HIROSHIMA MATH. J.
13 (1983), 1–28

On the extendibility of vector bundles over the lens spaces and the projective spaces

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(Received May 17, 1982)

§1. Introduction

Let X and A be a topological space and its subspace. Then a fibre bundle ζ over A is said to be *extendible* to X, if there is a fibre bundle α over X whose restriction $\alpha|A$ to A is equivalent to ζ.

R. L. E. Schwarzenberger ([9; Appendix I], [21]) and several authors studied the extendibility of vector bundles over the complex (resp. real) projective n-space CP^n (resp. RP^n) to CP^m (resp. RP^m) for $m>n$ (cf., e.g., the references of [24]).

For an integer $q\geqq 2$, let L_q^n denote the standard lens space mod q or its n-skeleton:

$$L_q^{2i+1} = L^i(q) = S^{2i+1}/Z_q \text{ or } L_q^{2i} = \pi(S^{2i})(\pi: S^{2i+1} \longrightarrow L_q^{2i+1} \text{ is the projection}),$$

where $L_2^n = RP^n$. The purpose of this paper is to study the extendibility of complex (or real) vector bundles over L_q^n to L_q^m for $m>n$, as a continuation of the previous papers [18], [14] and [15].

Let η be the canonical complex line bundle over L_q^n, i.e., the induced bundle $\pi^*\eta$ of the one η over CP^i by the natural projection $\pi: L_q^{2i+1}\rightarrow CP^i$ or its restriction $\pi^*\eta\,|\,L_q^{2i}$. Then the main results on complex bundles are stated as follows:

THEOREM 1.1. *Let ζ be a complex t-plane bundle over L_q^n. Then ζ is stably equivalent to a complex $t'(=\sum_{i=1}^{q-1} b_i)$-plane bundle $\zeta'=\sum_{i=1}^{q-1} b_i\eta^i$ over L_q^n for some integers $b_i\geqq 0$. Furthermore, we have the following* (i) *and* (ii):

(i) *If $t\geqq[n/2]$, then ζ is extendible to L_q^{2t+1}. If $t\geqq[(n+1)/2]$ and $t\geqq t'$, then ζ is extendible to L_q^m for any $m\geqq n$.*

(ii) *Take a prime factor p of q with $p\leqq[n/2]+1$, and put $a=[n/2(p-1)]$ and*

$$c_k \equiv \textstyle\sum_l b_{lp+k} \bmod p^a,\ 0\leqq c_k<p^a, \qquad \text{for} \quad 1\leqq k\leqq p-1.$$

If there is an integer m satisfying

$$t<m<p^a \text{ and } \textstyle\sum_{j_1+\cdots+j_{p-1}=m} \prod_{k=1}^{p-1} \binom{c_k}{j_k} k^{j_k} \not\equiv 0 \bmod p,$$

then $2m>n$ *and* ζ *is not extendible to* L_q^{2m}.

When q *is even, if* $c=c_1$ *for* $p=2$ *satisfies* $t<c$, *then* ζ *is not extendible to* L_q^{2N}, *where* $N=\min\{j+\nu_2(\binom{c}{j})|t<j\leqq c\}$($\nu_2(b)$ *is the exponent of* 2 *in the prime power decomposition of a positive integer* b).

In case of real bundles, we have the real restriction $r(\eta^i)$ of η^i over L_q^n, and the non-trivial real line bundle ρ over L_q^n when q is even. Furthermore, when q is odd and $n\equiv 1 \bmod 8$, we have the induced bundle β_n of the stably non-trivial real n-plane bundle over S^n by the projection $L_q^n\to L_q^n/L_q^{n-1}=S^n$.

THEOREM 1.2. *Let* ζ *be a real t-plane bundle over* L_q^n. *Then* ζ *is stably equivalent to a real* t'*-plane bundle* ζ' *over* L_q^n *such that*

$$\zeta' = \varepsilon\beta_n\oplus b\rho\oplus\sum_{i=1}^{u} b_i r(\eta^i) \quad \text{and} \quad t' = \varepsilon n+b+2\sum_{i=1}^{u} b_i \; (u=[(q-1)/2])$$

for some non-negative integers ε, b *and* b_i *with* $\varepsilon=0, 1$, *where* $\varepsilon\beta_n$ (*resp.* $b\rho$) *appears only when* q *is odd and* $n\equiv 1 \bmod 8$ (*resp.* q *is even*).

If $\varepsilon=1$, *then* ζ *is not extendible to* L_q^{n+1}. *Furthermore we have the following* (i) *and* (ii) *under the assumption that* $\varepsilon=0$ *or* $\varepsilon\beta_n$ *does not appear.*

(i) *If* $t\geqq n$, *then* ζ *is extendible to* L_q^t. *If* q *and* n *are odd and* $t>n$, *then* ζ *is extendible to* $L_q^{2t-(-1)^t}$. *If* $t>n$ *and* $t\geqq t'$, *then* ζ *is extendible to* L_q^m *for any* $m\geqq n$.

(ii) *Take an odd prime factor* p *of* q *with* $p\leqq[n/2]+1$, *and put* $a=[n/2(p-1)]$ *and*

$$d_k \equiv \sum_l (b_{lp+k}+b_{lp+p-k}) \bmod p^a \quad \text{and} \quad 0\leqq d_k<p^a \quad \text{for} \quad 1\leqq k\leqq v=(p-1)/2.$$

If there is an even integer m *satisfying*

$$t<m<2p^a \quad \text{and} \quad \sum_{j_1+\cdots+j_v=m/2}\prod_{k=1}^{v}\binom{d_k}{j_k}k^{2j_k}\not\equiv 0 \bmod p,$$

then $2m>n$ *and* ζ *is not extendible to* L_q^{2m}.

When q *is even, put*

$$d' \equiv b' + 2\sum_l b_{2l+1} \bmod 2^{\phi(n)} \quad \text{and} \quad 0\leqq d'<2^{\phi(n)},$$

where $b'=b$ *if* $q/2$ *is odd and* $b'=0$ *otherwise, and* $\phi(n)$ *is the number of integers* s *with* $0<s\leqq n$ *and* $s\equiv 0, 1, 2, 4 \bmod 8$. *If* $t<d'$, *then* ζ *is not extendible to* $L_q^{N'}$, *where* $N'=\min\{\min\{m\mid\phi(m)\geqq j+\nu_2(\binom{d'}{j}), t<j\leqq d'\}, \min\{j\mid t<j\leqq d', \nu_2(\binom{d'}{j})=0\}\}$.

Theorem 1.1 is proved in Lemma 3.5, Theorems 3.13 and 3.23, and Theorem 1.2 is proved in Lemma 5.4, Theorem 5.7 and Corollary 5.17, where the non-extendibility is shown by studying the γ-operations in K- and KO-theory and the Stiefel-Whitney classes.

As an application of these results, we study the extendibility of the higher order tangent bundle over L_q^n to L_q^m, and in particular, we obtain the following theorem, where $m(\zeta)$ denotes the maximum integer of m such that a bundle ζ over RP^n is extendible to RP^m.

THEOREM 1.3. *Let $\tau_k(RP^n)$ $(k \geqq 1)$ be the k-th order tangent bundle over the real projective space RP^n ($\tau_1(RP^n)$ is the tangent bundle of RP^n) and $c\tau_k(RP^n)$ be its complexification. Then*

$$m(\tau_k(RP^n)) = \begin{cases} \infty & \text{if } k \text{ is even or } C(n,k) \geqq 2^{\phi(n)}, \\ C(n,k)-1 & \text{otherwise;} \end{cases}$$

$$m(c\tau_k(RP^n)) = \begin{cases} \infty & \text{if } k \text{ is even or } C(n,k) \geqq 2^{[n/2]}, \\ 2C(n,k)-1 & \text{otherwise,} \end{cases}$$

where $C(n,k)=\binom{n+k}{k}$.

This theorem is proved in Theorem 6.10, and a result for the lens space $L^n(q)$ is proved in Theorems 6.16 and 6.17.

In §2, we study some conditions that a bundle over an n-skeleton X^n of a finite CW-complex X is extendible to an m-skeleton X^m. In §3, we prove Theorem 1.1. §4 is devoted to apply the results obtained in §§2–3 to the complexification of the tangent (or normal) bundle of $L^n(q)$ and to complex bundles over the complex projective space CP^n, and as a corollary, we obtain Schwarzenberger's result [9; p. 166] that the complex tangent bundle over CP^n $(n \geqq 2)$ is not extendible to CP^{n+1}. In §5, we prove Theorem 1.2 by using the KO-theory. By using these results, we study the higher tangent bundle of the lens space in §6.

The author wishes to express his hearty thanks to Professors M. Sugawara, T. Kobayashi and T. Yoshida for their useful advises and stimulating discussions during the preparation of this paper and also to Professor K. Fujii for his kind advises.

§2. Vector bundles over an n-skeleton

In this paper, let F denote the real field R or the complex field C, and set $f=\dim_R F=1$ or 2 according to $F=R$ or C. We denote simply by b the b-dimensional trivial F-vector bundle.

In this section, we consider a finite CW-complex X, and study some conditions that a given F-vector bundle ζ over the n-skeleton X^n of X is extendible to an m-skeleton $X^m \supset X^n$ for $m \geqq n$.

We notice the following (cf. [10; p. 100, Th. 1.5]):

(2.1) *If t- and t'-dimensional F-vector bundles ζ and ζ' over X^n are stably equivalent,* i.e., $\zeta \oplus s \cong \zeta' \oplus s'$ *(equivalent) for some non-negative integers s and s', and if $t \geqq t'$ and $t \geqq [(n+1)/f]$, then $\zeta \cong \zeta' \oplus (t-t')$.*

THEOREM 2.2. *Let ζ be a t-dimensional F-vector bundle over X^n, and assume that $t \geqq [(n+1)/f]$. Then ζ is extendible to X^m $(m>n)$ if and only if there exists a t'-dimensional F-vector bundle ζ' over X^n such that*

(1) *ζ is stably equivalent to ζ', and*

(2) *ζ' is extendible to a bundle α' over X^m with* $\mathrm{Span}(\alpha' \oplus k) \geqq t'-t+k$ *for some $k \geqq 0$.* ($\mathrm{Span}\,\alpha$ *denotes the maximum number of linearly independent cross-sections of an F-vector bundle α.*)

PROOF. The necessity is seen by taking $\zeta' = \zeta$. We prove the sufficiency.

If $t \geqq t'$, then (1) implies that $\zeta \cong \zeta' \oplus (t-t')$ by (2.1), and hence (2) implies that ζ is extendible to a bundle $\alpha' \oplus (t-t')$ over X^m.

If $t' > t$, then (1) implies that $\zeta' \cong \zeta \oplus (t'-t)$ by (2.1), and (2) implies that that $\alpha' \oplus k \cong \alpha \oplus (t'-t+k)$ for some α over X^m with $\dim \alpha = t$. Thus

$$\zeta \oplus (t'-t+k) \cong \zeta' \oplus k \cong (\alpha' \mid X^n) \oplus k \cong (\alpha \mid X^n) \oplus (t'-t+k),$$

which implies that $\zeta \cong \alpha \mid X^n$ by (2.1). q.e.d.

COROLLARY 2.3. *Let ζ (resp. ζ') be a t (resp. t')-dimensional F-vector bundle over X^n, and assume that ζ is stably equivalent to ζ' and that ζ' is extendible to X^m $(m>n)$. Then ζ is also extendible to X^m, if*

(1) *$t \geqq t'$ and $t \geqq [(n+1)/f]$, or* (2) *$t \geqq [m/f]$.*

PROOF. When (1) holds, then the result is clear by the above theorem.

