HIROSHIMA MATH. J.
13 (1983), 273–294

Numerical approximations to interface curves for a porous media equation

Kenji TOMOEDA and Masayasu MIMURA

(Received September 17, 1982)

1. Introduction

We are concerned with difference approximations to the initial value problem for the one dimensional porous media equation described by

$$(1.1) \qquad v_t = (v^m)_{xx}, \quad t \in (0, +\infty), \quad x \in \boldsymbol{R}^1 \qquad (m > 1)$$

with an initial value

$$(1.2) \qquad v(0, x) = v^0(x), \quad x \in \boldsymbol{R}^1,$$

where v represents the density of an ideal gas flowing in a homogeneous porous medium which occupies all of $\boldsymbol{R}^1$. (1.1) is obtained by combining the equation of state, conservation of mass and Darcy's law ([8]). Physically, v^{m-1} is the pressure of the gas and $(v^{m-1})_x$ is the velocity. From the reason that the diffusion rate of (1.1), mv^{m-1} vanishes at points where $v=0$ because of $m>1$, (1.1) exhibits an interesting phenomenon of the finite speed of propagation of disturbances. In other words, when $v^0(x)$ has compact support, a solution $v(t, x)$ of (1.1), (1.2) has also compact support for any $t>0$. As an example to illustrate this property, we may show an explicit solution of (1.1) due to Barenblatt and Pattle ([3], [10]). This is of the form

$$(1.3) \qquad v(t, x) = \begin{cases} \dfrac{1}{\lambda(t)}\left\{1-\left(\dfrac{x}{\lambda(t)}\right)^2\right\}^{1/(m-1)} & \text{for} \quad |x| \leqq \lambda(t),\ t \geqq 0, \\ 0 & \text{for} \quad |x| \geqq \lambda(t),\ t \geqq 0, \end{cases}$$

where

$$(1.4) \qquad \lambda(t) = \left\{\frac{2m(m+1)}{m-1}(t+1)\right\}^{1/(m+1)} \qquad \text{for} \quad t \geqq 0$$

(see Figs. 1 and 2).

We note here that the solution (1.3) is not a classical solution for $m \geqq 2$, in the sense that the trasition from the region of medium which contains gas ($v>0$) to the one which does not ($v=0$) is not smooth. The boundary, (1.4) in (1.3), is called the interface.

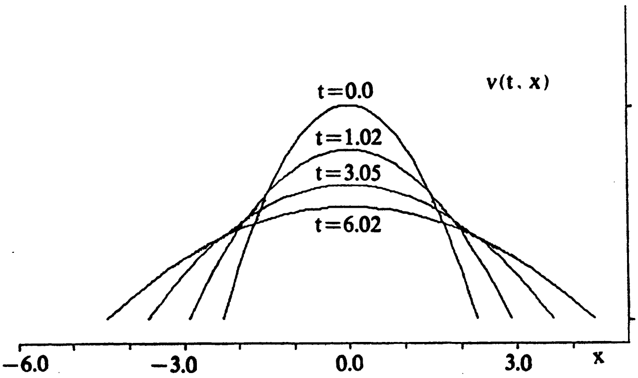

Fig. 1. Barenblatt-Pattle's solution (1.3) when $m=2$.

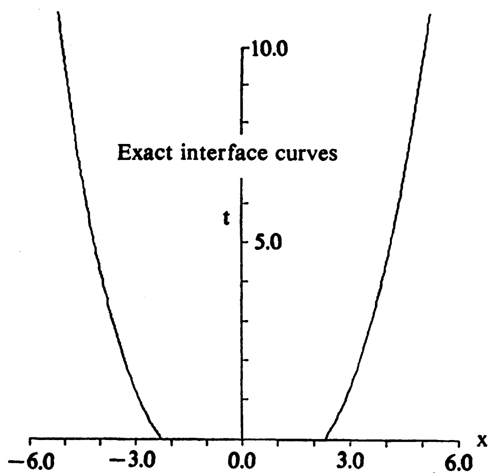

Fig. 2. Interface curves $|x|=\lambda(t)$ of Barenblatt-Pattle's solution (1.4) when $m=2$.

