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A system of elliptic variational inequalities associated with a stochastic switching game

Naoki YAMADA

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1. Introduction

This paper is concerned with a system of elliptic variational inequalities each of which is subject to constraints from upper and lower sides.

Let $\Omega \subset \boldsymbol{R}^N$ be a bounded domain with smooth boundary Γ, and let f^p, $p=1,\dots, m$, be given functions of $x \in \Omega$. Consider m second order elliptic differential operators

$$A^p v = -\sum_{i,j=1}^{N} a_{ij}^p(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^p(x)\frac{\partial v}{\partial x_i} + c^p(x)v$$

in Ω. The purpose of this paper is to investigate the existence and uniqueness of solutions to the Dirichlet problem for a system of variational inequalities of the form

$$\begin{aligned}
&u^{p+1}(x) - k \leqq u^p(x) \leqq u^{p+1}(x) + K, \quad x \in \Omega,\\
&A^p u^p = f^p \quad \text{if} \quad u^{p+1}(x) - k < u^p(x) < u^{p+1}(x) + K,\ x \in \Omega,\\
&A^p u^p \leqq f^p \quad \text{if} \quad u^p(x) = u^{p+1}(x) + K, \quad x \in \Omega,\\
&A^p u^p \geqq f^p \quad \text{if} \quad u^{p+1}(x) - k = u^p(x), \quad x \in \Omega,\\
&u^p(x) = 0 \quad \text{on} \quad \Gamma, \quad p = 1,\dots, m,
\end{aligned} \tag{1.1}$$

where we have put $u^{m+1} = u^1$, and k, K are two given positive constants.

The problem (1.1) has been motivated by a recent paper of L. C. Evans and A. Friedman [3] in which the Dirichlet problem for the Bellman equation

$$\begin{aligned}
&\sup_{p\geqq 1}(A^p u(x) - f^p(x)) = 0 \quad \text{a.e. in } \Omega,\\
&u = 0 \quad \text{on} \quad \Gamma,
\end{aligned} \tag{1.2}$$

was studied by an analytic method. In [3] they introduced the approximate system

$$\begin{aligned}
&A^p u_\varepsilon^p + \beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1}) = f^p \quad \text{in} \quad \Omega,\\
&u_\varepsilon^p = 0 \quad \text{on} \quad \Gamma,\ p = 1,\dots, m,
\end{aligned} \tag{1.3}$$

where $u_\varepsilon^{m+1}=u_\varepsilon^1$, and β_ε represents the penalty function: $\beta_\varepsilon(t)=0$ if $t\leqq 0$, $\beta_\varepsilon(t)\to\infty$ if $t>0$, $\varepsilon\to 0$, and obtained the solution of (1.2) as a limit of solutions of these systems. Furthermore, as a system which is the limit case, as $\varepsilon\to 0$, of the system derived from (1.3) with $\beta_\varepsilon(u_\varepsilon^p-u_\varepsilon^{p+1})$ replaced by $\beta_\varepsilon(u_\varepsilon^p-k^p-u_\varepsilon^{p+1})$, they studied [3, Sec. 7] the system of variational inequalities with unilateral constraints

$$
\begin{aligned}
&A^p u^p \leqq f^p, \quad u^p \leqq k^p + u^{p+1} \quad \text{in} \quad \Omega,\\
&(A^p u^p - f^p)(u^p - k^p - u^{p+1}) = 0 \quad \text{in} \quad \Omega, \qquad (1.4)\\
&u^p = 0 \quad \text{on} \quad \Gamma,\ p = 1,\dots, m,
\end{aligned}
$$

where $u^{m+1}=u^1$ and the k^p are positive constants. It was probabilistically interpreted that the solution component $u^p(x)$ of this system (1.4) represents the optimal cost starting at $x\in\Omega$ in state p of some cost functional.

Our system (1.1) is a natural extension of (1.4) to the case of bilateral constraints.

The outline of this paper is as follows:

In Section 2, after stating our notations and assumptions, we formulate the problem (1.1) in a weak form by using the corresponding bilinear forms on $H_0^1(\Omega)\times H_0^1(\Omega)$.

We shall solve the system (1.1) by the penalty method. For that purpose, in Section 3, we construct an approximate system by using a suitable penalty function and prove the solvability of this system.

Section 4 is devoted to deriving a priori estimates in $W^{1,\infty}(\Omega)$ for approximate solutions given in Section 3. The main idea is quite similar to that of L. C. Evans and A. Friedman [3], or P. L. Lions [4]. However, since the convexity of penalty functions cannot be expected in our case, their use needs some careful consideration.

Since these a priori estimates show the convergence of approximate solutions, we shall see in Section 5 that the limit functions satisfy the weak problem formulated in Section 2. Unfortunately, however, we know nothing yet about the uniqueness of solutions of this weakly formulated problem.

In Section 6 we shall prove that these limit functions have indeed $W^{2,r}(\Omega)$-regularity for any r, $r<\infty$. To do this, we derive a priori estimates in $W^{2,r}(\Omega)$ for approximate solutions, using the fact that the limit function is a solution of the weakly formulated problem. In this section, because of bilateral constraints, we impose a condition on the ratio k/K which did not appear in [3].

In Section 7 we study the uniqueness of our solution of the problem (1.1). We shall introduce a stochastic switching game and represent the solution component u^p as a value of this game.

In the case of unilateral constraints, the solution component admits a representation as the optimal cost of some cost functional ([3]). In our case, however, we shall associate with it a stochastic switching game in which two players compete for the value of some cost functional.

As a result of this representation, we obtain the uniqueness of solutions of (1.1) in the class $W^{2,r}(\Omega) \cap C(\bar{\Omega})$.

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2. Formulation of the weak problem and assumptions

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary Γ. We denote by $W^{j,r}(\Omega)$, $1 \leqq r \leqq \infty$, the usual Sobolev space of real functions with norm $\|\cdot\|_{j,r}$. The space $W_0^{j,r}(\Omega)$ denotes the closure in $W^{j,r}(\Omega)$ of the set of all C^∞-functions with compact support in Ω. As usual, we write $W^{0,r}(\Omega) = L^r(\Omega)$, $W^{j,2}(\Omega) = H^j(\Omega)$ and $W_0^{j,2}(\Omega) = H_0^j(\Omega)$.

Let $m > 1$ be a fixed integer. For $p = 1, \dots, m$, we consider the second order elliptic differential operator

$$A^p v = -\sum_{i,j=1}^N a_{ij}^p(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i^p(x) \frac{\partial v}{\partial x_i} + c^p(x) v \tag{2.1}$$

and the corresponding bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$:

$$\begin{aligned} a^p(u, v) = \int_\Omega \Big(\sum_{i,j=1}^N a_{ij}^p(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^N \tilde{b}_i^p(x) \frac{\partial u}{\partial x_i} v \\ + c^p(x) u v \Big) dx, \end{aligned} \tag{2.2}$$

where
$$\tilde{b}_i^p(x) = b_i^p(x) + \sum_{j=1}^N \frac{\partial}{\partial x_j} a_{ij}^p(x).$$

We make the following assumptions:

(A.1) *There exists a positive number α such that*

$$\sum_{i,j=1}^N a_{ij}^p(x) \xi_i \xi_j \geqq \alpha |\xi|^2$$

for all $p = 1, \dots, m$, $x \in \Omega$ and $\xi \in \boldsymbol{R}^N$.

(A.2) a_{ij}^p, b_i^p, $c^p \in C^1(\bar{\Omega})$ *for all $i, j = 1, \dots, N$ and $p = 1, \dots, m$.*

There exist positive constants M_1 and M_2 satisfying

$$|a_{ij}^p(x)|, \left| \frac{\partial}{\partial x_r} a_{ij}^p(x) \right| \leqq M_1,$$

$$|b_i^p(x)|, \left| \frac{\partial}{\partial x_r} b_i^p(x) \right| \leqq M_1,$$

and

$$|c^p(x)|, \quad \left|\frac{\partial}{\partial x_r} c^p(x)\right| \leqq M_2,$$

for all $i, j, r=1,\dots, N$, $p=1,\dots, m$ and $x \in \bar{\Omega}$.

(A.3) *There exists a positive constant c_0 such that*

$$c^p(x) \geqq c_0$$

for all $p=1,\dots, m$ and $x \in \bar{\Omega}$. Moreover, we assume that c_0 is a sufficiently large constant depending only on M_1 and α so that the inequality $c_0 > 4\bar{M}$ appearing in the proof of Lemma 4.3 is valid.

By (A.2), we can find a positive constant M_1' such that

$$|\tilde{b}_i^p(x)| \leqq M_1'$$

for all $i=1,\dots, N$ and $p=1,\dots, m$. On the other hand, by (A.3), the bilinear forms $a^p(\cdot, \cdot)$ are coercive i.e., there exists a positive constant α' such that

$$a^p(u, u) \geqq \alpha' \|u\|_{1,2}^2 \tag{2.3}$$

for all $p=1,\dots, m$ and $u \in H_0^1(\Omega)$.

