4.1), (4.2). In what follows, by using a series of lemmas mentioned below we shall show that F has a fixed point if T is sufficiently small.

At first some properties of $K[f]$ which will be used later are shown in Lemma 4.1, where K satisfies the assumtion (A.4) and has discontinuity points $\{c_1, c_2, \ldots, c_n\}$ of the first kind. The proof is so easy that it is omitted.

LEMMA 4.1. *Let $f \in H^{1+\alpha,\alpha/2}(\bar{Q}_T)$ $(0<\alpha\leq 1)$. Then $K[f]$, $(K[f])_x \in H^{1+\alpha,\alpha/2}(\bar{Q}_T)$ and $(K[f])_{xx} \in H^{\alpha,\alpha/2}(\bar{Q}_T)$. We also have the following estimates:*

(i) $\|K[f]\|_{\infty,Q_T} \leq \|K\|_1 \|f\|_{\infty,Q_T}$;

(ii) $\|(K[f])_x\|_{\infty,Q_T} \leq C(K)\|f\|_{\infty,Q_T}$;

(iii) $\|(K[f])_{xx}\|_{\infty,Q_T} \leq C(K)\|f_x\|_{\infty,Q_T}$;

(iv) $[K[f]]_{\alpha,Q_T} \leq \|K\|_1 [f]_{\alpha,Q_T}$;

(v) $[(K[f])_x]_{\alpha,Q_T} \leq C(K)\,[f]_{\alpha,Q_T}$,

where

$$C(K) = \textstyle\sum_{i=1}^n |K(c_i+0) - K(c_i-0)| + \|K'\|_1.$$

In what follows we always assume

$$f \in H^{1+\alpha,\alpha/2}(\bar{Q}_T)\ (0 < \alpha \leq 1).$$

LEMMA 4.2. *The Cauchy problem $P_\varepsilon(K, u_0, f)$ has a unique nonnegative solution $u \in H^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$ having the following property: u_{xxx} and u_{xt} exist in Q_T and are locally Hölder continuous in x and t with exponents α and $\alpha/2$ in Q_T, respectively.*

PROOF. Let $a(u)$ be a smooth function on $\boldsymbol{R}^1$ such that $a(u)=m(u+\varepsilon)^{m-1}$ for $u>0$, $a(u)\geq m(\varepsilon/2)^{m-1}$ on $\boldsymbol{R}^1$ and there are positive constants ν and μ satisfying

$$\nu(|u|+\varepsilon)^{m-1} \leq a(u) \leq \mu(|u|+\varepsilon)^{m-1} \quad \text{on} \quad \boldsymbol{R}^1.$$

Consider the Cauchy problem for the equation

$$u_t = (a(u)u_x)_x - (K[f]u)_x \quad \text{in} \quad Q_T \tag{4.4}$$

subject to the initial condition (4.2). Noting the regularity properties of $K[f]$ in Lemma 4.1, we obtain by Theorem 8.1 in [15; p. 495] that there exists a unique solution u of (4.4), (4.2) belonging to $H^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$. By the standard maximum principle we have

$$u(x, t) \geq 0 \quad \text{on} \quad \bar{Q}_T \quad \text{and} \quad u(x, t) > 0 \quad \text{on} \quad Q_T,$$

which imply that u is the nonnegative solution of $P_\varepsilon(K, u_0, f)$. By the regularity results of solutions for the parabolic equations (for example, see [7]), the second assertion can be shown. Thus the proof is completed.

LEMMA 4.3. *The solution u of $P_\varepsilon(K, u_0, f)$ has the property that $u(x, t)\to 0$ and $u_x(x, t)\to 0$ as $|x|\to\infty$ uniformly in $t\in[0, T]$.*

PROOF. We note that the assumption (A.3) on u_0 implies

$$u_0(x) \longrightarrow 0 \text{ and } u_0'(x) \longrightarrow 0 \text{ as } |x| \longrightarrow \infty \tag{4.5}$$

and that u satisfies the equation

$$\mathscr{L}u \equiv u_t - a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u = 0$$

where

$$a(x, t) = m(u+\varepsilon)^{m-1} \geq m\varepsilon^{m-1},$$
$$b(x, t) = -m(m-1)(u+\varepsilon)^{m-2}u_x + K[f],$$
$$c(x, t) = (K[f])_x.$$

The functions a, b and c are bounded and uniformly Hölder continuous in $\bar{Q}_T$. Hence, combining a fundamental solution of the operator $\mathscr{L}$ with (4.5), we obtain the statements.

LEMMA 4.4. *The solution u of $P_\varepsilon(K, u_0, f)$ satisfies*

$$\int_{-\infty}^{\infty} u(x, t)dx = \int_{-\infty}^{\infty} u_0(x)dx \qquad \text{for every} \quad t > 0.$$

It is easy to prove the lemma, so we omit it. In the sequel, for the sake of simplicity we put

$$c = \int_{-\infty}^{\infty} u_0(x)dx.$$

A priori bounds for the solution u of $P_\varepsilon(K, u_0, f)$ are given in the following lemmas by using Bernstein's method which was used in [1, 10] to prove the regularity result of solutions for the porous media equation.

LEMMA 4.5. *The solution u of $P_\varepsilon(K, u_0, f)$ satisfies*

$$u^2(x, t) \le \max[1, e^2\|u_0\|_\infty^2, M_1\|(K[f])_x\|_{\infty, Q_T}] \equiv L_1,$$

where M_1 is a constant depending only on m and c.

PROOF. Let us define the function v on $\bar{Q}_T$ by

$$v(x, t) = \int_{-\infty}^{x} u(y, t)dy.$$

Integrating the equation (4.3) with respect to x from $-\infty$ to x and using Lemma 4.3 yield that v satisfies the equation

$$v_t = m(v_x+\varepsilon)^{m-1}v_x - K[f]v_x \text{ in } Q_T. \tag{4.6}$$

Define the function $\varphi(w)$ by

$$\varphi(w) = -2c + 6ce\int_0^w e^{-\xi^q}d\xi \qquad \text{for} \quad w \ge 0,$$

in which q is a constant satisfying

$$q[1-(m-1)2^{-q}] = 2.$$

Here we note that $q>2$. Since the range of variation $[w_1, w_2]$ when $\varphi(w)$ varies from 0 to c is determined by $\varphi(w_1)=0$ and $\varphi(w_2)=c$, we obtain

$$1/(3e) < w_1 < w_2 < 1/2.$$

For $w \in [w_1, w_2]$ we have

$$\begin{aligned} &\varphi' = 6cee^{-w^q} > 0, \qquad \varphi'' = -6ceqw^{q-1}e^{-w^q} < 0, \\ &\varphi''/\varphi' = -qw^{q-1}, \qquad (\varphi''/\varphi')' = -q(q-1)w^{q-2} < 0. \end{aligned} \tag{4.7}$$

A function $w(x, t)$ on Q_T is defined by

$$v(x, t) = \varphi(w(x, t)). \tag{4.8}$$

Substitution of (4.8) into (4.6) yields that

$$w_t = m(\varphi' w_x+\varepsilon)^{m-1}w_{xx} + m(\varphi' w_x+\varepsilon)^{m-1}(\varphi''/\varphi')(w_x)^2 - K[f]w_x. \tag{4.9}$$

Differentiate (4.9) with respect to x and put $p=w_x\geq 0$. We then have

$$
\begin{aligned}
(4.10)\qquad & p_t - m(\varphi'p+\varepsilon)^{m-1}p_{xx} \\
&= m(m-1)(\varphi'p+\varepsilon)^{m-2}(\varphi'p_x+\varphi''p^2)p_x + 2m(\varphi''/\varphi')(\varphi'p+\varepsilon)^{m-1}pp_x \\
&\quad + m(m-1)(\varphi''/\varphi')(\varphi'p+\varepsilon)^{m-2}(\varphi'p_x+\varphi''p^2)p^2 \\
&\quad + m(\varphi''/\varphi')'(\varphi'p+\varepsilon)^{m-1}p^3 - K[f]p_x - (K[f])_x p.
\end{aligned}
$$

Since p attains the maximum on $\bar{Q}_T$ by Lemma 4.3, it allows us to consider a point (x_0, t_0) where p takes the maximum on $\bar{Q}_T$. At first suppose that $t_0=0$. We then have

$$0\leq p(x, t)\leq \|p(\cdot, 0)\|_\infty \quad \text{on} \quad \bar{Q}_T,$$

and hence

$$0\leq u(x, t)=v_x(x, t)\leq e\|u_0\|_\infty \quad \text{on} \quad \bar{Q}_T.$$

Next suppose that $0<t_0\leq T$. At this point (x_0, t_0) the function p satisfies

$$p_x=0, \quad p_{xx}\leq 0 \text{ and } p_t\geq 0.$$

Combining these with (4.10), we obtain

$$(4.11)\qquad -m[\{(m-1)(\varphi'')^2/\varphi'\}p + (\varphi''/\varphi')'(\varphi'p+\varepsilon)](\varphi'p+\varepsilon)^{m-2}p^3\leq K[f]p.$$

It follows from (4.7) and the choice of q that

$$
\begin{aligned}
(4.12)\qquad & \{(m-1)(\varphi'')^2/\varphi'\}p + (\varphi''/\varphi')'(\varphi'p+\varepsilon) \\
&\leq \varphi'p\{(m-1)(\varphi''/\varphi')^2 + (\varphi''/\varphi')'\} \\
&\leq -6cw_1^{q-2}p.
\end{aligned}
$$

From (4.11) and (4.12) we have

$$p^3\leq \{w_1^{2-q}/(6mc)\}\|(K[f])_x\|_{\infty,Q_T}(\varphi'p+\varepsilon)^{2-m}.$$

It is enough to assume $p\geq 1$. The relation $\varphi'(w)=6cee^{-w^q}$ yields that

$$
(4.14)\qquad (\varphi'p+\varepsilon)^{2-m}\leq
\begin{cases}
(6ce+1)p & \text{if } 1<m<2, \\
(6c)^{2-m} & \text{if } m\geq 2.
\end{cases}
$$

(4.13) and (4.14) give

$$p^2\leq M\|(K[f])_x\|_{\infty,Q_T},$$

where M is a constant depending only on m and c. As a result this inequality obtained just above implies

$$u^2(x, t) \leq (6ce)^2 M \|(K[f])_x\|_{\infty, Q_T} \quad \text{in} \quad \bar{Q}_T.$$

Thus we have obtained the desired inequality.