From analytical point of views, the existence and uniqueness of weak solutions of the problem (1.1), (1.2) were studied by many authors (for instance, [1], [7], [9]). The determination of interface curves is also an important problem in porous medium flow. When $v^0(x)>0$ on (a_1, a_2) and $v^0(x)=0$ on $\boldsymbol{R}^1\backslash(a_1, a_2)$, the curves $\lambda_j(t)$ $(j=1, 2)$ are governed by

$$(1.5)\qquad \frac{d}{dt}\lambda_j(t) = -(m/(m-1))\lim_{x\to\lambda_j(t)}(v^{m-1})_x(t, x) \qquad (j=1, 2),$$

$$(1.6)\qquad \lambda_j(0) = a_j \qquad (j=1, 2).$$

It is proved in [6] that there exist $t_j^*\in[0, +\infty)(j=1, 2)$ such that

$$\lambda_j(t)\begin{cases} = a_j & \text{for} \quad t\in[0, t_j^*], \\ \text{is strictly monotone} & \text{for} \quad t\in(t_j^*, +\infty). \end{cases}$$

The time t_j^* is called the waiting time when an initially stationary interface begins to move. The determination of the waiting time is interesting from both physical and mathematical points of view. Aronson, Caffarelli and Kamin [2] estimated the time t_j^* for rather general initial functions. Suppose that the initial function $v^0(x)$ is of the form

$$\frac{m}{m-1}(v^0)^{m-1} = \begin{cases} (1-\theta)\sin^2 x + \theta \sin^4 x & \text{for } x \in (-\pi, 0), \\ 0 & \text{for } x \notin (-\pi, 0), \end{cases}$$

where $\theta \in [0, 1]$. If $\theta \in [0, 1/4]$, then $t_1^* = t_2^* = t_m/(1-\theta)$ with $t_m = 1/2(m+1)$. On the other hand, if $\theta \in (1/4, 1)$, then $t_m/\beta \leqq t_1^* = t_2^* \leqq t_m/(1-\theta)$, where β is obtained by solving the nonlinear equations $\beta y^2 = v^0(y)$ and $2\beta y = (v^0(y))'$. Thus, in the case $\theta \in (1/4, 1)$, t_j^* can not be calculated. This motivates us to develop numerical methods to determine interface curves which enable us to compute numerically the waiting time.

Meanwhile, numerical methods for (1.1), (1.2) have been investigated ([4], [5]). In particular, the difference scheme by Graveleau and Jamet [5] (Graveleau-Jamet scheme) absorbs much interest, though they are not concerned with numerical interfaces. We let briefly mention their method. It approximates the following problem instead of (1.1), (1.2) by setting $u = v^{m-1}$:

$$u_t = muu_{xx} + \frac{m}{m-1}(u_x)^2, \tag{1.7}$$

$$u(0, x) = u^0(x) = (v^0(x))^{m-1}. \tag{1.8}$$

The difference scheme is constructed based on splitting of the operator $Au = muu_{xx} + a(u_x)^2$ with $a = m/(m-1)$ into two parts

$$Pu = muu_{xx}, \tag{1.9}$$

$$Hu = a(u_x)^2. \tag{1.10}$$

In the neighborhood of interface curves, one may expect that the hyperbolic term (1.10) in (1.7) is dominant rather than the parabolic term (1.9), because u is sufficiently small there. However, as Graveleau-Jamet scheme includes an artificial viscosity in approximating (1.10), it does not give good approximations in realizing interface curves (Figs. 3 and 4).

In this paper, in order to construct a difference scheme which approximates not only a solution but also interfaces, we present a scheme different from Graveleau-Jamet scheme in approximating the hyperbolic equation

$$u_t = Hu. \tag{1.11}$$

Since the equation (1.11) reduces to the Burgers equation by differentiating it with

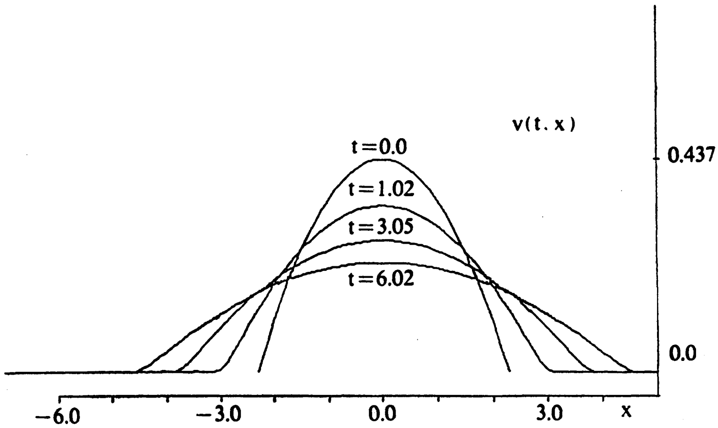

Fig. 3. Numerical solution of Graveleau-Jamet scheme with $h=0.14309$ when $m=2$.

