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Kaplansky's radical and Hilbert Theorem 90 II

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Let F be a pre-Hilbert field, $K=F(\sqrt{a})$ be a non-radical extension of F (i.e. $a\notin R(F)$ where $R(F)$ is Kaplansky's radical of F) and $N: K\to F$ be the norm map. In [2], we introduced topologies on the groups $\dot{F}/\dot{F}^2$ and $\dot{K}/\dot{K}^2$ so that the norm map N is continuous and $R(F)$ is closed. We showed there that $N^{-1}(R(F))=(\dot{F}\cdot R(K))^-$, where the bar means the topological closure of $\dot{F}\cdot R(K)$.

In this paper we discuss the case where $K=F(\sqrt{a})$ is a radical extension of a quasi-pythagorean field F. A field F is called quasi-pythagorean if $R(F)=D_F\langle 1, 1\rangle=\{x\in\dot{F};$ the form $\langle 1, 1\rangle$ represents $x\}$. The main purpose of this paper is to give some properties of a quasi-pythagorean field F and show that $N^{-1}(R(F))=\dot{F}\cdot R(K)$. In the last section of this paper, we shall give an example of a quasi-pythagorean field F with $\dim R(F)/\dot{F}^2=n$ for any natural number n and $\dim \dot{F}/R(F)=\infty$.

§1. Preliminaries

In this section, we state some basic facts on Scharlau's method of transfer. By a field F, we shall always mean a field of characteristic different from two. Let $\dot{F}$ denote the multiplicative group of F. For a quadratic form φ_F over F, we define $D_F(\varphi)=\{a\in\dot{F};\ \varphi_F$ represents $a\}$ and $G_F(\varphi)=\{a\in F;\ a\varphi\simeq\varphi\}$. Let K be an extension field of F, and φ_F be a form over F. We denote $\varphi_F\otimes K$ by φ_K for simplicity.

Let $K=F(\sqrt{a})$ be a quadratic extension of F and $x=b+c\sqrt{a}$ $(b, c\in F)$ be an element of K. We write $Im(x)=c$ and $\bar{x}=b-c\sqrt{a}$. For any element $y\in\dot{K}$, we define the map s_y: $K\to F$ with $s_y(x)=Im(y\bar{x})$. It is clear that s_y is a non-zero F-linear functional, and for any non-zero functional s: $K\to F$, there exists a unique element $y\in\dot{K}$ such that $s=s_y$. For a form q_K over K, we denote the transfer of q_K with respect to s_y by $s_y^*(q_K)$.

LEMMA 1.1. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. For $y\in\dot{K}$ and a form q_K over K, the following statements are equivalent:*

(1) *$s_y^*(q_K)$ is isotropic.*

(2) *$D_K(q_K)\cap y\dot{F}\neq\phi$.*

PROOF. We first assume that $s_y^*(q_K)$ is isotopic. Then there exists $x\in D_K(q_K)$

such that $s_y(x)=0$. This implies that $x \in Ker(s_y)=yF$ and we have $D_K(q_K) \cap y\dot{F} \neq \phi$.

Conversely let x be an element of $D_K(q_K) \cap y\dot{F}$. Then $s_y(x)=0$ and $s_y^*(q_K)$ is isotropic. Q. E. D.

LEMMA 1.2. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. For $y, z \in \dot{K}$ and a form q_K over K, we have*

$$s_y^*(z \cdot q_K) \cong s_{y\bar{z}}^*(q_K).$$

PROOF. Let V be the underlying quadratic space of q_K. Then for any element $x \in V$, we have $s_y^*(z \cdot q_K)(x)=s_y(z \cdot q_K(x))=Im(y \cdot \bar{z} \cdot \overline{q_K(x)})$ and $s_{y\bar{z}}^*(q_K)(x)=s_{y\bar{z}}(q_K(x))=Im(y \cdot \bar{z} \cdot \overline{q_K(x)})$. It follows from these relations that $s_y^*(z \cdot q_K) \cong s_{y\bar{z}}^*(q_K)$. Q. E. D.