In the following these constants M_1', α' will be denoted by the same letters M_1, α, respectively.

Let f^p, $p=1,\dots, m$, be given functions of $x \in \bar{\Omega}$ and satisfy the following assumptions:

(A.4) *$f^p \in C^1(\bar{\Omega})$ for $p=1,\dots, m$, and there exists a positive constant M_3 such that*

$$|f^p(x)|, \quad \left|\frac{\partial}{\partial x_r} f^p(x)\right| \leqq M_3$$

for all $r=1,\dots, N$, $p=1,\dots, m$ and $x \in \bar{\Omega}$.

Let k and K be two given positive constants. For these k, K, we impose the following condition:

$$\frac{k}{K} \neq \frac{m-q}{q} \qquad \textit{for all} \quad q = 1,\dots, m-1. \tag{A.5}$$

For $\psi \in H_0^1(\Omega)$, we define a set $\mathscr{K}(\psi)$ in $H_0^1(\Omega)$ by

$$\mathscr{K}(\psi) = \{v \in H_0^1(\Omega);\ \psi(x) - k \leqq v(x) \leqq \psi(x) + K, \quad \text{a.e. in } \Omega\}. \tag{2.4}$$

Using these notations, we may state our problem in a weak form as follows:

Weak problem: (1) *Find* $u^p \in \mathscr{K}(u^{p+1})$, $p=1,\dots, m$, *satisfying*

$$a^p(u^p, v-u^p) \geqq (f^p, v-u^p) \tag{2.5}$$

for all $v \in \mathscr{K}(u^{p+1})$. *Here we have set* $u^{m+1}=u^1$ *and* $(\cdot,\cdot)$ *on the right hand side denotes the inner product in* $L^2(\Omega)$.

(2) *Investigate the regularity and uniqueness of solutions of* (2.5).

For $\psi \in H^1(\Omega)$, we use the notations

$$\psi^+(x) = \max\{\psi(x), 0\}, \qquad \psi^-(x) = \max\{-\psi(x), 0\}.$$

In the following of this paper we denote various constants depending only on M_1, M_2, M_3, α, Ω, m and N by the same letter M.

3. Approximate system

To solve our problem (2.5), we use the penalty method. In this section we construct penalty equations which approximate (2.5) and prove the solvability of these equations. Throughout this section we always assume (A.1)–(A.4).

We choose the so-called penalty function β: $\boldsymbol{R}\to\boldsymbol{R}$ satisfying the following conditions:

$$\beta \in C^\infty(\boldsymbol{R}),$$

$$\beta(t) = \begin{cases} 0, & \text{if} \quad t \leqq 0, \\ t-1, & \text{if} \quad t \geqq 2, \end{cases} \tag{3.1}$$

$0<\beta(t)<1$, $\beta'(t)>0$ and $\beta''(t)>0$ if $0<t<2$.

For each $\varepsilon>0$, we put

$$\beta_\varepsilon(t) = \beta\left(\frac{t}{\varepsilon}\right). \tag{3.2}$$

We note that it holds

$$-1 \leqq \beta_\varepsilon(t) - t\beta'_\varepsilon(t) \leqq 0 \tag{3.3}$$

for all $\varepsilon>0$ and $t \in \boldsymbol{R}$.

For each $\varepsilon>0$, we consider the following approximate system:

$$\begin{gathered} u^p_\varepsilon \in W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega),\ 1<r<\infty, \\ A^p u^p_\varepsilon + \beta_\varepsilon(u^p_\varepsilon - u^{p+1}_\varepsilon - K) - \beta_\varepsilon(u^{p+1}_\varepsilon - k - u^p_\varepsilon) = f^p \quad \text{a.e.} \quad \text{in } \Omega, \\ p=1,\dots, m, \quad \text{where} \quad u^{m+1}_\varepsilon = u^1_\varepsilon. \end{gathered} \tag{3.4}$$

To prove the existence of solutions of (3.4), we apply a successive approximation method. In the following of this section we omit the subscript ε for simplicity.

First, we define $u_0^p=0$ for all $p=1,\dots,m$. For $n\geqq 1$ and $p=1,\dots,m$, we define u_n^p inductively by the solution of

$$(3.5)\quad \begin{aligned} &A^p u_n^p + \beta(u_n^p - u_{n-1}^{p+1} - K) - \beta(u_{n-1}^{p+1} - k - u_n^p) = f^p \quad \text{a.e. in } \Omega, \\ &u_n^p = 0 \quad \text{a.e. on } \Gamma, \text{ where } \quad u_{n-1}^{m+1} = u_{n-1}^1. \end{aligned}$$

It is well known that there exists a solution $u_n^p \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, $1\leqq r<\infty$ (for example, we refer to A. Bensoussan [2], Theorem IV. 2. 1, p. 143).

LEMMA 3.1. *Let u_n^p, $p=1,\dots,m$, be a solution of* (3.5). *We have*

$$\|u_n^p\|_{0,\infty} \leqq \sup_{1\leqq p\leqq m}\|f^p\|_{0,\infty}/c_0.$$

PROOF. Let $\gamma=\sup_{1\leqq p\leqq m}\|f^p\|_{0,\infty}/c_0$. First we show that $u_n^p(x)\leqq\gamma$ by induction on n. Clearly, the case $n=0$ is valid. We assume $u_{n-1}^p\leqq\gamma$ for $p=1,\dots,m$. Multiply the both sides of (3.5) by $(u_n^p-\gamma)^+$ and integrate over Ω.

Noting that $(u_n^p-\gamma)^+\in W_0^{1,r}(\Omega)$ in view of $(u_n^p-\gamma)|_\Gamma=-\gamma<0$, we have, by integration by parts,

$$\begin{aligned} &a^p(u_n^p-\gamma, (u_n^p-\gamma)^+) + (\beta(u_n^p-u_{n-1}^{p+1}-K), (u_n^p-\gamma)^+) \\ &\quad - (\beta(u_{n-1}^{p+1}-k-u_n^p), (u_n^p-\gamma)^+) = (f^p-c^p\gamma, (u_n^p-\gamma)^+). \end{aligned}$$

For each term, we get

(i) $\qquad a^p(u_n^p-\gamma, (u_n^p-\gamma)^+) \geqq \alpha\|(u_n^p-\gamma)^+\|_{1,2}^2 \qquad$ by (2.3);

(ii) $\qquad (\beta(u_n^p-u_{n-1}^{p+1}-K), (u_n^p-\gamma)^+) \geqq 0;$

since $u_{n-1}^{p+1}-k-u_n^p>0$ implies $(u_n^p-\gamma)^+=0$ by the assumption of induction, we get

(iii) $\qquad (\beta(u_{n-1}^{p+1}-k-u_n^p), (u_n^p-\gamma)^+) = 0;$

finally, by the definition of γ, $f^p-c^p\gamma\leqq 0$.

Combining these estimates, we obtain

$$\alpha\|(u_n^p-\gamma)^+\|_{1,2}^2 \leqq 0,$$

so that $u_n^p\leqq\gamma$.

It is quite similar to prove the part $-\gamma\leqq u_n^p$. Multiply the both sides of (3.5) by $(u_n^p+\gamma)^-$ and integrate over Ω. In this case, we have by the assumption of induction

$$(\beta(u_n^p-u_{n-1}^{p+1}-K), (u_n^p+\gamma)^-) = 0.$$

Using this and arguing as in the preceding case, we get $-\gamma\leqq u_n^p$. This completes the proof.

LEMMA 3.2. *For each* $\varepsilon>0$, *there exists a solution* u_ε^p *of* (3.4) *which belongs to* $C^{2+\delta}(\bar{\Omega})$, $\delta<1$.

PROOF. Write (3.5) as

$$A^p u_n^p = f^p - \beta(u_n^p - u_{n-1}^{p+1} - K) + \beta(u_{n-1}^{p+1} - k - u_n^p). \tag{3.6}$$

By Lemma 3.1, the right hand side of this equation belongs to $L^\infty(\Omega)$. Hence, by applying the linear elliptic theory, there exists a constant M_ε for each ε, such that

$$\|u_n^p\|_{2,r} \leqq M_\varepsilon.$$

Consequently, there exists a solution $u_\varepsilon^p \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ of (3.4) which is the limit of a suitable subsequence of $\{u_n^p\}$. Again, in (3.6), since the right hand side now belongs to $C^1(\bar{\Omega})$, we conclude $u_\varepsilon^p \in C^{2+\delta}(\bar{\Omega})$. This completes the proof.

4. A priori estimates

In this section we shall derive some a priori estimates which are independent of ε, on the solution u_ε^p, $p=1,\dots, m$, of (3.4). Let the assumptions (A.1)–(A.4) be always satisfied.