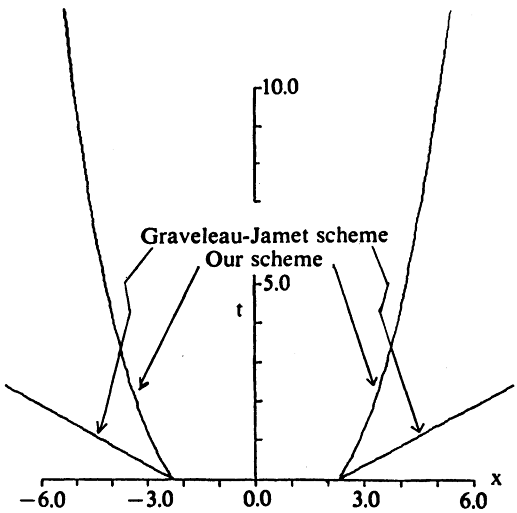

Fig. 4. Numerical interface curves when $m=2$.

respect to x, we apply the Rankine-Hugoniot jump condition so as to determine interface curves. As will be shown, this method excludes numerical viscosities so that numerical interface curves are realizable.

In Sections 3–5, we obtain stability and convergence of our scheme, following the argument by Graveleau and Jamet (Theorems 4.1 and 7.2 in [5]). Unfortunately, we have not been able to prove the convergence of numerical interface curves. Finally it is shown in Section 6 that some numerical solutions give good profiles to the exact ones as well as interface curves (Figs. 4 and 5).

Acknowledgement. The authors are grateful for the discussion with T. Nagai in the development of this work.

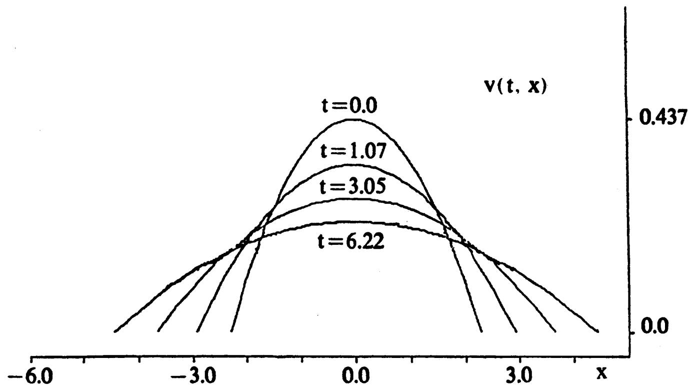

Fig. 5. Numerical solution of our scheme with $h=0.125$ when $m=2$.

2. Difference schemes

Let h be a positive number and $\{t_n\}$ be an increasing sequence. This sequence will be determined later. We denote by $u_i^n(\geqq 0)$ the difference approximation for the solution of (1.7), (1.8) at $(t, x)=(t_n, ih)$, where $n(\geqq 0)$ and i are integers. For $u_\cdot^n$, let l_n and r_n be points such that

$$u_i^n = 0 \qquad \text{for all} \quad ih \in \boldsymbol{R}^1 \backslash (l_n, r_n)$$

and $u_{L_n}^n \neq 0$, $u_{R_n}^n \neq 0$ with $L_n=[l_n/h]+1$, $R_n=-\{[-r_n/h]+1\}$, where $[x]$ means the greatest integer not exceeding x. We may define a *numerical left* (resp. *right*) *interface curve* by piecewise-linearly interpolating points $(t_n, l_n)(n\geqq 0)$ (resp. (t_n, r_n)). We introduce a piecewise linear function $u_h^n(x)$ constructed by

