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# Weak boundary components in $R^N$

Dedicated to Professor M. Ohtsuka for his 60th birthday

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## Introduction

Let $D$ be a bounded plane domain and $\gamma$ be a component of the boundary of $D$ consisting of a single point. It is called by Sario [7] weak if its image under any conformal mapping of $D$ consists of a single point. Jurchescu [3] gave a characterization of the weakness by means of extremal length.

In the $N$-dimensional euclidean space $R^N$ $(N \geqq 3)$, Sario [8] introduced the notion of the capacity $c_\gamma$ of a subboundary $\gamma$ of a domain in $R^N$ and posed the following question: Is a component $\gamma$ of a compact set $E$ in $R^N$ a point if and only if $c_\gamma = 0$ for the domain $R^N - E$ ([8, p. 110])? A boundary component $\gamma$ is called weak if $c_\gamma = 0$.

In the present paper we shall be concerned with this question. Let $D$ be a domain in $R^N$ and $E$ be a compact set such that $\gamma = \partial E$ is a subboundary of $D$. We shall give an example (Example 1) in which $\gamma$ is a point but $c_\gamma \neq 0$. Moreover, in case $\gamma$ is an isolated subboundary, we shall show (Theorem 2) that $c_\gamma = 0$ if and only if the Newtonian capacity $C_2(E) = 0$. Since there exists a continuum $E$ with $C_2(E) = 0$ (cf. [1]), it follows that even for a continuum $E$, $\gamma = \partial E$ can be weak.

In §4, we shall give a characterization of the weakness by means of the extremal length of order 2. Let $B$ be a ball in $D$ and $\hat{\Gamma}$ denote the family of curves in the Kerékjártó-Stoïlow compactification each of which connects $\gamma$ and $B$. We shall show (Theorem 4) that $c_\gamma = 0$ if and only if the extremal length $\lambda_2(\hat{\Gamma}) = \infty$. In §5, we shall derive the modular criterion of the weakness which is well known for Riemann surfaces (cf. [9]).

## § 1. Preliminaries

Let $R^N$ $(N \geqq 3)$ be the $N$-dimensional euclidean space. We shall denote by $x = (x_1, x_2, \dots, x_N)$ a point in $R^N$, and set $|x| = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$. For a set $E$ in $R^N$, we denote by $\partial E$ and $\bar{E}$ the boundary and the closure of $E$ with respect to the $N$-dimensional Möbius space $R^N \cup \{\infty\}$, respectively. Let $B(r, x)$ denote the open $N$-ball of radius $r$ and centered at $x$. The area of $\partial B(1, x)$ will be written as $\omega_N$. For a function $u$ defined in a domain $G$, we let $\nabla u$ denote the gradient of

$u$ in case it exists. We denote by $H(G)$ the class of all harmonic functions $u$ on $G$, and by $HD^2(G)$ the class of all $u$ in $H(G)$ such that its Dirichlet integral $\int_G |\nabla u|^2 dx$ is finite.

Let $D$ be a domain in $R^N$. Denote by $\hat{D}$ the Kerékjártó-Stoïlow compactification of $D$. Let $\hat{\gamma}$ be a closed subset of the ideal boundary $\hat{D}-D$ of $D$ and let $\hat{\beta}=(\hat{D}-D)-\hat{\gamma}$. Let $\{D_n\}$ be an exhaustion of $D$, that is, each $D_n$ is a bounded subdomain of $D$, each component of $D-D_n$ is noncompact in $D$, each $\partial D_n$ consists of a finite number of $C^1$-surfaces, $\bar{D}_n \subset D_{n+1}$ $(n=1, 2,\dots)$ and $\cup_{n=1}^{\infty} D_n=D$. Let $A_n$ be the union of the components of $\hat{D}-D_n$ each of which meets $\hat{\gamma}$, and $B_{ni}$ $(i=1,\dots, i(n))$ be the rest of the components of $\hat{D}-D_n$. Set $\gamma=\cap_{n=1}^{\infty} \bar{U}_n$, where $U_n=A_n \cap D$. We shall call $\gamma$ a subboundary of $D$. If $\hat{\gamma}$ is an ideal boundary component, then $\gamma$ is a boundary component of $D$. When there is no ambiguity, we shall identify $\gamma$ with $\hat{\gamma}$. A subboundary $\gamma$ is said to be isolated if there exists an $A_n$ with $A_n \cap \hat{\beta}=\emptyset$. We set $\gamma_n=\partial D_n \cap \partial A_n$ and $\beta_{ni}=\partial D_n \cap \partial B_{ni}$ $(i=1,\dots, i(n))$.

Take a point $x^0$ in $D$ and a ball $B=B(r, x^0)$ with $\bar{B}\subset D_n$ for all $n$. Denote by $P_n$ the class of functions $p$ on $\bar{D}_n$ having the following properties:

(1.1) $p \in H(D_n-\{x^0\}) \cap C^1(\bar{D}_n-\{x^0\})$;

(1.2) $p(x) = -|x-x^0|^{2-N}/(\omega_N(N-2)) + h(x)$ in $B$, where $h \in H(B)$ and $h(x^0) = 0$;

(1.3) $\int_{\beta_{ni}} \frac{\partial p}{\partial \nu} dS=0$ for $i=1,\dots, i(n)$ and $\int_{\gamma_n} \frac{\partial p}{\partial \nu} dS=1$, where $\frac{\partial}{\partial \nu}$ is the outer normal derivative on $D_n$ and $dS$ is the surface element.