Assume that (2) holds. If $t \geqq t'$, then (1) holds. If $t' > t$, then $t' > [m/f]$ and an extension α' of ζ' over X^m satisfies $\alpha' \cong \beta \oplus (t' - [m/f])$ for some β by [10; p. 99, Th. 1.2], and the condition $\mathrm{Span}\,\alpha' \geqq t'-t$ in (2) of the above theorem holds. Thus we see the corollary by the above theorem. q.e.d.

As typical examples of extendible bundles, we have the following

PROPOSITION 2.4. *If $n \geqq 3$, then any oriented real 2-plane bundle and any complex line bundle over X^n are extendible to X^m for each $m(\geqq n)$.*

PROOF. Let θ be a complex line bundle over X^n, and $f: X^n \to BU(1)$ be its classifying map. Then the obstructions for extending f to X^m are contained in the cohomology groups $H^{r+1}(X^m, X^n; \pi_r(BU(1)))$ for $n \leqq r < m$, which are 0 since $\pi_r(BU(1)) \cong \pi_{r-1}(S^1)=0$ for $r \geqq 3$. Thus f has an extension f': $X^m \to BU(1)$ and hence θ is extendible to X^m. The result for an oriented real 2-plane bundle is proved similarly in [14, Lemma 5.2] by considering $BSO(2)$ instead of $BU(1)$. q.e.d.

COROLLARY 2.5. *Assume that $n \geqq 3$, and a real (resp. complex) t-plane bundle ζ over X^n is stably equivalent to a sum of s oriented real 2-plane bundles (resp. s complex line bundles), where t and s are assumed to be $t \geqq n+1$ and $t \geqq 2s$ (rest. $t \geqq [(n+1)/2]$ and $t \geqq s$). Then ζ is extendible to X^m for each $m(\geqq n)$.*

PROOF. By the assumptions and (2.1), we have

$$\zeta = \theta_1 \oplus \cdots \oplus \theta_s \oplus \delta, \quad \delta = t-2s \text{ (resp. } t-s),$$

where θ_i $(1 \leqq i \leqq s)$ are oriented real 2-plane bundles (resp. complex line bundles). Thus the corollary follows immediately from Proposition 2.4. q.e.d.

§3. Complex bundles over the lens spaces

In this paper, we shall denote the standard lens space mod q by

(3.1) $$L_q^{2i+1} = L^i(q) = S^{2i+1}/Z_q \qquad \text{for a fixed integer} \quad q \geqq 2,$$

where $S^{2i+1} = \{(z_0,\dots, z_i) \in C^{i+1} \mid |z_0|^2 + \cdots + |z_i|^2 = 1\}$ is the $(2i+1)$-sphere, $Z_q = \{z \in C \mid z^q = 1\}$ is the cyclic subgroup of order q of the circle group $S^1 = \{z \in C \mid |z| = 1\}$, and the action is given by $z(z_0,\dots, z_i) = (zz_0,\dots, zz_i)$. We consider $L_q^{2j+1} \subset L_q^{2i+1}$ for $j<i$ by identifying $[z_0,\dots, z_j] \in L_q^{2j+1}$ with $[z_0,\dots, z_j, 0,\dots, 0] \in L_q^{2i+1}$, and set

(3.2) $$L_q^{2i} = L_0^i(q) = \{[z_0,\dots, z_i] \in L_q^{2i+1} \mid z_i \text{ is a non-negative real number}\}.$$

Then $L_q^n - L_q^{n-1}$ is an open n-cell and we have a CW-decomposition of L_q^N whose n-skeleton is L_q^n for $0 \leqq n \leqq N$.

If $q=2$, then L_2^n is the real projective space RP^n.

Let η_{2i+1} be the canonical complex line bundle over L_q^{2i+1}, i.e., the induced bundle of the one over the complex projective space CP^i by the projection $L_q^{2i+1} = S^{2i+1}/Z_q \to S^{2i+1}/S^1 = CP^i$. Then the restriction $\eta_{2i+1} \mid L_q^{2j+1}$ for $j<i$ is η_{2j+1}, and we denote η_{2i+1} and its restriction $\eta_{2i} = \eta_{2i+1} \mid L_q^{2i}$ by η simply.

If $q=2$, then η is the complexification of the canonical real line bundle ξ over RP^n.

To study the extendibility of a complex bundle over L_q^n to L_q^m $(m \geqq n)$, we use the following results on the K-ring of the lens space.

(3.3) (cf. [12; Prop. 2.6]) *The reduced K-ring $\tilde{K}(L_q^n)$ is generated by*

$\sigma=\eta-1$ and contains exactly $q^{[n/2]}$ elements. Furthermore $(1+\sigma)^q-1=\eta^q-1=0=\sigma^{[n/2]+1}$, and the order of $\sigma^{[n/2]}$ is equal to q.

(3.4) (J. F. Adams [1; Th. 7.3], T. Kambe [11; Th. 1]) *If q is a prime, then*

$$\tilde{K}(L_q^n) = \oplus_{i=1}^{q-1} Z_{r_i}\langle\sigma^i\rangle \ (\textit{direct sum}), \quad r_i = q^{1+[([n/2]-i)/(q-1)]},$$

where $Z_r\langle\alpha\rangle$ denotes the cyclic group of order r generated by α.

LEMMA 3.5. (i) *Any complex t-plane bundle ζ over L_q^n is stably equivalent to a complex t'-plane bundle ζ' over L_q^n, where*

(3.6) $$\zeta' = \sum_{i=1}^{q-1} b_i\eta^i \quad \textit{and} \quad t' = \sum_{i=1}^{q-1} b_i \quad \textit{for some integers } b_i \geqq 0.$$

(ii) *b_i in (3.6) can be reduced to the residue modulo $q^{[n/2]}$ or, more precisely, modulo the order of η^i-1 in $\tilde{K}(L_q^n)$.*

(iii) *If q is a prime, then b_i in (3.6) can be reduced to the residue modulo $r_1=q^{1+[([n/2]-1)/(q-1)]}$.*

(iv) *Let q be a prime p. If $[n/2]\geqq p-1$ and if $\sum_{i=1}^{p-1} b_i\eta^i$ and $\sum_{i=1}^{p-1} b_i'\eta^i$ over L_p^n are stably equivalent, then*

$$b_i \equiv b_i' \mod p^a, \quad a = [n/2(p-1)] (\geqq 1), \qquad \textit{for} \quad 1\leqq i\leqq p-1.$$

PROOF. (i), (ii) $\zeta-t\in\tilde{K}(L_q^n)$ is equal to $\sum_{i=1}^{q-1} a_i\sigma^i=\sum_{i=1}^{q-1} b_i(\eta^i-1)$ for some a_i and $0\leqq b_i<q^{[n/2]}$ by (3.3). Thus ζ is stably equivalent to $\zeta'=\sum_{i=1}^{q-1} b_i\eta^i$.

(iii) If q is a prime, then the order of $\eta^i-1=(1+\sigma)^i-1=\sum_{j=1}^{i}\binom{i}{j}\sigma^j\in\tilde{K}(L_q^n)$ is equal to r_1 for $1\leqq i<q$ by (3.4). Thus we have (iii) by (ii).

(iv) Since $\eta=\sigma+1$, we have

$$0 = \sum_{i=1}^{p-1} (b_i-b_i')(\eta^i-1) = \sum_{j=1}^{q-1}\Big(\sum_{i=j}^{p-1}\binom{i}{j}(b_i-b_i')\Big)\sigma^j \text{ in } \tilde{K}(L_p^n)$$

by assumption, and hence

$$\sum_{i=j}^{p-1}\binom{i}{j}(b_i-b_i') \equiv 0 \mod r_j \qquad \text{for} \quad 1\leqq j\leqq p-1$$

by (3.4). Since r_i is a power of p and $r_i\,|\,r_{i-1}$, this implies that

$$b_i-b_i' \equiv 0 \mod r_{p-1} \qquad \text{for} \quad 1\leqq i\leqq p-1 \quad (r_{p-1}=p^a)$$

by the induction on $p-i$. q. e. d.

We now study the extendibility of a complex t-plane bundle ζ over L_q^n to L_q^m for $m\geqq n$, by using the notation

(3.7) $$m(\zeta) = \max\{m \,|\, \zeta \text{ is extendible to } L_q^m\ (m\geqq n)\},$$

where $m(\zeta)=\infty$ means that ζ is extendible to L_q^m for any $m\geqq n$.

THEOREM 3.8. *Let ζ be a complex t-plane bundle over L_q^n and assume that ζ is stably equivalent to a t'-plane bundle ζ' in* (3.6) *by Lemma* 3.5 (i).

(i) *If $t\geqq[n/2]$, then $m(\zeta)\geqq 2t+1$.*

(ii) *If $t\geqq[(n+1)/2]$ and $t\geqq t'$, then $m(\zeta)=\infty$.*

(iii) *If $t\geqq[(n+1)/2]$ and $t\geqq(q-1)(q^{[n/2]}-1)$, then $m(\zeta)=\infty$.*

(iv) *If q is a prime and $t\geqq(q-1)(r_1-1)$ where r_1 is the integer in Lemma* 3.5 (iii), *then $m(\zeta)=\infty$.*

PROOF. (i) By definition, $m(\eta)=\infty$ and hence $m(\zeta')=\infty$ by (3.6). Thus Corollary 2.3 (2) implies (i).

(ii) Corollary 2.3(1) implies (ii) in the same way as above.

(iii) By Lemma 3.5(ii), (iii) is a special case of (ii).

(iv) If $n=1$, then (iv) is a special case of (iii). If $q=2$ and $t=1$, then ζ is η or 1 since complex line bundles are classified by their first Chern classes. Thus $m(\zeta)=\infty$. Assume that q is a prime, $n\geqq 2$ and $t\geqq 2$ if $q=2$. Then t' can be taken so that $(q-1)(r_1-1)\geqq t'$ by Lemma 3.5 (iii), and we see easily that $(q-1)(r_1-1)\geqq[(n+1)/2]$ if $q\neq 2$ or $n\neq 3$. Thus we have (iv) by (ii). q.e.d.

To study the upper bound of $m(\zeta)$, we use the γ-operation in $K(L_q^n)$.

For a given integer $q\geqq 2$ and integers $b_i\geqq 0$ $(1\leqq i\leqq q-1)$, we have

$$\prod_{i=1}^{q-1}\{1+((\sigma+1)^i-1)t\}^{b_i}=\sum_{j\geqq 0}\{\sum_{k\geqq 0}A_k(b_1,\dots,b_{q-1};j)\sigma^{j+k}\}t^j \tag{3.9}$$

for some coefficients $A_k(b_1,\dots,b_{q-1};j)$, where

$$\begin{aligned} A_0(b_1,\dots,b_{q-1};j) &= \sum_{j_1+\cdots+j_{q-1}=j}\prod_{i=1}^{q-1}\binom{b_i}{j_i}i^{j_i},\\ A_1(b_1,\dots,b_{q-1};j) &= \sum_{j_1+\cdots+j_{q-1}=j}\{\prod_{i=1}^{q-1}\binom{b_i}{j_i}i^{j_i}\}\{\sum_{i=1}^{q-1}j_i(i-1)\}/2. \end{aligned} \tag{3.10}$$

LEMMA 3.11. *Assume that a complex t-plane bundle ζ over L_q^n is stably equivalent to a $t'(=\sum_{i=1}^{q-1}b_i)$-plane bundle $\zeta'=\sum_{i=1}^{q-1}b_i\eta^i(b_i\geqq 0)$ in* (3.6), *and that*

$$\gamma^j(\zeta-t)=0 \text{ in } \tilde{K}(L_q^n) \text{ for some positive integer } j\leqq[n/2], \tag{3.12}$$

where γ^j denotes the γ-operation. Then we have the following (i)–(iii) *for $A_k(b_1,\dots,b_{q-1};j)$ in* (3.10):

(i) $A_0(b_1,\dots,b_{q-1};j)\equiv 0 \mod q$.

(ii) *If q is an odd prime and $j<[n/2]$ in* (3.12), *then* $A_1(b_1,\dots,b_{q-1};j)\equiv 0 \mod q$.