$$(2.1)\quad \begin{cases} u_h^n(ih+\theta h) = (1-\theta)u_i^n + \theta u_{i+1}^n \\ \qquad\qquad\qquad\qquad \text{for all} \quad \theta\in[0, 1] \quad \text{and} \quad i\in\{L_n,\dots, R_n-1\}, \\ u_h^n(l_n+\theta(L_nh-l_n)) = \theta u_{L_n}^n \qquad \text{for all} \quad \theta\in[0, 1], \\ u_h^n(R_nh+\theta(r_n-R_nh)) = (1-\theta)u_{R_n}^n \qquad \text{for all} \quad \theta\in[0, 1], \\ u_h^n(x) = 0 \qquad \text{for all} \quad x\in \boldsymbol{R}^1\backslash(l_n, r_n). \end{cases}$$

Here we construct a difference approximation for (1.7) with the initial value at $t=t_n$, $u(t_n, x)=u_h^n(x)$. First we determine the sequence $\{t_n\}$ as follows: Let us consider the equation (1.7) in the absence of the parabolic term Pu, so that, putting $w=u_x$, we obtain

$$(2.2)\qquad w_t = a(w^2)_x \qquad (t > t_n),$$

which is called the Burgers equation. The initial value at $t=t_n$ is

$$w(t_n, x) = w^n(x) \equiv (u_h^n(x))_x. \tag{2.3}$$

It is already known that for small $t-t_n(>0)$ the solution $w(t, x)$ of (2.2) consists of constant states 0, $w^n(l_n+0)$, $w^n(L_n h+0),\dots, w_n(R_n h+0)$, 0, which are separated by shock waves and are connected by rarefaction waves. More precisely, if $w^n(ih-0)<w^n(ih+0)$ for some $i\in\{L_n,\dots, R_n\}$, w has a shock on the line

$$y_i(t) = ih - a(w^n(ih-0) + w^n(ih+0))(t-t_n). \tag{2.4}$$

This is well known as the Rankine-Hugoniot jump condition. If $w^n(jh-0)>w^n(jh+0)$ for some $j\in\{L_n,\dots, R_n\}$, w has a rarefaction wave which connects two states $w^n(jh-0)$ and $w^n(jh+0)$ on the wedge determined by two characteristics

$$z_{j1}(t) = jh - 2aw^n(jh-0)(t-t_n) \text{ and } z_{j2}(t) = jh - 2aw^n(jh+0)(t-t_n). \tag{2.5}$$

It is clear that $w(t, x)$ has shocks on two lines

$$y_{l_n}(t) = l_n - aw^n(l_n+0)(t-t_n) \text{ and } y_{r_n}(t) = r_n - aw^n(r_n-0)(t-t_n). \tag{2.6}$$

Under the above consideration we take the variable time step k_{n+1} as the largest number of k satisfying the following two conditions:

i) The lines $y_i(t)$, $z_{j1}(t)$, $z_{j2}(t)$, $y_{l_n}(t)$ and $y_{r_n}(t)$ do not intersect each other on (t_n, t_n+k) and

$$\begin{cases} |y_i(t_n+k) - ih| \leqq h, \quad |z_{js}(t_n+k) - jh| \leqq h, \qquad (s=1, 2), \\ |y_{l_n}(t_n+k) - l_n| \leqq h, \quad |y_{r_n}(t_n+k) - r_n| \leqq h; \end{cases} \tag{2.7}$$

ii) $k \leqq \sqrt{h}$,

which indicates that k_{n+1} tends to zero as $h\to 0$.

Along this way, we define the sequence $\{t_n\}$ by

$$t_{n+1} = t_n + k_{n+1} \qquad (n \geqq 0), \tag{2.8}$$

and put $T_\infty = \lim_{n\to\infty} t_n$.

Thus, integrating the solution $w(t, x)$ with respect to x, we obtain the difference approximation to the solution of (1.11).

We next construct the difference scheme for the nonlinear parabolic equation $u_t=Pu$ by modifying Graveleau-Jamet scheme in the neighborhood of interface curves. Let μ_{n+1} be some positive integer and $\tilde{k}_{n+1}=k_{n+1}/\mu_{n+1}$ (see (3.9)).

Suppose u_i^n are known for all $i\in \boldsymbol{Z}$, where $\boldsymbol{Z}$ denotes the set of integers.