We know (cf. [8]) that there exists a unique function $p_{n\gamma}$ in $P_n$ having the following properties:

(1.4) $$p_{n\gamma} = k_{n\gamma} \quad \text{on} \quad \gamma_n;$$

(1.5) $$p_{n\gamma} = k_{ni} \quad \text{on} \quad \beta_{ni}\ (i = 1,\dots, i(n)),$$

where $k_{n\gamma}$ and $k_{ni}$ are constants. In reference to the pole $x^0$, we also use the notation $p_{n\gamma}=p_{n\gamma}(\cdot, x^0)$ and $k_{n\gamma}=k_{n\gamma}(x^0)$.

The following lemmas are known:

LEMMA 1 ([8, the proof of Theorem 25]).

$$\int_{\partial D_n} p_{n\gamma} \frac{\partial p_{n\gamma}}{\partial \nu} dS=k_{n\gamma}$$

*and*

$$\int_{D_n} |\nabla (p-p_{n\gamma})|^2 dx = \int_{\partial D_n} p \frac{\partial p}{\partial \nu} dS - \int_{\partial D_n} p_{n\gamma} \frac{\partial p_{n\gamma}}{\partial \nu} dS$$

*for every* $p \in P_n$.

LEMMA 2 (cf. [4, p. 20] and [9, Theorem III. 2E]). *The sequence* $\{p_{n\gamma}\}$ *is uniformly bounded on every compact subset of* $D-\{x^0\}$.

By Lemma 2 we see that the sequence $\{p_{n\gamma}\}$ contains a subsequence, denoted by $\{p_{n\gamma}\}$ again, converging to a harmonic function $p_\gamma$, which is called a capacity function of $\gamma$, uniformly on every compact subset of $D-\{x^0\}$.

Since $k_{n\gamma}$ increases with $n$ by Lemma 1, the limit $k_\gamma=\lim_{n\to\infty} k_{n\gamma}$ exists. The capacity $c_\gamma$ of $\gamma$ is defined by $c_\gamma=k_\gamma^{1/(2-N)}$. A subboundary $\gamma$ is called weak if $c_\gamma=0$, that is, if $k_\gamma=\infty$. We note that the capacity $c_\gamma$ does not depend on the choice of exhaustion.

Take any $x^1\in D$ with $x^0\neq x^1$. By using Green's formula we have the following symmetry property (cf. [9, Theorem V. 2A])

$$k_{n\gamma}(x^1) - p_{n\gamma}(x^0, x^1) = k_{n\gamma}(x^0) - p_{n\gamma}(x^1, x^0).$$

This implies that the weakness of $\gamma$ does not depend on the choice of the pole $x^0$ in $D$.

## § 2. Weak boundary components

Denote by $P=P(D)$ the class of functions $p$ on $D$ having the following properties:

(2.1) $p\in H(D-\{x^0\}) \cap HD^2(D-\bar{B})$;

(2.2) $p(x) = -|x-x^0|^{2-N}/(\omega_N(N-2)) + h(x)$ in $B$, where $h\in H(B)$ and $h(x^0) = 0$;

(2.3) $\int_\tau \frac{\partial p}{\partial \nu} dS=0$ for every compact $C^1$-surface $\tau$ in $D-\{x^0\}$ which divides $R^N$ into a bounded domain and an unbounded domain, and which does not separate $\gamma$ from $\{x^0\}$.

THEOREM 1 (cf. [9, Theorem III. 3B]). *$\gamma$ is weak if and only if $P=\emptyset$.*

PROOF. Suppose $P\neq\emptyset$. Since the restriction of $p\in P$ to $D_n$ belongs to $P_n$, by Lemma 1 we have

$$k_{n\gamma} \leqq \int_{\partial D_n} p\frac{\partial p}{\partial \nu} dS.$$

By Green's formula and (2.1) we obtain

$$\left|\int_{\partial D_n} p\frac{\partial p}{\partial \nu} dS\right| \leqq \int_{D_n-\bar{B}} |\nabla p|^2 dx + \left|\int_{\partial B} p\frac{\partial p}{\partial \nu} dS\right|$$
$$< \int_{D-\bar{B}} |\nabla p|^2 dx + \left|\int_{\partial B} p\frac{\partial p}{\partial \nu} dS\right| < \infty.$$

This implies $k_\gamma < \infty$.

Next we suppose $k_\gamma < \infty$. We shall show that the capacity function $p_\gamma$ belongs to $P$. Obviously, $p_\gamma \in H(D-\{x^0\})$ and it satisfies (2.2). It is easy to verify that $p_\gamma$ has property (2.3). Therefore it is enough to show that $p_\gamma \in HD^2(D-\bar{B})$. Since $\int_{D_m-\bar{D}_n} |\nabla p_{m\gamma}|^2 dx > 0$ for $m > n$, Green's formula gives

$$\int_{\partial D_n} p_{m\gamma} \frac{\partial p_{m\gamma}}{\partial \nu} dS \leqq \int_{\partial D_m} p_{m\gamma} \frac{\partial p_{m\gamma}}{\partial \nu} dS.$$

By Lemma 1 we have

$$\begin{aligned} \lim_{n\to\infty} \int_{\partial D_n} p_\gamma \frac{\partial p_\gamma}{\partial \nu} dS &= \lim_{n\to\infty} \lim_{m\to\infty} \int_{\partial D_n} p_{m\gamma} \frac{\partial p_{m\gamma}}{\partial \nu} dS \\ &\leqq \lim_{m\to\infty} \int_{\partial D_m} p_{m\gamma} \frac{\partial p_{m\gamma}}{\partial \nu} dS \\ &= \lim_{m\to\infty} k_{m\gamma} = k_\gamma. \end{aligned}$$