We write $\partial/\partial x_i = \partial_i$ and use the summation convention for simplicity.

LEMMA 4.1. *We have*

$$\|u_\varepsilon^p\|_{0,\infty} \leqq \sup_{1\leqq p\leqq m} \|f^p\|_{0,\infty}/c_0.$$

PROOF. This is nothing but Lemma 3.1.

LEMMA 4.2. *We have*

$$\|u_\varepsilon^p\|_{1,\infty,\Gamma} \leqq M,$$

where $\|\cdot\|_{1,\infty,\Gamma}$ *is the norm in the space* $W^{1,\infty}(\Gamma)$.

PROOF. Since Γ is assumed to be smooth, the exterior sphere property holds, i.e., there exists a positive number ρ such that for each $y\in\Gamma$ we can find $\hat{y}\in \boldsymbol{R}^N\setminus\bar{\Omega}$ satisfying

$$\{z\in \boldsymbol{R}^N;\ |z-\hat{y}|\leqq\rho\} \cap \bar{\Omega} = \{y\}.$$

Let $\mu>0$ be a number to be determined later and consider

$$w(x) = e^{-\mu\rho^2} - e^{-\mu|x-\hat{y}|^2}.$$

Since a simple calculation yields

$$A^p w(x) \geqq \{-2\mu a^p_{ij}\delta_{ij} + 4\mu^2\alpha\rho^2 + 2\mu b^p_i(x_i - \hat{y}_i)\}e^{-\mu|x-\hat{y}|^2},$$

where δ_{ij} is Kronecker's delta, we can take $\mu>0$ so large that the inequality

$$A^p w(x) \geqq \lambda$$

holds for some $\lambda>0$ and for all $p=1,\dots, m$, $x \in \bar{\Omega}$.

Hence we have

$$\text{(4.1)} \qquad \begin{aligned} &A^p(-Mw) < f^p \text{ on } \bar{\Omega}, \quad p = 1,\dots, m, \\ &-Mw \leqq 0 \text{ on } \Gamma \text{ and } -Mw(y) = 0 \end{aligned}$$

and

$$\text{(4.1)}' \qquad \begin{aligned} &A^p(Mw) > f^p \text{ on } \bar{\Omega}, \quad p=1,\dots, m, \\ &Mw \geqq 0 \text{ on } \Gamma \text{ and } Mw(y) = 0 \end{aligned}$$

for some constant $M>0$.

Next, we show

$$\text{(4.2)} \qquad |u^p_\varepsilon(x)| \leqq Mw(x) \quad \text{for} \quad x \in \bar{\Omega}, \quad p = 1,\dots, m.$$

In fact, to prove the part $u^p_\varepsilon(x) \geqq -Mw(x)$, let p_0, $1 \leqq p_0 \leqq m$, $x_0 \in \bar{\Omega}$ be such that

$$\min_{p,x}(u^p_\varepsilon(x) + Mw(x)) = u^{p_0}_\varepsilon(x_0) + Mw(x_0).$$

If $x_0 \in \Gamma$, then we immediately have $u^p_\varepsilon(x) \geqq -Mw(x)$ by (4.1)$'$ and $u^{p_0}_\varepsilon|_\Gamma = 0$. Consider the case $x_0 \in \Omega$ and suppose, on the contrary, that $u^{p_0}_\varepsilon(x_0) + Mw(x_0) < 0$. Applying the maximum principle, we get

$$\text{(4.3)} \qquad \begin{aligned} 0 &\geqq A^{p_0}(u^{p_0}_\varepsilon + Mw)(x_0) \\ &\geqq f^{p_0}(x_0) - A^{p_0}(-Mw)(x_0) - \beta_\varepsilon(u^{p_0}_\varepsilon - u^{p_0+1}_\varepsilon - K). \end{aligned}$$

Since $u^{p_0}_\varepsilon(x_0) \leqq u^{p_0+1}_\varepsilon(x_0)$ from the definition of p_0 and x_0, it follows $\beta_\varepsilon(u^{p_0}_\varepsilon - u^{p_0+1}_\varepsilon - K) = 0$. Using this and (4.1), we see that the right hand side of (4.3) is strictly positive. This is a contradiction, and so we have $u^p_\varepsilon(x) \geqq -Mw(x)$ for $p=1,\dots, m$, $x \in \Omega$.

By a similar argument, we can show that $u^p_\varepsilon(x) \leqq Mw(x)$ for $p=1,\dots, m$, $x \in \Omega$. In this case, we may choose p_0 and x_0 in such a way that

$$\max_{p,x}(u^p_\varepsilon(x) - Mw(x)) = u^{p_0}_\varepsilon(x_0) - Mw(x_0),$$

whence we have (4.2).

Moreover, we note that $|\operatorname{grad} u_\varepsilon^p(y)| = |(\partial/\partial n)u_\varepsilon^p(y)|$ at $y \in \Gamma$, where n is the outer normal vector at $y \in \Gamma$. Using (4.2) and that $w(y)=0$, we get

$$\left|\frac{\partial}{\partial n} u_\varepsilon^p(y)\right| \leqq M\left|\frac{\partial}{\partial n} w(y)\right| \leqq M,$$

which concludes the proof.

LEMMA 4.3. *We have*

$$\|u_\varepsilon^p\|_{1,\infty} \leqq M.$$

PROOF. It is sufficient to show the boundedness of the function

$$z_\varepsilon^p(x) = |\operatorname{grad} u_\varepsilon^p|^2. \tag{4.4}$$

Choose p_0, $1 \leqq p_0 \leqq m$, $x_0 \in \bar{\Omega}$ such that

$$z_\varepsilon^{p_0}(x_0) = \max_{p,x} z_\varepsilon^p(x).$$

If $x_0 \in \Gamma$, then the assertion is clear by Lemma 4.2. Consider the case $x_0 \in \Omega$. In the following of the proof, we omit the subscripts p_0, ε and write $v = u_\varepsilon^{p_0+1}$ for simplicity.

We have

$$\begin{aligned} Az = &- 2a_{\xi\zeta}(\partial_\xi\partial_i u)(\partial_\zeta\partial_i u) - 2a_{\xi\zeta}(\partial_\xi\partial_\zeta\partial_i u)(\partial_i u) \\ &+ 2b_\xi(\partial_\xi\partial_i u)(\partial_i u) + c(\partial_i u)(\partial_i u) \\ \leqq &- 2\alpha(\partial_j\partial_i u)(\partial_j\partial_i u) + 2(A(\partial_i u))(\partial_i u) - c_0(\partial_i u)(\partial_i u). \end{aligned} \tag{4.5}$$

Differentiating the both sides of (3.4) with respect to x_i, we get

$$A(\partial_i u) + \beta'(u-v-K)(\partial_i u - \partial_i v) - \beta'(v-k-u)(\partial_i v - \partial_i u) = \partial_i f + \dot{A}u, \tag{4.6}$$

where

$$\dot{A}u = (\partial_i a_{\xi\zeta})(\partial_\xi\partial_\zeta u) - (\partial_i b_\xi)(\partial_\xi u) - (\partial_i c)u. \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5), we have

$$\begin{aligned} Az \leqq &- 2\alpha(\partial_j\partial_i u)(\partial_j\partial_i u) + 2(\partial_i f)(\partial_i u) \\ &+ 2(\partial_i a_{\xi\zeta})(\partial_\xi\partial_\zeta u)(\partial_i u) - 2(\partial_i b_\xi)(\partial_\xi u)(\partial_i u) \\ &- 2(\partial_i c)(\partial_i u)u - 2\beta'(u-v-K)(\partial_i u - \partial_i v)(\partial_i u) \\ &+ 2\beta'(v-k-u)(\partial_i v - \partial_i u)(\partial_i u) - c_0(\partial_i u)(\partial_i u). \end{aligned}$$

Let the right hand side be equal to $(1)+(2)+\cdots+(8)$.

Concerning the assumption (A.3), we denote by $\overline{M}$ various constants which depends only on M_1 and α.