Step 1: Compute

$$\begin{cases} l_{n+1} = y_{l_n}(t_{n+1}), \quad r_{n+1} = y_{r_n}(t_{n+1}), \\ u_i^{n+1,0} = \displaystyle\int_{-\infty}^{ih} w(t_{n+1}, x)dx \qquad \text{for all} \quad i \in \{L_{n+1},\dots, R_{n+1}\}, \end{cases} \tag{2.9}$$

where $L_{n+1}=[l_{n+1}/h]+1$ and $R_{n+1}=-\{[-r_{n+1}/h]+1\}$. These are computed by numerically solving $u_t=Hu$.

Step 2: Compute μ_{n+1} intermediate functions $u_{\cdot}^{n+1,r}(1\leqq r\leqq\mu_{n+1})$ by the relations

$$(2.10)\quad \begin{cases} (u_i^{n+1,r+1}-u_i^{n+1,r})/\tilde{k}_{n+1}=(P_h u_{\cdot}^{n+1,r})_i & (0\leqq r\leqq \mu_{n+1}-1) \\ \qquad\qquad\qquad\qquad\qquad \text{for all} \quad i\in\{L_{n+1}+1,\dots, R_{n+1}-1\}, \\ u_{L_{n+1}}^{n+1,r}=u_{L_{n+1}}^{n+1,0}, \quad u_{R_{n+1}}^{n+1,r}=u_{R_{n+1}}^{n+1,0} & (1\leqq r\leqq \mu_{n+1}), \\ u_i^{n+1,r}=0 \qquad (1\leqq r\leqq \mu_{n+1}) & \text{for all} \quad i\in \boldsymbol{Z}\backslash\{L_{n+1},\dots, R_{n+1}\}, \end{cases}$$

where

$$(P_h u_{\cdot}^{n+1,r})_i = m u_i^{n+1,r}(u_{i+1}^{n+1,r}-2u_i^{n+1,r}+u_{i-1}^{n+1,r})/h^2.$$

Step 3: Put

$$(2.11)\qquad u_i^{n+1}=u_i^{n+1,\mu_{n+1}} \qquad \text{for all} \quad i\in \boldsymbol{Z}.$$

Thus, it turns out that u_i^{n+1} can be computed by u_i^n for all $i\in \boldsymbol{Z}$.

3. Stability

By the difference approximations $u_i^{n+1,r}(0\leqq r\leqq\mu_{n+1})$ computed in Steps 1 and 2, let us introduce piecewise linear functions $u_h^{n+1,r}(x)$ defined by the similar way to (2.1), replacing l_n, r_n and u_h^n by l_{n+1}, r_{n+1} and $u_h^{n+1,r}$, respectively.

To show the stability of the scheme consisting of (2.9)–(2.11) (Steps 1–3), we impose Condition A on the initial value and then show two lemmas.

Condition A. i) $u^0(x)=(v^0(x))^{m-1}$ is a continuous function satisfying

$$(3.1)\qquad \begin{cases} u^0(x)>0 & \text{on } (a_1, a_2), \\ u^0(x)=0 & \text{on } \boldsymbol{R}^1\backslash(a_1, a_2); \end{cases}$$

ii) There exist constants $C_j(j=0, 1, 2, 3)$ such that

$$(3.2)\qquad \begin{cases} \|u^0\|_\infty \leqq C_0, & \|u_x^0\|_\infty \leqq C_1, \\ V(u^0)\leqq C_2, & V(u_x^0)\leqq C_3, \end{cases}$$

where $\|\cdot\|_\infty=\|\cdot\|_{L^\infty(\boldsymbol{R}^1)}$ and $V(f)$ denotes the total variation of $f(x)$.

We first take

$$t_0=0, \quad l_0=a_1, \quad r_0=a_2 \quad \text{and} \quad u_i^0=u^0(ih).$$

Lemma 3.1. *Assume Condition* A. *Then it holds that for all* $h>0$ *and* $n\geqq 0$, (3.3) $$0 < u_h^{n+1,0}(x) \leqq C_0 \quad on \ (l_{n+1}, r_{n+1});$$

(3.4) $$u_h^{n+1,0}(x) = 0 \quad on \ \boldsymbol{R}^1 \backslash (l_{n+1}, r_{n+1});$$

(3.5) $$\|(u_h^{n+1,0})_x\|_\infty \leqq C_1;$$

(3.6) $$\|(u_h^{n+1,0})_x\|_1 \leqq C_2;$$

(3.7) $$V((u_h^{n+1,0})_x) \leqq C_3;$$

(3.8) $$\|(u_h^{n+1,0} - u_h^n)/k_{n+1}\|_1 \leqq aC_1C_2,$$

where $\|\cdot\|_1 = \|\cdot\|_{L^1(\boldsymbol{R}^1)}$.