LEMMA 4.6. *For the solution u of $P_\varepsilon(K, u_0, f)$, put $w=(u+\varepsilon)^{m-1}$. Then w satisfies the following relations*:

(i) $|w_x(x, t)|^2 \leq L_2$ *on* $\bar{Q}_T$,

where

$$L_2 = \max\,[1,\ e^2\|w_x(\cdot, 0)\|_\infty^2,\ 12e^2 L_3(\|(K[f])_x\|_{\infty, Q_T} + \|(K[f])_{xx}\|_{\infty, Q_T})]$$

and

$$L_3 = (\|u\|_{\infty, Q_T} + 1)^{m-1};$$

(ii) *For each τ with $0<\tau<T$ there exists a positive constant C_τ depending only on τ such that for every $(x, t) \in Q_{\tau,T} = \mathbf{R}^1 \times [\tau, T]$*

$$|w_x(x, t)|^2 \leq \max\,[1,\ 12e^2 L_3(\|(K[f])_x\|_{\infty, Q_{\tau,T}} + \|(K[f])_{xx}\|_{\infty, Q_{\tau,T}}) + C_\tau].$$

PROOF. The proof is done by using the same method as in the proof of Lemma 4.5. We take the function φ defined by

$$\varphi(z) = -2L_3 + 6L_3 e \int_0^z e^{-\xi} d\xi.$$

Let us define the function $z(x, t)$ on $\bar{Q}_T$ by $w(x, t) = \varphi(z(x, t))$ and let us put $p(x, t) = (\chi(t) z_x(x, t))^2$, where $\chi \equiv 1$ in the case of our proving the statement (i), and $\chi \in C^\infty(\mathbf{R}^1)$, $0 \leq \chi \leq 1$, $\chi(t) = 1$ on $[\tau, \infty)$ and $\chi(t) = 0$ on $(-\infty, \tau/2]$ in the case of our proving the statement (ii). We then consider a point where p attains its maximum on $\bar{Q}_T$. By making use of the same argument as in the proof of Lemma 4.5, we can establish Lemma 4.6.

LEMMA 4.7. *The solution u of $P_\varepsilon(K, u_0, f)$ satisfies*

$$|u_x(x, t)| \leq L_4 \quad on \quad \bar{Q}_T,$$

where

$$L_4 = \begin{cases} \{(L_1)^{1/2} + 1\}^{2-m}(L_2)^{1/2}/(m-1) & \text{if} \quad 1 < m \leq 2, \\ \varepsilon^{2-m}(L_2)^{1/2}/(m-1) & \text{if} \quad m > 2, \end{cases}$$

PROOF. Since $u_x = (u+\varepsilon)^{2-m} w_x/(m-1)$, Lemma 4.6 implies

$$|u_x| \leq (u+\varepsilon)^{2-m}(L_2)^{1/2}/(m-1) \leq L_4.$$

In order to obtain the Hölder continuity of u and $w=(u+\varepsilon)^{m-1}$ with respect to t we use the following result due to Gilding [8].

LEMMA 4.8. *Let $z \in C^{2,1}((a, b)\times(\tau, T)) \cap C([a, b]\times[\tau, T])$ be a solution of the equation*

$$z_t = A(x, t)z_{xx} + B(x, t)z_x + g(x, t) \quad in \quad (a, b) \times (\tau, T),$$

where $-\infty < a < b < \infty$, $0 \le \tau < T < \infty$ and A, B and g are continuous functions on $[a, b]\times[\tau, T]$ satisfying

$$0 < A(x, t) \le \mu,\ |B(x, t)| \le \mu \quad and \quad |g(x, t)| \le \mu \quad on \quad [a, b] \times [\tau, T]$$

for some positive constant μ. If z is Hölder continuous with respect to x in $[a, b]\times[\tau, T]$ with an exponent $\alpha \in (0, 1]$ and a Hölder constant N_1, then for any $0 < d < (b-a)/2$ it holds that for $\tau \le s < t \le s+\delta \le T$ and $a+d \le x \le b-d$

$$|z(x, s) - z(x, t)| \le N_2|s - t|^{\alpha/2},$$

where

$$\delta = d^2/(4\mu(1+d)) \quad and \quad N_2 = 2[N_1\{2\mu(1+d)^{1/2}\}^\alpha + \mu\delta^{1-\alpha/2}].$$

By virtue of Lemma 4.8 we have the following lemma.

LEMMA 4.9. *For the solution u of $P_\varepsilon(K, u_0, f)$, put $w=(u+\varepsilon)^{m-1}$. Then w and u satisfy the following relations, respectively:*

(i) *For $x, y \in \boldsymbol{R}^1$ and $t, s \in [0, T]$*

$$|w(x, t) - w(y, s)| \le L_5(|x - y| + |t - s|^{1/2}),$$

where

$$L_5 = 16\mu(1+\mu)$$

and

$$\mu = \max[mL_3,\ m(L_2)^{1/2}/(m-1) + \|K[f]\|_{\infty,Q_T},\ (m-1)L_3\|(K[f])_x\|_{\infty,Q_T}];$$

(ii) *For $x, y \in \boldsymbol{R}^1$ and $t, s \in [0, T]$*

$$|u(x, t) - u(y, s)| \le L_6(|x - y| + |t - s|^{1/2}),$$

where

$$L_6 = 8(1+L_4)(1+\mu)$$

and

$$\mu = \max[mL_3,\ m(L_2)^{1/2} + \|K[f]\|_{\infty,Q_T},\ L_4\|(K[f])_x\|_{\infty,Q_T}].$$

PROOF. We shall prove only the statement (i) because the proof of the statement (ii) is done by the same way as in the statement (i). It follows from the equation (4.3) that w satisfies the equation

$$\begin{aligned} &w_t - mww_{xx} - \{mw_x/(m-1) - K[f]\}w_x \\ &= (1-m)(K[f])_x(u+\varepsilon)^{m-2}u \quad \text{in} \quad Q_T. \end{aligned}$$

Let x_0 be an arbitrarily fixed point in $\boldsymbol{R}^1$ and let us apply Lemma 4.8 as $[a, b] = [x_0-2, x_0+2]$ and $[\tau, T]=[0, T]$. We take μ, α, N_1, d and δ in Lemma 4.8 as follows:

$$\begin{aligned} &\mu = \max\,[mL_3,\ m(L_2)^{1/2}/(m-1)+\|K[f]\|_{\infty,Q_T},\ (m-1)L_3\|(K[f])_x\|_{\infty,Q_T}], \\ &\alpha = 1, \quad d = 1,\ \delta = 1/(8\mu), \quad N_1 = (L_3)^{1/2}, \end{aligned}$$

where L_2 and L_3 are the same constants as in Lemma 4.6. Then, in the case where $0 \le s < t \le s+\delta \le T$ we can apply Lemma 4.8 to get

$$|w(x, t) - w(x, s)| \le N_2|t - s|^{1/2},$$

where

$$N_2 = 2\{2^{3/2}M_1\mu + (\mu/8)^{1/2}\}.$$

In the case where $|t-s| \ge \delta$ we get

$$|w(x, t) - w(x, s)| \le 2\|w\|_{\infty,Q_T}\delta^{-1/2}|t - s|^{1/2} \le 2^{5/2}L_3\mu^{1/2}|t - s|^{1/2}.$$

Hence, taking account of Lemma 4.6, from the inequalities obtained above we have the desired inequality.

By making use of Lemma 4.1 into Lemmas 4.5–4.7 and 4.9, we obtain that a priori estimates of the solution u for $P_\varepsilon(K, u_0, f)$ and $w=(u+\varepsilon)^{m-1}$ are given in terms of m, u_0 and f.

PROPOSITION 4.1. *The following relations hold for the solution u of $P_\varepsilon(K, u_0, f)$ and $w=(u+\varepsilon)^{m-1}$:*

(i) $\|u\|_{\infty,Q_T} \le \max\,[1, e\|u_0\|_\infty, M_1C(K)\|f\|_{\infty,Q_T}] = \bar{L}_1$,

where M_1 and $C(K)$ are the same constants as in Lemmas 4.1 *and* 4.5, *respectively*;

(ii) $\|w_x\|_{\infty,Q_T} \le \max\,[1, e\|w_x(\cdot, 0)\|_\infty, 4e\bar{L}_3C(K)(\|f\|_{\infty,Q_T} + \|f_x\|_{\infty,Q_T})]$
$= \bar{L}_2$

where $\bar{L}_3 = (\bar{L}_1+1)^{m-1}$;

(iii) $\|u_x\|_{\infty,Q_T} \le \bar{L}_4$

where

$$\bar{L}_4 = \begin{cases} \{(\bar{L}_1)^{1/2} + 1\}^{2-m}(\bar{L}_2)^{1/2}/(m-1) & \text{if } 1 < m \leq 2, \\ \varepsilon^{2-m}(\bar{L}_2)^{1/2}/(m-1) & \text{if } m > 2; \end{cases}$$

(iv) $[w]_{1/2,Q_T} \leq \bar{L}_5$,

where

$$\bar{L}_5 = \bar{L}_3 + 16\mu(1+\mu)$$

and

$$\mu = \max [m\bar{L}_3,\ m(\bar{L}_2)^{1/2}/(m-1) + \|K\|_1\|f\|_{\infty,Q_T},\ (m-1)\bar{L}_2C(K)\|f\|_{\infty,Q_T}];$$

(v) $[u]_{1/2,Q_T} \leq \bar{L}_6$,

where

$$\bar{L}_6 = \bar{L}_1 + 8(1+\bar{L}_4)(1+\mu)$$

and

$$\mu = \max [m\bar{L}_3,\ m(\bar{L}_2)^{1/2} + \|K\|_1\|f\|_{\infty,Q_T},\ \bar{L}_4C(K)\|f\|_{\infty,Q_T}].$$

In the sequel we shall give the estimates of u and $w=(u+\varepsilon)^{m-1}$, in which u is the solution of $P_\varepsilon(K, u_0, f)$.