We may estimate each term as follows:

$$2(\partial_i f)(\partial_i u) \leqq \frac{c_0}{4}(\partial_i u)(\partial_i u) + M, \tag{2}$$

$$2(\partial_i a_{\xi\zeta})(\partial_\xi \partial_\zeta u)(\partial_i u) \leqq \alpha(\partial_j \partial_i u)(\partial_j \partial_i u) + \overline{M}(\partial_i u)(\partial_i u), \tag{3}$$

$$-2(\partial_i b_\xi)(\partial_\xi u)(\partial_i u) \leqq \overline{M}(\partial_i u)(\partial_i u), \tag{4}$$

$$\begin{aligned} -2(\partial_i c)(\partial_i u)u &\leqq \frac{c_0}{4}(\partial_i u)(\partial_i u) + M|u|^2 \\ &\leqq \frac{c_0}{4}(\partial_i u)(\partial_i u) + M, \end{aligned} \tag{5}$$

$$\begin{aligned} &-2\beta'(u-v-K)(\partial_i u - \partial_i v)(\partial_i u) \\ &\leqq \beta'(u-v-K)((\partial_i v)(\partial_i v) - (\partial_i u)(\partial_i u)) \\ &= \beta'(u-v-K)(z_\varepsilon^{p_0+1} - z_\varepsilon^{p_0}) \leqq 0, \end{aligned} \tag{6}$$

(7) similarly as in (6),

$$\begin{aligned} &2\beta'(v-k-u)(\partial_i v - \partial_i u)(\partial_i u) \\ &\quad \leqq \beta'(v-k-u)(z_\varepsilon^{p_0+1} - z_\varepsilon^{p_0}) \leqq 0. \end{aligned}$$

Combining these estimates, we have

$$Az \leqq -\alpha(\partial_j \partial_i u)(\partial_j \partial_i u) - \frac{c_0}{2}(\partial_i u)(\partial_i u) + \overline{M}(\partial_i u)(\partial_i u) + M.$$

Hence if c_0 is large enough, say $c_0 > 4\overline{M}$, we get

$$Az \leqq -\frac{c_0}{4}(\partial_i u)(\partial_i u) + M.$$

On the other hand, since $z_\varepsilon^{p_0}$ attains its maximum at x_0, we have by the maximum principle

$$0 \leqq A^{p_0} z_\varepsilon^{p_0}(x_0) \leqq -\frac{c_0}{4} z_\varepsilon^{p_0}(x_0) + M.$$

Consequently, we get $z_\varepsilon^p(x) \leqq M$ and the proof is complete.

REMARK 4.4. In our proof of Lemma 4.3, we had to choose c_0 large enough in accordance with the boundedness of derivatives of a_{ij}^p and b_i^p, and α. We need not such requirement if a_{ij}^p and b_i^p are constants.

In the paper of P. L. Lions [4] in which the Bellman equation was considered, he assumed that c_0 was sufficiently large when he extended the result of L. C.

Evans and A. Friedman [3] to variable coefficient case. However, in that paper, the assumption was needed to derive $W^{2,\infty}(\Omega)$-estimate for approximate solutions.

5. Passage to the limit

In this section we shall prove the existence of solutions of (2.5) which belong to $W^{1,\infty}(\Omega)$. For that purpose, we need the stability of convex sets $\mathscr{K}(u_\varepsilon^p)$ defined by approximate solutions u_ε^p.

LEMMA 5.1. *Let ψ_ε, $\varepsilon>0$, and ψ be functions in $H_0^1(\Omega)$ and let $\delta(\varepsilon)=\|\psi_\varepsilon-\psi\|_{0,\infty}$. If $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$, then for any $v\in\mathscr{K}(\psi)$, we can find $v_\varepsilon\in\mathscr{K}(\psi_\varepsilon)$ for sufficiently small ε such that v_ε converges to v in the strong topology of $H^1(\Omega)$ as $\varepsilon\to 0$.*

PROOF. By the assumption, it is sufficient to consider $\psi_\varepsilon(x)$ only for ε such that

$$\psi_\varepsilon(x) - k \leqq \psi(x) \leqq \psi_\varepsilon(x) + K \qquad \text{in} \quad \Omega.$$

For $v\in\mathscr{K}(\psi)$, we define

$$v_\varepsilon(x) = \begin{cases} \dfrac{K-\delta(\varepsilon)}{K}(v(x)-\psi(x)) + \psi(x) & \text{if} \quad v(x) \geqq \psi(x), \\[2ex] \dfrac{k-\delta(\varepsilon)}{k}(v(x)-\psi(x)) + \psi(x) & \text{if} \quad v(x) < \psi(x). \end{cases}$$

It is obvious that $v_\varepsilon\in\mathscr{K}(\psi_\varepsilon)$ for all small ε. Moreover, we have by a simple calculation

$$\int_\Omega |v(x) - v_\varepsilon(x)|^2 dx \leqq \max\left\{\frac{\delta(\varepsilon)^2}{K^2}, \ \frac{\delta(\varepsilon)^2}{k^2}\right\}$$
$$\times \int_\Omega |v(x) - \psi(x)|^2 dx,$$

and

$$\int_\Omega \left|\frac{\partial v}{\partial x_i}(x) - \frac{\partial v_\varepsilon}{\partial x_i}(x)\right|^2 dx \leqq \max\left\{\frac{\delta(\varepsilon)^2}{K^2}, \ \frac{\delta(\varepsilon)^2}{k^2}\right\}$$
$$\times \int_\Omega \left|\frac{\partial v}{\partial x_i}(x) - \frac{\partial \psi}{\partial x_i}(x)\right|^2 dx$$

for all $i=1,\dots, N$. These relations show us the strong convergence of v_ε to v in $H^1(\Omega)$, so that the proof is complete.

THEOREM 5.2. *Suppose* (A.1)–(A.4). *There exists a solution u^p, $p=1,\dots, m$, of* (2.5) *belonging to $W^{1,\infty}(\Omega)\cap C(\overline{\Omega})$.*

PROOF. By a priori estimates obtained in the preceding section, we can find subsequences (denoted again by ε) of u_ε^p and $u^p \in W^{1,\infty}(\Omega)$, $p=1,\dots, m$, such that for every $p=1,\dots, m$, u_ε^p converges to u^p in the weak* topology of $W^{1,\infty}(\Omega)$.

By virtue of Sobolev's imbedding theorem, we see that $u^p \in C(\bar{\Omega})$ and u_ε^p converges to u^p in the strong topology of $L^\infty(\Omega)$.

We shall show that these u^p, $p=1,\dots, m$, satisfy the inequalities (2.5).

Let $v \in \mathscr{K}(u^{p+1})$ and fix it. By Lemma 5.1 there exists $v_\varepsilon \in \mathscr{K}(u_\varepsilon^{p+1})$ such that v_ε converges to v in the strong topology of $H^1(\Omega)$ as $\varepsilon \to 0$.

Multiplying the both sides of (3.4) by $u_\varepsilon^p - v_\varepsilon$ which belongs to $H_0^1(\Omega)$, integrating over Ω and using integration by parts, we have

$$\begin{aligned} &a^p(u_\varepsilon^p, u_\varepsilon^p - v_\varepsilon) + (\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), u_\varepsilon^p - v_\varepsilon) \\ &\quad - (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^p - v_\varepsilon) = (f^p, u_\varepsilon^p - v_\varepsilon). \end{aligned} \tag{5.1}$$

Since v_ε, u_ε^p are bounded in $H_0^1(\Omega)$, there exists a constant M such that

$$\begin{aligned} &(\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), u_\varepsilon^p - v_\varepsilon) \\ &\quad - (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^p - v_\varepsilon) \leqq M. \end{aligned}$$

Since

$$\begin{aligned} &(\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), u_\varepsilon^p - u_\varepsilon^{p+1} - K) \\ &\quad + (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^{p+1} - k - u_\varepsilon^p) \\ &= (\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), u_\varepsilon^p - v_\varepsilon) \\ &\quad + (\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), v_\varepsilon - u_\varepsilon^{p+1} - K) \\ &\quad - (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^p - v_\varepsilon) \\ &\quad + (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^{p+1} - k - v_\varepsilon) \\ &\qquad \leqq (\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), u_\varepsilon^p - v_\varepsilon) \\ &\quad - (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^p - v_\varepsilon) \\ &\qquad \leqq M, \end{aligned}$$

we have

$$(\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K), u_\varepsilon^p - u_\varepsilon^{p+1} - K) \leqq M$$

and

$$(\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p), u_\varepsilon^{p+1} - k - u_\varepsilon^p) \leqq M.$$

Noting that u_ε^p and u_ε^{p+1} converge uniformly to u^p and u^{p+1}, respectively, we obtain

$$u^p - u^{p+1} - K \leqq 0 \quad \text{and} \quad u^{p+1} - k - u^p \leqq 0 \quad \text{in} \quad \Omega$$

and so $u^p \in \mathscr{K}(u^{p+1})$ for every $p=1,\dots, m$.