(iii) *If $q=2$, then* $\binom{t'}{j}=A_0(b_1;j)\equiv 0 \mod 2^{1+[n/2]-j}$ $(t'=b_1)$.

PROOF. (i) By the first assumption and the fundamental properties of the γ-operation (cf. [3]), we see that

$$\gamma_t(\zeta-t)=\gamma_t(\zeta'-t')=\gamma_t(\textstyle\sum_{i=1}^{q-1}b_i(\eta^i-1))=\prod_{i=1}^{q-1}\{1+((1+\sigma)^i-1)t\}^{b_i}.$$

This equality and (3.9) show that

$$\gamma^j(\zeta-t)=\textstyle\sum_{k\geqq 0}A_k(j)\sigma^{j+k}\quad (A_k(j)=A_k(b_1,\dots,b_{q-1};j)).$$

Therefore the assumption (3.12) implies that $A_0(j)\sigma^{[n/2]}=\gamma^j(\zeta-t)\sigma^{[n/2]-j}=0$ and $A_0(j)\equiv 0 \bmod q$ by (3.3). Thus we have (i).

(ii) In the same way, we see that $A_0(j)\sigma^{[n/2]-1}+A_1(j)\sigma^{[n/2]}=0$ since $j<[n/2]$, and that $A_1(j)\equiv 0 \bmod q$ by (i) and the relation $q\sigma^{[n/2]-1}=0$ (cf. [12; Th. 1.1]).

(iii) When $q=2$, $\zeta'=t'\eta(t'=b_1)$ and $\gamma^j(\zeta-t)=\binom{t'}{j}\sigma^j$ by the first equality in the proof of (i). Thus we see (iii) by (3.4) and the equality $\sigma^2=-2\sigma$. q.e.d.

By the above lemma, we have the following non-extendibility theorem.

THEOREM 3.13. *Assume that a complex t-plane bundle* ζ *over* L_q^n *is stably equivalent to* $\zeta'=\sum_{i=1}^{q-1}b_i\eta^i$ $(b_i\geqq 0)$ *by Lemma* 3.5 (i). *Furthermore,*

(3.14) *take a prime factor p of q with* $p\leqq[n/2]+1$, *and let* a, c_k $(1\leqq k\leqq p-1)$ *and c be the integers given by*

$a=[n/2(p-1)]$ $(\geqq 1)$, $c_k\equiv\sum_l b_{lp+k} \bmod p^a$ *and* $0\leqq c_k<p^a$, $c=\sum_{k=1}^{p-1}c_k$.

(i) *Assume that* $t+1<p^a$ *and there is an integer m satisfying*

(3.15) $t<m<p^a$ *and*

(3.16) $A_0(c_1,\dots,c_{p-1};m)\left(=\sum_{j_1+\cdots+j_{p-1}=m}\prod_{k=1}^{p-1}\binom{c_k}{j_k}k^{j_k}\right)\not\equiv 0 \bmod p$.

Then $2m>n$ *and* $m(\zeta)<2m$, i.e., ζ *is not extendible to* L_q^{2m}.

(ii) (cf. [15; Th. 1.1]) *If the integer c in* (3.14) *satisfies* $t<c<p^a$, *then* $n\leqq m(\zeta)<2c$.

(iii) *If* $t+1<p^a$ *and* $m=t+1$ *satisfies* (3.16), *e.g., if* $c=t+1<p^a$, *then* $m(\zeta)=2t+1\geqq n$.

(iv) *Assume that p in* (3.14) *is odd, and that there is an integer m satisfying* (3.15) *and*

$$A_1(c_1,\dots,c_{p-1};m)\ (\textit{the integers given in } (3.10))\not\equiv 0 \bmod p.$$

Then $n\leqq m(\zeta)<2m+2$.

PROOF (i) In general, we see easily that

(3.17) $\binom{c+p^a}{j} \equiv \binom{c}{j}$ mod p for any integers c and j with $0 \leqq j < p^a$,

where p is a prime. Therefore, by the definition of A_0, we have the following

(3.18) *If* $b_k \equiv c_k$ mod p^a $(1 \leqq k \leqq p-1)$ *and if* $m < p^a$, *then*

$$A_0(b_1,\dots, b_{p-1}; m) \equiv A_0(c_1,\dots, c_{p-1}; m) \bmod p.$$

In the first place, we prove (i) by assuming

(*) $$q = p \text{ in addition.}$$

Since ζ is a t-plane bundle and $t < m$ by (3.15), we have $\gamma^m(\zeta - t) = 0$ in $\tilde{K}(L_q^n)$. Therefore, if $2m \leqq n$, then $A_0(b_1,\dots, b_{p-1}; m) \equiv 0$ mod p by Lemma 3.11(i). This shows that (3.16) does not hold by (3.18), since $m < p^a$ by (3.15) and $b_k \equiv c_k$ mod p^a $(1 \leqq k \leqq p-1)$ by (3.14) with $q = p$. Thus $2m > n$.

To prove $m(\zeta) < 2m$, suppose contrariwise that $m(\zeta) \geqq 2m$, i.e., ζ has an extension α over L_p^{2m}. Then α is stably equivalent to $\alpha' = \sum_{k=1}^{p-1} s_k \eta_{2m}^k$ over L_p^{2m} for some integers $s_k \geqq 0$ by Lemma 3.5(i). Since α is a t-plane bundle and $t < m$ by (3.15), $\gamma^m(\alpha - t) = 0$ in $\tilde{K}(L_p^{2m})$ and hence Lemma 3.11(i) implies that

(**) $$A_0(s_1,\dots, s_{p-1}; m) \equiv 0 \quad \bmod p.$$

On the other hand, $\zeta (\cong \alpha \mid L_p^n)$ is stably equivalent to $\alpha' \mid L_p^n = \sum_{k=1}^{p-1} s_k \eta^k$ and also to $\sum_{k=1}^{p-1} b_k \eta^k$ by assumption. Hence

$$s_k \equiv b_k \equiv c_k \quad \bmod p^a \quad \text{for} \quad 1 \leqq k \leqq p-1,$$

by Lemma 3.5 (iv) and (3.14) with $q = p$. Therefore

$$A_0(c_1,\dots, c_{p-1}; m) \equiv A_0(s_1,\dots, s_{p-1}; m) \equiv 0 \quad \bmod p$$

by (3.15), (3.18) and (**), which contradicts (3.16). Thus $m(\zeta) < 2m$ and we have proved (i) when $q = p$.

In general, p is a factor of q and we have the natural map

(3.19) $$\pi\colon L_p^i \longrightarrow L_q^i \text{ induced by the inclusion } Z_p \subset Z_q,$$

which is the projection $L_p^{2i+1} = S^{2i+1}/Z_p \to S^{2i+1}/Z_q = L_q^{2i+1}$ or its restriction $L_p^{2i} \to L_q^{2i}$. Then $\pi^* \eta \cong \eta$ is clear by definition. Therefore, by the assumption that ζ is stably equivalent to $\sum_{i=1}^{q-1} b_i \eta^i$ and by the equality $\eta^p - 1 = 0$ in $\tilde{K}(L_p^n)$ of (3.3), we see that

(3.20) *the induced bundle* $\pi^* \zeta$ *over* L_p^n *is stably equivalent to*

$$\sum_{k=1}^{p-1} b'_k \eta^k, \quad \textit{where} \quad b'_k = \sum_l b_{lp+k} \ (1 \leqq k \leqq p-1).$$

On the other hand, if ζ has an extension α over L_q^m, then $\pi^*\alpha$ over L_p^m is an extension of $\pi^*\zeta$. Thus

$$m(\zeta) \leqq m(\pi^*\zeta). \tag{3.21}$$

For $\pi^*\zeta$ over L_p^n in (3.20), we have $n \leqq m(\pi^*\zeta) < 2m$ by (i) with $q=p$. Therefore $n \leqq m(\zeta) < 2m$ in general by (3.21).

(ii) Take $m=c=\sum_{k=1}^{p-1} c_k$ in (i). Then we have $A_0(c_1,\dots,c_{p-1}; c)=\prod_{k=1}^{p-1} k^{c_k} \not\equiv 0 \bmod p$, since p is a prime. Thus (ii) is a special case of (i).

(iii) (i) shows that $n \leqq m(\zeta) < 2t+2$ and hence $t \geqq [n/2]$. Thus $m(\zeta) \geqq 2t+1$ by Theorem 3.8 (i), and we see (iii).

(iv) In the same way as the proof of (i), we can prove (iv) by using Lemma 3.11 (ii) instead of Lemma 3.11 (i). q.e.d.

If q is even, then we can take $p=2$ in the above theorem. In this case, (i) of the above theorem can be sharpened by the following theorem, where

(3.22) $\nu_2(a)$ denotes the exponent of 2 in the prime power decomposition of a positive integer a, and

$$N(t, c)=\min\left\{j+\nu_2\left(\binom{c}{j}\right) \mid t+1 \leqq j < c\right\} \quad \text{for} \quad t<c.$$

Theorem 3.23. *Let q be even, and assume that a complex t-plane bundle ζ over L_q^n $(n \geqq 2)$ is stably equivalent to $\zeta'=\sum_{i=1}^{q-1} b_i\eta^i$ $(b_i \geqq 0)$ by Lemma 3.5(i), and consider the integer c in (3.14) for $p=2$, which is given by*

$$c = c_1 \equiv \sum_l b_{2l+1} \bmod 2^{[n/2]} \quad \textit{and} \quad 0 \leqq c < 2^{[n/2]}. \tag{3.24}$$

(i) *If $t<c$, then $n \leqq m(\zeta) < 2N(t, c)$.*

(ii) *Especially, if $t<c$ and $\binom{c}{1+t}$ is odd, then $t \geqq [n/2]$ and $m(\zeta)=2t+1$.*

Proof. (i) We prove (i) by assuming $q=2$. Then (i) can be proved in general, in the same way as the latter half of the proof of Theorem 3.13(i) by taking $p=2$.

Assume that $q=2$, i.e., $L_q^k=RP^k$. Suppose that $m(\zeta) \geqq 2N(t, c)(>n)$, i.e., ζ has an extension α over RP^{2m}, where

$$m = j + \nu_2(a),\ a = \binom{c}{j}, \quad \text{for some } j \text{ with} \quad t<j \leqq c, \tag{$*$}$$

by the definition (3.22) of $N(t, c)$. Then, in the same way as the first half of the proof of Theorem 3.13(i) and by using Lemma 3.11(iii) instead of Lemma 3.11(i), we see that $\gamma^j(\alpha-t)=0$ in $\tilde{K}(RP^{2m})$ where $j \leqq m$, and that

$$\binom{s}{j} \equiv 0 \bmod 2^{1+m-j} \quad \text{for some integer} \quad s \geqq 0 \quad \text{with} \quad s \equiv c \bmod 2^{[n/2]}. \tag{$**$}$$

On the other hand, we see easily that (cf. [6; Lemma 4.8])

$$\nu_2(b!) = b-\mu_2(b) \quad \text{and} \quad \nu_2(\binom{b}{j}) = \mu_2(j) + \mu_2(b-j) - \mu_2(b),$$

where $\mu_2(a)$ denotes the number of 1's in the dyadic expansion of a. Therefore

$$\text{(3.25)} \quad s \equiv c \bmod 2^k \quad \text{and} \quad 0 \leqq j \leqq c < 2^k \quad \text{imply that} \quad \nu_2(\binom{s}{j}) = \nu_2(\binom{c}{j}).$$

Thus (**) and (*) lead a contradiction $\nu_2(a) \geqq 1+m-j=1+\nu_2(a)$; and $m(\zeta) < 2N(t, c)$ is proved. (If $2N(t, c) \leqq n$, then we can take an integer m in (*) with $2m \leqq n$, and we have a contradiction in the same way as the above proof by taking $\alpha=\zeta$.)