Hence, by using Green's formula we have

$$\begin{aligned} \int_{D-B} |\nabla p_\gamma|^2 dx &= \lim_{n\to\infty} \int_{D_n - B} |\nabla p_\gamma|^2 dx \\ &\leqq \lim_{n\to\infty} \int_{\partial D_n} p_\gamma \frac{\partial p_\gamma}{\partial \nu} dS + \left| \int_{\partial B} p_\gamma \frac{\partial p_\gamma}{\partial \nu} dS \right| \\ &\leqq k_\gamma + \left| \int_{\partial B} p_\gamma \frac{\partial p_\gamma}{\partial \nu} dS \right| < \infty. \end{aligned}$$

Therefore $p_\gamma \in HD^2(D-\bar{B})$. The proof is completed.

COROLLARY 1. *If $\gamma$ contains the point at infinity, then $\gamma$ is not weak.*

PROOF. Let $p(x) = -|x-x^0|^{2-N}/(\omega_N(N-2))$. Then $p \in P$, so that $k_\gamma < \infty$.

EXAMPLE 1. We shall give an example of $D$ which has a boundary component $\gamma$ consisting of a single point and satisfying $k_\gamma < \infty$. We introduce the polar coordinates $(r, \theta_1, \dots, \theta_{N-1})$ in $R^N$, that is, $r = |x|$, $x_1 = r\cos\theta_1, \dots, x_{N-1} = r\sin\theta_1 \cdots \sin\theta_{N-2}\cos\theta_{N-1}$, $x_N = r\sin\theta_1 \cdots \sin\theta_{N-2}\sin\theta_{N-1}$, for $x = (x_1, \dots, x_N)$. Consider sequences $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ defined by

$$a_n = (n + \textstyle\sum_{k=2}^{n} k^{-2})^{1/(2-N)}, \quad b_n = (n + \textstyle\sum_{k=1}^{n} k^{-2})^{1/(2-N)}$$

and

$$\int_{\{x;|x|=1, 0 \leqq \theta_1 < \delta_n\}} dS = n^{-2}.$$

Set

$$E_n = \{x;\, b_n \leqq |x| \leqq a_n\} - \{x;\, 0 \leqq \theta_1 < \delta_n\}$$

and

$$D = R^N - \cup_{n=1}^{\infty} E_n - \{0\}.$$

Let $\gamma = \{0\}$. Then $\gamma$ is a boundary component of $D$. It is easily verified that $|x|^{2-N}$ has a finite Dirichlet integral on $D$. Let $x^0 \in D$. Then the function

$$(-|x-x^0|^{2-N} + |x|^{2-N} - |x^0|^{2-N})/(\omega_N(N-2))$$

belongs to $P$, so that by Theorem 1 we have $k_\gamma < \infty$.

## § 3. An isolated subboundary and Newtonian capacity

Let $E$ be a compact set in $R^N$. The Newtonian capacity of $E$ is defined as

$$C_2(E) = \inf \int |\nabla f|^2 dx,$$

where the infimum is taken over all functions $f \in C^\infty$ that have compact support and are identically equal to 1 on $E$. Let $G$ be a bounded domain containing $E$. We say that $E$ is removable for $HD^2$ if every function in $HD^2(G-E)$ can be extended to a function in $HD^2(G)$. It is well known that $E$ is removable for $HD^2$ if and only if $C_2(E)=0$ (see, e.g., [1, §VII, Theorem 1]).

THEOREM 2 (cf. [9, Theorem X. 3A]). *Let $E$ be a compact set such that $R^N - E$ is a domain. Let $D$ be a subdomain of $R^N - E$ and $\gamma = \partial E$ be an isolated subboundary of $D$. Then $C_2(E)=0$ if and only if $\gamma$ is weak.*

PROOF. Suppose $C_2(E)=0$. By assumption we can take a bounded domain $G$ such that $G \supset E$, $D \supset G - E$ and $\partial G$ separates $\gamma$ from $\beta \cup \{x^0\}$, where $\beta = \partial D - \gamma$. We may assume that $G \supset \gamma_n$ for all $n$. Since every $u$ in $HD^2(G-E)$ can be extended to a function $\tilde{u}$ in $HD^2(G)$, we have

$$\int_{\gamma_n} \frac{\partial u}{\partial \nu} dS = \int_{\gamma_n} \frac{\partial \tilde{u}}{\partial \nu} dS = 0$$

for all $n$. This implies $P(D) = \emptyset$, so that $\gamma$ is weak by Theorem 1.

Conversely we suppose $C_2(E) > 0$. Let $\mu$ be the equilibrium mass-distribution on $E$ and consider the potential

$$U_2^\mu(x) = \int_E \frac{d\mu(y)}{|x-y|^{N-2}}.$$

It is known that $U_2^\mu \in HD^2(R^N - E)$ and

$$\int_\tau \frac{\partial U_2^\mu}{\partial \nu} dS \neq 0$$

for every compact $C^1$-surface $\tau$ in $R^N - E$ which separates the point at infinity from $E$. Therefore we can take a non-zero constant $\ell$ satisfying

$$\ell \int_{\gamma_n} \frac{\partial U_2^\mu}{\partial \nu} dS = 1$$

for all $n$. Let $x^0 \in D$. Then the function

$$-|x-x^0|^{2-N}/(\omega_N(N-2)) + \ell(U_2^\mu(x) - U_2^\mu(x^0))$$

belongs to $P(D)$. From Theorem 1 it follows that $\gamma$ is not weak. Thus our theorem is proved.