PROPOSITION 1.3. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. The for $x, y \in \dot{K}$, the following statements hold.*

(1) *If $y \in x\dot{F}$ (i.e. $Im(y\bar{x})=0$), then $s_y^*(\langle x \rangle) \cong H=\langle 1, -1 \rangle$.*

(2) *If $y \notin x\dot{F}$ (i.e. $Im(y\bar{x} \neq 0)$, then $s_y^*(\langle x \rangle) \cong Im(y\bar{x})\langle 1, -N(xy) \rangle$.*

PROOF. Case 1. We first consider the case $x=1$. The underlying quadratic space of $s_y^*(\langle 1 \rangle)$ is K. If $y \notin x\dot{F}=\dot{F}$, then the 2×2 symmetric matrix of the quadratic form $s_y^*(\langle 1 \rangle)$, relative to the F-basis $\{1, y\}$ on K, is of the form

$$\begin{pmatrix} Im(y) & 0 \\ 0 & -Im(y)N(y) \end{pmatrix}.$$

Hence we have

$$s_y^*(\langle 1 \rangle) \cong \langle Im(y), -Im(y)N(y) \rangle \cong Im(y)\langle 1, -N(y) \rangle.$$

If $y \in x\dot{F}=\dot{F}$, then $D_K(\langle 1 \rangle) \cap y\dot{F} \neq \phi$; therefore it follows from Lemma 1.1 that $s_y^*(\langle 1 \rangle)$ is isotropic and $s_y^*(\langle 1 \rangle) \cong H$.

Case 2. Next we consider the case $x \neq 1$. By Lemma 1.2, we obtain $s_y^*(\langle x \rangle)$ $s_{y\bar{x}}^*(\langle 1 \rangle)$. The result of Case 1 shows that if $y\bar{x} \in \dot{F}$ (i.e. $y \in x\dot{F}$), then $s_y^*(\langle x \rangle) \cong H$ and if $y\bar{x} \notin \dot{F}$ (i.e. $y \notin x\dot{F}$), then we have

$$s_y^*(\langle x \rangle) \cong s_{y\bar{x}}^*(\langle 1 \rangle) \cong Im(y\bar{x})\langle 1, -N(y\bar{x}) \rangle.$$

Since $N(xy)=N(y\bar{x})$, $s_y^*(\langle x \rangle)$ is isometric to $Im(y\bar{x})\langle 1, -N(xy) \rangle$. Q. E. D.

§2. Radical extensions of quasi-pythagorean fields

A field F is called *pythagorean* if the sum of two squares in F is always a square. We shall now define the term in the title of this section.

DEFINITION 2.1. A field F is called *quasi-pythagorean* if $R(F)=D_F\langle 1, 1\rangle$, where $R(F)$ is Kaplansky's radical of F.

It is clear that pythagorean fields are quasi-pythagorean fields. An important example of quasi-pythagorean fields is a formally real pre-Hilbert field. In fact, let F be a formally real pre-Hilbert field. Then $-1 \notin R(F)$ and it implies that $\dot{F} \supsetneqq D_F\langle 1, 1\rangle \supseteqq R(F)$. We have $D_F\langle 1, 1\rangle = R(F)$ by the fact $|\dot{F}/R(F)|=2$. So F is quasi-pythagorean.

LEMMA 2.2. *Let F be a quasi-pythagorean field. Then $R(F)=D_F(\infty)$.*

PROOF. Let $x=x_1^2+x_2^2+x_3^2$ be any element of $D_F(3)=D_F\langle 1, 1, 1\rangle$. Since $x_1^2+x_2^2$ is an element of $R(F)$, x belongs to the group $D_F\langle x_1^2+x_2^2, 1\rangle = D_F\langle 1, 1\rangle$ by [2], Proposition 2.1. It is easy to show that $D_F(n)=D_F(n+1)$ for any $n \geqq 2$ and we have $D_F(2)=D_F(\infty)$. Q.E.D.

For a field F, we write $W_t(F)$ to denote the torsion subgroup of the Witt group $W(F)$.

PROPOSITION 2.3. *For a field F, the following statements are equaivalent:*

(1) *F is a quasi-pythagorean field.*

(2) $W_t(F)=\{\langle 1, -a\rangle \in W(F);\ a \in R(F)\}$.

Moreover if F is a quasi-pythagorean field, then $W_t(F) \cong R(F)/\dot{F}^2$.