Next, consider the form

$$\begin{aligned} X_\varepsilon^p &= a^p(u_\varepsilon^p - v_\varepsilon, u_\varepsilon^p - v_\varepsilon) \\ &\quad + (\beta_\varepsilon(u_\varepsilon^p - u_\varepsilon^{p+1} - K) - \beta_\varepsilon(v_\varepsilon - u_\varepsilon^{p+1} - K), u_\varepsilon^p - v_\varepsilon) \\ &\quad - (\beta_\varepsilon(u_\varepsilon^{p+1} - k - u_\varepsilon^p) - \beta_\varepsilon(u_\varepsilon^{p+1} - k - v_\varepsilon), u_\varepsilon^p - v_\varepsilon). \end{aligned}$$

Clearly we have $X_\varepsilon^p \geqq 0$. On the other hand, substituting (5.1) we get

$$X_\varepsilon^p = (f^p, u_\varepsilon^p - v_\varepsilon) - a^p(v_\varepsilon, u_\varepsilon^p - v_\varepsilon),$$

so that

$$a^p(v_\varepsilon, v_\varepsilon - u_\varepsilon^p) \geqq (f^p, v_\varepsilon - u_\varepsilon^p).$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$a^p(v, v - u^p) \geqq (f^p, v - u^p) \qquad \text{for all} \quad v \in \mathscr{K}(u^{p+1}). \tag{5.2}$$

To derive the inequality (2.5) from (5.2), we can use a standard argument in the theory of variational inequalities. Namely, for $v \in \mathscr{K}(u^{p+1})$ and $\theta \in]0, 1[$, we substitute $\theta v + (1-\theta)u^p \in \mathscr{K}(u^{p+1})$ into v in (5.2). Dividing the both sides of this inequality by θ and taking the limit as $\theta \to 0$, we obtain

$$a^p(u^p, v - u^p) \geqq (f^p, v - u^p)$$

for all $v \in \mathscr{K}(u^{p+1})$. This completes the proof.

REMARK 5.1. We know nothing yet about uniqueness of these solutions of (2.5) belonging to the space $W^{1,\infty}(\Omega)$.

6. The $W^{2,r}(\Omega)$-regularity

In this section we shall prove $W^{2,r}(\Omega)$-regularity of the solution which we have constructed in the preceding section. For that purpose, we use the assumption (A.5).

THEOREM 6.1. *Assume* (A.1)–(A.5). *The solution* u^p, $p=1,\dots, m$, *of* (2.5) *whose existence was proved in Theorem* 5.2 *belongs to* $W^{2,r}(\Omega)$ *for any* r, $1 < r < \infty$.

PROOF. Let $1<r<\infty$. First we note that for any $x_0 \in \Omega$ there exists p, $1 \leqq p \leqq m$, such that

$$u^{p+1}(x_0) - k < u^p(x_0) < u^{p+1}(x_0) + K. \tag{6.1}$$

Indeed, it not so, we have for all $p=1,\dots,m$

$$u^p(x_0) = u^{p+1}(x_0) + \kappa^p,$$

where $\kappa^p = K$ or $-k$. Summing up these equations from 1 to m with respect to p, we get $\sum_{p=1}^m \kappa^p = 0$. But this contradicts the assumption (A.5), and so we have (6.1) for some p.

Changing the number p if necessary, we may assume

$$u^2(x_0) - k < u^1(x_0) < u^2(x_0) + K. \tag{6.2}$$

Since u^1 and u^2 are continuous, we may suppose that (6.2) is valid in $G_\delta = \{x \in \Omega; |x-x_0| < \delta\}$ for some $\delta > 0$.

If $w \in C^\infty(\Omega)$ satisfies $\operatorname{supp} w \subset G_\delta$, $u^1 \pm \lambda w \in \mathscr{K}(u^2)$ for sufficiently small $\lambda > 0$. Substituting this into (2.5), we have

$$a^1(u^1, w) = (f^1, w). \tag{6.3}$$

Since the totality of such w forms a dense subset in $H_0^1(G_\delta)$, (6.3) is valid for all $w \in H_0^1(G_\delta)$. Applying the regularity theorem for linear elliptic equations, we obtain $u^1 \in W^{2,r}(G_\delta)$.

Next, we shall prove the regularity of u^m which is a solution of a variational inequality whose obstacles are determined by u^1.

Since the sequence of approximate solutions u_ε^p converges uniformly to u^p, respectively, we may assume

$$u_\varepsilon^2(x) - k < u_\varepsilon^1(x) < u_\varepsilon^2(x) + K \qquad \text{in} \quad G_\delta$$

for sufficiently small ε. For such u_ε^p we have

$$\beta_\varepsilon(u_\varepsilon^2 - k - u_\varepsilon^1) = \beta_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2 - K) = 0,$$

so that the approximate equation (3.4) becomes

$$A^1 u_\varepsilon^1 = f^1 \qquad \text{in} \quad G_\delta.$$

Hence, $A^1 u_\varepsilon^1$ and u_ε^1 are bounded in $L^r(G_\delta)$ and $W^{2,r}(G_\delta)$, respectively.

Put $\phi_\varepsilon = u_\varepsilon^1 - k$, $\Phi_\varepsilon = u_\varepsilon^1 + K$, and note that $A^m\phi_\varepsilon$, $A^m\Phi_\varepsilon$ are also bounded in $L^r(G_\delta)$.

We shall show that u_ε^m are bounded in $W_{loc}^{2,r}(G_\delta)$.

Let $\zeta \in C^\infty(\Omega)$ be such that $\operatorname{supp} \zeta \subset G_\delta$, $\zeta \geqq 0$, and fix it.

Multiplying the both sides of (3.4) for $p=m$ by $\zeta^r \beta_\varepsilon^{r-1}(u_\varepsilon^m - \Phi_\varepsilon)$ and integrating over Ω, we have

$$
\begin{aligned}
(6.4)\qquad & \int_\Omega (A^m u_\varepsilon^m)\zeta^r \beta_\varepsilon^{r-1}(u_\varepsilon^m-\Phi_\varepsilon)dx + \int_\Omega \zeta^r \beta_\varepsilon^r(u_\varepsilon^m-\Phi_\varepsilon)dx \\
& - \int_\Omega \zeta^r \beta_\varepsilon^{r-1}(u_\varepsilon^m-\Phi_\varepsilon)\beta_\varepsilon(\phi_\varepsilon-u_\varepsilon^m)dx \\
& = \int_\Omega \zeta^r \beta_\varepsilon^{r-1}(u_\varepsilon^m-\Phi_\varepsilon)f^m dx.
\end{aligned}
$$

The third term on the left hand side is equal to zero.

To estimate the first term on the left hand side, we write

$$
\begin{aligned}
(6.5)\qquad & \int_\Omega (A^m u_\varepsilon^m)\zeta^r \beta_\varepsilon^{r-1}(u_\varepsilon^m-\Phi_\varepsilon)dx \\
& = a^m(u_\varepsilon^m-\Phi_\varepsilon, \zeta^r\beta_\varepsilon^{r-1}(u_\varepsilon^m-\Phi_\varepsilon)) \\
& \quad + \int_\Omega (A^m\Phi_\varepsilon)\zeta^r\beta_\varepsilon^{r-1}(u_\varepsilon^m-\Phi_\varepsilon)dx.
\end{aligned}
$$

The first term of (6.5) can be estimated as follows:
We put $u_\varepsilon^m-\Phi_\varepsilon=\psi$ and use abbreviated notations as in Section 4. It follows that

$$
\begin{aligned}
& a(\psi, \zeta^r\beta^{r-1}(\psi)) \\
& = \int_\Omega (a_{ij}(\partial_i\psi)(\partial_j\zeta^r\beta^{r-1}(\psi)) + \tilde{b}_i(\partial\psi)\zeta^r\beta^{r-1}(\psi) \\
& \quad + c\psi\zeta^r\beta^{r-1}(\psi))dx \\
& = r\int_\Omega a_{ij}\beta^{r-1}\zeta^{r-1}(\partial_i\psi)(\partial_j\zeta)dx \\
& \quad + \int_\Omega ((r-1)a_{ij}\zeta^r\beta^{r-2}\beta'(\partial_i\psi)(\partial_j\psi) + \tilde{b}_i\zeta^r\beta^{r-1}(\partial_i\psi) \\
& \quad + c\psi\zeta^r\beta^{r-1})dx \\
& = r\int_\Omega \beta^{r-1}\zeta^{r-1}a_{ij}(\partial_i\psi)(\partial_j\zeta)dx \\
& \quad + \int_\Omega (\beta^{r-2}\beta'\zeta^r)((r-1)a_{ij}(\partial_i\psi)(\partial_j\psi) + \tilde{b}_i(\partial_i\psi)\psi \\
& \quad + c\psi^2)dx \\
& \quad - \int_\Omega \beta^{r-2}\zeta^r(\beta'\psi-\beta)(\tilde{b}_i(\partial_i\psi) + c\psi)dx \\
& = I + II + III.
\end{aligned}
$$

Easily we get

$$
I \geqq -M\int_\Omega \zeta^{r-1}|\mathrm{grad}\,\zeta|\beta^{r-1}dx,
$$

and $II \geqq 0$ from coercivity.

Applying (3.3) we have

$$III \geqq -M\int_{\Omega}\beta^{r-2}(\psi)\zeta^{r}dx$$
$$\geqq -\eta\int_{\Omega}\beta^{r}(\psi)\zeta^{r}dx - M$$

for any $\eta>0$, where M may depend on η.