LEMMA 3.2. *Under Condition* A, *let* $h>0$ *and* $\tilde{k}_{n+1}>0$ *satisfy*

(3.9) $$2mC_0(\tilde{k}_{n+1}/h^2) \leqq 1 \qquad for\ all \quad n \geqq 0.$$

Then $u_h^{n+1,r+1}(x)$ $(n \geqq 0;\ 0 \leqq r \leqq \mu_{n+1}-1)$ *satisfy the estimates* (3.3)–(3.7) *in which* $u_h^{n+1,0}$ *are replaced by* $u_h^{n+1,r+1}$ *and*

(3.10) $$\|(u_h^{n+1,r+1} - u_h^{n+1,r})/\tilde{k}_{n+1}\|_1 \leqq mC_0C_3 \qquad for\ all \quad n \geqq 0.$$

The proofs of Lemmas 3.1 and 3.2 will be shown in Section 5.

The stability of the scheme (2.9)–(2.11) easily follows from these lemmas.

THEOREM 3.1. *Assume Condition* A, *and let* h *and* $\tilde{k}_{n+1}$ $(n \geqq 0)$ *satisfy* (3.9). *Then* $u_h^n(x)$ $(n \geqq 0)$ *satisfy the estimates* (3.3)–(3.7) *in which* $u_h^{n+1,0}$ *are replaced by* u_h^n *and*

(3.11) $$\|(u_h^{n+1} - u_h^n)/k_{n+1}\|_1 \leqq mC_0C_3 + aC_1C_2 \qquad for\ all \quad n \geqq 0.$$

Moreover,

(3.12) $$\lim_{n\to\infty} l_n = -\infty \quad and \quad \lim_{n\to\infty} r_n = +\infty,$$

and

(3.13) $$T_\infty = +\infty.$$

PROOF. The estimates (3.3)–(3.7) with $u_h^{n+1,0} = u_h^n$ follows from Lemmas 3.1 and 3.2. By (3.8) and (3.10), we have

$$\|(u_h^{n+1} - u_h^n)/k_{n+1}\|_1 \leqq (1/\mu_{n+1})\sum_{r=0}^{\mu_{n+1}-1}\|(u_h^{n+1,r+1} - u_h^{n+1,r})/\tilde{k}_{n+1}\|_1$$
$$+ \|(u_h^{n+1,0} - u_h^n)/k_{n+1}\|_1 \leqq mC_0C_3 + aC_1C_2,$$

which yields (3.11).

We prove (3.12). We put

$$l_\infty = \lim_{n\to\infty} l_n, \qquad r_\infty = \lim_{n\to\infty} r_n.$$

Since $\{l_n\}$ (resp. $\{r_n\}$) is a strictly monotone decreasing (resp. increasing) sequence, it suffices to show that the following three cases do not occur:

Case (a) $l_\infty > -\infty$ and $r_\infty < +\infty$;

Case (b) $l_\infty > -\infty$ and $r_\infty = +\infty$;

Case (c) $l_\infty = -\infty$ and $r_\infty < +\infty$.

In Case (a) there exists an integer N such that

$$(3.14)\qquad (L_N - 1)h \leqq l_\infty < l_n < L_N h$$

and

$$(3.15)\qquad (R_N + 1)h \geqq r_\infty > r_n > R_N h \qquad \text{for all} \quad n \geqq N,$$

which mean

$$(3.16)\qquad L_n = L_N \quad \text{and} \quad R_n = R_N \qquad \text{for all} \quad n \geqq N.$$

Taking the properties (3.14)–(3.16) into consideration, we find from (2.9)–(2.11) that

$$(3.17)\qquad u^n_{L_n} \geqq u^N_{L_N} \quad \text{and} \quad u^n_{R_n} \geqq u^N_{R_N} \qquad \text{for all} \quad n \geqq N \text{ (see (5.1)–(5.5))}.$$