PROOF. (1)⇒(2): If $a \in R(F)=D_F(2)$, then $\langle a, a\rangle \cong a\langle 1, 1\rangle \cong \langle 1, 1\rangle$ and we have $2\langle 1, -a\rangle = 0 \in W(F)$. On the other hand, let q_F be any torsion element of $W(F)$. We may assume that q_F is anisotropic. By [4], Satz 22, we can find $b_i \in \dot{F}$ and $a_i \in D_F(\infty)$ $(i=1,\dots, n)$ such that $q \sim \sum_{i=1,\dots,n} b_i\langle 1, -a_i\rangle$. Lemma 2.2 shows that $a_i \in R(F)$ and $\langle 1, -a_i\rangle$ is universal. So $b_i\langle 1, -a_i\rangle \cong \langle 1, -a_i\rangle$ and, since q is anisotropic, $n=1$; therefore $q \cong \langle 1, -a\rangle$ for some $a \in R(F)$.

(2)⇒(1): Let b be an element of $D_F(2)$. Then $2\langle 1, -b\rangle = 0 \in W(F)$ and $\langle 1, -b\rangle$ is a torsion element in $W(F)$. Hence it follows from the assumption that there exists $a \in R(F)$ such that $\langle 1, -b\rangle \cong \langle 1, -a\rangle$ and we have $b=a \in R(F)$. This shows $R(F)=D_F(2)$ and F is quasi-pythagorean.

Finally we assume that F is a quasi-pythagorean field. We define the map $f\colon R(F)/\dot{F}^2 \to W_t(F)$ by $f(a)=\langle 1, -a\rangle$, $a \in R(F)$. For any $a, b \in R(F)$, we have $\langle 1, -a\rangle \perp \langle 1, -b\rangle \cong \langle 1, -ab\rangle \perp H$ and this shows that the map f is a group homomorphism. We can readily see that f is injective and moreover f is surjective by the statement (2). This settles our assertion. Q.E.D.

REMARK 2.4. It is well-known that if F is a pythagorean field, then $W(F)$ is torsion free. If F is quasi-pythagorean, then I^2F is torsion free by Proposition 2.3.

According to the definition in [2], we say that a quadratic extension $K=$

$F(\sqrt{a})$ is a radical extension of F, if $a \in R(F)$. For $x \in \dot{F}$, we write $D_F\langle 1, -x\rangle = I_F(x)$ and for a subset $B \subset \dot{F}$, we write $\bigcap_{x \in B} D_F\langle 1, -x\rangle = I_F(B)$.

PROPOSITION 2.5. *Let* $K = (\sqrt{a})$ *be a radical extension of* F. *Then for any n-fold Pfister form* $\rho_F (n \geqq 1)$, *we have* $D_K(\rho_K) \cap \dot{F} = D_F(\rho_F)$.

PROOF. It is clear that $D_K(\rho_K) \cap \dot{F} \supseteq D_F(\rho_F)$. Conversely we take an element $x \in D_K(\rho_K) \cap \dot{F}$. It is sufficient to show that the $(n+1)$-fold Pfister form $\rho_F \otimes \langle\langle -x \rangle\rangle$ is isotropic. Suppose $\rho_F \otimes \langle\langle -x \rangle\rangle$ is anisotropic. Since $(\rho_F \otimes \langle\langle -x \rangle\rangle) \otimes K \cong \rho_K \otimes \langle\langle -x \rangle\rangle$ is isotropic, [3], p. 200, Lemma 3.1 implies that $\rho_F \otimes \langle\langle -x \rangle\rangle$ contains a subform $b\langle 1, -a\rangle$ for some $b \in \dot{F}$. Since $a \in R(F)$, $b\langle 1, -a\rangle \cong \langle 1, -a\rangle$ is universal and the fact $\dim(\rho_F \otimes \langle\langle -x \rangle\rangle) \geqq 4$ implies that $\rho_F \otimes \langle\langle -x \rangle\rangle$ is isotropic. This is a contradiction. Q.E.D.

PROPOSITION 2.6. *Let* F *be a quasi-pythagorean field and* $K = F(\sqrt{a})$ *be a radical extension of* F. *The for* $b, c \in \dot{F}$, *the following statements are equivalent*:

(1) $I_F(b) \subseteq I_F(c)$.

(2) $I_K(b) \subseteq I_K(c)$.

PROOF. (2)⇒(1): For any $x \in \dot{F}$, it follows from Proposition 2.5 that $I_K(x) \cap \dot{F} = I_F(x)$, and the assertion follows immediately.