Applying Hölder's inequality to the second term on the right hand side of (6.5), we conclude

$$\begin{aligned}
&\int_{\Omega}(A^{m}u_{\varepsilon}^{m})\zeta^{r}\beta_{\varepsilon}^{r-1}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx\\
&\geqq -M\int_{\Omega}\zeta^{r-1}|\mathrm{grad}\,\zeta|\beta_{\varepsilon}^{r-1}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx\\
&\quad -\eta\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx - M - \eta\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx\\
&\quad -M\int_{\Omega}\zeta^{r}|A^{m}\Phi_{\varepsilon}|^{r}dx\\
&\geqq -M - 3\eta\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx
\end{aligned}$$

for any $\eta>0$.

Hence we have by (6.4)

$$\begin{aligned}
&-M + (1-3\eta)\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx\\
&\leqq M\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r-1}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx,
\end{aligned}$$

which shows

$$\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})dx \leqq M. \tag{6.6}$$

Next, multiply the both sides of (3.4) for $p=m$ by $\zeta^{r}\beta_{\varepsilon}^{r-1}(\phi_{\varepsilon}-u_{\varepsilon}^{m})$ and integrate over Ω. Then we have

$$\begin{aligned}
&\int_{\Omega}(A^{m}u_{\varepsilon}^{m})\zeta^{r}\beta_{\varepsilon}^{r-1}(\phi_{\varepsilon}-u_{\varepsilon}^{m})dx\\
&\quad +\int_{\Omega}\zeta^{r}\beta_{\varepsilon}(u_{\varepsilon}^{m}-\Phi_{\varepsilon})\beta_{\varepsilon}^{r-1}(\phi_{\varepsilon}-u_{\varepsilon}^{m})dx - \int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r}(\phi_{\varepsilon}-u_{\varepsilon}^{m})dx\\
&=\int_{\Omega}\zeta^{r}\beta_{\varepsilon}^{r-1}(\phi_{\varepsilon}-u_{\varepsilon}^{m})f^{m}dx.
\end{aligned}$$

In this case, the estimation of the first term of the left hand side becomes

$$\int_{\Omega}(A^m u_\varepsilon^m)\zeta^r\beta_\varepsilon^{r-1}(\phi_\varepsilon-u_\varepsilon^m)dx$$
$$= -a^m(\phi_\varepsilon-u_\varepsilon^m, \zeta^r\beta_\varepsilon^{r-1}(\phi_\varepsilon-u_\varepsilon^m)) + \int_{\Omega}(A^m\phi_\varepsilon)\zeta^r\beta_\varepsilon^{r-1}(\phi_\varepsilon-u_\varepsilon^m)dx$$
$$\leqq M + 3\eta\int_{\Omega}\zeta^r\beta_\varepsilon^r(\phi_\varepsilon-u_\varepsilon^m)dx.$$

Arguing similarly for other terms as in the preceding case, we have

$$\int_{\Omega}\zeta^r\beta_\varepsilon^r(\phi_\varepsilon-u_\varepsilon^m)dx \leqq M. \tag{6.7}$$

It follows from (6.6), (6.7) and (3.4) that $A^m u_\varepsilon^m$ are bounded in $L_{loc}^r(G_\delta)$ with respect to ε. Hence, we have the boundedness of u_ε^m in $W_{loc}^{2,r}(G_\delta)$ by using the standard elliptic theory.

Taking the limit as $\varepsilon\to 0$, we obtain

$$u^m \in W_{loc}^{2,r}(G_\delta) \qquad \text{for} \quad r,\ 1 < r < \infty.$$

Repeating this argument inductively with respect to the parameter p, we can show that for any $x_0 \in \Omega$, there exists $\delta > 0$ such that

$$u^p \in W_{loc}^{2,r}(G_\delta) \tag{6.8}$$

for all $p=1,\dots, m$ and r, $1<r<\infty$.

Since, for any precompact subset Ω' in Ω, the totality of such G_δ, $x_0 \in \Omega'$, forms an open covering of Ω', we may select a finite open subcovering of Ω' from G_δ's. Hence we obtain

$$u^p \in W_{loc}^{2,r}(\Omega) \qquad \text{for} \quad p = 1,\dots, m \text{ and } r,\ 1 < r < \infty. \tag{6.9}$$

On the other hand, since $u^p|_\Gamma=0$, (6.1) is satisfied in some neighborhood of Γ for all $p=1,\dots, m$. Hence, each u^p satisfies the second order linear elliptic equation $A^p u^p = f^p$ there, so that u^p belongs to $W^{2,r}$ in this neighborhood.

From this and (6.9), we obtain $u^p \in W^{2,r}(\Omega)$.

The proof is complete.

REMARK 6.1. Since u^p belongs to $W^{2,r}(\Omega)\cap C(\bar{\Omega})$, the u^p satisfy the inequalities (1.1) for almost all $x \in \Omega$.

7. Stochastic representation and uniqueness of the solution

In this section we shall introduce a stochastic switching game and represent the solution component u^p of our problem as the value of this game. As a result of such representation, we shall obtain the uniqueness of regular solutions.

Let $(\hat{\Omega}, \mathscr{F}, P)$ be a probability space and let $w(t)$ be an N-dimensional Brownian motion on it. We denote by $\mathscr{F}_t$ the σ-field $\sigma(w(s), 0 \leqq s \leqq t)$ in $\mathscr{F}$ which is generated by $\{w(s); 0 \leqq s \leqq t\}$.

Let $\sigma^p = [\sigma^p_{ij}]$ be a non-negative matrix satisfying $a^p = (1/2)\sigma^p(\sigma^p)^*$ for each $p = 1,\ldots, m$, where $(\sigma^p)^*$ is the transposed matrix of σ^p and $a^p = [a^p_{ij}]$ is the coefficient matrix of the principal part of A^p. Let $b^p = (b^p_1,\ldots, b^p_N)$ be a N-vector of coefficients of the first order terms of A^p.

We may assume that σ^p and b^p are extended to the whole of $\boldsymbol{R}^N$ preserving Lipschitz continuity.

Consider a system of stochastic differential equations

$$d\xi^p(t) = -b^p(\xi^p(t))dt + \sigma^p(\xi^p(t))dw(t), \quad p = 1,\ldots, m. \tag{7.1}$$

Let $\eta = (\eta_1, \eta_2,\ldots, \eta_n,\ldots)$ be a sequence of $\mathscr{F}_t$- stopping times such that

$$0 < \eta_1 < \eta_2 < \cdots < \eta_n < \cdots .$$

For $x \in \boldsymbol{R}^N$ and for this increasing sequence of stopping times η, by making use of solutions of (7.1), we define a continuous process $\xi(t) = \xi_{\eta,x}(t)$ starting at x as follows:

$$\xi_{\eta,x}(t) = \xi^1(t) \quad \text{with} \quad \xi^1(0) = x \quad \text{if} \quad 0 \leqq t \leqq \eta_1,$$
$$\xi_{\eta,x}(t) = \xi^2(t) \quad \text{with} \quad \xi^2(\eta_1) = \xi^1(\eta_1) \quad \text{if} \quad \eta_1 \leqq t \leqq \eta_2,$$

in general, for any integer $l \geqq 0$ and $1 \leqq p \leqq m$,

$$\xi_{\eta,x}(t) = \xi^p(t) \quad \text{with} \quad \xi^p(\eta_{lm+p-1}) = \xi^{p-1}(\eta_{lm+p-1}) \quad \text{if} \quad \eta_{lm+p-1} \leqq t \leqq \eta_{lm+p}, \tag{7.2}$$

where we put $\xi^0 = \xi^m$, $\eta_0 = 0$.

The process $\xi_{\eta,x}$ is a continuous process starting at $x \in \boldsymbol{R}^N$ with path ξ^1, and whenever it hits the next stopping time η_{lm+p-1}, the path of $\xi_{\eta,x}$ switches from the path of ξ^{p-1} to the path of ξ^p.

We define

$$\begin{aligned} f(\xi_{\eta,x}(t)) &= f^p(\xi^p(t)) \\ c(\xi_{\eta,x}(t)) &= c^p(\xi^p(t)) \end{aligned} \tag{7.3}$$

when $\xi_{\eta,x}(t) = \xi^p(t)$.

Let T be the exit time of the process $\xi_{\eta,x}$ from the domain Ω.