PROPOSITION 4.2. *u and w satisfy the following relations*:

(i) $[u]_{1+\alpha,Q_T} \leq [u_0]_{1+\alpha} + \bar{M}_1 T^{(1-\alpha)/2}$

and

$$[w]_{1+\alpha,Q_T} \leq [w_0]_{1+\alpha} + \bar{M}_1 T^{(1-\alpha)/2},$$

where $w_0(x)=w(x, 0)$ *on* $\boldsymbol{R}^1$ *and* $\bar{M}_1=\bar{M}_1(\varepsilon, |u_0|_2, |w_0|_2, |f|_{1,0,Q_T}, T)$ *increases in all variables but* ε:

(ii) $[u]_{2+\alpha,Q_T} \leq \bar{M}_2[u_0]_{2+\alpha}$,

where $\bar{M}_2=\bar{M}_2(\varepsilon, [u_0]_{2+\alpha}, [f]_{1+\alpha,Q_T}, T)$ *increases in all variables but* ε,

PROOF. Let us prove the statement (i). We shall prove only the statement for u because the proof of the estimate for w is done by the same method as in the proof for u. Since u satisfies

$$u_t - mwu_{xx} = (mw_x - K[f])u_x - (K[f])_x u \text{ in } Q_T,$$

the function $v=u-u_0$ satisfies

$$\begin{cases} v_t - mwv_{xx} = (mw_x - K[f])u_x - (K[f])_x u + mwu_0'' \equiv g \text{ in } Q_T, \\ v(x, 0) = 0 \text{ on } \boldsymbol{R}^1. \end{cases}$$

Hence, applying Proposition 2.5, we get

$$[v]_{1+\alpha,Q_T} \leq MT^{(1-\alpha)/2}\|g\|_{\infty,Q_T}, \tag{4.15}$$

where $M = M(\varepsilon, [w]_{\alpha/2,Q_T}, T)$ increases in all variables but ε. Therefore, combining (4.15) with Lemma 4.1 and Proposition 4.1, we obtain the first part of the statement (i).

Next let us prove the statement (ii). The regularity results for parabolic equations (see, for example [15]) give

$$[u]_{2+\alpha,Q_T} \leq C[u_0]_{2+\alpha}, \tag{4.16}$$

where $C = C(\varepsilon, [w]_{1+\alpha,Q_T}, [K[f]]_{1+\alpha,Q_T}, T)$ increases in all variables but ε. Hence, with the aid of Lemma 4.1 and the statement (i) in this proposition the statement (ii) follows from (4.16). Thus the proof is completed.

PROPOSITION 4.3. *For a given* $f_i \in H^{1+\alpha,\alpha/2}(\bar{Q}_T)$ $(i=1, 2)$ *let* u_i $(i=1, 2)$ *be a solution of* $P_\varepsilon(K, u_0, f_i)$ *and let us put* $w_i = (u_i + \varepsilon)^{m-1}$. *Then we have*

$$[u_1 - u_2]_{1+\alpha,Q_T} \leq \bar{M}_3 T^{(1-\alpha)/2}\|f_1 - f_2\|_{\infty,Q_T} \max\,[|u|_{1,0,Q_T}, |u_2|_{1,0,Q_T}],$$

where $\bar{M}_3 = \bar{M}_3(\varepsilon, [u_0]_{2+\alpha,Q_T}, [f_i]_{1+\alpha,Q_T}, T)$ *increases in all variables but* ε.

PROOF. Let us put $v = u_1 - u_2$. The function v satisfies

$$v_t - a(x, t)v_{xx} - b(x, t)v_x - c(x, t)v = g \text{ in } Q_T,$$

where

$$a = m(\bar{u}+\varepsilon)^{m-1} \geq m\varepsilon^{m-1},\ b = 2m(m-1)(\bar{u}+\varepsilon)^{m-2} - K[f_1],$$
$$c = m(m-1)(\bar{u}+\varepsilon)^{2m-2}\bar{q} + m(m-1)(m-2)(\bar{u}+\varepsilon)^{m-3}(\bar{p})^2 - (K[f_1])_x,$$
$$g = K[f_1 - f_2](u_2)_x + (K[f_1 - f_2])_x u_2$$

and $(\bar{u}, \bar{p}, \bar{q})$ is a point between $(u_1, (u_1)_x, (u_1)_{xx})$ and $(u_2, (u_2)_x, (u_2)_{xx})$. Making use of Proposition 2.5, we have

$$[u_1 - u_2]_{1+\alpha,Q_T} \leq MT^{(1-\alpha)/2}\|g\|_{\infty,Q_T}, \tag{4.17}$$

where $M = M(\varepsilon, [a]_{\alpha,Q_T}, \|b\|_{\infty,Q_T}, \|c\|_{\infty,Q}, T)$ increases in all variables but ε. Taking Lemma 4.1 and Proposition 4.2 into account, by (4.17) we obtain the desired estimate. Thus we have proved the proposition.

Now we are in a position to prove Theorem 4.1 under the assumptions (A.3) and (A.4) by using a series of lemmas and propositions obtained above.

PROOF OF THEOREM 4.1: For a given $f \in H^{1+\alpha,\alpha/2}(\bar{Q}_T)$, let u be a unique nonnegative solution of $P_\varepsilon(K, u_0, f)$ satisfying the property in Lemma 4.2. Define the mapping F from $H^{1+\alpha,\alpha/2}(\bar{Q}_T)$ into itself by

$$(Ff)(x, t) = u(x, t) \text{ on } \bar{Q}_T.$$

We choose a constant A satisfying

$$[u_0]_{2+\alpha} \leq A/2$$

and fix it. Define the set X_T by

$$X_T = \{f \in H^{1+\alpha,\alpha/2}(\bar{Q}_T);\ [f]_{1+\alpha,Q_T} \leq A\}.$$

Then, by virtue of Propositions 4.2 and 4.3 we can take a sufficiently small positive time T so that

$$F(X_T) \subset X_T$$

and that there exists a constant $k \in (0, 1)$ satisfying

$$[Ff_1 - Ff_2]_{1+\alpha,Q_T} \leq k[f_1 - f_2]_{1+\alpha,Q_T} \qquad \text{for every } f_1, f_2 \in X_T.$$

Hence the application of the fixed point theorem yields that there exists a function $u \in X_T$ satisfying

$$u = Fu,$$

which means that u is a nonnegative solution of the problem (4.1), (4.2). Thus the proof of Theorem 4.1 is completed.

4.2. Global existence of nonnegative solutions for the problem (4.1), (4.2)

In the previous subsection we have shown the local existence of nonnegative classical solutions in time for the problem (4.1), (4.2). However, it is expected that the existence of nonnegative solutions for (4.1), (4.2) is global in time. The result about this is stated as follows.

THEOREM 4.2. *Under the assumptions* (A.3) *and* (A.4) *there exists a unique nonnegative function u on $\mathbf{R}^1 \times [0, \infty)$ which belongs to $H^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$ for any $T>0$ and is a solution of the problem* (4.1), (4.2) *satisfying*

$$\|u\|_\infty \leq \max[1, e\|u_0\|_\infty, M_1 C(K)], \tag{4.18}$$

where M_1 is a constant depending only on m and $\|u_0\|_1$ and $C(K)$ is the same constant as in Lemma 4.1.

The uniqueness of solutions for the problem (4.1), (4.2) is an immediate consequence of Proposition 4.3 in the previous subsection, since a solution u of (4.1), (4.2) is taken for a solution of $P_\varepsilon(K, u_0, u)$. In order to obtain the global existence of solutions in time it sufficies to show a priori estimates of solutions for the problem (4.1), (4.2), since we have obtained the local existence of solutions for (4.1), (4.2) in the previous subsection. Hence, in what follows we shall give a priori estimates of solutions for the problem (4.1), (4.2).

Throughout this subsection let u be a nonnegative solution of the problem (4.1), (4.2) belonging to $H^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$ and let us put $w=(u+\varepsilon)^{m-1}$ and $w_0=w(\cdot, 0)$. We note that u is considered as a solution of the problem $P_\varepsilon(K, u_0, u)$.

LEMMA 4.10. *u satisfies*

$$\|u\|_{\infty,Q_T} \leq \tilde{L}_1,$$

where $\tilde{L}_1$ is the same constant as the one appearing in the right hand side of (4.18).

PROOF. Since u is considered as a solution of $P_\varepsilon(K, u_0, u)$, it follows from Lemma 4.5 that

$$\|u\|^2_{\infty,Q_T} \leq \max\,[1,\ e^2\|u_0\|^2_\infty,\ M_1\|(K[u])_x\|_{\infty,Q_T}].$$

Combining this inequality with Lemma 4.1, we can assume

$$\|u\|^2_{\infty,Q_T} \leq M_1 C(K)\|u\|_{\infty,Q_T},$$

which yields the desired inequality. Thus the proof is completed.

REMARK. The constant $\tilde{L}_1$ does not depend on T.