(ii) We see (ii) by (i) and Theorem 3.8(i) or by Theorem 3.13 (iii). q.e.d.

By the above theorem, we have the following corollary which gives some necessary conditions that there exists a complex t-plane bundle ζ over RP^n being stably equivalent to $t'\eta$.

COROLLARY 3.26. *Assume that a complex t-plane bundle ζ over the real projective space RP^n is stably equivalent to $\zeta'=t'\eta$ with $0 \leqq t' < 2^{[n/2]}$ by Lemma* 3.5(ii).

(i) *If $t<t'$, then $n<2N(t, t')$ for $N(t, t')$ in* (3.22). *Especially*

$$t' > [n/2] \quad \textit{and} \quad t + \nu_2(\binom{t'}{t+1}) \geqq [n/2] \quad \textit{if} \quad t < t'.$$

(ii) *If $T(\geqq t)$ satisfies $m(\zeta) \geqq 2N(t, s)$ (e.g., $n \geqq 2N(t, s)$) for any s with $T<s<2^{[n/2]}$, then $t' \leqq T$.*

(iii) *If $T'(<t')$ satisfies $m(\zeta) \geqq 2N(T', t')$ (e.g., $n \geqq 2N(T', t')$), then $t>T'$.*

(iv) *If $m(\zeta) \geqq 2^{[n/2]+1}-2$ (e.g., $n \leqq 3$), then $t' \leqq t$.*

PROOF. (i) In this case, c in the above theorem is t'. Thus

$$(*) \qquad n \leqq m(\zeta) < 2N(t, t') \quad \text{if} \quad t < t'.$$

(i) is an immediate consequence of (*) and the definition (3.22) of $N(t, t')$.

(ii) If $t' \leqq t$, then there is nothing to prove. If $t<t'$, then $m(\zeta)<2N(t, t')$ by (*) and hence $N(t, s)<N(t, t')$ for any s with $T<s<2^{[n/2]}$ by assumption. Thus $t' \leqq T$.

(iii) If $t' \leqq t$, then there is nothing to prove. If $t<t'$, then $m(\zeta)<2N(t, t')$ by (*) and hence $N(T', t')<N(t, t')$. Thus $t>T'$ by the definition (3.22).

(iv) If $t<t'$, then $m(\zeta)<2N(t, t') \leqq 2t' \leqq 2^{[n/2]+1}-2$ by (*), since $t'<2^{[n/2]}$. If $n \leqq 3$, then $2^{[n/2]+1}-2 \leqq n \leqq m(\zeta)$. Thus we see (iv). q.e.d.

REMARK 3.27. For example, we have the following under the assumption of the above corollary:

(i) *If n is even and $t'=2^s-1\geqq n/2$ for some $s\geqq 1$, then $t\geqq n/2$ and*

$$m(\zeta) = 2t+1 \text{ when } t<t', \quad m(\zeta) = \infty \text{ when } t\geqq t'.$$

(ii) *If $n=8$ and $t'=8$, then $t\geqq 2$ and*

$$m(\zeta) \leqq 9 \text{ when } t=2, 3, \ 2t+1 \leqq m(\zeta) \leqq 15 \text{ when } 4\leqq t\leqq 7, \ m(\zeta) = \infty \text{ when } t>7.$$

In fact, $t\geqq n/2$ in (i) and $t\geqq 2$ in (ii) follow from Corollary 3.26 (iii), since $N(T', t')=T'+1$ ($t'=2^s-1$) and $N(1, 8)=4$.

§4. The complexification of the tangent bundle of the lens space and complex bundles over the complex projective space

As applications of the results obtained in the previous sections, we have the following theorems on the complexification of the tangent bundle of the lens space.

THEOREM 4.1. *Let $\tau(RP^n)$ be the tangent bundle of the real projective space RP^n, and $c\tau(RP^n)$ be its complexification.*

(i) *$c\tau(RP^n)$ is extendible to RP^{2n+1} and is not to RP^{2n+2} if $n=6$ or $n\geqq 8$.*

(ii) *$c\tau(RP^n)$ is extendible to RP^m for any $m\geqq n$ if $n\leqq 5$ or $n=7$.*

PROOF. Put $\tau=\tau(RP^n)$. Then it is well known that

(4.2) $\tau\oplus 1\cong(n+1)\xi$ where ξ is the canonical real line bundle over RP^n, and that $c\xi\cong\eta$. Therefore

$$(*) \qquad c\tau \text{ is stably equivalent to } \zeta' = (n+1)\eta.$$

Assume that $n=6$ or $n\geqq 8$, which is equivalent to $n+1<2^{[n/2]}$. Then Theorem 3.13(iii) for $\zeta=c\tau$, $t=n$, ζ' in (*) and $c=n+1$ shows that $m(c\tau)=2n+1$.

Assume that $n=7$ or $n\leqq 5$, i.e., $n+1\geqq 2^{[n/2]}$. Then $m(c\tau)=\infty$ by Theorem 3.8(iii). q.e.d.

THEROEM 4.3. *Assume that $q\geqq 3$ and $n=2n'+1$ is odd, and let $\tau=\tau(L^{n'}(q))$ be the tangent bundle of the lens space $L_q^n=L^{n'}(q)$.*

(i) *Then the complexification $c\tau$ of τ is extendible to $L_q^{2n+1}=L^n(q)$.*

(ii) *Let p be the least prime factor of q, and assume that $n'\geqq 2(p-1)$ when $p\geqq 5$, and $n'\geqq 2p$ when $p=2, 3$.*
Then $c\tau$ is not extendible to L_q^{2n+2}.

PROOF. (i) Since $c\tau$ is a complex n-plane bundle and $n\geqq[n/2]$, (i) is an immediate consequence of Theorem 3.8(i).

(ii) It is known that ([25; Cor. 3.2])

(4.4) $\tau\oplus 1\cong(n'+1)r\eta$ where $r\eta$ is the real restriction of η.

Since $cr=1+t$ (t denotes the conjugation) and $\eta^q-1=0$ in $\tilde{K}(L_q^n)$ by (3.3), this shows that

(*) $$c\tau \text{ is stably equivalent to } \zeta' = (n'+1)(\eta\oplus\eta^{q-1}).$$

By assumption, we see that

$$p\leqq n'+1 = [n/2]+1 \quad \text{and} \quad n+1 = 2(n'+1) < p^a \quad \text{where} \quad a = [n/2(p-1)].$$

Therefore, the integer $c_k(1\leqq k\leqq p-1)$ and c in (3.14) for $\zeta=c\tau$ and ζ' in (*) are given by $c_1=c_{p-1}=n'+1$, $c_k=0$ $(k\neq 1,\ p-1)$ and $c=n+1$ when $p\geqq 3$, and by $c_1=c=n+1$ when $p=2$. Thus $m(c\tau)<2c=2n+2$ as desired by Theorem 3.13(ii). q.e.d.

REMARK 4.5. In the above theorem, we see that *$c\tau$ is extendible to L_q^m for any $m\geqq n$ if q is an odd prime and $n'=q-1$.*

In fact, $c\tau$ is stably trivial by (*) in the above proof and by Lemma 3.5(iii) since $r_1=q=n'+1$. Thus $m(c\tau)=\infty$ by Theorem 3.8(ii).

Now, assume that

(4.6) $L_q^n = L^{n'}(q)$ when $q\geqq 3$ and $n = 2n'+1$, or $L_q^n = RP^n$ when $q=2$,

can be (differentiably) immersed in the Euclidean space $R^{n+t}(t\geqq 1)$, e.g.,

(4.7) $t \geqq n-1$, or $t \geqq 2[n/4]+1$ when q is an odd prime ([22; Th. C(i)]).

Then we can consider

(4.8) the normal bundle $\nu(f)$ over L_q^n of an immersion $f\colon L_q^n\subseteq R^{n+t}$ $(t\geqq 1)$.

PROPOSITION 4.9. (i) *The complexification $c\nu(f)$ over L_q^n of $\nu(f)$ in* (4.8) *is extendible to L_q^{2t+1} if $t\geqq[n/2]$.*

(ii) *Assume that an integer m and a prime factor p of q satisfy the conditions that $p\leqq[n/2]+1$ and $t<m<p^a$ $(a=[n/2(p-1)])$ and that m is even and $\binom{-[n/2]-1}{m/2}\not\equiv 0 \bmod p$ if p is odd, or $\binom{-n-1}{m}\not\equiv 0 \bmod 2$ if $p=2$. Then $c\nu(f)$ is not extendible to L_q^{2m}.*

(iii) *Especially, if we can take $m=t+1$ in* (ii), *then $t\geqq[n/2]$ and $c\nu(f)$ is extendible to L_q^{2t+1} and not to L_q^{2t+2}*

PROOF. (i) Since $c\nu(f)$ is a t-plane bundle, we see (i) by Theorem 3.8(i).

(ii) It is well known that $\nu(f)\oplus\tau(L_q^n)\cong n+t$. Thus we have

(4.10) $\nu(f)\oplus(n'+1)r\eta\cong n+t+1$ and $c\nu(f)\oplus(n'+1)(\eta\oplus\eta^{q-1})\cong n+t+1$,

by (4.4) (and (4.2) where $2\xi\cong r\eta$ when $q=2$). This equivalence and (3.3) imply that the t-plane bundle $\zeta=c\nu(f)$ is stably equivalent to $\zeta'=b_1\eta\oplus b_{q-1}\eta^{q-1}$, where

$$b_1 = b_{q-1} \equiv -n'-1 \bmod q^{[n/2]} \quad\text{and}\quad b_1,\ b_{q-1}\geqq 0.$$

Therefore the integers c_k $(1\leqq k\leqq p-1)$ in (3.14) for these bundles are given by

$c_1 = c_{p-1} \equiv -n'-1 \bmod p^a$, $0\leqq c_k<p^a$ and $c_k=0$ if $k \neq 1, p-1$, when $p\geqq 3$;

$c_1 \equiv -2n'-2 = -n-1 \bmod p^a$ and $0\leqq c_1<p^a$, when $p=2$.

Thus the integer $A_0(c_1,\dots, c_{p-1}; m)$ in (3.16) satisfies that

$$A_0(c_1,\dots, c_{p-1}; m) = \sum_{j=0}^m \binom{c_1}{m-j}\binom{c_{p-1}}{j}(p-1)^j \equiv \sum_{j=0}^m \binom{c_1}{m-j}\binom{c_1}{j}(-1)^j$$

$$= (-1)^{m/2}\binom{c_1}{m/2} \equiv (-1)^{m/2}\binom{-n'-1}{m/2} \bmod p, \text{ when } p\geqq 3 \text{ and } m \text{ is even};$$

$$A_0(c_1; m) = \binom{c_1}{m} \equiv \binom{-n-1}{m} \bmod p, \text{ when } p=2,$$

since (3.17) is also valid when $c<0$. Hence (ii) follows from Theorem 3.13(i).

(iii) $n\leqq m(c\nu(f))<2t+2$ by (ii), which shows $t\geqq[n/2]$. Thus $m(c\nu(f))=2t+1$. q.e.d.

In the rest of this section, we consider complex bundles over the complex projective space CP^n. The canonical complex line bundle over $CP^n=S^{2n+1}/S^1$ is also denoted by η, which is the restriction $\eta\,|\,CP^n$ of the one η over CP^m for any $m\geqq n$.

THEOREM 4.11. *Let ζ be a complex t-plane bundle over CP^n.*

(i) *Then $\zeta-t=\sum_{k=1}^n b_k(\eta^k-1)$ in $\tilde{K}(CP^n)$ for some integers b_k.*

(ii) *If $b_k\geqq 0$ $(1\leqq k\leqq n)$ in* (i) *and $t\geqq n$, then ζ is extendible to CP^t.*

If $t\geqq \sum_{k=1}^n b_k$ in addition, then ζ is extendible to CP^m for any $m\geqq n$.