For two increasing sequences $\theta = (\theta_n)$, $\tau = (\tau_n)$ of stopping times, we consider the following cost functional:

$$J_x^1(\theta,\tau) = E_x\Big[\int_0^T \exp\Big(-\int_0^t c(\xi(s))ds\Big)f(\xi(t))dt$$
$$+ K\textstyle\sum_{n=1}^{\infty} \exp\Big(-\displaystyle\int_0^{\eta_n\wedge T} c(\xi(s))ds\Big)\chi\{\eta_n\wedge T=\tau_n\}$$
$$- k\textstyle\sum_{n=1}^{\infty} \exp\Big(-\displaystyle\int_0^{\eta_n\wedge T} c(\xi(s))ds\Big)\chi\{\eta_n\wedge T=\theta_n\}\Big],$$

where $\eta=(\eta_n)$ is an increasing sequence of stopping times defined by $\eta_n=\theta_n\wedge\tau_n=\min\{\theta_n,\tau_n\}$, and $\chi\{\eta=\tau\}=1$ if $\eta=\tau$, $\chi\{\eta=\tau\}=0$ if $\eta\neq\tau$.

The cost functional $J_x^1(\theta,\tau)$ may be interpreted as follows: The first term of the cost functional shows that the running cost per unit time of $\xi(t)$ is given by $f(\xi(t))$ with discounting term $c(\xi(t))$. When $\xi(t)$ switches from ξ^p to ξ^{p+1} at $\eta_n=\eta_{lm+p}$, positive switching cost K or negative switching cost $-k$ or both of them with discounting term are imposed according as $\eta_n=\tau_n<\theta_n$ or $\eta_n=\theta_n<\tau_n$ or $\eta_n=\theta_n=\tau_n$, respectively.

Let us consider a stochastic switching game such that two players compete for the value of the cost functional $J_x^1(\theta,\tau)$. Player 1 wants to maximize the value of $J_x^1(\theta,\tau)$ by operating the stopping times $\theta=(\theta_n)$, while player 2 tries to minimize the value of $J_x^1(\theta,\tau)$ by operating the stopping times $\tau=(\tau_n)$.

We are interested in the value of this stochastic switching game:

$$\sup_\theta \inf_\tau J_x^1(\theta,\tau) \quad \text{or} \quad \inf_\tau \sup_\theta J_x^1(\theta,\tau)$$

where θ, τ range over the set of all increasing sequences of stopping times.

Let $u^p\in W^{2,r}(\Omega)\cap C(\bar\Omega)$, $p=1,\dots,m$, be any solution of

$$(7.5)\qquad \begin{aligned} &u^{p+1}-k\leqq u^p\leqq u^{p+1}+K \quad \text{in} \quad \Omega,\\ &A^pu^p=f^p \quad \text{if} \quad u^{p+1}-k<u^p<u^{p+1}+K,\\ &A^pu^p\leqq f^p \quad \text{if} \quad u^p=u^{p+1}+K,\\ &A^pu^p\geqq f^p \quad \text{if} \quad u^{p+1}-k=u^p,\\ &u^p|_\Gamma=0,\ u^{m+1}=u^1.\end{aligned}$$

Let, for each $p=1,\dots,m$,

$$(7.6)\qquad \hat S^p=\{x\in\bar\Omega;\ u^p(x)=u^{p+1}(x)-k\}$$

and define an increasing sequence $\hat\theta=(\hat\theta_n)$ of stopping times associated with these $\hat S^p$, $p=1,\dots,m$, as follows:

$$\hat\theta_1=\inf\{t\geqq 0;\ \xi^1(t)\in\hat S^1,\ \xi^1(0)=x\},$$
$$\hat\theta_2=\inf\{t\geqq\hat\theta_1;\ \xi^2(t)\in\hat S^2,\ \xi^2(\hat\theta_1)=\xi^1(\hat\theta_1)\},$$

in general,

(7.7) $$\hat{\theta}_{lm+p} = \inf\{t \geqq \hat{\theta}_{lm+p-1};\ \xi^p(t) \in \hat{S}^p,\ \xi^p(\hat{\theta}_{lm+p-1}) = \xi^{p-1}(\hat{\theta}_{lm+p-1})\}$$

for any nonnegative integer l and $p=1,\dots, m$.

Similarly, let

$$\hat{T}^p = \{x \in \bar{\Omega};\ u^p(x) = u^{p+1}(x) + K\}$$

and define an increasing sequence $\hat{\tau}=(\hat{\tau}_n)$ of stopping times associated with $\hat{T}^p$, $p=1,\dots, m$, as follows:

$$\hat{\tau}_1 = \inf\{t \geqq 0;\ \xi^1(t) \in \hat{T}^1,\ \xi^1(0) = x\},$$

in general,

(7.8) $$\hat{\tau}_{lm+p} = \inf\{t \geqq \hat{\tau}_{lm+p-1};\ \xi^p(t) \in \hat{T}^p,\ \xi^p(\hat{\tau}_{lm+p-1}) = \xi^{p-1}(\hat{\tau}_{lm+p-1})\}$$

for any nonnegative integer l and $p=1,\dots, m$.

The next theorem asserts that any regular solution component u^1 of (7.5) can be represented as the value of our stochastic switching game and that the increasing sequences $\hat{\theta}$, $\hat{\tau}$ of stopping times constructed above are the saddle points of this game.

THEOREM 7.1. *Assume (A.1)–(A.5). For any solution component $u^p \in W^{2,r}(\Omega) \cap C(\bar{\Omega})$, $p=1,\dots, m$, of (7.5), we have*

(7.9) $$u^1(x) = J_x^1(\hat{\theta}, \hat{\tau}),$$

(7.10) $$J_x^1(\theta, \hat{\tau}) \leqq u^1(x) \leqq J_x^1(\hat{\theta}, \tau)$$

for all increasing sequences θ, τ of stopping times.

PROOF. In the following we assume $c^p(x)=c$ (a constant) for simplicity of notations.

First of all, we note that the cost functional is rewritten as

$$\begin{aligned} J_x^1(\theta, \tau) = \textstyle\sum_{l=0}^{\infty} \sum_{p=1}^{m} E_x \Big[\displaystyle\int_{\eta_{lm+p-1}\wedge T}^{\eta_{lm+p}\wedge T} & e^{-ct} f^p(\xi^p(t))\, dt \\ & + Ke^{-c\eta_{lm+p}} \chi\{\eta_{lm+p}\wedge T = \tau_{lm+p}\} \\ & - ke^{-c\eta_{lm+p}} \chi\{\eta_{lm+p}\wedge T = \theta_{lm+p}\}\Big]. \end{aligned} \tag{7.11}$$

Let $\hat{\eta} = \hat{\theta}\wedge\hat{\tau}$. We shall show (7.9).

When $\hat{\eta}_{lm+p-1} < t < \hat{\eta}_{lm+p}$, we have

(7.12) $$A^p u^p(\xi^p(t)) = f^p(\xi^p(t))$$

since $\xi^p(t) \notin \hat{S}^p \cup \hat{T}^p$ in this time interval.

Because $u^p \in W^{2,r}(\Omega) \cap C(\bar{\Omega})$, we may apply Ito's formula to have

$$
(7.13)\qquad
\begin{aligned}
&E_x\left[\int_{\hat\eta_{lm+p-1}\wedge T}^{\hat\eta_{lm+p}\wedge T} e^{-ct} f^p(\xi^p(t))\,dt\right]\\
&= E_x\left[\int_{\hat\eta_{lm+p-1}\wedge T}^{\hat\eta_{lm+p}\wedge T} e^{-ct} A^p u^p(\xi^p(t))\,dt\right]\\
&= E_x[-u^p(\xi^p(\hat\eta_{lm+p}\wedge T))e^{-c(\hat\eta_{lm+p}\wedge T)}\\
&\quad + u^p(\xi^p(\hat\eta_{lm+p-1}\wedge T))e^{-c(\hat\eta_{lm+p-1}\wedge T)}].
\end{aligned}
$$

Substituting this into (7.11), we obtain

$$
(7.14)\qquad
\begin{aligned}
&J_x^1(\hat\theta, \hat\tau) = E_x[u^1(\xi^1(\hat\eta_0\wedge T))]\\
&+ \textstyle\sum_{l=0}^{\infty} \sum_{p=1}^{m} E_x[\{-u^p(\xi^p(\hat\eta_{lm+p}\wedge T)) + u^{p+1}(\xi^{p+1}(\hat\eta_{lm+p}\wedge T))\\
&+ K\chi\{\hat\eta_{lm+p}\wedge T = \hat\tau_{lm+p}\} - k\chi\{\hat\eta_{lm+p}\wedge T = \hat\theta_{lm+p}\}\}e^{-c(\hat\eta_{lm+p}\wedge T)}],
\end{aligned}
$$

where $\hat\eta_0 = 0$.