(1)⇒(2): Let x be an element of $I_K(b)$. We must show that $x \in I_K(c)$. Norm principle ([1], Proposition 2.13) shows that $\dot{F} \cdot I_K(b) \subseteq \dot{F} \cdot I_K(c)$, and so there exists $f \in \dot{F}$ such that $fx \in I_K(c)$. Thus, $f \in xI_K(c) \cap \dot{F} \subseteq I_K(b) \cdot I_K(c) \cap \dot{F} \subseteq D_K(\langle\langle -b, -c \rangle\rangle) \cap \dot{F}$, and by Proposition 2.5, we have $f \in D_F(\langle\langle -b, -c \rangle\rangle)$. The fact $I_F(b) \subseteq I_F(c)$ implies $-b \in D_F\langle 1, -c\rangle = G_F\langle 1, -c\rangle$ and so

$$\begin{aligned} \langle\langle -b, -c \rangle\rangle &\cong \langle 1, -c\rangle \perp (-b)\langle 1, -c\rangle \\ &\cong \langle 1, -c\rangle \perp \langle 1, -c\rangle \cong \langle 1, 1\rangle \perp (-c)\langle 1, 1\rangle. \end{aligned}$$

Since F is quasi-pythagorean, it follows from [2], Proposition 2.1 that $D_F\langle\langle -b, -c \rangle\rangle = D_F\langle 1, -c\rangle$. Hence $f \in D_F\langle 1, -c\rangle = I_F(c) \subseteq I_K(c)$ and we have $x \in I_K(c)$. Q.E.D.

COROLLARY 2.7. *Let* F *be a quasi-pythagorean field and* $K = F(\sqrt{a})$ *be a radical extension of* F. *Then*, $D_K\langle 1, 1\rangle \subseteq I_K(\dot{F})$.

PROOF. For any $x \in \dot{F}$, we have $R(F) = D_F\langle 1, 1\rangle = I_F(-1) \subseteq I_F(x)$. It follows from Proposition 2.6 that $D_K\langle 1, 1\rangle = I_K(-1) \subseteq I_K(x)$. So, $D_K\langle 1, 1\rangle \subseteq \bigcap_{x \in \dot{F}} I_K(x) = I_K(\dot{F})$. Q.E.D.

DEFINITION 2.8. Let $K = F(\sqrt{a})$ be a quadratic extension of F. We denote by $\bar{R}(K)$ the set $\{x \in \dot{K};\ \dot{F} \cdot D_K\langle 1, -x\rangle = \dot{K}\}$.

It is clear that $R(K) \subseteq \bar{R}(K)$. In general $\bar{R}(K)$ is not a subgroup of $\dot{K}$.

LEMMA 2.9. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. Then $\bar{R}(K) \cap I_K(\dot{F})=R(K)$.*

PROOF. It is clear that $\bar{R}(K) \cap I_K(\dot{F}) \supseteq R(K)$. Conversely, suppose $x \in \bar{R}(K) \cap I_K(\dot{F})$. Since $x \in I_K(\dot{F})$, we have $x \in I_K(b)$ for any $b \in \dot{F}$ and this implies $b \in I_K(x)$ for any $b \in \dot{F}$ by [2], Lemma 4.1; therefore we have $\dot{F} \subseteq I_K(x)$. On the other hand, since $x \in \bar{R}(K)$, $\dot{F} \cdot I_K(x)=\dot{K}$ and the assertion follows. Q.E.D.

PROPOSITION 2.10. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. Then for $x \in \dot{K}$, the following statements are equivalent:*

(1) $x \in \bar{R}(K)$.

(2) *For any $y \in \dot{K}-(\dot{F} \cup x\dot{F})$, the form over F,*

$$\langle 1, -N(y)\rangle - Im(y\bar{x})/Im(y)\langle 1, -N(xy)\rangle$$

is isotropic.

PROOF. First we note that if $y \in \dot{K}-(\dot{F} \cup x\dot{F})$, then $Im(y) \neq 0$ and $Im(y\bar{x}) \neq 0$.