(iii) *Take a prime $p\leqq n+1$ and put*

$$c_i \equiv \textstyle\sum_l b_{lp+i} \bmod p^{a'} \text{ and } 0\leqq c_i<p^{a'}\ (1\leqq i\leqq p-1),\quad c = \sum_{i=1}^{p-1} c_i,$$

where b_k's are the integers in (i) *and $a'=[n/(p-1)]$. If there is an integer m satisfying $t<m<p^{a'}$ and* (3.16), *then $m>n$ and ζ is not extendible to CP^m.*

(iv) *If the integer c in* (iii) *satisfies $t<c<p^{a'}$, then ζ is not extendible to CP^c.*

(v)' *Take* $p=2$ *in* (iv). *Then* ζ *is not extendible to* $CP^{N(t,c)}$ *for* $N(t, c)$ *in* (3.22).

PROOF. (i) It is known (cf. [1; Th. 7.2]) that the K-ring $K(CP^n)$ is the truncated polynomial ring $Z[\sigma]/(\sigma^{n+1})$ with one generator $\sigma=\eta-1$. Thus we see (i).

(ii) Since $b_k\geqq 0$, ζ is stably equivalent to the bundle $\sum_{k=1}^{n} b_k\eta^k$ by (i), which is extendible to CP^m for any $m\geqq n$. Thus (ii) follows immediately from Corollary 2.3.

(iii)–(v) Consider the natural projection $\pi\colon L_p^{2n+1}=S^{2n+1}/Z_p\to S^{2n+1}/S^1=CP^n$. Then $\pi^*\eta$ is the canonical complex line bundle η over L_p^{2n+1} by definition, and we see that

(*) $\pi^*\zeta$ is stably equivalent to $\sum_{i=1}^{p-1} b'_i\eta^i$ where $b'_i\equiv\sum_l b_{lp+i} \bmod p^n$ and $b'_i\geqq 0$,

by (i) and (3.3). Furthermore, if ζ is extendible to CP^m, then so is $\pi^*\zeta$ to L_p^{2m+1}. Thus (iii)–(v) follow immediately from the non-extendibility of $\pi^*\zeta$ in (*), which is shown by Theorems 3.13(i), (ii) and 3.23(i). q.e.d.

COROLLARY 4.12. *Assume that a complex t-plane bundle* ζ *over* CP^n *satisfies* $\zeta-t=b(\eta^k-1)$ *in* $\tilde{K}(CP^n)$ *for some integers k and b with* $1\leqq k\leqq n$.

(i) *Assume that there are a prime p and an integer m satisfying*

$$k\not\equiv 0 \bmod p,\quad t<m<p^{a'}\ (a'=[n/(p-1)])\quad and\quad \binom{c}{m}\not\equiv 0 \bmod p,$$

where $c\equiv b \bmod p^{a'}$ *and* $0\leqq c<p^{a'}$. *Then* $m>n$ *and* ζ *is not extendible to* CP^m.

(ii) *In case that* $t\geqq n$, $k\not\equiv 0 \bmod p$ *and* $n<b<p^{[n/(p-1)]}$ *for some prime p,* ζ *is extendible to* CP^m *for any* $m\geqq b$ *if and only if* $b\leqq t$.

PROOF. (i) is an immediate consequence of Theorem 4.11(iii).

(ii) The sufficiency is seen by Theorem 4.11(ii). If $b>t$, then (i) shows that ζ is not extendible to CP^b. q.e.d.

COROLLARY 4.13 (cf. [9; p. 166]). *The complex tangent bundle* $\tau_c(CP^n)$ *over* CP^n *with* $n\geqq 2$ *is not extendible to* CP^{n+1}, *and* $\tau_c(CP^1)$ *is extendible to* CP^m *for any* $m\geqq 1$.

PROOF. It is known that $\tau_c(CP^n)\oplus 1\cong(n+1)\eta$ (cf. [17]). Thus we see the desired result for $n\geqq 2$ by Corollary 4.12(i) for $\zeta=\tau_c(CP^n)$, $t=n$, $k=1$, $b=c=n+1$, $p=2$ and $m=n+1$, since $n+1<2^n$ if $n\geqq 2$. The result for $n=1$ is proved in [15; Remark 5.3] by the same proof as that of Proposition 2.4 and by noticing that $H^{r+1}(CP^m, CP^1; \pi_r(BU(1)))=0$ for $r\geqq 2$. q.e.d.

REMARK 4.14. The extendibility of a complex bundle over CP^n to CP^m is investigated by several authors (cf. e.g., the references of [24]). Especially, A. Thomas [26; Prop. 3.5] determined a necessary and sufficient condition for a complex n-plane bundle over CP^n to be extendible to CP^{n+1}.

§5. Real bundles over the lens spaces

In this section, we consider real vector bundles over L_q^n of (3.1–2).

When q is even, let $\rho=\rho_n$ be the non-trivial real line bundle over $L_q^n(n\geqq 1)$, i.e., the one whose first Stiefel-Whitney class $w_1(\rho)\in H^1(L_q^n; Z_2)=Z_2$ is non-zero. If $q=2$, then ρ is the canonical real line bundle ξ over RP^n.

Consider the additive homomorphism

(5.1) $$r\colon \tilde{K}(L_q^n) \longrightarrow \widetilde{KO}(L_q^n) \text{ given by the real restriction } r$$

between the reduced K- and KO-rings. Then we have the following

LEMMA 5.2. (i) (cf. [12; Prop. 2.11, Th. 1.1(ii)]) *When q is odd,*

$$\widetilde{KO}(L_q^n) = \begin{cases} r(\tilde{K}(L_q^n)) & \text{if } n \not\equiv 1 \mod 8, \\ r(\tilde{K}(L_q^n)) \oplus Z_2,\ Z_2 \cong \widetilde{KO}(S^n), & \text{otherwise,} \end{cases}$$

where the last isomorphism is induced by the projection $L_q^n\to L_q^n/L_q^{n-1}=S^n$, and $r(\tilde{K}(L_q^n))$ is the subring of $\widetilde{KO}(L_q^n)$ generated by $r\sigma(\sigma=\eta-1$ is the one in (3.3)) and contains exactly $q^{[n/4]}$ elements. Furthermore, if q is an odd prime p, then the order of $r\sigma$ is equal to $r_2=p^{1+[([n/2]-2)/(p-1)]}$ and hence $r_2\alpha=0$ for any $\alpha\in r(\tilde{K}(L_p^n))$.

(ii) *If q is even, then $\widetilde{KO}(L_q^n)/r(\tilde{K}(L_q^n))\cong Z_2$ and the element*

$$\kappa = \rho-1\in \widetilde{KO}(L_q^n)$$

does not belong to $r(\tilde{K}(L_q^n))$, and $c\rho\cong\eta^{q/2}$, $2\rho\cong r(\eta^{q/2})$ over L_q^n, and $2\kappa=r(\eta^{q/2}-1)$.

(iii) *For any q, $r(\eta^i-\eta^{q-i})=0$ in $\widetilde{KO}(L_q^n)$.*

PROOF. (i) is proved in [12]. (iii) follows from $rt=r$ (t is the conjugation) and $\eta^q-1=0$ in (3.3). We prove (ii).

The last equalities in (ii) follow from [16; Prop. 3.3]. Let $q=2^rq'$ where $r\geqq 1$ and q' is odd. Then we have the natural projections or their restrictions $\pi\colon L_{2^r}^n\to L_q^n$ and $\pi'\colon L_{q'}^n\to L_q^n$, induced by the inclusions $Z_{2^r}\subset Z_q$ and $Z_{q'}\subset Z_q$, and the commutative diagram

$$\begin{array}{ccc} \tilde{K}(L_q^n) & \xrightarrow{\pi^*+\pi'^*} & \tilde{K}(L_{2^r}^n)\oplus \tilde{K}(L_{q'}^n) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r\oplus r} \\ \widetilde{KO}(L_q^n) & \xrightarrow{\pi^*+\pi'^*} & \widetilde{KO}(L_{2^r}^n)\oplus r(\tilde{K}(L_{q'}^n)), \end{array}$$

where the two $\pi^*+\pi'^*$ are isomorphic and

$$\pi^*(\sigma) = \sigma, \quad \pi'^*(\sigma) = \sigma, \quad \pi^*(\kappa) = \kappa, \quad \pi'^*(\kappa) = 0.$$

In fact, these equalities are clear by definition and hence we see that the upper $\pi^*+\pi'^*$ is isomorphic by (3.3). The lower one is so by [8; Prop. 2.2] and (i).

Now consider the ring $\widetilde{KO}(L^n_{2^r})$. Then this is generated by κ and $r\sigma=r(\eta-1)$, and $\kappa^2=-2\kappa$, $(r\sigma)^i$ and $\kappa(r\sigma)^i$ $(i\geqq 1)$ are contained in $r(\tilde{K}(L^n_{2^r}))$ by [7; Prop. 1.1] and [6; Lemma 2.12]. Furthermore $\kappa\notin r(\tilde{K}(L^n_{2^r}))$ since $w_1(\rho)\neq 0$. Therefore we see (ii) by the above diagram. q.e.d.

In case that $q=2$ and $L^n_q=RP^n$, we have the following

(5.3) (J. F. Adams [1; Th. 7.4]) *$\widetilde{KO}(RP^n)$ is a cyclic group of order $2^{\phi(n)}$ generated by $\kappa=\rho-1$ $(\rho\cong\xi)$, where $\phi(n)$ is the number of integers s with $0<s\leqq n$ and* $s\equiv 0, 1, 2, 4 \bmod 8$.

When $n\equiv 1 \bmod 8$, let β_n be the real n-plane bundle over the sphere S^n such that the stable class $\beta_n-n\in\widetilde{KO}(S^n)=Z_2$ is non-zero, and denote by the same letter β_n the induced bundle of β_n by the projection $L^n_q\to L^n_q/L^{n-1}_q=S^n$. Then we have immediately the following lemma by Lemmas 5.2, 3.5 and (5.3), in the same way as Lemma 3.5(i)–(iii).

LEMMA 5.4. (i) *Any real t-plane bundle ζ over L^n_q is stably equivalent to a real t'-plane bundle ζ' over L^n_q such that*

(5.5) $\zeta' = \varepsilon\beta_n\oplus b\rho\oplus\sum_{i=1}^u b_i r(\eta^i)$ *and* $t'=\varepsilon n+b+2\sum_{i=1}^u b_i$ $(u=[(q-1)/2])$

for some non-negative integers ε, b and b_i with $\varepsilon=0, 1$, where $\varepsilon\beta_n$ (resp. $b\rho$) appears only when q is odd and $n\equiv 1$ mod 8 (resp. q is even).

(ii) *b (resp. b_i) in (5.5) can be reduced to the residue modulo the order of $\kappa=\rho-1$ (resp. $r(\eta^i-1)$) in $\widetilde{KO}(L^n_q)$ and, especially, to the one modulo $2^{\phi(n)}$ (resp. $r_2=p^{1+[([n/2]-2)/(p-1)]}$) when $q=2$ (resp. q is an odd prime p).*

We now study the extendibility of a real t-plane bundle ζ over L^n_q to L^m_q for $m\geqq n$ by using the same notation

(5.6) $m(\zeta)=\max\{m\,|\,\zeta\cong\alpha\,|\,L^n_q$ for some real bundle α over L^m_q $(m\geqq n)\}$

as (3.7) for complex bundles.

THEOREM 5.7. *Let ζ be a real t-plane bundle over L^n_q and assume that ζ is stably equivalent to a real t'-plane bundle ζ' over L^n_q in (5.5) by Lemma 5.4.*

(i) *When q is odd and $n\equiv 1$ mod 8, if*

$$\varepsilon = 1, \text{ i.e., } \zeta' = \beta_n\oplus\sum_{i=1}^u b_i r(\eta^i) \text{ in } (5.5),$$

then ζ *is not extendible to* L_q^{n+1}, *i.e.,* $m(\zeta)=n$.