COROLLARY 2. *Let $E$ be a compact set such that $R^N - E$ is a domain. Suppose $\partial E = \gamma$ is a subboundary of a domain $D$. If $\gamma$ is weak, then $C_2(E) = 0$.*

PROOF. Let $G = R^N - E$. If $C_2(E) > 0$, then $P(G) \neq \emptyset$ by Theorems 1 and 2. Since the restriction of $p \in P(G)$ to $D$ belongs to $P(D)$, we have $P(D) \neq \emptyset$. It follows that $\gamma$ is not weak from Theorem 1.

REMARK 1. If $N \geqq 3$, then there exists a continuum $E$ with $C_2(E) = 0$ (see, e.g., [1, §IV, Theorem 1]). Hence there exists a continuum $E$ in $R^N$ $(N \geqq 3)$ such that $\gamma = \partial E$ is weak for the domain $R^N - E$. Thus Example 1 and Theorem 2 give a negative answer to the problem 11 in [8].

REMARK 2. By the inversion with respect to $B(1, 0)$, a line segment $E = \{x = (x_1, 0,\dots, 0);\ 0 \leqq x_1 \leqq 1\}$ is mapped to $E_0 = \{x = (x_1, 0,\dots, 0);\ 1 \leqq x_1 < \infty\} \cup \{\infty\}$. Since $C_2(E) = 0$, $\gamma = \partial E$ is weak for the domain $R^N - E$. But $\gamma_0 = \partial E_0$ is not weak for the domain $R^N - E_0$ by Corollary 1. Thus we see that the weakness in $R^N$ $(N \geqq 3)$ is not invariant under quasiconformal mappings.

## § 4. Extremal length criterion

Let $D$ be a domain in $R^N$. By a locally rectifiable chain in $D$ we mean a countable formal sum $c = \sum c_i$, where each $c_i$ is a locally rectifiable curve in $D$. If $f$ is a non-negative Borel measurable function defined in $D$ and $c = \sum c_i$ is a locally rectifiable chain in D, then we set $\int_c f ds = \sum \int_{c_i} f ds$, where $ds$ is the line element. Let $\Gamma$ be a family of locally rectifiable chains in $D$. A non-negative Borel measurable function $f$ defined in $D$ is called admissible in association with $\Gamma$ if $\int_c f ds \geqq 1$ for every $c \in \Gamma$. The module $M_2(\Gamma)$ of $\Gamma$ is defined by $\inf_f \int_D f^2 dx$, where the infimum is taken over all admissible functions $f$ in association with $\Gamma$, and the extremal length $\lambda_2(\Gamma)$ of $\Gamma$ is defined by $1/M_2(\Gamma)$. In case $\hat{\Gamma}$ is a family of curves in $\hat{D}$ such that the restriction $c|_D$ is a locally rectifiable chain in $D$ for each $c \in \hat{\Gamma}$, we denote by $\lambda_2(\hat{\Gamma})$ the extremal length of $\{c|_D; c \in \hat{\Gamma}\}$. Hereafter, by a curve we shall mean a locally rectifiable curve. The following properties are well known (see, e.g., [2, Chapter I]):

(4.1) If every $c_1 \in \Gamma_1$ contains a $c_2 \in \Gamma_2$, then $\lambda_2(\Gamma_1) \geqq \lambda_2(\Gamma_2)$.

(4.2) If $\Gamma \subset \bigcup_n \Gamma_n$, then $M_2(\Gamma) \leqq \sum_n M_2(\Gamma_n)$.

(4.3) Let $\{G_n\}$ be mutually disjoint open sets and $\Gamma_n$ be a family of curves in $G_n$. If $\Gamma$ is a family of curves such that each $c \in \Gamma$ contains at least one $c_n \in \Gamma_n$ for every $n$, then $\lambda_2(\Gamma) \geqq \sum_n \lambda_2(\Gamma_n)$.

Let $\alpha_0$, $\alpha_1$ be subboundaries of $D$ with $\alpha_0 \cap \alpha_1 = \emptyset$. Denote by $\Gamma(\alpha_0, \alpha_1; D)$ (resp. $\hat{\Gamma}(\alpha_0, \alpha_1; D)$) the family of curves in $D$ (resp. $\hat{D}$) each of which connects $\alpha_0$ and $\alpha_1$. (A subboundary of $D$ will be identified with the corresponding closed subsets of $\hat{D} - D$.) Suppose that $\alpha_0$ is an isolated subboundary consisting of a finite number of compact $C^1$-surfaces. Let $\{D_n\}$ be an approximation of $D$ towards $\partial D - \alpha_0$, that is, each $D_n$ is a bounded subdomain of $D$, each $\partial D_n$ consists of $\alpha_0$ and a finite number of compact $C^1$-surfaces, $\bar{D}_n \subset D_{n+1} \cup \alpha_0$ ($n = 1, 2,\dots$) and $\bigcup_{n=1}^{\infty} D_n = D$. Let $A_{1n}$ be the union of the components of $\hat{D} - D_n$ each of which meets $\alpha_1$. Set $\alpha_{1n} = \partial D_n \cap \partial A_{1n}$. The following lemma follows in the same manner as in [10, Lemma 4].

LEMMA 3. $\lim_{n\to\infty} \lambda_2(\hat{\Gamma}(\alpha_0, \alpha_{1n}; D_n)) = \lambda_2(\hat{\Gamma}(\alpha_0, \alpha_1; D))$.