LEMMA 4.11. *The function w satisfies the following relations*:

(i) $\|w_x\|_{\infty,Q_T} \leq \max\,[1,\ e\|w_0'\|_\infty,\ 12e^2\tilde{L}_3 C(K)M_2]=\tilde{L}_2$, *where* $\tilde{L}_3=(\tilde{L}_1+1)^{m-1}$ *and*

$$M_2 = \begin{cases} \tilde{L}_1 + (\tilde{L}_1+1)^{2-m}/(m-1) & \text{if } 1<m\leq 2, \\ \tilde{L}_1 + \varepsilon^{2-m}/(m-1) & \text{if } m>2; \end{cases}$$

(ii) *For each $\tau>0$ there exists a constant C_τ depending only on τ such that for every $(x, t)\in \boldsymbol{R}^1\times[\tau, T]$*

$$|w_x(x, t)| \leq \max\,[1,\ 12e^2\tilde{L}_3 C(K)M_2 + C_\tau].$$

PROOF. We shall prove only the statement (ii) because the statement (i) is verified by the same method as in the statement (ii). Let $(x_0, t_0)\in \boldsymbol{R}^1\times[\tau, T]$ be a point where $|w_x|$ takes the maximum on $\bar{Q}_T$. Using Lemmas 4.6 and 4.1, we can assume

(4.19) $$1 \leq |w_x(x_0, t_0)|^2 \leq 12e^2 \tilde{L}_3(C(K)\tilde{L}_1 + C(K)\|u_x\|_{\infty,Q_{\tau,T}}) + C_\tau,$$

where $Q_{\tau,T} = \boldsymbol{R}^1 \times (\tau, T)$. Let us consider a point $(x_1, t_1) \in \bar{Q}_{\tau,T}$ where $|u_x|$ takes the maximum on $\bar{Q}_{\tau,T}$. At this point we have

(4.20) $$|u_x| = (u+\varepsilon)^{2-m}|w_x|/(m-1) \leq (u+\varepsilon)^{2-m}|w_x(x_0, t_0)|/(m-1).$$

Substituting (4.20) into (4.19), we obtain

$$|w_x(x_0, t_0)| \leq 12e^2\tilde{L}_3C(K)\{\tilde{L}_1 + (u(x_1, t_1) + \varepsilon)^{2-m}/(m-1)\} + C_\tau,$$

which implies the desired inequality. Thus the proof is completed.

The following lemma is an immediate consequence of Lemma 4.11.

LEMMA 4.12. $\|u_x\|_{\infty,Q_T} \leq \tilde{L}_4$,

where

$$\tilde{L}_4 = \begin{cases} \{(\tilde{L}_1)^{1/2} + 1\}^{2-m}(\tilde{L}_2)^{1/m}/(m-1) & \text{if } 1 < m \leq 2, \\ \varepsilon^{2-m}(\tilde{L}_2)^{1/2}/(m-1) & \text{if } m > 2. \end{cases}$$

By making use of Lemmas 4.10–4.12 obtained above and the same calculation as in the proofs of Propositions 4.1 and 4.2, we obtain a priori estimates of u and $w = (u+\varepsilon)^{m-1}$ which imply that the existence of nonngegative solutions for (4.1), (4.2) is global in time.

PROPOSITION 4.4. *The functions u and w satisfy the following relations*:

(i) $[w]_{1,Q_T} \leq \tilde{L}_5$,

where $\tilde{L}_5 = \tilde{L}_3 + 16\mu(1+\mu)$ *and*

$$\mu = \max[m\tilde{L}_3,\ m(\tilde{L}_2)^{1/2}/(m-1) + \|K\|_1\tilde{L}_1,\ (m-1)\tilde{L}_1\tilde{L}_2C(K)];$$

(ii) $[u]_{1,Q_T} \leq \tilde{L}_6$,

where $\tilde{L}_6 = \tilde{L}_1 + 8(1+\tilde{L}_4)(1+\mu)$ *and*

$$\mu = \max[m\tilde{L}_3,\ m(\tilde{L}_2)^{1/2} + \|K\|_1\tilde{L}_1,\ \tilde{L}_1\tilde{L}_4C(K)];$$

(iii) $[u]_{1+\alpha,Q_T} \leq [u_0]_{1+\alpha} + M_1T^{(1-\alpha)/2}$

and

$$[w]_{1+\alpha,Q_T} \leq [w_0]_{1+\alpha} + M_1T^{(1-\alpha)/2},$$

where $M_1 = M_1(\varepsilon, |u_0|_2, |w_0|_2, \tilde{L}_6, T)$ *increases in all variables but* ε;

(iv) $[u]_{2+\alpha,Q_T} \leq M_2[u_0]_{2+\alpha}$,

where $M_2 = M_2(\varepsilon, [u_0]_{2+\alpha}, T)$ *increases in all variables but* ε.

5. Existence of solutions for the problem $P(K, u_0)$

In this section it is shown that under the assumptions (A.1) and (A.2) a solution of $P(K, u_0)$ is constructed as a limit function of a sequence of solutions for the Cauchy problem studied in Section 4. The result is as follows.

THEOREM 5.1. *Under the assumptions* (A.1) *and* (A.2) *on* u_0 *and* K *there exists a solution* u *of* $P(K, u_0)$ *which is bounded on* $\boldsymbol{R}^1 \times (0, \infty)$ *and satisfies*

$$\text{ess. sup}_{0<t<\infty} (t \wedge 1) \int_{-\infty}^{\infty} |(u^m)_x(x, t)|^2 dx < \infty \tag{5.1}$$

and

$$\sup_{0<s<\infty} \int_s^{s+1} \int_{-\infty}^{\infty} (t \wedge 1)|(u^m)_t(x, t)|^2 dx dt < \infty, \tag{5.2}$$

where $t \wedge 1 = \min(t, 1)$. *Also* (5.1) *and* (5.2) *hold without* $t \wedge 1$ *if* $((u_0)^m)' \in L^2$.

The proof of Theorem 5.1 will be done by using a series of lemmas mentioned below.

For sufficiently small $\varepsilon > 0$ take a sequence of functions $\{u_{0\varepsilon}\}$ such that:

(i) $u_{0\varepsilon} \in B^3$ and $0 \leq u_{0\varepsilon}(x) \leq 2\|u_0\|_\infty$ on $\boldsymbol{R}^1$

(ii) $\|u_{0\varepsilon}\|_1 = \|u_0\|_1$;

(iii) $u_{0\varepsilon} \longrightarrow u_0$ strongly in $L^p(\boldsymbol{R}^1)$ $(1 \leq p < \infty)$ as $\varepsilon \longrightarrow 0$;

(iv) $\lim_{x\to-\infty} \int_{-\infty}^{x} u_{0\varepsilon}(y)dy = 0$ and $\lim_{x\to\infty} \int_x^{\infty} u_{0\varepsilon}(y)dy = 0$ uniformly in ε;

(v) $\|((u_{0\varepsilon}+\varepsilon)^m)'\|_2 \leq 2\|((u_0)^m)'\|_2$ if $((u_0)^m)' \in L^2$.

For K let us define the function K_ε on $\boldsymbol{R}^1$ by

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } |x| < 1/\varepsilon, \\ 0 & \text{if } |x| > 1/\varepsilon. \end{cases}$$

It follows from the assumption (A.2) on K that K_ε satisfies the assumption (A.4) mentioned in Section 4. For each $\varepsilon > 0$ we consider the equation

$$u_t = (m(u+\varepsilon)^{m-1}u_x)_x - (K_\varepsilon[u]u)_x \text{ in } \boldsymbol{R}^1 \times (0, \infty) \tag{5.3}$$

subject to the initial condition

(5.4) $\qquad u(x, 0) = u_0(x) \quad \text{on} \quad \boldsymbol{R}^1.$

By virtue of Theorem 4.2 there exists a nonnegative function u_ε on $R^1 \times [0, \infty)$ which belongs to $H^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)$ for any $T>0$, where $Q_T = \boldsymbol{R}^1 \times (0, T)$, and is a solution of the problem (5.3), (5.4). In the sequel we shall construct a solution of $P(K, u_0)$ as a limit function of the sequence $\{u_\varepsilon\}$. For this purpose we have to give a priori estimates of u_ε.

LEMMA 5.1. *u_ε satisfies*

$$\int_{-\infty}^{\infty} u_\varepsilon(x, t)dx = \int_{-\infty}^{\infty} u_0(x)dx \qquad \text{for any} \quad t > 0.$$

PROOF. Since u_ε is taken for a solution of $P_\varepsilon(K_\varepsilon, u_{0\varepsilon}, u_\varepsilon)$, this lemma is an immediate consequence of Lemma 4.4.

LEMMA 5.2. *u_ε satisfies*

(5.5) $\quad 0 \le u_\varepsilon(x, t) \le \max[1, e\|u_0\|_\infty, M_1\bar{C}(K)] = C_1$ on $\boldsymbol{R}^1 \times [0, \infty)$.

Here M_1 is a positive constant depending only on m and $\|u_0\|_1$ and $\bar{C}(K)$ is determined by

(5.6) $\qquad \bar{C}(K) = \sum_{i=1}^{n} |K(c_i+0) - K(c_1-0)| + \|K\|_\infty + \|K'\|_1$

where $\{c_i\}$ is the set of all discontinuity points of the first kind for K.

PROOF. By Theorem 4.2 we have

$$\|u_\varepsilon\|_\infty \le \max[1, e\|u_0\|_\infty, M_1 C(K_\varepsilon)],$$

where M_1 is a positive constant depending only on m and $\|u_{0\varepsilon}\|_1$. We note that $\|u_{0\varepsilon}\|_1 = \|u_0\|_1$ and

$$C(K)_\varepsilon \le \sum_{|c_i|<1/\varepsilon} |K(c_i+0) - K(c_i-0)| + \|K'\|_1 + 2\|K\|_\infty \le 2\bar{C}(K).$$

Hence we obtain the desired inequality.