(1)⇒(2): The fact $\dot{F} \cdot D_K\langle 1, -x\rangle=\dot{K}$ implies $D_K\langle 1, -x\rangle \cap y\dot{F} \neq \phi$ for any $y \in \dot{K}$. Hence $s_y^*(\langle 1, -x\rangle)$ is isotropic for any $y \in \dot{K}$ by Lemma 1.1. If $y \in \dot{K}-(\dot{F} \cup x\dot{F})$, then $Im(y) \neq 0$ and $Im(y\bar{x}) \neq 0$; therefore it follows from Proposition 1.3 that

$$\begin{aligned} s_y^*(\langle 1, -x\rangle) &\cong s_y^*(\langle 1\rangle) - s_y^*(\langle x\rangle) \\ &\cong Im(y)\langle 1, -N(y)\rangle - Im(y\bar{x})\langle 1, -N(xy)\rangle . \end{aligned}$$

Thus $Im(y)\langle 1, -N(y)\rangle - Im(y\bar{x})\langle 1, -N(xy)\rangle$ is isotropic and we obtain the assertion (2).

(2)⇒(1): By Lemma 1.1, it is sufficient to show that for any $y \in \dot{K}$, $s_y^*(\langle 1, -x\rangle)$ is isotropic. If $y \in \dot{F} \cup x\dot{F}$, then $s_y^*(\langle 1\rangle)$ or $s_y^*(\langle x\rangle)$ is hyperbolic by Proposition 1.3; thus $s_y^*(\langle 1, -x\rangle)$ is isotropic in this case. If $y \in \dot{K}-(\dot{F} \cup x\dot{F})$, then we have $Im(y)\langle 1, -N(y)\rangle - Im(y\bar{x})\langle 1, -N(xy)\rangle \cong s_y^*(\langle 1, -x\rangle)$ and the assumption (2) implies $s_y^*(\langle 1, -x\rangle)$ is isotropic. Q.E.D.

Let $K=F(\sqrt{a})$ be a quadratic extension of F. Let y be an element of $\dot{K}-\dot{F}$; then by using the F-basis $\{1, y\}$ of K, any element $x \in K$ can be written as $x=b+cy$ $(b, c \in F)$. Here the element b is uniquely determined and so we put $b=f_y(x)$. By a straightfoward computation, we have the following

LEMMA 2.11. *In the above situation, we have $f_y(x)=Im(y\bar{x})/Im(y)$.*

LEMMA 2.12. *Let $K=F(\sqrt{a})$ be a quadratic extension of F and x be an element of $\dot{K}$. If $f_y(x) \in D_F\langle 1, -N(y)\rangle$ for any $y \in \dot{K}-(\dot{F} \cup x\dot{F})$, then $x \in \bar{R}(K)$.*

PROOF. Since $Im(y\bar{x})/Im(y)=f_y(x)\in D_F\langle 1, -N(y)\rangle$, the form $\langle 1, -N(y)\rangle - Im(y\bar{x})/Im(y)\langle 1, -N(xy)\rangle$, which is isometric to $\langle 1, -N(y)\rangle - f_y(x)\langle 1, -N(xy)\rangle$, is isotropic. By Proposition 2.9, we have $x\in \bar{R}(K)$. Q.E.D.

THEOREM 2.13. *Let F be a quasi-pythagorean field, and $K=F(\sqrt{a})$ be a radical extension of F. Then we have $N^{-1}(R(F)=\dot{F}\cdot R(K)$, where $N\colon K\to F$ is the norm map.*

PROOF. Norm principle ([1], Proposition 2.13) says that $N^{-1}(R(F))=N^{-1}(D_F\langle 1, 1\rangle)=\dot{F}\cdot D_K\langle 1, 1\rangle$. So it is sufficient to show that $R(K)=D_K\langle 1, 1\rangle$. By Corollary 2.7 and Lemma 2.9, we have only to show that $D_K\langle 1, 1\rangle\subseteq\bar{R}(K)$. We take an element $x\in D_K\langle 1, 1\rangle$. Then for any $y\in\dot{K}-(\dot{F}\cup x\dot{F})$, we can write $x=(b_1+c_1y)^2+(b_2+c_2y)^2 (b_i, c_i\in F)$. Then $x=(b_1^2+b_2^2)+(c_1^2+c_2^2)y^2+2(b_1c_1+b_2c_2)y$. By Lemma 2.11, we have $f_y(y^2)=Im(y\cdot\bar{y}^2)/Im(y)=N(y)Im(\bar{y})/Im(y)=-N(y)$, and this implies that there exists $\alpha\in F$ such that $y^2=-N(y)+\alpha y$, and hence there exists $\beta\in F$ such that $x=(b_1^2+b_2^2)+(c_1^2+c_2^2)(-N(y))+\beta y$. Namely $f_y(x)=(b_1^2+b_2^2)+(c_1^2+c_2^2)(-N(y))\in D_F(\langle\langle 1, -N(y)\rangle\rangle)$. Since F is quasi-pythagorean, we have $D_F(\langle\langle 1, -N(y)\rangle\rangle)=D_F(\langle 1, -N(y)\rangle)$ and $x\in\bar{R}(K)$ by Lemma 2.12. Q.E.D.