(ii) *Assume that* $\zeta'=b\rho \oplus \sum_{i=1}^u b_i r(\eta^i)$ *in* (5.5). *Then*

(a) $m(\zeta)\geqq t$ *if* $t\geqq n$.

(b) $m(\zeta)=\infty$ *if* $t>n$ *and* $t\geqq t'$.

(c) ([14; Th. 4.2]) $m(\zeta)\geqq 2t-(-1)^t$ *if* q *is odd* ($b\rho$ *does not appear*), n *is odd and* $t>n$.

PROOF. (i) Suppose that ζ is extendible to a real bundle α over L_q^{n+1}. Then α is stably equivalent to $\alpha'=\sum_{i=1}^u c_i r(\eta_{n+1}^i)$ for some $c_i\geqq 0$ by the above lemma. Thus $\alpha' \mid L_q^n=\sum_{i=1}^v c_i r(\eta^i)$ is stably equivalent to ζ and hence to $\varepsilon\beta_n \oplus \sum_{i=1}^u b_i r(\eta^i)$ in (5.5). Therefore their stable classes in $\widetilde{KO}(L_q^n)$ are equal to each other, and we see that $\varepsilon=0$ by the direct sum decomposition of Lemma 5.2(i) and the definition of β_n. Hence $m(\zeta)\leqq n$ if $\varepsilon=1$.

(ii) By definition, $m(\rho)=\infty=m(r(\eta^i))$ and hence $m(\zeta')=\infty$. Thus (a) and (b) follow immediately from Corollary 2.3. If $t\geqq t'$, then (c) holds by (b). If $t<t'$, then (c) is proved in [14; Th. 4.2]. q.e.d.

To study the non-extendibility, we use the γ-operation in KO-theory (cf. [4]).

LEMMA 5.8. *Let* q *be odd and assume that a real t-plane bundle* ζ *over* L_q^n *is stably equivalent to* $\zeta'=\sum_{i=1}^u b_i r(\eta^i)$ *with* $b_i\geqq 0$ ($u=(q-1)/2$). *If*

$$\gamma^{2j}(\zeta-t) = 0 \text{ in } \widetilde{KO}(L_q^n) \quad \text{for some positive integer } j\leqq[n/4],$$

where γ^{2j} *is the* γ*-operation in KO-theory, then*

$$B_0(b_1,\dots, b_u; j) = \sum_{j_1+\dots+j_u=j} \prod_{i=1}^u \binom{b_i}{j_i} i^{2j_i} \equiv 0 \bmod q. \tag{5.9}$$

PROOF. By assumption and by [13; Prop. 3.2], we see that

$$\gamma_t(\zeta-t) = \gamma_t(\textstyle\sum_{i=1}^u b_i r(\eta^i-1))$$

$$= \sum_l \left\{\sum_{j_1+\dots+j_u=l} \prod_{i=1}^u \binom{b_i}{j_i}\left(\sum_{s=1}^i (i/s)\binom{i+s-1}{2s-1}(r\sigma)^{s-1}\right)^{j_i}\right\}(r\sigma)^l(t-t^2)^l,$$

where $\sigma=\eta-1$. By taking the coefficient of t^{2j}, we have

$$\gamma^{2j}(\zeta-t) = \sum_{k\geqq 0} B_k(r\sigma)^{j+k} \text{ for some coefficients } B_k,$$

where $(-1)^j B_0$ is $B_0(b_1,\dots, b_u; j)$ in (5.9). On the other hand, we see that

$$(r\sigma)^{[n/4]+1} = 0 \text{ and the order of } (r\sigma)^{[n/4]} \text{ is } q \text{ in } \widetilde{KO}(L_q^n), \tag{5.10}$$

by using [12; Prop. 2.11 and 2.6]. Therefore the assumption $\gamma^{2j}(\zeta-t)=0$ implies that $B_0(r\sigma)^{[n/4]}=\gamma^{2j}(\zeta-t)(r\sigma)^{[n/4]-j}=0$ and $B_0\equiv 0 \bmod q$. q.e.d.

In the same way as the proof of Theorem 3.13 by using Lemma 5.8 instead of Lemma 3.11(i), we can prove the following

THEOREM 5.11. *Let q be odd and assume that a real t-plane bundle ζ over L_q^n is stably equivalent to $\zeta' = \sum_{i=1}^{u} b_i r(\eta^i)$ with $b_i \geqq 0 (u=(q-1)/2)$. Furthermore*

(5.12) *take a prime factor p of q with $p \leqq [n/2]+1$, and let $d_k (1 \leqq k \leqq v = (p-1)/2)$ and d be the integers given by*

$$d_k \equiv \sum_l (b_{lp+k} + b_{lp+p-k}) \bmod p^a \text{ and } 0 \leqq d_k < p^a, \quad d = 2\sum_{k=1}^{v} d_k,$$

where $a = [n/2(p-1)]$.

(i) *Assume that there is an even integer m satisfying*

(5.13) $\quad t < m < 2p^a$ *and*

(5.14) $\quad B_0(d_1, \ldots, d_v; m/2) \left(= \sum_{j_1+\cdots+j_v=m/2} \prod_{k=1}^{v} \binom{d_k}{j_k} k^{2j_k}\right) \not\equiv 0 \bmod p.$

Then $2m > n$ and $m(\zeta) < 2m$, i.e., ζ is not extendible to L_q^{2m}.

(ii) (cf. [14; Th. 1.1]) *If d in (5.12) satisfies $t < d < 2p^a$, then $n \leqq m(\zeta) < 2d$.*

(iii) *When n is odd and $n < t$, if t is odd $< 2p^a - 1$ and $m = t+1$ satisfies* (5.14), *e.q., if $t+1 = d < 2p^a$, then $m(\zeta) = 2t+1$.*

PROOF. (i) Assume that $q = p$ $(u = v)$ in addition.

Suppose that $m(\zeta) \geqq 2m (> n)$, i.e., ζ has an extension α over L_p^{2m}. Then α is stably equivalent to $\alpha' = \sum_{k=1}^{v} s_k r(\eta_{2m}^k)$ for some $s_k \geqq 0$ by Lemma 5.4. Since $t < m$ by (5.13), $\gamma^m(\alpha - t) = 0$ in $\widetilde{KO}(L_p^{2m})$ and Lemma 5.8 shows that

$$(*) \qquad B_0(s_1, \ldots, s_v; m/2) \equiv 0 \bmod p \qquad (m \text{ is even}).$$

On the other hand, $\zeta (\cong \alpha | L_p^n)$ is stably equivalent to $\zeta' = \sum_{k=1}^{v} b_k r(\eta^k)$ and to $\alpha' | L_p^n = \sum_{k=1}^{v} s_k r(\eta^k)$. Therefore $c\zeta' \cong \sum_{k=1}^{v} b_k (\eta^k \oplus \eta^{p-k})$ is so to $c\alpha' | L_p^n = \sum_{k=1}^{v} s_k (\eta^k \oplus \eta^{p-k})$. Hence Lemma 3.5(iv) and the definition of d_k in (5.12) for $q = p$ imply that

$$s_k \equiv b_k \equiv d_k \mod p^a \qquad \text{for} \quad 1 \leqq k \leqq v.$$

Since $m/2 < p^a$ by (5.13), this and the definition of B_0 imply that

$$B_0(s_1, \ldots, s_v; m/2) \equiv B_0(d_1, \ldots, d_v; m/2) \mod p,$$

in the same way as the proof of (3.18). Thus $(*)$ contradicts (5.14). (If $2m \leqq n$, then we have a contradiction in the same way as the above proof by taking $\alpha = \zeta$.) Therefore (i) is proved when $q = p$.

In general, consider the natural map $\pi: L_p^i \to L_q^i$ in (3.19). Then the assumption, $\eta^p - 1 = 0$ in $\widetilde{K}(L_p^n)$ and $r(\eta^k - \eta^{p-k}) = 0$ in $\widetilde{KO}(L_p^n)$ of Lemma 5.2(iii) show that the induced bundle $\pi^*\zeta$ over L_p^n is stably equivalent to

$$\sum_{k=1}^{v} b_k'' r(\eta^k) \text{ where } b_k'' = \sum_l (b_{lp+k}+b_{lp+p-k}) \qquad \text{for } 1\leqq k\leqq v.$$

Thus $n\leqq m(\pi^*\zeta)<2m$ by the above proof, and we see (i) in general since $m(\zeta)\leqq m(\pi^*\zeta)$ in (3.21) is also valid for a real bundle ζ.

(ii) By taking $m=d=2\sum_{k=1}^{v} d_k$ in (i), we have (ii).

(iii) follows immediately from (i) and Theorem 5.7(ii) (c). q.e.d.

In case that $q=2$ and $L_q^n=RP^n$, we have the following theorem by using the γ-operation in the same way as Theorem 3.23 and by using the Stiefel-Whitney class, where

(5.15)
$$N_1(t, s) = \min\{m|\phi(m)\geqq j+v_2(\binom{s}{j}) \text{ for some } t<j\leqq s\},$$

$$N_2(t, s) = \min\{j\,|\,t<j\leqq s \text{ and } v_2(\binom{s}{j})=0\},$$

$$N'(t, s) = \min\{N_1(t, s), N_2(t, s)\}$$

for $t<s$, ($\phi(m)$ and $v_2(a)$ are the integers given in (5.3) and (3.22) respectively).

THEOREM 5.16. *Assume that a real t-plane bundle ζ over the real projective space RP^n is stably equivalent to $\zeta'=t'\rho$ with $0\leqq t'<2^{\phi(n)}$ by Lemma 5.4.*

(i) *If $t<t'$, then $n\leqq m(\zeta)<N'(t, t')$ and especially $n\leqq m(\zeta)<t'$.*

(ii) *If $t<t'$ and $\binom{t'}{1+t}$ is odd, then $t\geqq n$ and $m(\zeta)=t$.*

(iii) *If $T(\geqq t)$ satisfies that $m(\zeta)\geqq N'(t, s)$ (e.g., $n\geqq N'(t, s)$) for any s with $T<s<2^{\phi(n)}$, then $t'\leqq T$.*

(iv) *If $T'(<t')$ satisfies $m(\zeta)\geqq N'(T', t')$ (e.g., $n\geqq N'(T', t')$), then $t>T'$.*

(v) ([14; Th. 6.5]) *If $m(\zeta)\geqq 2^{\phi(n)}-1$, then $t\geqq t'$.*

PROOF. (i) Suppose that $m(\zeta)\geqq N_2(t, t')(>n)$, i.e., ζ has an extension α over RP^j for some integer j with

$$(*) \qquad t<j\leqq t' \text{ and } v_2(a)=0 \text{ (i.e., } a\not\equiv 0 \bmod 2) \text{ where } a=\binom{t'}{j}.$$

Then α is stably equivalent to $s'\rho$ over RP^j for some integers s' with $0\leqq s'<2^{\phi(j)}$ by Lemma 5.4. Therefore

$$\binom{s'}{j}\equiv 0 \bmod 2, \text{ i. e., } v_2(\binom{s'}{j})\neq 0,$$

because $0=w_j(\alpha)=w_j(s'\rho)=\binom{s'}{j}y^j$ in $H^*(RP^j; Z_2)$. On the other hand, ζ is stably equivalent to $t'\rho$ and also to $s'\rho\,|\,RP^n=s'\rho$, and we see that

(**) $t' \equiv s' \bmod 2^{\phi(n)}$ by (5.3), and $\nu_2(a)=\nu_2(\binom{t'}{j})=\nu_2(\binom{s'}{j})$ by (3.25).

These show that $\nu_2(a) \neq 0$ which contradicts (*). (If $N_2(t, t') \leqq n$, then we have also a contradiction by taking $\alpha=\zeta$ and j in (*) with $j \leqq n$ in the above proof.) Thus $n \leqq m(\zeta) < N_2(t, t')$.