Let $G$ be a bounded domain such that $\partial G$ consists of a finite number of compact $C^1$-surfaces $\alpha_0$, $\alpha_1$ and $\beta_j$ ($j = 1,\dots, k$). We know (cf. [6]) that there exists the principal function $h$ with respect to $\alpha_0$, $\alpha_1$ and $G$, which is characterized by the following properties:

(1) $h \in H(G) \cap C^1(\bar{G})$;

(2) $h = 0$ on $\alpha_0$ and $h = 1$ on $\alpha_1$;

(3) $h = $ const. on each $\beta_j$ and $\int_{\beta_j} \frac{\partial h}{\partial \nu} dS = 0$ for $j = 1,\dots, k$.

The following property is known ([10, Theorems 5 and 12]):

$$M_2(\hat{\Gamma}(\alpha_0, \alpha_1; G)) = \int_G |\nabla h|^2 dx. \tag{4.4}$$

Let $\gamma$ be a subboundary of $D$ and $\{D_n\}$ be an exhaustion of $D$. Consider the capacity function $p_{n\gamma}$ of $\gamma_n$ with pole at $x^0 \in D$. Let $B_r = B(r, x^0)$ with $\bar{B}_r \subset D_n$ for all $n$. Set $a_{n,r}^0 = \max_{x\in\partial B_r} p_{n\gamma}(x)$ and $a_{n,r}^1 = \min_{x\in\partial B_r} p_{n\gamma}(x)$.

LEMMA 4. *There exists an $r_0 > 0$ such that, for every $r$ with $0 < r < r_0$, the following inequalities hold*:

$$k_{n\gamma} - a_{n,r}^0 \leqq \lambda_2(\hat{\Gamma}(\partial B_r, \gamma_n; D_n - \bar{B}_r)) \leqq k_{n\gamma} - a_{n,r}^1.$$

PROOF. Let $E_{n,r}^i=\{x;\ p_{n\gamma}(x)\leqq a_{n,r}^i\}$ $(i=0, 1)$. Then, there exists an $r_0>0$ such that $D_n-E_{n,r}^i$ $(i=0, 1)$ is a domain for every $r$ with $0<r<r_0$. Since $(p_{n\gamma}-a_{n,r}^i)/(k_{n\gamma}-a_{n,r}^i)$ is the principal function with respect to $\partial E_{n,r}^i$, $\gamma_n$ and $D_n-E_{n,r}^i$, by Green's formula and (4.4) we have

$$\lambda_2(\hat{\Gamma}(\partial E_{n,r}^i, \gamma_n;\ D_n-E_{n,r}^i)) = k_{n\gamma} - a_{n,r}^i \qquad (i = 0, 1).$$

Since

$$\lambda_2(\hat{\Gamma}(\partial E_{n,r}^0, \gamma_n;\ D_n-E_{n,r}^0)) \leqq \lambda_2(\hat{\Gamma}(\partial B_r, \gamma_n;\ D_n-\bar{B}_r)) \leqq \lambda_2(\hat{\Gamma}(\partial E_{n,r}^1, \gamma_n;\ D_n-E_{n,r}^1))$$

by (4.1), we obtain the required inequalities.

THEOREM 3 (cf. [9, Theorem IV. 3G]). *Let $\gamma$ be a subboundary of $D$ with $k_\gamma<\infty$ and let $B_r=B(r, x^0)$ with $\bar{B}_r\subset D$. Then*

$$k_\gamma = \lim_{r\to 0}\{\lambda_2(\hat{\Gamma}(\partial B_r, \gamma;\ D-\bar{B}_r)) - r^{2-N}/(\omega_N(N-2))\}.$$

PROOF. By Lemmas 3 and 4, we obtain

$$k_\gamma - \lim_{n\to\infty} a_{n,r}^0 \leqq \lambda_2(\hat{\Gamma}(\partial B_r, \gamma;\ D-\bar{B}_r)) \leqq k_\gamma - \lim_{n\to\infty} a_{n,r}^1.$$

The capacity function $p_\gamma$ has the property

$$p_\gamma(x) = -|x-x^0|^{2-N}/(\omega_N(N-2)) + h(x) \qquad \text{in} \quad B_r,$$

where $h\in H(B_r)$ and $h(x^0)=0$. Since $\{p_{n\gamma}\}$ converges to $p_\gamma$ uniformly on $\partial B_r$, we have

$$\lim_{n\to\infty} a_{n,r}^0 = -r^{2-N}/(\omega_N(N-2)) + \max_{x\in\partial B_r} h(x)$$

and

$$\lim_{n\to\infty} a_{n,r}^1 = -r^{2-N}/(\omega_N(N-2)) + \min_{x\in\partial B_r} h(x).$$

Therefore we see that

$$\begin{aligned} k_\gamma - \max_{x\in\partial B_r} h(x) &\leqq \lambda_2(\hat{\Gamma}(\partial B_r, \gamma;\ D-\bar{B}_r)) - r^{2-N}/(\omega_N(N-2)) \\ &\leqq k_\gamma - \min_{x\in\partial B_r} h(x). \end{aligned}$$

Since $h(x^0)=0$, letting $r\to 0$ we obtain the theorem.