LEMMA 5.3. *There is a constant C_2 depending only on m, $\|u_0\|_1$, $\|u_0\|_\infty$ and K such that u_ε* satisfies

(5.7) $$\sup_{0<s<\infty} \int_s^{s+1} \int_{-\infty}^{\infty} ((u_\varepsilon+\varepsilon)^m)_x (u_\varepsilon)_x dxdt \le C_2$$

and

(5.8) $$\sup_{0<s<\infty} \int_s^{s+1} \int_{-\infty}^{\infty} |((u_\varepsilon+\varepsilon)^m)_x|^2 dxdt \le C_2.$$

PROOF. Let x_1 and x_2 be arbitrary points in $\boldsymbol{R}^1$ satisfying $x_1 < x_2$. Multiply

(5.3) by u_ε and integrate the resulting relation on the interval $[x_1, x_2]\times[0, s]$. Then an integration by parts yields that

$$\begin{aligned}\int_{x_1}^{x_2}(u_\varepsilon)^2(x, s+1)dx + \int_s^{s+1}\int_{x_1}^{x_2}((u_\varepsilon+\varepsilon)^m)_x(u_\varepsilon)_x dxdt \\ = \int_{x_1}^{x_2}(u_\varepsilon)^2(x, s)dx - 1/2\int_s^{s+1}\int_{x_1}^{x_2}(K_\varepsilon[u_\varepsilon])_x(u_\varepsilon)^2 dxdt \\ + \int_s^{s+1}\left[((u_\varepsilon+\varepsilon)^m)_x u_\varepsilon\Big|_{x=x_1}^{x_2} - (1/2)K_\varepsilon[u_\varepsilon](u_\varepsilon)^2\Big|_{x=x_1}^{x_2}\right]dt.\end{aligned}$$

We note that $\lim_{x\to\pm\infty} u_\varepsilon(x, t)=\lim_{x\to\pm\infty}(u_\varepsilon)_x(x, t)=0$ uniformly in $t\in[s, s+1]$ by Lemma 4.3 and that Lemmas 4.1 and 5.2 give

$$\int_{x_1}^{x_2}(K_\varepsilon[u_\varepsilon])_x(u_\varepsilon)^2 dx \le \bar{C}(K)\|u_\varepsilon\|_\infty^2\|u_{0\varepsilon}\|_1 \le \bar{C}(K)C_1^2\|u_0\|_1,$$

where $\bar{C}(K)$ is the constant determined by (5.6). Hence, letting $x_1\to-\infty$ and $x_2\to+\infty$ in the relation obtained above, we have

$$\begin{aligned}\int_{-\infty}^{\infty}(u_\varepsilon)^2(x, s+1)dx + \int_s^{s+1}\int_{-\infty}^{\infty}((u_\varepsilon+\varepsilon)^m)_x(u_\varepsilon)_x dxdt \\ \le \int_{-\infty}^{\infty}(u_\varepsilon)^2(x, s)dx + \{\bar{C}(K)C_1^2/2\}\|u_0\|_1. \\ \le C_1(1+\bar{C}(K)C_1)\|u_0\|_1,\end{aligned}$$

which implies (5.7).

Next, using $(u_\varepsilon)_x=m(u_\varepsilon+\varepsilon)^{1-m}((u_\varepsilon+\varepsilon)^m)_x$ in (5.7), we obtain (5.8). Thus the proof is completed.

LEMMA 5.4. *There is a constant C_3 depending only on m, $\|u_0\|_1$, $\|u_0\|_\infty$ and K such that*

$$\sup_{0<.<\infty}(t\wedge 1)\int_{-\infty}^{\infty}|((u_\varepsilon+\varepsilon)^m)_x|^2 dx \le C_3 \tag{5.9}$$

and

$$\sup_{0<s<\infty}\int_s^{s+1}\int_{-\infty}^{\infty}(t\wedge 1)|((u_\varepsilon+\varepsilon)^m)_t|^2 dxdt \le C_3. \tag{5.10}$$

If $((u_0)^m)'\in L^2$, then (5.9) *and* (5.10) *hold without $t\wedge 1$ and C_3 depends on m, $\|u_0\|_1$, K and $\|((u_0)^m)'\|_2$.*

PROOF. Let us put $v=(u+\varepsilon)^m$. Multiply (5.3) by v_t and integrate with respect to x on an interval $[x_1, x_2]$. Using an integration by parts, we have

$$\int_{x_1}^{x_2}(u_\varepsilon)_t v_t dx + 1/2(d/dt)\int_{x_1}^{x_2}|v_x|^2 dx$$
(5.11)
$$= v_x v_t\Big|_{x=x_1}^{x_2} - \int_{x_1}^{x_2}(K_\varepsilon[u_\varepsilon])_x u_\varepsilon v_t dx - \int_{x_1}^{x_2}K_\varepsilon[u_\varepsilon](u_\varepsilon)_x v_t dx.$$

The second term in the right hand side of (5.11) is bounded by

$$(5.12)\qquad (1/2\delta)\|(K_\varepsilon[u_\varepsilon])_x\|_\infty\int_{-\infty}^{\infty}(u_\varepsilon)^2 dx + \delta/2\int_{x_1}^{x_2}(v_t)^2 dx$$

for any $\delta>0$. For the third term in the right hand side of (5.11) using $(u_\varepsilon)_x v_t = (u_\varepsilon)_t v_x$ and (5.3) and then integrating by parts, we obtain

$$-\int_{x_1}^{x_2}K_\varepsilon[u_\varepsilon](u_\varepsilon)_x v_t dx$$
$$= -(1/2)K_\varepsilon[u_\varepsilon](v_x)^2\Big|_{x=x_1}^{x_2} + 1/2\int_{x_1}^{x_2}(K_\varepsilon[u_\varepsilon])_x(v_x)^2 dx$$
(5.13)
$$+\int_{x_1}^{x_2}(K_\varepsilon[u_\varepsilon])^2(u_\varepsilon)v_x dx + \int_{x_1}^{x_2}K_\varepsilon[u_\varepsilon](K_\varepsilon[u_\varepsilon])_x u_\varepsilon v_x dx$$
$$\le -(1/2)K_\varepsilon[u_\varepsilon](v_x)^2\Big|_{x=x_1}^{x_2} + (1/2)\|K_\varepsilon[u_\varepsilon]\|_\infty^2\|(K_\varepsilon[u_\varepsilon])_x\|_\infty^2\int_{-\infty}^{\infty}(u_\varepsilon)^2 dx$$
$$+ (1/2)(\|(K_\varepsilon[u_\varepsilon])_x\|_\infty + 1)\int_{x_1}^{x_2}(v_x)^2 dx + \|(K_\varepsilon[u_\varepsilon])_x\|^2\int_{-\infty}^{\infty}(u_\varepsilon)_x v_x dx.$$

The first term in the left side of (5.11) is estimated as follows:

$$\int_{x_1}^{x_2}(u_\varepsilon)_t v_t dx = (1/m)\int_{x_1}^{x_2}(u_\varepsilon+\varepsilon)^{1-m}(v_t)^2 dx$$
(5.14)
$$\ge \{(C_1+1)^{1-m}/m\}\int_{x_1}^{x_2}(v_t)^2 dx.$$

Choosing $\delta=(C_1+1)^{1-m}/m$ and using the fact that

$$\|K_\varepsilon[u_\varepsilon]\|_\infty \le \|K\|_\infty\|u_0\|_1 \quad\text{and}\quad \|(K_\varepsilon[u_\varepsilon])_x\| \le \bar{C}(K)C_1,$$

by (5.11)–(5.14) we obtain

$$\int_{x_1}^{x_2}(v_t)^2 dx + (d/dt)\int_{x_1}^{x_2}(v_x)^2 dx$$
(5.15)
$$\le C\left\{\left|K_\varepsilon[u_\varepsilon](v_x)^2\Big|_{x=x_1}^{x_2}\right| + 1 + \int_{-\infty}^{\infty}(u_\varepsilon)_x v_x dx\right\} \equiv F(x_1, x_2, t),$$

where C is a constant depending only on m, $\|u_0\|_1$, $\|u_0\|_\infty$, K and C_1, but not ε.

In the case of $s\in[0, 1]$, we integrate (5.15) multiplied by t on $[0, s]$ and use an integration by parts to get

$$
(5.16)\qquad \begin{aligned}&\int_0^s\int_{x_1}^{x_2} t(v_t)^2dxdt + s\int_{x_1}^{x_2}(v_x)^2(x, s)dx\\ &\le \int_0^1\int_{-\infty}^{\infty}(v_x)^2dxdt + \int_0^1 F(x_1, x_2, t)dt.\end{aligned}
$$

We note that $v_x(x, t)\to 0$ as $|x|\to\infty$ uniformly in $t\in[0, T]$ for any $T>0$ by Lemma 4.3. Letting $x_1\to-\infty$ and $x_2\to+\infty$ in (5.16) and using Lemma 5.3, we obtain

$$
(5.17)\qquad \int_0^s\int_{-\infty}^{\infty} t(v_t)^2dxdt + s\int_{-\infty}^{\infty}(v_x)^2(x, s)dx \le C_2 + C(1+C_2),
$$

which implies (5.9) and (5.10) for $s\in[0, 1]$. Next we assume $((u_0)^m)'\in L^2$. Integrating (5.15) on the interval $[0, 1]$ and using $\|(v_x)^2(\cdot, 0)\|_2\le 2\|((u_0)^m)'\|_2$, we obtain

$$
\begin{aligned}&\int_0^1\int_{-\infty}^{\infty}(v_t)^2dxdt + \int_{-\infty}^{\infty}(v_x)^2(x, t)dx\\ &\le \int_{-\infty}^{\infty}(v_x)^2(x, 0)dx + C(1+C_2)\\ &\le 2\|((u_0)^m)'\|_2 + C(1+C_2).\end{aligned}
$$

Let us consider the case when $s\in(1, \infty)$. By virtue of (5.15) we have

$$
(d/dt)\left\{(t-s+1)\int_{x_1}^{x_2}(v_x)^2dx\right\}\le \int_{x_1}^{x_2}(v_x)^2dx + (t-s+1)F(x_1, x_2, t) \quad\text{for } t > s,
$$

from which, integrating with respect to t on $(s-1, s)$, we get

$$
\int_{x_1}^{x_2}(v_x)^2(x, s)dx \le \int_{s-1}^{s}\int_{-\infty}^{\infty}(v_x)^2dxdt + \int_{s-1}^{s}F(x_1, x_2, t)dt.
$$

Hence, letting $x_1\to-\infty$ and $x_2\to+\infty$, we have

$$
\int_{-\infty}^{\infty}(v_x)^2(x, s)dx \le C(2C_2+1) \qquad \text{for} \quad s\in(1, \infty).
$$

Next, in order to get (5.10) we integrate (5.15) on $(s, s+1)$ and let $x_1\to-\infty$ and $x_2\to+\infty$. Taking account of (5.9) with $s=1$, we can conclude that (5.10) holds.