Now suppose that $m(\zeta) \geqq N_1(t, t')(>n)$, i.e., ζ has an extension α over RP^m for some integer m with

$$(***) \qquad \phi(m) \geqq j+\nu_2(a),\ a=\binom{t'}{j}, \quad \text{for some } j \text{ with } t<j \leqq t'.$$

Then α is stably equivalent to $s'\rho$ over RP^m for some $s' \geqq 0$ by Lemma 5.4. Therefore

$$0=\gamma^j(\alpha-t)=\gamma^j(s'\kappa)=\binom{s'}{j}\kappa^j=(-2)^{j-1}\binom{s'}{j}\kappa \quad \text{in} \quad \widetilde{KO}(RP^m)\ (\kappa=\rho-1)$$

in the same way as the proof of Lemma 3.11(iii). Thus

$$2^{j-1}\binom{s'}{j} \equiv 0 \bmod 2^{\phi(m)}, \text{ i.e., } \quad \nu_2(\binom{s'}{j}) \geqq \phi(m)-j+1,$$

by (5.3). Thus $\nu_2(a) \geqq \phi(m)-j+1$ by (**), which contradicts (***). (If $N_1(t, t') \leqq n$, then we have also a contradiction by taking $\alpha=\zeta$ and m in (***) with $m \leqq n$.) Hence $m(\zeta)<N_1(t, t')$ and (i) is proved.

(ii) $N_2(t, t')=t+1$ by (5.15), since $\nu_2(\binom{t'}{t+1})=0$. Thus $n \leqq m(\zeta)<t+1$ by (i), and $m(\zeta) \geqq t$ by Theorem 5.7(a). These prove (ii).

(iii)–(v) By using (i), we see (iii)–(v) by the same proof as that of Corollary 3.26(ii)–(iv). q.e.d.

COROLLARY 5.17. *Let q be even, and assume that a real t-plane bundle ζ over L_q^n is stably equivalent to $\zeta'=b\rho\oplus\sum_{i=1}^{u} b_i r(\eta^i)$ for some $b \geqq 0$ and $b_i \geqq 0$ $(u=q/2-1)$ by Lemma* 5.4.

(i) *Then* (i) *and* (ii) *of Theorem* 5.11 *are also valid when p is odd in* (5.12).

(ii) *Let d' be the integer given by*

$$d' \equiv b'+2\textstyle\sum_l b_{2l+1} \bmod 2^{\phi(n)} \quad \text{and} \quad 0 \leqq d' < 2^{\phi(n)},$$

where $b'=b$ if $q/2$ is odd and $b'=0$ otherwise. If $t<d'$, then $m(\zeta)<N'(t, d')$ for $N'(t, d')$ in (5.15). *In particular, if $t<d'$ and $\binom{d'}{t+1}$ is odd, e.q., if $d'=t+1$, then $t \geqq n$ and $m(\zeta)=t$.*

PROOF. Consider the natural map $\pi\colon L_p^n \to L_q^n$ of (3.19). Then $\pi^*\rho \cong \rho$ if $p=2$ and $q/2$ is odd, and $\pi^*\rho \cong 1$ otherwise, by the definition of ρ, because $\pi^*\colon H^1(L_q^n; Z_2) \to H^1(L_p^n; Z_2)$ is isomorphic or trivial in each cases. Furthermore

$2\rho \cong r\eta$ over L_2^n (see Lemma 5.2(ii)). Therefore, by using Theorems 5.11(ii) and 5.16(i), we see the corollary in the same way as the last part of the proof of Theorem 5.11(i). q.e.d.

REMARK 5.18. We can obtain a theorem similar to Theorem 4.11 on the extendibility of a real bundle ζ over the complex projective space CP^n whose stable calss $\zeta - t$ is equal to $\sum_{k=1}^n b_k r(\eta^k - 1)$ in $\widetilde{KO}(CP^n)$, in the same way as the above corollary.

§6. The higher order tangent bundles

Throughout this section, we continue to use the notation $m(\zeta)$ in (5.6) or (3.7), which denotes the maximum integer m such that a bundle ζ over L_q^n is extendible to L_q^m $(m \geqq n)$.

In the first place, we consider the tangent (or normal) bundle of

(6.1) $L_q^n = L^{n'}(q)$ when $q \geqq 3$ and $n = 2n'+1$, or $L_q^n = RP^n$ when $q=2$.

(6.2) ([14; Th. 5.1, 5.3, 6.6]) *For the tangent bundle $\tau(L_q^n)$ of L_q^n in* (6.1).

$$m(\tau(L_q^n)) = \begin{cases} \infty & \textit{if } n = 1, 3 \textit{ or } 7, \\ n & \textit{otherwise.} \end{cases}$$

In fact, if $n=1$, 3 or 7, then L_q^n is parallelizable and $m(\tau(L_q^n)) = \infty$ except for L_q^7 with $q \geqq 3$. L_q^7 has a tangent 5-field by [27]. Therefore $\tau(L_q^7) \cong \beta \oplus 5$ for some oriented 2-plane bundle β, which implies $m(\tau(L_q^7)) = \infty$ by Corollary 2.4. Conversely, suppose that $\tau(L_q^n)$ has an extension α over L_q^{n+1}. Then, by considering the natural projection π: $S^m \to L_q^m$, we see that

$$\tau(S^n) \cong \pi^*\tau(L_q^n) \cong \pi^*(\alpha | L_q^n) \cong (\pi^*\alpha) \,|\, S^n \cong i^*(\pi^*\alpha),$$

where the inclusion i: $S^n \subset S^{n+1}$ is homotopic to the constant map. Thus $\tau(S^n)$ is trivial and hence $n=1$, 3 or 7.

In the same way as the above proof, we can prove the following

(6.3) *The real tangent bundle $\tau(CP^n)$ of the complex projective space CP^n is not extendible to CP^{n+1} if and only if $n \neq 0$, 1 and* 3.

In fact, consider the differentiable fibre bundle π: $S^{2m+1} \to CP^m$ with fibre S^1. Then, on the tangent bundles of these manifolds, it is well known that

$$\tau(S^{2n+1}) \cong \pi^*\tau(CP^n) \oplus \alpha, \text{ where } \alpha \text{ is the bundle along the fibre.}$$

Here α is a line bundle and orientable. Thus $\alpha \cong 1$. Therefore, if $\tau(CP^n)$ has an extension β over CP^{n+1}, then

$$\tau(S^{2n+1}) \cong \pi^*\tau(CP^n)\oplus 1 \cong \pi^*(\beta\oplus 1)|S^{2n+1} \cong 2n+1,$$

since the inclusion $S^{2n+1}\subset S^{2n+3}$ is homotopic to the constant map. Thus $n=0$, 1 or 3. Conversely, the obstructions for extending the classifying map of $\tau(CP^3)$ to CP^4 are contained in the cohomology groups $H^{i+1}(S^8;\pi_{i-1}(SO(6)))$ for $i=6$, 7, and these groups are 0 because $H^7(S^8)=0$ and $\pi_6(SO(6))=0$. Thus $\tau(CP^3)$ is extendible to CP^4. $\tau(CP^1)=r\tau_c(CP^1)$ is so to CP^2 by the latter half of Corollary 4.13.

We now consider the normal bundle $\nu(f)$ in (4.8).

PROPOSITION 6.4. *Let $\nu(f)$ be the normal bundle over L_q^n in* (6.1) *of an immersion $f: L_q^n\subseteq R^{n+t}$* $(t\geqq 1)$.

(i) *$m(\nu(f))\geqq t$ if $t\geqq n$, and $m(\nu(f))\geqq 2t-(-1)^t$ if q is odd and $t>n$.*

(ii) *Assume that q is odd. If there is an even integer m satisfying*

(6.5) $t<m<2p^{[n/2(p-1)]}$ *and* $\binom{-[n/2]-1}{m/2}\not\equiv 0 \bmod p$ *for some prime factor p of q,*

then $m(\nu(f))<2m$. Especially, if t is odd$>n$ and $m=t+1$ satisfies (6.5), *then $m(\nu(f))=2t+1$.*

(iii) *Assume that q is even.*

(a) *If the integer t', given by $t'\equiv t+n+1 \bmod 2^{\phi(n)}$ and $0\leqq t'<2^{\phi(n)}$, satisfies $t'>t$, then $m(\nu(f))<N'(t, t')$ for $N'(t, t')$ in* (5.15).

(b) *If there is an integer m satisfying*

$$t<m<2^{\phi(n)} \text{ and } \binom{t+n+1}{m}\not\equiv 0 \bmod 2, \tag{6.6}$$

then $m(\nu(f))<m$. Especially, if (6.6) *holds for $m=t+1$, then $t\geqq n$ and $m(\nu(f))=t$.*

PROOF. We see that the t-plane bundle $\zeta=\nu(f)$ over L_q^n is stably equivalent to

(∗) $\zeta' = b_1 r\eta$, where $b_1\equiv -n'-1 \bmod q^{[n/4]}$ and $b_1 \geqq 0$ $(n=2n'+1)$,

by (4.10) and Lemma 5.2 (i).

(i) is a consequence of Theorem 5.7(ii).

(ii) We can prove the first half in the same way as the proof of Proposition 4.9(ii) by using Theorem 5.11(i). If t is odd$>n$, then $m(\nu(f))\geqq 2t+1$ by (i). Thus we see the latter half.

(iii) Consider the projection $\pi: RP^n=L_2^n\to L_q^n$ (q is even). Then

$$\pi^*\nu(f) \oplus (n+1)\rho \cong n+t+1 \quad \text{over} \quad RP^n$$

by (4.10), since $2\rho \cong r\eta$ ($\rho \cong \xi$) over RP^n. Further $\rho^2 \cong 1$ over RP^n. Thus (**) $(\pi^*\nu(f))\otimes\rho$ over RP^n is stably equivalent to $(n+t+1)\rho$ and hence to $t'\rho$, by Lemma 5.4(ii), where t' is the integer given in (a). Therefore Theorem 5.16(i) shows that

$$m(\pi^*\nu(f)\otimes\rho) < N'(t, t') \qquad \text{if} \quad t<t'.$$

On the other hand, since $\rho^2 \cong 1$ over RP^n, we see easily that

$$m(\zeta) \leqq m(\pi^*\zeta) = m((\pi^*\zeta)\otimes\rho) \qquad (\zeta=\nu(f)).$$

Therefore (a) is proved.

Assume that m satisfies (6.6). Then (3.17) implies that

$$\binom{t'}{m} \equiv \binom{t+n+1}{m} \not\equiv 0 \bmod 2, \text{ and hence } t' \geqq m > t.$$

Thus $m(\nu(f)) < N'(t, t') \leqq m$ by (a) and the definition (5.15). Especially, if $m=t+1$ satisfies (6.6), then $n \leqq m(\nu(f)) < t+1$ and hence $m(\nu(f)) \geqq t$ by (i). Therefore $m(\nu(f))=t$ and (b) is proved. q.e.d.

In the rest of this section, we study the extendibility of the higher order tangent bundles over the lens spaces.

For each smooth manifold M, let

(6.7) $$\tau_k(M) = \cup_{x\in M} \tau_k(M)_x \qquad \text{for} \quad k = 1,2,3,\dots$$

denote the k-th order tangent bundle over M, where the k-th order tangent space $\tau_k(M)_x$ at $x\in M$ is the real vector space spanned by the linear functionals

$$\{\partial^j/\partial x_{i_1}\cdots\partial x_{i_j}|_x,\ 1\leqq j\leqq k,\ 1\leqq i_1\leqq\cdots\leqq i_j\leqq n\} \quad (n=\dim M)$$

with respect to the local coordinate $(x_1, x_2,\dots, x_n)$ of x, (see [20], [5] for the detailed definition). Thus

(6.8) *$\tau_k(M)$ is a real $t(n, k)$-plane bundle over M ($n=\dim M$), where*

$$t(n, k) = \binom{n}{1}+\binom{n+1}{2}+\cdots+\binom{n+k-1}{k} = C(n, k)-1,\ C(n, k) = \binom{n+k}{k};$$

and $\tau_1(M)$ is the tangent bundle of M.