In the proof of Theorem 2.13, we have shown that $D_K\langle 1, 1\rangle=R(K)$. Thus any raidcal extension of a quasi-pythagorean field is also quasi-pythagorean.

§3. Application

Throughout this section, we assume that F is a quasi-pythagorean field with a non-trivial radical (i.e. $\dot{F}^2\subsetneq R(F)\subsetneq\dot{F}$), unless otherwise stated. Let $K=F(\sqrt{a})$ be a radical extension of F. By [2], Proposition 4.7, we have $R(K)\cap\dot{F}=R(F)$, and this implies $K\supsetneq R(K)$. On the other hand, Theorem 2.13 says that $N^{-1}(R(F))=\dot{F}\cdot R(K)$, and the norm map is surjective since $K=F(\sqrt{a})$ is a radical extension. It follows from the fact $N^{-1}(\dot{F}^2)=\dot{F}\cdot\dot{K}^2$ that $\dot{F}\cdot R(K)\neq\dot{F}\cdot\dot{K}^2$, which implies $R(K)\neq\dot{K}^2$. Namely $K=F(\sqrt{a})$ is a quasi-pythagorean field with a non-trivial radical. Let L be a field and S be a multiplicative subgroup of $\dot{L}$ which contains $\dot{L}^2$. Then $S/\dot{L}^2$ has the structure of $\boldsymbol{Z}_2$-vector space, and we denote its dimension by $\dim S/\dot{L}^2$. In case when $\dim\dot{F}/\dot{F}^2<\infty$, we have the following

LEMMA 3.1. *Let $K=F(\sqrt{a})$ be a radical extension of F. If $\dim\dot{F}/\dot{F}^2$ is finite, then $\dim\dot{K}/\dot{K}^2=2n-1$ and $\dim R(K)/\dot{K}^2=2m-1$ where $n=\dim\dot{F}/\dot{F}^2$ and $m=\dim R(F)/\dot{F}^2$.*

PROOF. Hilbert Theorem 90 (or [3], p. 202, Theorem 3.4) says that the sequence

$$1 \longrightarrow \dot{F}/\langle \dot{F}^2, a\rangle \xrightarrow{\varepsilon} \dot{K}/\dot{K}^2 \xrightarrow{N} \dot{F}/\dot{F}^2 \longrightarrow 1$$

is exact. This exactness implies $\dim \dot{K}/\dot{K}^2 = 2n-1$. As for $\dim R(K)/\dot{K}^2$, we have the exact sequence

$$1 \longrightarrow \dot{F}/R(F) \xrightarrow{\bar{\varepsilon}} \dot{K}/R(K) \xrightarrow{\bar{N}} \dot{F}/R(F) \longrightarrow 1$$

by Theorem 2.13 and [2], Proposition 5.3. Hence we have $\dim \dot{K}/R(K) = 2(n-m)$ and $\dim R(K)/\dot{K}^2 = (2n-1) - 2(n-m) = 2m-1$. Q.E.D.

Starting from the quasi-pythagorean field F, we define a sequence of fields $\{K_i\}_{i=0,1,2,\dots}$ inductively as follows: $K_0 = F$ and K_{i+1} is a radical extension of K_i. Note that each K_i is a quasi-pythagorean field with a non-trivial radical. We let $K = \text{ind lim } K_i = \cup K_i$.

In the remainder of this paper, we use these notations unless otherwise stated. It is clear that if $i<j$, then $R(K_j) \cap K_i = R(K_i)$.

PROPOSITION 3.2. *K is a quasi-pythagorean field, and $R(K) \cap K_i = R(K_i)$ for any i.*

PROOF. Step 1. First we show that $R(K) \cap K_i \supseteq R(K_i)$. It is sufficient to show that $y \in D_K\langle 1, -x\rangle$ for any $x \in R(K_i)$ and any $y \in \dot{K}$. There exists $j (j \geqq i)$ such that $y \in K_j$. Since $x \in R(K_i) \subseteq R(K_j)$, we have $y \in D_{K_j}\langle 1, -x\rangle \subseteq D_K\langle 1, -x\rangle$.