LEMMA 5.5. *For each $\tau>0$ there is a constant C_4, depending only on m, $\|u_0\|_1$, $\|u_0\|_\infty$ and K but not on ε, such that for any $(x_i, t_i)\in R^1\times[\tau, \infty)$ $(i=1, 2)$*

$$
(5.18)\qquad \begin{aligned}&|(u_\varepsilon+\varepsilon)^m(x_1, t_1) - (u_\varepsilon+\varepsilon)^m(x_2, t_2)|\\ &\le \{C_4/(\tau\wedge 1)^{1/2}\}(|x_1-x_2|^{1/2} + |t_1 - t_2|^{1/4})\end{aligned}
$$

and

$$
(5.19)\qquad |u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \le \{C_4/(\tau\wedge 1)^{1/2}\}(|x_1 - x_2|^{m/2} + |t_1 - t_2|^{m/4}).
$$

PROOF. We shall prove only (5.18) because (5.19) is an immediate consequence of (5.18). For the sake of simplicity let us put $f=(u_\varepsilon+\varepsilon)^m$.

For any $x_1, x_2 \in \boldsymbol{R}^1$ and $t>\tau$ we have

$$|f(x_1, t) - f(x_2, t)| \leq |x_1 - x_2|^{1/2} \left\{\int_{x_1}^{x_2} |f_x(x, t)|^2 dx\right\}^{1/2}$$
$$\leq \{C_3/\tau \wedge 1\}^{1/2}|x_1 - x_2|^{1/2}.$$

For any $x \in \boldsymbol{R}^1$, $t_1, t_2 \in [\tau, \infty)$ with $t_1<t_2$ and $\lambda>0$, we have

$$\lambda|f(x, t_1)-f(x, t_2)|$$
$$\leq \int_x^{x+\lambda}\int_x^{\xi} |f_x(\eta, t_1) - f_x(\eta, t_2)|d\eta d\xi + \int_x^{x+\lambda}\int_{t_1}^{t_2} |f_t(\xi, \sigma)|d\sigma d\xi$$
$$\leq 2\lambda^{3/2} \sup_{t_1<t<t_2} \left\{\int_{-\infty}^{\infty} |f_x(\eta, t)|^2 d\eta\right\}^{1/2}$$
$$+ \lambda^{1/2}|t_1 - t_2|^{1/2} \left\{\int_{t_1}^{t_2}\int_{-\infty}^{\infty} |f_t(\xi, \sigma)|^2 d\xi d\sigma\right\}^{1/2}$$
$$\leq \{1/(\tau \wedge 1)^{1/2}\}C_3^{1/2}(2\lambda^{3/2} + \lambda^{1/2}|t_1 - t_2|^{1/2}),$$

from which, taking $\lambda=|t_1-t_2|^{1/2}/2$, we obtain

$$|f(x, t_1) - f(x, t_2)| \leq \{4/(\tau \wedge 1)^{1/2}\}C_3^{1/2}|t_1 - t_2|^{1/4}.$$

Thus we have established (5.18).

LEMMA 5.6. *For any $T>0$ we have*

$$\lim_{x\to-\infty}\int_{-\infty}^{x} u_\varepsilon(\xi, t)d\xi = \lim_{x\to+\infty}\int_{x}^{\infty} u_\varepsilon(\xi, t)d\xi = 0$$

uniformly with respect to $t \in [0, T]$ and $\varepsilon \in (0, 1)$.

PROOF. For any $\delta>0$ there exists a positive constant M independetn of ε such that for any $x \in \boldsymbol{R}^1$

$$0 \leq \int_{-\infty}^{x} u_{0\varepsilon}(\xi)d\xi \leq \delta + Me^{x} \quad\text{and}\quad 0 \leq \int_{x}^{\infty} u_{0\varepsilon}(\xi)d\xi \leq \delta + Me^{-x}.$$

Define the function v on $\boldsymbol{R}^1 \times [0, \infty)$ by

$$v(x, t) = \int_{-\infty}^{x} u_\varepsilon(\xi, t)d\xi,$$

which satisfies the equation

$$v_t = m(u_\varepsilon+\varepsilon)^{m-1}v_{xx} - K_\varepsilon[u_\varepsilon]v_x \quad\text{in}\quad \boldsymbol{R}^1 \times (0, \infty).$$

We consider the function w on $\boldsymbol{R}^1 \times [0, \infty)$ defined by

$$w(x, t) = \delta + Me^{x+\gamma t} - v(x, t),$$

where γ is determined by

$$\gamma > m(C_1+1)^{m-1} + \|K\|_\infty \|u_0\|_1. \tag{5.20}$$

We then see that

$$\begin{cases} \mathscr{L}w \equiv w_t - m(u_\varepsilon+\varepsilon)^{m-1}w_{xx} + K_\varepsilon[u_\varepsilon]w_x > 0 \text{ in } \boldsymbol{R}^1 \times (0, \infty), \\ w(x, 0) > 0 \text{ on } \boldsymbol{R}^1, \\ |w(x, t)| \le C_T \exp(C_T|x|^2) \text{ on } \bar{Q}_T \text{ for any } T > 0, \end{cases}$$

where C_T is a positive constant depending on T. Hence, by using the comparison theorem we have

$$w(x, t) \ge 0 \text{ on } \boldsymbol{R}^1 \times [0, \infty),$$

which implies that

$$0 \le \int_{-\infty}^x u_\varepsilon(\xi, t)d\xi \le \delta + Me^{x+\gamma t} < 2\delta$$

for sufficiently large $-x$. Therefore we get

$$\lim_{x\to-\infty} \int_{-\infty}^x u_\varepsilon(\xi, t)d\xi = 0$$

uniformly with respect to $t \in [0, T]$ and $\varepsilon \in (0, 1)$.

Next, using the function w on $\boldsymbol{R}^1 \times [0, \infty)$ defined by

$$w(x, t) = \delta + Me^{-x+\gamma t} - (\|u_0\|_1 - v(x, t)),$$

where v is the function defined above and γ is the same constant as in (5.20), analogously we obtain

$$0 \le \int_x^\infty u_\varepsilon(\xi, t)d\xi \le \delta + Me^{-x+\gamma t},$$

which implies that

$$\lim_{x\to\infty} \int_x^\infty u_\varepsilon(\xi, t)d\xi = 0$$

uniformly with respect to $t \in [0, T]$ and $\varepsilon \in (0, 1)$. Thus the proof is completed.

We are ready to prove Theorem 5.1 by means of the lemmas established above.

PROOF OF THEOREM 5.1: We use Lemmas 5.3 and 5.4 and $|((u_\varepsilon+\varepsilon)^m)_x| \geq |((u_\varepsilon)^m)_x|$ to obtain

(5.21) $$\sup_{0<s<\infty} \int_s^{s+1} \int_{-\infty}^{\infty} |((u_\varepsilon)^m)_x|^2 dxdt \leq C_2$$

and

(5.22) $$\sup_{0<s<\infty} (s \wedge 1) \int_{-\infty}^{\infty} |((u_\varepsilon)^m)_x|^2 dx \leq C_3.$$

Analogously we obtain

(5.23) $$\sup_{0<t<\infty} \int_t^{t+1} \int_{-\infty}^{\infty} (s \wedge 1) |((u_\varepsilon)^m)_t|^2 dxds \leq C_3.$$

Also (5.22) and (5.23) hold without $s \wedge 1$ if $((u_0)^m)' \in L^2$. Making use of Lemmas 5.2 and 5.5 and Ascoli-Arzela's theorem shows that from $\{u_\varepsilon\}$ we can select a subsequence which converges to a limit function u uniformly on every compact set in $\boldsymbol{R}^1 \times (0, \infty)$. We reindex this subsequence if necessary and also denote it by $\{u_\varepsilon\}$. By (5.21)–(5.23) we can assume that for each $T>0$

$$((u_\varepsilon)^m)_x \longrightarrow (u^m)_x \quad \text{weakly in} \quad L^2(0, T: L^2(\boldsymbol{R}^1))$$
$$\text{and weak star in} \quad L^\infty_{loc}((0, \infty): L^2(\boldsymbol{R}^1)),$$

and

$$((u_\varepsilon)^m)_t \longrightarrow (u^m)_t \quad \text{weakly in} \quad L^2_{loc}((0, \infty): L^2(\boldsymbol{R}^1)),$$

as ε tends to zero. Hence, we see that u has the following properties:

(i) $u \in C(\boldsymbol{R}^1 \times (0, \infty))$ and $0 \leq u \leq C_1$ on $\boldsymbol{R}^1 \times (0, \infty)$;

(ii) $\int_{-\infty}^{\infty} u(x, t)dx = \int_{-\infty}^{\infty} u_0(x)dx$ for every $t > 0$;

(iii) $\sup_{0<s<\infty} \int_s^{s+1} \int_{-\infty}^{\infty} |(u^m)_x|^2 dxdt \leq C_2$;

(iv) $\text{ess.}\sup_{0<s<\infty} (s \wedge 1) \int_{-\infty}^{\infty} |(u^m)_x|^2 dx \leq C_3$

and

$$\sup_{0<t<\infty} \int_t^{t+1} \int_{-\infty}^{\infty} (s \wedge 1) |(u^m)_t|^2 dxds \leq C_3.$$

If $((u_0)^m)' \in L^2$, the relations mentioned just above hold without $s \wedge 1$.