For the real projective space RP^n ($n\geqq 1$), we have the following

LEMMA 6.9. *$\tau_k(RP^n)$ is stably equivalent to $t'\rho$, where $t'=0$ if k is even, $t'=C(n, k)$ if k is odd.*

PROOF. H. Suzuki [23; p. 274] proved that

$$\tau_k(RP^n) - t(n, k) = C(n, k)(\rho^k - 1) \quad \text{in} \quad \widetilde{KO}(RP^n).$$

This shows the lemma since $\rho^2 - 1 = 0$ in $\widetilde{KO}(RP^n)$. q.e.d.

THEOREM 6.10. *For the k-th order tangent bundle $\tau_k(RP^n)$ and its complexification $c\tau_k(RP^n)$ over RP^n, we have the following*

$$\text{(i)} \quad m(\tau_k(RP^n)) = \begin{cases} \infty & \text{if } k \text{ is even, or } C(n, k) \geqq 2^{\phi(n)}, \\ C(n, k) - 1 & \text{otherwise,} \end{cases}$$

where $C(n, k) = \binom{n+k}{k}$ and $\phi(n)$ is the integer given in (5.3).

$$\text{(ii)} \quad m(c\tau_k(RP^n)) = \begin{cases} \infty & \text{if } k \text{ is even, or } C(n, k) \geqq 2^{[n/2]}, \\ 2C(n, k) - 1 & \text{otherwise.} \end{cases}$$

In case that $k=1$, i.e., that $\tau_1(RP^n)$ is the tangent bunle $\tau(RP^n)$, (i) of this theorem is contained in (6.2) for $q=2$ and (ii) is Theorem 4.1.

PROOF OF THEOREM 6.10 (i) Assume $k \geqq 2$. Then $t(n, k) = C(n, k) - 1 > n$ in (6.8). Thus, by (6.8) and Lemma 6.9, the result for even k follows immediately from Theorem 5.7(ii) (b), and the one for odd k with $C(n, k) < 2^{\phi(n)}$ from Theorem 5.16(ii). If k is odd and $C(n, k) \geqq 2^{\phi(n)}$, then $\tau_k(RP^n)$ is stably equivalent to $t''\rho$, where $t'' = C(n, k) - 2^{\phi(n)} \leqq t(n, k)$, by (5.3). Thus the result follows from Theorem 5.7(ii) (b).

(ii) By Lemma 6.9, $c\tau_k(RP^n)$ is stably equivalent to $t'c\rho \cong t'\eta$. Therefore (ii) is proved in the same way as the above proof, by using Theorems 3.8(ii), 3.13(iii) and (3.3). q.e.d.

Now, we consider the k-th order tangent bundle $\tau_k(L^{n'}(q))$ of the lens space $L^{n'}(q) = L_q^n$ $(n = 2n'+1)$. The extendibility of the tangent bundle $\tau(L^{n'}(q)) = \tau_1(L^{n'}(q))$ or its complexification is given in (6.2) or Theorem 4.3.

To study the case that $k \geqq 2$, we use the following

LEMMA 6.11. *$\tau_k(L^{n'}(q))$ is stably equivalent to*

$\zeta' = 2b_{u+1}\rho \oplus \sum_{i=1}^{u} b_i r(\eta^i)$ *if q is even,* $= \sum_{i=1}^{u} b_i r(\eta^i)$ *if q is odd*$(u = [(q-1)/2])$, *where*

$$\text{(6.12)} \quad b_i = b_i(n', k; q) = \textstyle\sum_{j \in D_i} C(n', j)C(n', k-j) \quad \left(C(a, b) = \binom{a+b}{b}\right),$$
$$D_i = \{j \mid 0 \leqq 2j < k,\ k-2j \equiv \pm i \mod q\} \quad \textit{for} \quad 1 \leqq i \leqq [q/2].$$

PROOF. H. Ôike [19; Th. 2.8] proved that

$$\tau_k(L^{n'}(q)) - t(n, k) = \sum_{0\leqq 2j<k} C(n', j)C(n', k-j)\Psi^{k-2j}(r\sigma) \quad \text{in} \quad \widetilde{KO}(L^{n'}(q)),$$

where $\sigma=\eta-1$ and Ψ^l denotes the Adams operation on $\widetilde{KO}(L^{n'}(q))$. Since $\Psi^l(r\sigma)=r\Psi^l_C(\eta-1)=r(\eta^l-1)$ in $\widetilde{KO}(L^{n'}(q))$ ([2; Lemma A2]) and $\eta^q-1=0$ in $\tilde{K}(L^{n'}(q))$, the above equality implies the lemma by Lemma 5.2(ii) and (iii).

q.e.d.

LEMMA 6.13. *The bundle ζ' in Lemma 6.11 is a real t'-plane bundle, where*

$$(6.14) \qquad t' = t'(n', k; q) = \sum_{i=1}^{[q/2]} 2b_i = 2\sum_{j\in D} C(n', j)C(n', k-j),$$

$$D = D_1 \cup \cdots \cup D_{[q/2]} = \{j \mid 0\leqq 2j<k,\ k-2j \not\equiv 0 \bmod q\};$$

and $t'(n', k; q)$ satisfies the following properties $(n=2n'+1)$:

$$(6.15) \quad \begin{aligned} &t'(n', k; q) \leqq t(n, k)(=C(n, k)-1) \text{ if } k \text{ is even or } q \text{ is odd} \leqq k, \\ &t'(n', k; q) = t(n, k)+1 \quad \text{otherwise.}\end{aligned}$$

PROOF. (6.14) is clear by (6.12). By comparing the coefficients of x^k in the both sides of $(1-x)^{-n-1}=(1-x)^{-n'-1}(1-x)^{-n'-1}(n=2n'+1)$ and by (6.14), we see that

$$C(n, k) = \sum_{j=0}^{k} C(n', j)C(n', k-j) = t'(n', k; q)+2d_0+d,$$

where $d_0=d_0(n', k; q)=\sum_{j\in D_0} C(n', j)C(n', k-j)$ $(D_0=\{j \mid 0\leqq 2j<k,\ k-2j\equiv 0 \bmod q\})$ and

$$d = (C(n', k/2))^2 \text{ if } k \text{ is even}, \quad =0 \text{ if } k \text{ is odd}.$$

Therefore

$$t'(n', k; q) = C(n, k) \text{ if } d = d_0 = 0, \quad t'(n', k; q) < C(n, k) \text{ otherwise};$$

and $d=0$ if and only if k is odd, and $d_0=0$ if and only if $D_0=\emptyset$. When k is odd, we see easily that $D_0\neq\emptyset$ if and only if q is odd$\leqq k$. Thus (6.15) holds.

q.e.d.

THEOREM 6.16. *Let $\tau_k=\tau_k(L^{n'}(q))$ $(k\geqq 2)$ be the k-th order tangent bundle of the lens space $L^{n'}(q)=L^n_q(q\geqq 3,\ n=2n'+1)$.*

(i) *$m(\tau_k)=\infty$ if one of the following (1)–(4) holds:*

(1) *k is even.* (2) *q is odd$\leqq k$.*

(3) *b_i in (6.12) is not smaller than the order of $r(\eta^i-1)$ in $\widetilde{KO}(L^n_q)$ for some i with $1\leqq i\leqq[q/2]$.*

(4) *q is an odd prime and $b_i\geqq q^{1+[(n'-2)/(q-1)]}$ for some i with $1\leqq i\leqq[q/2]$.*

(ii) *$m(\tau_k)\geqq C(n, k)-1$; and*

$m(\tau_k)=C(n, k)-1$ if k is odd$\geqq 3$, q is even and $C(n, k)<2^{\phi(n)}$ ($\phi(n)$ is the integer given in (5.3)).

(iii) $m(\tau_k) \geqq 2C(n, k)-1$ *if* k *is odd* >3 *and* q *is odd* $>k$; *and* $m(\tau_k)=2C(n, k)-1$ *if* $p>k$ *and* $C(n, k)<2p^{[n'/(q-1)]}$ *for some prime factor* p *of* q, *in addition.*

PROOF. We notice that $t(n, k)=C(n, k)-1>n$ in (6.8) since $k \geqq 2$.

(i) If (1) or (2) holds, then $t'(n', k; q) \leqq t(n, k)$ by (6.15). Thus $m(\tau_k)=\infty$ by (6.8), Lemmas 6.11, 6.13 and Theorem 5.7(ii) (b). If (3) or (4) holds, then τ_k is stably equivalent to ζ'' which is obtained from ζ' in Lemma 6.11 by reducing b_i to the residue modulo the order of $r(\eta^i-1)$ in $\widetilde{KO}(L_q^n)$ by Lemma 5.4(ii), and ζ'' is a t''-plane bundle with $t'' \leqq t'(n', k; q)-1 \leqq t(n, k)$ by (6.15). Thus $m(\tau_k)=\infty$ in the same way as above.

(ii) $m(\tau_k) \geqq C(n, k)-1$ is a consequence of Theorem 5.7(ii) (a). If k is odd$\geqq 3$ and q is even, then $D_{2l}=\emptyset$ and $b_{2l}=0$ in (6.12), and d' in Corollary 5.17 (ii) for $\zeta=\tau_k$ and ζ' in Lemma 6.11 is equal to

$$2\sum_l b_{2l+1} = t'(n', k; q) = C(n, k) = t(n, k)+1$$

by (6.14–15). Thus $m(\tau_k)<C(n, k)$ if $C(n, k)<2^{\phi(n)}$ in addition, by Corollary 5.17(ii).

(iii) If k is odd$\geqq 3$ and q is odd$>k$, then $t(n, k)=C(n, k)-1$ is odd and $m(\tau_k) \geqq 2C(n, k)-1$ by Theorem 5.7(ii)(c). If there is a prime factor p of q with $p>k$, then $D_{pl}=\emptyset$ and $b_{pl}=0$ in (6.12), and d in (5.12) for $\zeta=\tau_k$ and ζ' in Lemma 6.11 is equal to $t'(n', k; q)=C(n, k)=t(n, k)+1$ by (6.14–15). Thus $m(\tau_k)< 2C(n, k)$ if $C(n, k)<2p^{[n'/p-1)]}$ in addition, by Theorem 5.11(ii). q.e.d.

THEOREM 6.17. *For the complexification* $c\tau_k$ *of* τ_k *in Theorem* 6.16, *we have the following*

(i) $m(c\tau_k) \geqq m(\tau_k)$ *for* $m(\tau_k)$ *in the above theorem, and hence* $m(c\tau_k)=\infty$ *if* $m(\tau_k)=\infty$, *e.g., if* k *is even or* q *is odd*$\leqq k$.

(ii) $m(c\tau_k) \geqq 2C(n, k)-1$ *if* k *is odd, and* q *is odd*$>k$ *or* q *is even*; *and* $m(c\tau_k)=2C(n, k)-1$ *if* $p>k$ *and* $C(n, k)<p^{[n'/(p-1)]}$ *for some prime factor* p *of* q, *in addition.*

PROOF. (i) If τ_k is extendible to L_q^m, then so is $c\tau_k$. Thus $m(c\tau_k) \geqq m(\tau_k)$.

(ii) $c\tau_k$ is a complex $t(n, k)$-plane bundle and is stably equivalent to $\sum_{i=1}^{[q/2]} b_i \cdot (\eta^i \oplus \eta^{q-i})$ where b_i's are the integers given in (6.12), by Lemma 6.11. Thus we see (ii) in the same way as the proof of Theorem 6.16(iii), by using Theorems 3.8(i) and 3.13(ii). q.e.d.

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