Each term

$$
(7.15)\qquad
\begin{aligned}
U(l, p; \hat\theta, \hat\tau) &= -u^p(\xi^p(\hat\eta_{lm+p}\wedge T)) + u^{p+1}(\xi^{p+1}(\hat\eta_{lm+p}\wedge T))\\
&+ \mathfrak{K}\chi\{\hat\eta_{lm+p}\wedge T = \hat\tau_{lm+p}\} - k\chi\{\hat\eta_{lm+p}\wedge T = \hat\theta_{lm+p}\},
\end{aligned}
$$

appearing in (7.14) may be estimated as follows according to the case of $\hat\eta_n$:

(i) In the case of $\hat\eta_{lm+p}\wedge T = \hat\tau_{lm+p} < \hat\theta_{lm+p}$, we have

$$U(l, p; \hat\theta, \hat\tau) = 0.$$

Indeed, since $\xi^p(\hat\tau_{lm+p}) = \xi^{p+1}(\hat\tau_{lm+p}) \in \hat T^p$, it follows

$$-u^p(\xi^p(\hat\eta_{lm+p}\wedge T)) + u^{p+1}(\xi^{p+1}(\hat\eta_{lm+p}\wedge T)) + K = 0$$

and $\chi\{\hat\eta_{lm+p}\wedge T = \hat\theta_{lm+p}\} = 0$.

(ii) In the case of $\hat\eta_{lm+p}\wedge T = \hat\theta_{lm+p} < \hat\tau_{lm+p}$, we have also

$$U(l, p; \hat\theta, \hat\tau) = 0$$

since $\xi^p(\hat\theta_{lm+p}) = \xi^{p+1}(\hat\theta_{lm+p}) \in \hat S^p$.

(iii) In the case of $\hat\eta_{lm+p}\wedge T = T < \hat\theta_{lm+p}\wedge\hat\tau_{lm+p}$, we get

$$U(l, p; \hat\theta, \hat\tau) = 0$$

because $\chi\{\hat\eta_{lm+p}\wedge T = \hat\theta_{lm+p}\} = \chi\{\hat\eta_{lm+p}\wedge T = \hat\tau_{lm+p}\} = 0$ and $u^p|_\Gamma = 0$.

(iv) Since $\hat S^p \cap \hat T^p = \hat S^p \cap \Gamma = \hat T^p \cap \Gamma = \phi$, there occur no other cases.

Accordingly, $U(l, p; \hat\theta, \hat\tau) = 0$ in all cases, and so we obtain (7.9) from (7.14).

Next, we show

(7.16) $$J_x^1(\theta, \hat{\tau}) \leqq u^1(x)$$

for any increasing sequence θ of stopping times.

Let $\eta=\theta\hat{}\hat{\tau}$. In the interval $\eta_{lm+p-1}<t<\eta_{lm+p}$, we have $\xi^p(t)\notin \hat{T}^p$ since $t<\hat{\tau}_{lm+p}$. Hence it follows from (7.5) that

(7.12)′ $$A^p u^p(\xi^p(t)) \geqq f^p(\xi^p(t)).$$

Therefore, a calculation similar to (7.13) yields

(7.14)′ $$J_x^1(\theta, \hat{\tau}) \leqq E_x[u^1(\xi^1(\eta_0))] + \sum_{l=0}^{\infty} \sum_{p=1}^{m} E_x[U(l, p; \theta, \hat{\tau})e^{-c(\eta_{lm+p}\hat{}T)}],$$

where $\eta_0=0$ and $U(l, p; \theta, \hat{\tau})$ was defined by (7.15).

We may estimate each of $U(l, p; \theta, \hat{\tau})$ as follows:

(i)′ In the case of $\eta_{lm+p}\hat{}T=\hat{\tau}_{lm+p}<\theta_{lm+p}$, we have

$$U(l, p; \theta, \hat{\tau}) = 0$$

by the same reason as in the case (i).

(ii)′ In the case of $\eta_{lm+p}\hat{}T=\theta_{lm+p}<\hat{\tau}_{lm+p}$, we have

$$U(l, p; \theta, \hat{\tau}) = - u^p(\xi^p(\eta_{lm+p})) + u^{p+1}(\xi^{p+1}(\eta_{lm+p})) - k \leqq 0,$$

since $\xi^p(\eta_{lm+p})=\xi^{p+1}(\eta_{lm+p})\notin \hat{T}^p$.

(iii)′ In the case of $\eta_{lm+p}\hat{}T=T<\theta_{lm+p}\hat{}\hat{\tau}_{lm+p}$, we have

$$U(l, p; \theta, \hat{\tau}) = 0$$

similarly as in (iii).

(iv)′ In the case of $\eta_{lm+p}\hat{}T=\theta_{lm+p}=\hat{\tau}_{lm+p}$ or $\eta_{lm+p}\hat{}T=\theta_{lm+p}=T$, we get

$$U(l, p; \theta, \hat{\tau}) = - u^p(\xi^p(\eta_{lm+p})) + u^{p+1}(\xi^{p+1}(\eta_{lm+p})) + K - k = - k < 0.$$

Finally, there does not occur the case $\hat{\tau}_n=T$.

Therefore, since we have $U(l, p; \theta, \hat{\tau})\leqq 0$ in each case, we conclude (7.16) from (7.14)′.

We proceed quite similarly to get

(7.17) $$u^1(x) \leqq J_x^1(\hat{\theta}, \tau)$$

for any increasing sequence τ of stopping times.

In this case we have

(7.14)″ $$J_x^1(\hat{\theta}, \tau) \geqq E_x[u^1(\xi^1(\eta_0))] + \sum_{l=0}^{\infty} \sum_{p=1}^{m} E_x[U(l, p; \hat{\theta}, \tau)e^{-c(\eta_{lm+p}\hat{}T)}]$$

and

$$U(l,\ p;\ \hat{\theta},\ \tau) \geqq 0$$

in each case of η_n.

Combining (7.16) and (7.17) we obtain (7.10) and the proof is complete.

For general $p=1,\dots,m$, we can represent the solution component u^p as a value of a stochastic switching game.

Define a continuous process $\xi^{(p)}_{\eta,x}(t)$ starting at x as in (7.2); the path of $\xi^{(p)}_{\eta,x}$ coincides with that of $\xi^p(t)$ with $\xi^p(0)=x$ in the first interval $0<t<\eta_1$, and it switches at η_n cyclically.

The cost functional $J^p_x(\theta,\tau)$ is defined as (7.4) by using $\xi^{(p)}_{\eta,x}(t)$. Then, obviously, we have the same conclusion about u^p as Theorem 7.1. Saddle points $\hat{\theta}^{(p)}$, $\hat{\tau}^{(p)}$ are also defined analogously by (7.7) and (7.8) starting from p.

The uniqueness result of solutions to our problem in the class $W^{2,r}(\Omega) \cap C(\bar{\Omega})$ is now an immediate consequence of this representation.

Our main theorem is the following:

THEOREM 7.2. *Suppose (A.1)–(A.5). There exists one and only one set of solutions u^p, $p=1,\dots,m$, of the problem* (2.5) *or* (1.1) *which belong to $W^{2,r}(\Omega) \cap C(\bar{\Omega})$ for any r, $1 \leqq r < \infty$.*

REMARK 7.1. We can also treat the more general system

$$(7.18)\qquad \begin{aligned} & u^{p+1}(x) - k^p \leqq u^p(x) \leqq u^{p+1}(x) + K^p,\ x \in \Omega, \\ & A^p u^p = f^p \quad \text{if} \quad u(x^{p+1}) - k^p < u^p < u^{p+1}(x) + K^p,\ x \in \Omega, \\ & A^p u^p \leqq f^p \quad \text{if} \quad u^p(x) = u^{p+1}(x) + K^p,\ x \in \Omega, \\ & A^p u^p \geqq f^p \quad \text{if} \quad u^{p+1}(x) - k^p = u^p(x),\ x \in \Omega, \\ & u^p = 0 \quad \text{on} \quad \Gamma,\ p = 1,\dots, m,\ u^{m+1} = u^1 \end{aligned}$$

in which the given positive constants k^p and K^p may different for $p=1,\dots,m$.

It is clear that, under the same assumptions (A.1)–(A.4), our arguments in Sections 3, 4 and 5 are still valid for this system. Hence we can find a solution u^p, $p=1,\dots,m$, such that

$$a^p(u^p,\ v-u^p) \geqq (f^p,\ v-u^p)$$

for all $v \in \mathscr{K}^p(u^{p+1})$, where we put

$$\mathscr{K}^p(u^{p+1}) = \{v \in H^1_0(\Omega);\ u^{p+1}(x) - k^p \leqq v(x) \leqq u^{p+1}(x) + K^p,\ \text{a.e. in } \Omega\}.$$

Moreover, if we make the assumption

(A.5)′ $\quad \sum_{p=1}^m \kappa^p \neq 0$ *for every system* κ^p, *where* $\kappa^p = -k^p$ *or* K^p, instead of (A.5), then we can prove the $W^{2,r}(\Omega)$-regularity of these solutions by a similar method as in Section 6. The arguments in the proof of Theorem 7.1 are also valid if we make suitable modifications.

Consequently, we can prove the same conclusion as in Theorem 7.2 for the system (7.18) under the assumptions (A.1)–(A.4) and (A.5)$'$.

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Department of Mathematics,
Faculty of Science,
Kobe University