From the determination of k_{n+1}, it follows that

$$(3.18)\qquad k_{n+1} \geqq k^*_n \qquad \text{for all} \quad n \geqq 0,$$

where

$$k^*_n = \min\{(L_n h - l_n)/(4aC_1),\ (r_n - R_n h)/(4aC_1),\ \sqrt{h}\},$$

and that by (3.16)

$$(3.19)\qquad k^*_n \geqq k^*_N \qquad \text{for all} \quad n \geqq N.$$

Then, by using (2.6), (2.9) and (3.17)–(3.19), we obtain

$$(3.20)\quad l_{n+1} - l_n = -a\{u^n_{L_n}/(L_n h - l_n)\}k_{n+1} \leqq -a(u^N_{L_N}/h)k^*_N < 0,$$

$$(3.21)\quad r_{n+1} - r_n = a\{u^n_{R_n}/(r_n - R_n h)\}k_{n+1} \geqq a(u^N_{R_N}/h)k^*_N > 0 \qquad \text{for all} \quad n \geqq N,$$

which contradict the fact that $l_{n+1} - l_n$ and $r_{n+1} - r_n$ tend to zero as $n \to \infty$.

In Case (b), let N be an integer satisfying (3.14). From the definition of r_n we may have

$$0 < r_{n+1} - r_n \leqq h/2,$$

which means that there exists a subsequence $\{r_{n_\nu}\}$ of $\{r_n\}$ such that

$$|r_{n_\nu} - R_{n_\nu}| > h/2 \qquad \text{for all} \quad \nu.$$

Putting

$$k_N^* = \min\{(L_N h - l_N)/(4aC_1),\ h/(8aC_1),\ \sqrt{h}\},$$

we have

$$k_{n_\nu+1} \geqq k_N^* \qquad \text{for all} \quad n_\nu \geqq N.$$

Hence it follows

$$l_{n_\nu+1} - l_{n_\nu} \leqq -a(u_{L_N}^N/h)k_N^* < 0 \qquad \text{for all} \quad n_\nu \geqq N, \tag{3.22}$$

which is a contradiction. Similarly we can show that Case (c) also does not occur. Thus (3.12) is shown. Finally we show (3.13). By (2.6) and (2.9) we have

$$l_0 - l_{n+1} = \sum_{i=0}^n (l_i - l_{i+1}) \leqq aC_1 \sum_{i=0}^n k_{i+1} = aC_1 T_{n+1},$$

which leads to (3.13) by (3.12). Thus Theorem 3.1 is proved.

4. Convergence

In this section we show the convergence of difference approximations u_i^{n+1} to the exact solution $u(t, x)$. We first introduce definitions of weak solutions of the problems (1.1), (1.2) and (1.7), (1.8).

DEFINITION 1. A function $v(x, t) \geqq 0$ defined on $\mathscr{H} = \{(t, x) \in (0, \infty) \times \boldsymbol{R}^1\}$ is a weak solution of the problem (1.1), (1.2) if $v \in C^0(\bar{\mathscr{H}}) \cap L^\infty(\mathscr{H})$, $(v^m)_x \in L^\infty(\mathscr{H})$, $v(0, x) = v^0(x)$ and

$$\int v\phi_t dxdt = \int (v^m)_x \phi_x dxdt \qquad \text{for all} \quad \phi \in \mathscr{D}(\mathscr{H}). \tag{4.1}$$

DEFINITION 2. A function $u(t, x) \geqq 0$ defined on $\mathscr{H}$ is a weak solution of the problem (1.7), (1.8) if $u \in C^0(\bar{\mathscr{H}}) \cap L^\infty(\mathscr{H})$, $u_x \in L^\infty(\mathscr{H})$, $u(0, x) = u^0(x)$ and

$$\int u\phi_t dxdt = \int \{muu_x\phi_x + (m-a)(u_x)^2\phi\} dxdt \qquad \text{for all} \quad \phi \in \mathscr{D}(\mathscr{H}). \tag{4.2}$$

To state convergence theorems, we extend the region of definition of the difference approximations u_i^n computed by (2.9)–(2.11) to the region $\{(t, x) \in [0, T_\infty) \times \boldsymbol{R}^1\}$ in a way that

$$u_h(t, x) = u_h^n(x) \qquad \text{for all} \quad t \in [t_n, t_{n+1}) \text{ and } n \geqq 0.$$