THEOREM 4. *Let $\gamma$ be a subboundary of $D$. Let $G$ be a subdomain of $D$ such that $\partial G\cap D$ is a compact $C^1$-surface, $D-\bar{G}$ is a domain and $\partial(D-\bar{G})$ contains $\gamma$. Then $\gamma$ is weak if and only if $\lambda_2(\hat{\Gamma}(\partial G, \gamma;\ D-\bar{G}))=\infty$.*

PROOF. From Lemmas 2, 3 and 4 it follows that $k_\gamma=\infty$ if and only if $\lambda_2(\hat{\Gamma}(\partial B, \gamma;\ D-\bar{B}))=\infty$ for some, as well as for any, $x\in D$ and for sufficiently small $r>0$, where $B=B(r, x)$.

Suppose $\lambda_2(\hat{\Gamma}(\partial G, \gamma; D-\bar{G}))=\infty$. Take a ball $B=B(r, x)$ with $\bar{B}\subset G$. By (4.1) we conclude that $\lambda_2(\hat{\Gamma}(\partial B, \gamma; D-\bar{B}))=\infty$, so that $k_\gamma=\infty$.

Conversely suppose $k_\gamma=\infty$. We can take a finite number of balls $B^i=B(r, x^i)$ $(i=1,\dots, j)$ in $D$ with the following properties:

(1) $x^i\in\partial G\cap D$ $(i=1,\dots, j)$ and $U=\cup_{i=1}^{j} B^i$ contains $\partial G\cap D$;

(2) $\partial D\cap\bar{B}^i=\emptyset$ $(i=1,\dots, j)$ and $\Omega=D-\bar{G}-\bar{U}$ is a subdomain of $D-\bar{G}$;

(3) $\lambda_2(\hat{\Gamma}(\partial B^i, \gamma; D-\bar{B}^i))=\infty$ $(i=1,\dots, j)$.

Since

$$\hat{\Gamma}(\partial\Omega\cap\partial U, \gamma; \Omega)\subset\cup_{i=1}^{j}\hat{\Gamma}(\partial B^i, \gamma; D-\bar{B}^i),$$

by (4.1) and (4.2) we have

$$\begin{aligned} M_2(\hat{\Gamma}(\partial G, \gamma; D-\bar{G})) &\leqq M_2(\hat{\Gamma}(\partial\Omega\cap\partial U, \gamma; \Omega)) \\ &\leqq \textstyle\sum_{i=1}^{j} M_2(\hat{\Gamma}(\partial B^i, \gamma; D-\bar{B}^i)) = 0. \end{aligned}$$

Thus we see that $\lambda_2(\hat{\Gamma}(\partial G, \gamma; D-\bar{G}))=\infty$. The proof is completed.

Corollary 3. *Let $\gamma$, $\gamma_0$ be subboundaries of $D$ such that $\gamma\supset\gamma_0$. If $\gamma$ is weak, then so is $\gamma_0$.*

## §5. Modular criterion

Let $\gamma$ be a subboundary of $D$ and $\{D_n\}$ be an exhaustion of $D$. We note that $A_n$ consists of a finite number of mutually disjoint components $A_n^1,\dots, A_n^{k(n)}$ of $\hat{D}-D_n$ each of which meets $\gamma$. Set $\Omega_n=(D_{n+1}-\bar{D}_n)\cap A_n$. Then $\Omega_n$ consists of a finite number of mutually disjoint domains $\Omega_n^1,\dots, \Omega_n^{k(n)}$. Set $\alpha_n^i=\partial\Omega_n^i\cap\gamma_n$, $\beta_n^i=\partial\Omega_n^i\cap\gamma_{n+1}$ $(i=1,\dots, k(n))$, and define the values $\hat{\mu}_{n\gamma}$ by

$$\log\hat{\mu}_{n\gamma}=\{\textstyle\sum_{i=1}^{k(n)} M_2(\hat{\Gamma}(\alpha_n^i, \beta_n^i; \Omega_n^i))\}^{-1}.$$

Theorem 5 (cf. [9, Theorem XI. 1A]). *A subboundary $\gamma$ of $D$ is weak if and only if there exists an exhaustion $\{D_n\}$ of $D$ for which $\prod_{n=1}^{\infty}\hat{\mu}_{n\gamma}=\infty$.*

Proof. Suppose such an exhaustion $\{D_n\}$ exists. We may assume that $\bar{B}_r\subset D_1$. Set $\hat{\Gamma}_n=\cup_{i=1}^{k(n)}\hat{\Gamma}(\alpha_n^i, \beta_n^i; \Omega_n^i)$. Since $\Omega_n^1,\dots, \Omega_n^{k(n)}$ are mutually disjoint, we see easily that $M_2(\hat{\Gamma}_n)=\sum_{i=1}^{k(n)} M_2(\hat{\Gamma}(\alpha_n^i, \beta_n^i; \Omega_n^i))$. By (4.1) and (4.3) we have

$$\lambda_2(\hat{\Gamma}(\partial B_r, \gamma_n; D_n-\bar{B}_r))\geqq\textstyle\sum_{m=1}^{n-1}\lambda_2(\hat{\Gamma}_m)=\log(\prod_{m=1}^{n-1}\hat{\mu}_{m\gamma}).$$

By assumption and Lemma 3, letting $n\to\infty$ we see that $\lambda_2(\hat{\Gamma}(\partial B_r, \gamma; D-\bar{B}_r))=\infty$. From Theorem 4 it follows that $\gamma$ is weak.