Step 2. Next we show that K is a quasi-pythagorean field. Let y be an element of $D_K\langle 1, 1\rangle$. There exist $y_1, y_2 \in K$ such that $y = y_1^2 + y_2^2$. We may assume that $y_1, y_2 \in K_j$ for some j. Then the fact $y \in D_{K_j}\langle 1, 1\rangle = R(K_j)$ implies $y \in R(K)$ by Step 1.

Step 3. Finally we show $R(K) \cap K_i = R(K_i)$. Let x be an element of $R(K) \cap K_i = D_K\langle 1, 1\rangle \cap K_i$. We may assume that $x = x_1^2 + x_2^2$, $(x_1, x_2 \in K_j)$ for some $j \geqq i$. Then $x \in D_{K_j}\langle 1, 1\rangle = R(K_j)$ and the fact $R(K_j) \cap K_i = R(K_i)$ implies $x \in R(K_i)$. Thus we have $R(K) \cap K_i \subseteq R(K_i)$. Q.E.D.

REMARK 3.3. Proposition 3.2 shows that K is a quasi-pythagorean field and $\dot{K} \supsetneqq R(K)$. More strictly, we have $\dim \dot{K}/R(K) = \infty$. In fact, $R(K) \cap K_i = R(K_i)$ implies that the canonical homomorphism $\dot{K}_i/R(K_i) \to \dot{K}/R(K)$ is injective for any i. Hence if $\dim \dot{F}/R(F) = \dim \dot{K}_0/R(K_0) = \infty$, then it is clear that $\dim \dot{K}/R(K) = \infty$. If $\dim \dot{F}/R(F) = t < \infty$, then we have $\dim \dot{K}_i/R(K_i) = 2^i t$ by Lemma 3.1; hence $\dim \dot{K}/R(K) \geqq 2^i t$ for any i and we obtain $\dim \dot{K}/R(K) = \infty$.

PROPOSITION 3.4. *If $\dim R(F)/\dot{F}^2 = 1$, then $R(K) = \dot{K}^2$.*

PROOF. We write $K_{i+1} = K_i(\sqrt{a_i})$, $a_i \in R(K_i) - \dot{K}_i^2$. Then $\dot{K}_{i+1}^2 \cap K_i = \langle \dot{K}_i^2, a_i\rangle$ for any i and $\dim R(K_i)/\dot{K}_i^2 = 1$ by Lemma 3.1. Hence we have $R(K_i) = \langle \dot{K}_i^2, a_i\rangle \subseteq \dot{K}_{i+1}^2$, and this implies $R(K) = \cup R(K_i) = \cup \dot{K}_i^2 = \dot{K}^2$. Q.E.D.

For any natural number n, K. Szymiczek ([5], p. 207) gave an example of a formally real pre-Hilbert (hence quasi-pythagorean) field F such that $\dim R(F)/\dot{F}^2=n$. In the following proposition, we shall give an example of a quasi-pythagorean field K such that $\dim \dot{K}/R(K)=\infty$ and $\dim R(K)/\dot{K}^2=n$ for any positive integer n. First we need a lemma.

LEMMA 3.5. *Let $L=k(\sqrt{x})$ be a quadratic extension of k. If $\{y_1,\dots, y_n, x\} \subset \dot{k}$ is lenearly independent in $\dot{k}/\dot{k}^2$ (as a $\boldsymbol{Z}_2$-vector space), then $\{y_1,\dots, y_n\}$ is linearly independent in $\dot{L}/\dot{L}^2$.*

PROOF. Since the canonical injection ε: $\dot{k}/\langle \dot{k}^2, x\rangle \rightarrow \dot{L}/\dot{L}^2$ is $\boldsymbol{Z}_2$-linear, we have $\dim(\langle y_1,\dots, y_n, L^2\rangle/\dot{L}^2)=\dim(\langle y_1,\dots, y_n, x, \dot{k}^2\rangle/\langle x, \dot{k}^2\rangle)=n$. This fact implies that $\{y_1,\dots, y_n\}$ is linearly independent in $\dot{L}/\dot{L}^2$. Q. E. D.