The fact that $\lim_{\varepsilon \to 0} K_\varepsilon[u_\varepsilon] = K[u]$ uniformly on every compact set in $\boldsymbol{R}^1 \times (0, \infty)$ is shown as follows. Let us take τ and T with $0<\tau<T$ and fix them. It follows from Lemma 5.6 that for an arbitrarily small positive number δ there is

a positive number N such that for any $t \in [\tau, T]$

$$0 \leq \int_{|x|>N} u_\varepsilon(\xi, t)d\xi \leq \delta/2 \qquad \text{for} \quad 0 < \varepsilon < 1$$

and

$$0 \leq \int_{|x|>N} u(\xi, t)d\xi < \delta/2.$$

We choose a positive number ε_0 so that if $0<\varepsilon<\varepsilon_0$ then

$$|u_\varepsilon(x, t) - u(x, t)| \leq \delta/(2N\|K\|_\infty) \text{ for every } |x| \leq N \text{ and } t \in [\tau, T].$$

Then, for $t \in [\tau, T]$ we have

$$\begin{aligned}
&|K_\varepsilon[u_\varepsilon](x, t) - K[u](x, t)| \\
&\leq \left|\int_{|y|>N} K_\varepsilon(x-y)u_\varepsilon(y, t)dy\right| + \left|\int_{|y|>N} K(x-y)u(y, t)dy\right| \\
&\quad + \int_{-N}^{N} |K_\varepsilon(x-y)|\,|u_\varepsilon(y, t) - u(y, t)|dy + \int_{-N}^{N} |K_\varepsilon(x-y) - K(x-y)|u(y, t)dy \\
&\leq \|K\|_\infty\delta + \delta + \int_{-N}^{N} |K_\varepsilon(x-y) - K(x-y)|u(y, t)dy.
\end{aligned}$$

For an arbitrary positive number M let $|x|<M$. If we choose ε so that $\varepsilon<1/(M+N)$, then we see that $K_\varepsilon(x-y)=K(x-y)$ for $|x|<M$ and $|y|<N$. Consequently, for $|x|<M$ and $t \in [\tau, T]$ we obtain

$$|K_\varepsilon[u_\varepsilon](x, t) - K[u](x, t)| \leq (\|K\|_\infty + 1)\delta$$

whenever $0<\varepsilon<\min[\varepsilon_0, 1/(M+N)]$. Thus we have proved our assertion.

The property $u \in C((0, \infty): L^1(\boldsymbol{R}^1))$ follows from $u \in C(\boldsymbol{R}^1 \times (0, \infty))$ and

$$\int_{-\infty}^{\infty} u(x, t)dx = \int_{-\infty}^{\infty} u(x, s)dx \qquad \text{for any} \quad s, t \in (0, \infty).$$

It is easy to prove the integral identity (iv) for u, so we omit the proof.

Finally we note that the original sequence $\{u_\varepsilon\}$ conveges to u by using the uniqueness of solutions for $P(K, u_0)$.

6. Some properties of solutions for the problem $P(K, u_0)$

In the case of the porous media equation, that is, the equation (1.1) with $K \equiv 0$, one of the most important properties of solutions is that the solution u has a finite speed of propagation which means that for each $t>0$ the support of $u(\cdot, t)$ in $\boldsymbol{R}^1$ is compact in $\boldsymbol{R}^1$ if the initial datum has compact support in $\boldsymbol{R}^1$.

It is natural for us to expect that solutions of our problem also have such an interesting property.

THEOREM 6.1. *Let u be a solution of $P(K, u_0)$, where the assumptions* (A.1) *and* (A.2) *are imposed on u_0 and K, respectively. If the support of u_0 is compact in $\boldsymbol{R}^1$, then for each $t>0$ the support of $u(\cdot, t)$ in $\boldsymbol{R}^1$ is also compact.*

PROOF. Define the functions v on $\boldsymbol{R}^1 \times (0, \infty)$ and v_0 on $\boldsymbol{R}^1$ by

$$v(x, t) = \int_{-\infty}^{x} u(y, t)dy \quad \text{and} \quad v_0(x) = \int_{-\infty}^{x} u_0(y)dy,$$

respectively. The function v holds the properties (i)–(v) in Proposition 2.3. Let ω be a solution of the problem

$$(6.1) \qquad \begin{cases} ((\omega')^m)' + (2\omega - c)\omega' = 0 \ \text{ on } \ \boldsymbol{R}^1, \\ \omega(-\infty) = 0, \quad \omega(+\infty) = c, \\ \omega'(x) \geq 0 \ \text{ on } \ \boldsymbol{R}^1, \end{cases}$$

where $c = \|u_0\|_1$. As was shown in [18], there exists a solution ω of (6.1) satisfying $\omega(x)=0$ on $(-\infty, \alpha]$, $0<\omega(x)<c$ on (α, β) and $\omega(x)=c$ on $[\beta, \infty)$ for some constants α and β. We note that for an arbitrarily fixed point $x_0 \in \boldsymbol{R}^1$ the function $\omega(x+x_0)$ is also a solution of (6.1) because of the translation invariance of (6.1) with respect to x.

Since the support of u_0 is compact in $\boldsymbol{R}^1$, we can take a solution ω of (6.1) so that $v_0(x) \leq \omega(x)$ on $\boldsymbol{R}^1$, and define the function $\bar{\omega}$ by

$$\bar{\omega}(x, t) = \omega(x + \lambda t) \ \text{ on } \ \boldsymbol{R}^1 \times [0, \infty),$$

where λ is determined by

$$(6.2) \qquad \lambda = c + \|K[u]\|_\infty.$$

The function $\bar{\omega}$ satisfies

$$(6.3) \qquad \bar{\omega}_t = ((\bar{\omega}_x)^m)_x + (2\bar{\omega} - c + \lambda)\bar{\omega}_x \ \text{ in } \ \boldsymbol{R}^1 \times (0, \infty)$$

and

$$\bar{\omega}(x, 0) \geq v_0(x) \ \text{ on } \ \boldsymbol{R}^1.$$

In what follows it will be shown that

$$(6.4) \qquad v(x, t) \leq \bar{\omega}(x, t) \ \text{ on } \ \boldsymbol{R}^1 \times [0, \infty).$$

At first we note that from Proposition 2.3 it follows that $((v_x)^m)_x, v_t \in L^2(Q_T)$ for any $T>0$, where $Q_T = \boldsymbol{R}^1 \times (0, T)$, and

(6.5) $$v_t = ((v_x)^m)_x - K[u]v_x \quad \text{a.e. in} \quad \boldsymbol{R}^1 \times (0, \infty).$$

Subtracting (6.3) from (6.5) and using (6.2) for the resulting relation yield the following relation

(6.6) $$(v-\varpi)_t \leq [(v_x)^m - (\varpi_x)^m]_x - K[u](v-\varpi)_x \quad \text{a.e. in} \quad \boldsymbol{R}^1 \times (0, \infty).$$

Let $h(s)$ be a nonnegative bounded function on $\boldsymbol{R}^1$ such that h is continuously differentiable, $h' \geq 0$ on $\boldsymbol{R}^1$, $h(s)=0$ on $(-\infty, 0]$ and $h(s) \geq 0$ on $(0, \infty)$. Multiply (6.6) by $h(v(x, t)-\varpi(x, t))\chi_N(x)$ and integrate on $\boldsymbol{R}^1 \times (0, s)$ for every $s>0$. Here for each $N=1, 2, \ldots, \chi_N \in C^\infty$, $0 \leq \chi_N \leq 1$, $\chi_N(x)=1$ on $[-N, N]$, $\chi_N(x)=0$ on $\boldsymbol{R}^1 \backslash [-N-1, N+1]$ and $\|\chi_N'\|_\infty \leq M$, where M is a constant independent of N. Integrating the resulting relation by parts and using

$$\int_0^s \int_{-\infty}^\infty \{(v_x)^m - (\varpi_x)^m\}(v_x - \varpi_x) h'(v-\omega)\chi_N dx dt \geq 0,$$

we get

$$\begin{aligned} &\int_{-\infty}^\infty H(v(x, s) - \varpi(x, s))\chi_N(x) dx \\ (6.7) \quad &\leq \int_{-\infty}^\infty H(v_0(x) - \omega(x))\chi_N(x) dx - \int_0^s \int_{-\infty}^\infty \{(v_x)^m - (\varpi_x)^m\} h(v-\varpi)\chi_N' dx dt \\ &\quad - \int_0^s \int_{-\infty}^\infty K[u](v-\varpi)_x h(v-\varpi)\chi_N dx dt \end{aligned}$$

for every $s>0$, where

$$H(w) = \int_0^w h(\sigma) d\sigma \text{ for } w \in \boldsymbol{R}^1.$$

It follows from $v_0 \leq \omega$ on $\boldsymbol{R}^1$ that $H(v_0 - \omega) \equiv 0$ on $\boldsymbol{R}^1$. Since $(v_x)^m - (\varpi_x)^m$, $K[u](v-\varpi)_x \in L^1(\boldsymbol{R}^1) \cap L^\infty(\boldsymbol{R}^1)$, letting $N \to \infty$ in (6.7) we obtain

(6.8) $$\int_{-\infty}^\infty H(v(x, s) - \varpi(x, s)) dx \leq -\int_0^s \int_{-\infty}^\infty K[u](v-\varpi)_x h(v-\varpi) dx dt$$

for any $s > 0$. The right hand side of (6.8) is rewritten as

$$\int_0^s \int_{-\infty}^\infty (K[u])_x H(v-\varpi) dx dt,$$

which is bounded by

$$\|(K[u])_x\|_\infty \int_0^s \int_{-\infty}^\infty H(v-\varpi) dx dt.$$

As a result of the estimates we conclude that

$$\int_{-\infty}^{\infty} H(v(x, s) - \bar{\omega}(x, s))dx \leq \|(K[u])_x\|_\infty \int_0^s \int_{-\infty}^{\infty} H(v-\bar{\omega})dxdt$$

for any $s > 0$, which yields that

$$H(v(x, s) - \bar{\omega}(x, s)) \equiv 0 \text{ for } x \in \boldsymbol{R}^1 \text{ and } s > 0.$$

Hence we have obtained the desired inequality (6.4).