Conversely suppose that $\gamma$ is weak. Let $\{D_n\}$ be any exhaustion of $D$. Set $\tilde{\gamma}_n^i=A_n^i\cap\gamma$ $(i=1,\dots, k(n))$. We note that each $\tilde{\gamma}_n^i$ is a subboundary of $D$, $\gamma=$

$\cup_{i=1}^{k(n)} \tilde{\gamma}_n^i$ and $\tilde{\gamma}_n^i \cap \tilde{\gamma}_n^j = \emptyset$ for $i \neq j$. Set $\tilde{\Omega}_n^i = (A_n^i - \gamma_n) \cap D$, $\tilde{\alpha}_n^i = \partial\tilde{\Omega}_n^i \cap \gamma_n$. Since $\tilde{\gamma}_n^i$ is weak by Corollary 3, we have $\lambda_2(\hat{\Gamma}(\tilde{\alpha}_n^i, \tilde{\gamma}_n^i; \tilde{\Omega}_n^i)) = \infty$ by Theorem 4. Set $\tilde{\Omega}_{n,m}^i = \tilde{\Omega}_n^i \cap D_m$, $\tilde{\alpha}_{n,m}^i = \partial\tilde{\Omega}_{n,m}^i \cap \gamma_m$ for $m > n$. Then $\{\tilde{\Omega}_{n,m}^i\}_{m=n+1}^{\infty}$ is an approximation of the domain $\tilde{\Omega}_n^i$ towards $\partial\tilde{\Omega}_n^i - \tilde{\alpha}_n^i$. By Lemma 3 we see that

$$\lim_{m\to\infty} \lambda_2(\hat{\Gamma}(\tilde{\alpha}_n^i, \tilde{\alpha}_{n,m}^i; \tilde{\Omega}_{n,m}^i)) = \lambda_2(\hat{\Gamma}(\tilde{\alpha}_n^i, \tilde{\gamma}_n^i; \tilde{\Omega}_n^i)) = \infty.$$

Hence, for $n=1$ we can take $m(1)$ with $m(1)>1$ such that $\lambda_2(\hat{\Gamma}(\tilde{\alpha}_1^i, \tilde{\alpha}_{1,m(1)}^i; \tilde{\Omega}_{1,m(1)}^i)) \geqq k(1)$ for all $i=1,\dots, k(1)$. Next, for $n=m(1)$, take $m(2)$ with $m(2) > m(1)$ such that $\lambda_2(\hat{\Gamma}(\tilde{\alpha}_{m(1)}^i, \tilde{\alpha}_{m(1),m(2)}^i; \tilde{\Omega}_{m(1),m(2)}^i)) \geqq k(m(1))$ for all $i=1,\dots, k(m(1))$. We continue this process and obtain a subsequence $\{D_{m(j)}\}_{j=1}^{\infty}$ of $\{D_n\}_{n=1}^{\infty}$. Since $\log \hat{\mu}_{m(j)\gamma} \geqq 1$ $(j=1, 2,\dots)$, we obtain an exhaustion $\{D_{m(j)}\}$ with $\prod_{j=1}^{\infty} \hat{\mu}_{m(j)\gamma} = \infty$. The proof is completed.

The modulus $\mu_{n\gamma}$ of $\Omega_n$ is defined by

$$\log \mu_{n\gamma} = \{\textstyle\sum_{i=1}^{k(n)} M_2(\Gamma(\alpha_n^i, \partial\Omega_n^i - \alpha_n^i; \Omega_n^i))\}^{-1}$$

(cf. [5]). Since $\log \mu_{n\gamma} \leqq \log \hat{\mu}_{n\gamma}$ by (4.1), we have

COROLLARY 4 ([5, Theorem 1]). *If there exists an exhaustion $\{D_n\}$ of $D$ for which $\prod_{n=1}^{\infty} \mu_{n\gamma} = \infty$, then $\gamma$ is weak.*

A bounded domain $R$ is called a ring domain if its complement consists of two components.

THEOREM 6 (cf. [9, Theorem XI. 1C]). *Let $\gamma$ be a subboundary of $D$ consisting of a single compact continuum. In order that $\gamma$ be weak, it is necessary and sufficient that, for any positive number $\ell$, there exist a finite number of ring domains $R_1, R_2,\dots, R_m$ in $D - \bar{B}_r$ satisfying the following conditions*:

(1) *$R_1,\dots, R_m$ are mutually disjoint*;

(2) *Each $R_i$ separates $\gamma$ from $B_r$ $(i=1, 2,\dots, m)$ and separates $R_{i-1}$ from $R_{i+1}$ $(i=2, 3,\dots, m-1)$*;

(3) *$\sum_{i=1}^{m} \lambda_2(\Gamma_i) \geqq \ell$, where $\Gamma_i$ is the family of all curves in $R_i$ each of which connects two boundary components of $R_i$.*

PROOF. Suppose such a finite number of ring domains $R_1, R_2,\dots, R_m$ exist. By (4.3) we have

$$\lambda_2(\hat{\Gamma}(\partial B_r, \gamma; D - \bar{B}_r)) \geqq \textstyle\sum_{i=1}^{m} \lambda_2(\Gamma_i) \geqq \ell.$$

This implies $\lambda_2(\hat{\Gamma}(\partial B_r, \gamma; D - \bar{B}_r)) = \infty$, so that $\gamma$ is weak by Theorem 4.