PROPOSITION 3.6. *If* $\dim R(F)/\dot{F}^2\geqq 2$, *then for any natural number n, we can construct a suitable sequence of radical extensions $\{K_i\}_{i=1,2,3\dots}$ such that* $\dim R(K)/\dot{K}^2=n$.

PROOF. Since $\dim R(F)/\dot{F}^2\geqq 2$, Lemma 3.1 shows that there exists $i(1)$ such that $\dim R(K_{i(1)})/\dot{K}_{i(1)}>n$. Let $\{b_1,\dots, b_n, a_{i(1)+1}, a_{i(1)+2},\dots, a_{i(2)}\}$ be a basis of $R(K_{i(1)})/\dot{K}^2_{i(1)}$, where $\dim R(K_{i(1)})/\dot{K}^2_{i(1)}=n+i(2)-i(1)$. We fix the field K_i and the set of elements $\{b_1,\dots, b_n\}$, and we put $K_{i(1)+1}=K_{i(1)}(\sqrt{a_{i(1)+1}})$, $K_{i(1)+2}=K_{i(1)+1}(\sqrt{a_{i(1)+2}}),\dots, K_{i(2)}=K_{i(2)-1}(\sqrt{a_{i(2)}})$. Then we have $a_j\in \dot{K}^2_{i(2)}$ for any $j=i(1)+1, i(1)+2,\dots, i(2)$ and its implies that $\langle b_1,\dots, b_n, \dot{K}^2_{i(2)}\rangle=\langle R(K_{i(1)}), \dot{K}^2_{i(2)}\rangle$. Lemma 3.1 shows that $\dim R(K_{i(2)})/\dot{K}^2_{i(2)}>n$, and Lemma 3.5 shows that $\{b_1,\dots, b_n\}$ is linearly independent in $R(K_{i(2)})/\dot{K}^2_{i(2)}$. Let $\{b_1,\dots, b_n, a_{i(2)+1}, a_{i(2)+2},\dots, a_{i(3)}\}$ be a basis of $R(K_{i(2)})/\dot{K}^2_{i(2)}$, where $\dim R(K_{i(2)})/\dot{K}^2_{i(2)}=n+i(3)-i(2)$. We put $K_{i(2)+1}=K_{i(2)}(\sqrt{a_{i(2)+1}})$, $K_{i(2)+2}=K_{i(2)+1}(\sqrt{a_{i(2)+2}}),\dots, K_{i(3)}=K_{i(3)-1}(\sqrt{a_{i(3)}})$. The sequence of fields $K_{i(3)+1}, K_{i(3)+2},\dots, K_{i(4)},\dots$ is defined similarly.

We shall now show that $K=\operatorname{ind}\lim K_i$ has the required property. First we show that $\{b_1,\dots, b_n\}$ is linearly independent in $R(K)/\dot{K}^2$. Suppose $\{b_1,\dots, b_n\}$ is linearly dependent in $R(K)/\dot{K}^2$. Then there exists a partial product b of $\{b_1,\dots, b_n\}$ such that $b\in \dot{K}^2$. Since $\dot{K}^2=\cup \dot{K}^2_j$, there exists j such that $b\in \dot{K}^2_j$, and this means that $\{b_1,\dots, b_n\}$ is linearly dependent in $R(K_j)/\dot{K}^2_j$. This is a contradiction, and hence $\{b_1,\dots, b_n\}$ is linearly independent in $R(K)/\dot{K}^2$.

Next we show that $R(K)/\dot{K}^2$ is generated as a $\boldsymbol{Z}_2$-vector space by $\{b_1,\dots, b_n\}$. Let x be an element of $R(K)$. There exists j such that $x\in K_j$. Since $R(K)\cap K_j=R(K_j)$, we have $x\in R(K_j)$. Let $i(s)$ be a number which is larger than j. Then we have $x\in\langle R(K_{i(s)}), \dot{K}^2_{i(s+1)}\rangle=\langle b_1,\dots, b_n, \dot{K}^2_{i(s+1)}\rangle\subseteq\langle b_1,\dots, b_n, \dot{K}^2\rangle$, and this shows that $\{b_1,\dots, b_n\}$ generates $R(K)/\dot{K}^2$. Thus we see that K is a quasi-pythagorean field with $\dim \dot{K}/R(K)=\infty$ and $\dim R(K)/\dot{K}^2=n$. Q. E. D.

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