Next we choose a solution ω of (4.1) such that $\omega(x) \leq v_0(x)$ on $\boldsymbol{R}^1$. Define the function ω by

$$\omega(x, t) = \omega(x-\lambda t).$$

Here λ is the constant determined by (6.2). By using the same calculation as in the proof of (6.4), we obtain

$$\omega(x, t) \leq v(x, t) \text{ on } \boldsymbol{R}^1 \times [0, \infty). \tag{6.9}$$

Taking account of the property of ω and the definition of v, by (6.4) and (6.9) we obtain that for each $t>0$ the support of $u(\cdot, t)$ on $\boldsymbol{R}^1$ is compact. Thus the proof is completed.

REMARK. In the case when $K(x)=k_1$ on $(-\infty, 0]$ and $K(x)=-k_2$ on $(0, \infty)$ for some positive constants k_1 and k_2, it has been shown in [18] that a stronger result than Theorem 6.1 holds: There are constants α and β, depending on the amount of the support of u_0, such that

$$u(x, t) = 0 \text{ outside of } \alpha \leq x - kt \leq \beta,$$

where

$$k = \{(k_1 - k_2)/2\} \int_{-\infty}^{\infty} u_0(x)dx.$$

As concerns the regularity of solutions, in the case of the porous media equation more precise estimates for the smoothness of the solution u have been obtained by Aronson [1] and Gilding [8]. The former has shown that u is Hölder continuous with respect to x with exponent $\alpha=\min[1, 1/(m-1)]$. It is shown from the exact solution obtained by [4] and [21] that this exponent α is the best possible. The latter has shown that u is Hölder continuous with respect to t with exponent $\alpha/2$. In the case of our problem $P(K, u_0)$, we have not obtained the same results as that mentioned just above when $m>2$.

THEOREM 6.2. *Let u be the solution of $P(K, u_0)$, where it is assumed that u_0 and K satisfy the assumptions* (A.1) *and* (A.2), *respectively. Then u holds the following properties*:

(i) *u is a classical solution of the equation* (1.1) *in a neighbourhood of a point in $\boldsymbol{R}^1\times(0, \infty)$ where u is positive;*

(ii) *If $1<m\leq 2$, then for each $\tau\in(0, \infty)$ there exists a positive constant C_τ depending only on τ, m, $\|u_0\|_1$ and $\|u_0\|_\infty$ such that for $x, y\in \boldsymbol{R}^1$ and $\tau\leq s, t<\infty$*

$$|u^{m-1}(x, t) - u^{m-1}(y, s)| \leq C_\tau(|x - y| + |s - t|^{1/2});$$

(iii) *The derivative $(u^m)_x$ exists and is continuous on $\boldsymbol{R}^1\times(0, \infty)$ if $1<m\leq 2$, and moreover u_x exists and is continuous on $\boldsymbol{R}^1\times(0, \infty)$ if $1<m<2$.*

PROOF. At first we shall prove (i). Let $\{u_\varepsilon\}$ be the sequence of approximate solutions of $P(K, u_0)$ constructed in the previous section. We define the function v_ε by

$$v_\varepsilon(x, t) = \int_{-\infty}^{x} u_\varepsilon(y, t)dy \text{ for } (x, t)\in \boldsymbol{R}^1 \times [0, \infty).$$

v_ε satisfies the euqation

$$(v_\varepsilon)_t = m(u_\varepsilon+\varepsilon)^{m-1}(v_\varepsilon)_{xx} - K_\varepsilon[u_\varepsilon](v_\varepsilon)_x \text{ in } Q_\infty = \boldsymbol{R}^1 \times (0, \infty).$$

$[v_\varepsilon]_{1,Q_\infty}$ is estimated independently of ε by using Lemma 4.8, and it follows from Lemma 5.5 that for each $\tau>0$

$$[u_\varepsilon]_{m/2,Q_{\tau,\infty}} \leq C_\tau \text{ and } [(u_\varepsilon+\varepsilon)^{m-1}]_{(1\,\text{[illegible]}\,2m),Q_{\tau,\infty}} \leq C_\tau,$$

where C_τ is a constant depending on τ but not on ε, and $Q_{\tau,\infty}=R^1\times[\tau, \infty)$. We note that

$$\begin{aligned} K_\varepsilon[u_\varepsilon](x, t) &= K(1/\varepsilon)v_\varepsilon(x-1/\varepsilon) - K(-1/\varepsilon)v_\varepsilon(x+1/\varepsilon) \\ &+ \textstyle\sum_{i=1}^{n}(K(c_i-0) - K(c_i+0))v_\varepsilon(x-c_i) + \displaystyle\int_{-1/\varepsilon}^{1/\varepsilon} K'(x-y)v_\varepsilon(x-y)dy, \end{aligned}$$

where $\{c_i\}$ is the set of discontinuity points of the first kind for K, from which it follows that

$$[K_\varepsilon[u_\varepsilon]]_{1,Q_\infty} \leq C \text{ and } [(K_\varepsilon[u_\varepsilon])_x]_{1/2,Q_{\tau,\infty}} \leq C_\tau,$$

where C_τ is a constant not depending on ε and C_τ depends on τ but not on ε. Hence, by the method similar to that used in the proof of the statement (ii) of Theorem 3 in [10] we can establish the statement (i).

It follows from Lemma 4.11 that if $1<m\leq 2$ then for each $\tau>0$ there is a constant C_τ, depending on τ but not on ε, such that

$$|(u_\varepsilon+\varepsilon)^{m-1}(x, t) - (u_\varepsilon+\varepsilon)^{m-1}(y, s)| \leq C_\tau(|x - y| + |t - s|^{1/2})$$

for $x, y \in \boldsymbol{R}^1$ and $\tau \leq s, t < \infty$. Letting $\varepsilon \to 0$, we obtain the statement (ii).

The statement (iii) is obtained by using the same method as in the proof of the statement (iii) of Theorem 3 in [10]. Thus the proof is completed.

REMARK. In the case when $K(x)=k_1$ on $(-\infty, 0]$ and $K(x)=-k_2$ on $(0, \infty)$ for some positive constants k_1 and k_2, the statement (ii) and the first part of statement (iii) in Theorem 6.2 are valid without the condition on m [17].

Finally we state the dependency of solutions for $P(K, u_0)$ on K.

THEOREM 6.3. *Let u_0 satisfy the assumption* (A.1) *and let K and $\{K_\eta\}$ be a function and a sequence of functions satisfying the assumption* (A.2), *respectively. Assume that*

(i) *there is a positive constant L such that $\|K\|_\infty \leq L$, $\|K_\eta\|_\infty \leq L$, $C(K) \leq L$ and $C(K_\eta) \leq L$, where $C(K)$ and $C(K_\eta)$ are the constants defined by* (3.2);

(ii) $\lim_{\eta\to 0} K_\eta(x) = K(x)$ a.e. in $\boldsymbol{R}^1$.

Then, for the solution u of $P(K, u_0)$ and the sequence of solutions u_η of $P(K_\eta, u_0)$ we have

$$\lim_{\eta\to 0} u_\eta(x, t) = u(x, t) \quad \textit{uniformly on every compact set in } \boldsymbol{R}^1 \times (0, \infty).$$

PROOF. Using Lemma 5.2, by the condition (i) we obtain that $\sup\{\|u_\eta\|_p; \eta\}$ is finite for every $p \in [1, \infty]$. We then use Lemma 3.1 to obtain

$$\|v_\eta(\cdot, t) - v(\cdot, t)\|_2 \leq e^{Mt} \int_0^t \|(K-K_\eta)[u](\cdot, s)u(\cdot, s)\|_2 e^{-Ms} ds \tag{6.10}$$

for every $t>0$, where v and v_η are the functions on $\boldsymbol{R}^1 \times (0, \infty)$ defined by

$$v(x, t) = \int_{-\infty}^x u(y, t)dy \quad \text{and} \quad v_\eta(x, t) = \int_{-\infty}^x u_\eta(y, t)dy$$

respectively, and

$$M = L\max[\sup\{\|u_\eta\|_\infty; \eta\}, \|u\|_\infty].$$

Since $u \in L^1(\boldsymbol{R}^1 \times (0, T)) \cap L^\infty(\boldsymbol{R}^1 \times (0, \infty))$ for any $T>0$, by using the conditions (i) and (ii), Lebesgue's convergence theorem guarantees that

$$(K-K_\eta)[u] \longrightarrow 0 \text{ as } \eta \longrightarrow 0 \quad \text{a.e. in } \boldsymbol{R}^1$$

for each $t \in (0, T)$. We note that

$$\|(K-K_\eta)[u]\|_\infty \leq 2L\|u_0\|_1.$$

Hence, from (6.10) it follows that for every $T>0$

$$\lim_{\eta\to 0} \sup_{0<t<T} \|v_\eta(\cdot, t) - v(\cdot, t)\|_2 = 0.$$

Noting that $v_\eta(x, t)-v(x, t)\to 0$ as $|x|\to\infty$, we obtain

$$(v_\eta(x, t) - v(x, t))^2 = 2\int_{-\infty}^{x} (v_\eta(y, t) - v(y, t))(u_\eta(y, t) - u(y, t))dy$$

$$\leq 2\|v_\eta(\cdot, t) - v(\cdot, t)\|_2 \sup_{0<t<T}\|u_\eta(\cdot, t) - u(\cdot, t)\|_2,$$

which yields that

$$\lim_{\eta\to 0} \sup_{x\in \boldsymbol{R}^1, 0<t<T} |v_\eta(x, t)-v(x, t)|=0$$

for every $T>0$. Since $\{u_\eta\}$ is bounded and equi-continuous on $R^1\times[\tau, T]$ for any $0<\tau<T<\infty$, by using Ascoli-Arzela's theorem and $u_\eta=(v_\eta)_x$, we have

$$\lim_{\eta\to 0} u_\eta(x, t) = u(x, t) \text{ uniformly on every compact set in } \boldsymbol{R}^1 \times (0, \infty).$$

Thus we have established the proof of Theorem 6.3.

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