Next suppose that $\gamma$ is weak. By Theorem 5 we see that there exists an exhaustion $\{D_n\}$ of $D$ with $\prod_{n=1}^{\infty} \hat{\mu}_{n\gamma} = \infty$. Since $\gamma$ is a single compact continuum, we see that $\Omega_n = (D_{n+1} - \bar{D}_n) \cap A_n$ is a domain. For given $\ell > 0$, take an $n_0$ such that $\sum_{n=1}^{no} \log \hat{\mu}_{n\gamma} \geqq \ell + 1$. Set $G = (A_1 - \gamma_1) \cap D_{no+1}$. By (4.1) and (4.3) we have

$$\lambda_2(\hat{\Gamma}(\gamma_1, \gamma_{no+1}; G)) \geqq \sum_{n=1}^{no} \log \hat{\mu}_{n\gamma} \geqq \ell + 1.$$

We note that $\partial G$ consists of a finite number of $C^1$-surfaces $\gamma_1, \gamma_{no+1}, \beta_1, \dots, \beta_{i_0}$ each of which is a component of $\partial G$. Let $\tilde{u}$ be the principal function with respect to $\gamma_{no+1}, \gamma_1$ and $G$, which is characterized by the following properties:

(1) $\tilde{u} \in H(G) \cap C^1(\bar{G})$;

(2) $\tilde{u} = 0$ on $\gamma_{no+1}$ and $\tilde{u} = 1$ on $\gamma_1$;

(3) $\tilde{u} = \tilde{c}_i$ on $\beta_i$ and $\int_{\beta_i} \frac{\partial \tilde{u}}{\partial \nu} dS = 0$ $(i = 1, \dots, i_0)$, where each $\tilde{c}_i$ is a constant with $0 < \tilde{c}_i < 1$.

Set $\ell_0 = \lambda_2(\hat{\Gamma}(\gamma_1, \gamma_{no+1}; G))$ and $u(x) = \ell_0 \tilde{u}(x)$. Let $c_1 < c_2 < \cdots < c_{j_0}$ be all the different values of $\ell_0 \tilde{c}_1, \dots, \ell_0 \tilde{c}_{i_0}$. Take an $\varepsilon > 0$ such that

(1) $c_{j-1} + \varepsilon < c_j - \varepsilon$ $(j = 1, \dots, j_0 + 1)$, where $c_0 = 0$ and $c_{j_0+1} = \ell_0$,

(2) $\sum_{j=1}^{j_0+1} (c_j - c_{j-1} - 2\varepsilon) \geqq \ell_0 - 1$,

(3) $u$ has no critical points on level surfaces $\{x;\, u(x) = c_{j-1} + \varepsilon\}$ and $\{x;\, u(x) = c_j - \varepsilon\}$ $(j = 1, \dots, j_0 + 1)$.

Set $R_j = \{x;\, c_{j-1} + \varepsilon < u(x) < c_j - \varepsilon\}$ $(j = 1, \dots, j_0 + 1)$. Since $u$ has no critical points on the level surface $\alpha = \{x;\, u(x) = c_{j-1} + \varepsilon\}$, it consists of a finite number of mutually disjoint analytic surfaces. We see easily that each component of $\alpha$ divides $R^N$ into a bounded domain containing $\gamma_{no+1}$ and an unbounded domain containing $\gamma_1$. Since $u = \text{const.}$ on $\beta_i$ and $\int_{\beta_i} \frac{\partial u}{\partial \nu} dS = 0$, by using Green's formula we see that $\alpha$ consists of a single analytic surface such that $R^N - \alpha$ consists of a bounded domain $\Omega_0$ containing $\gamma_{no+1}$ and an unbounded domain containing $\gamma_1$. By similar arguments we see that the level surface $\alpha' = \{x;\, u(x) = c_j - \varepsilon\}$ is a single analytic surface such that $R^N - \alpha'$ consists of a bounded domain $\Omega'_0$ containing $\gamma_{no+1}$ and an unbounded domain containing $\gamma_1$. Since $c_{j-1} + \varepsilon < u(x) < c_j - \varepsilon$ for any $x \in \Omega'_0 - \bar{\Omega}_0$, we conclude that $R_j$ is a ring domain. It is clear that the sequence $\{R_j\}_{j=1}^{j_0+1}$ satisfies the conditions (1) and (2) in theorem.

Since $u_0 = (u - c_{j-1} - \varepsilon)/(c_j - c_{j-1} - 2\varepsilon)$ is harmonic on $R_j$, $u_0 = 0$ on $\alpha$ and $u_0 = 1$ on $\alpha'$, we have

$$M_2(\Gamma_j) = \int_{R_j} |\nabla u_0|^2 dx = \int_{\alpha'} \frac{\partial u_0}{\partial \nu} dS = (c_j - c_{j-1} - 2\varepsilon)^{-1} \int_{\alpha'} \frac{\partial u}{\partial \nu} dS$$

(see, e.g., [10, Theorem 4] and [11, Theorem 3.8]). On the other hand, by (4.4) we have

$$\ell_0^{-1} = \int_G |\nabla \tilde{u}|^2 dx.$$

By using Green's formula we see that

$$\int_{\alpha'} \frac{\partial u}{\partial \nu} dS = 1.$$

Therefore we have $M_2(\Gamma_j)=(c_j-c_{j-1}-2\varepsilon)^{-1}$. From this we derive that

$$\sum_{j=1}^{j_0+1}\lambda_2(\Gamma_j)=\sum_{j=1}^{j_0+1}(c_j-c_{j-1}-2\varepsilon)\geqq \ell_0-1\geqq \ell,$$

so that we obtain the required results.

EXAMPLE 2. Set $R_n=\{x; (2n+1)^{1/(2-N)}<|x|<(2n)^{1/(2-N)}\}$ $(n=1, 2,\dots)$. Let $D$ be a domain such that $D\supset R_n$ for all $n$ and $\gamma=\{0\}$ is a boundary component of $D$. It is well known that $\lambda_2(\Gamma_n)=(\omega_N(N-2))^{-1}$. By Theorem 6, $\gamma$ is weak.

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