Then we have

THEOREM 4.1. *Assume Condition* A, *and let* $\{h\}$ *be the sequence which tends to zero, and assume that the stability condition* (3.9) *holds for each* h *and* $\tilde{k}_{n+1}$ $(n \geqq 0)$. *Then there exist a subsequence* $\{h'\}$ *of* $\{h\}$ *and a function* U *with the following properties.*

i) $U \in C^0(\mathscr{H}) \cap L^\infty(\mathscr{H})$, $U_x \in L^\infty(\mathscr{H})$;

ii) $U(0, x) = u^0(x)$ *for all* $x \in \boldsymbol{R}^1$;

iii) *As* $h' \to 0$,

$$\|u_{h'} - U\|_{L^\infty(G)} \longrightarrow 0, \tag{4.3}$$

$$\|(u_{h'})_x - U_x\|_{L^p(G)} \longrightarrow 0 \qquad (1 \leqq p < +\infty), \tag{4.4}$$

where G is any bounded subdomain of $\mathscr{H}$.

iv) *U is a weak solution of the problem* (1.7), (1.8).

PROOF. By following Graveleau and Jamet (Theorem 6.1 in [5]), the properties i), ii) and iii) can be proved. We only note here that the estimates obtained in Theorem 3.1 play an important role in proving these properties.

We now show the property iv). For this end, it suffices to prove that U satisfies the integral relation (4.2). In the following, let h take the value belonging to the extracted subsequence. From (2.9)–(2.11) we have

$$(u_i^{n+1} - u_i^n)/k_{n+1} = (1/\mu_{n+1}) \textstyle\sum_{r=0}^{\mu_{n+1}-1} (u_i^{n+1,r+1} - u_i^{n+1,r})/\tilde{k}_{n+1}$$
$$+ (u_i^{n+1,0} - u_i^n)/k_{n+1}.$$

Let $\phi \in \mathscr{D}(\mathscr{H})$. Multiplying both sides by $hk_{n+1}\phi_i^n$ and summing for all $i \in \boldsymbol{Z}$ and $n \geqq 0$, we have by summation by parts

$$-\textstyle\sum_{n,i} hk_{n+1} u_i^{n+1}(\phi_i^{n+1} - \phi_i^n)/k_{n+1} = A_h + B_h \tag{4.5}$$

with $\phi_i^n = \phi(t_n, ih)$, where A_h and B_h are represented by

$$A_h = \textstyle\sum_{n,i} hk_{n+1}\phi_i^n(1/\mu_{n+1})\{\sum_{r=0}^{\mu_{n+1}-1} (u_i^{n+1,r+1} - u_i^{n+1,r})/\tilde{k}_{n+1}\} \tag{4.6}$$

and

$$B_h = \textstyle\sum_{n,i} hk_{n+1}\phi_i^n(u_i^{n+1,0} - u_i^n)/k_{n+1}, \tag{4.7}$$

respectively. In order to show that U satisfies the integral relation (4.2), we prepare the following Lemmas 4.1–4.4, and then obtain the theorem. We first define $\tilde{u}_h^n(x)$ by

$$\tilde{u}_h^n(ih + \theta h) = (1-\theta)u_i^n + \theta u_{i+1}^n \qquad \text{for all} \quad i \in \boldsymbol{Z} \text{ and } \theta \in [0, 1),$$

and then define $\tilde{u}_h(t, x)$ by

$$\tilde{u}_h(t, x) = \tilde{u}_h^n(x) \qquad \text{for all} \quad t \in [t_n, t_{n+1}) \text{ and } n \geqq 0.$$

LEMMA 4.1. *It holds that*

$$\|\tilde{u}_h - u_h\|_{L^\infty(\mathscr{H})} \longrightarrow 0 \quad as \ h \longrightarrow 0, \tag{4.8}$$

(4.9) $$|\sum_{n,i} hk_{n+1}u_i^{n+1}(\phi_i^{n+1}-\phi_i^n)/k_{n+1} - \int \tilde{u}_h\phi_t dxdt| \longrightarrow 0 \quad as \quad h \longrightarrow 0,$$

(4.10) $$\int \tilde{u}_h\phi_t dxdt \longrightarrow \int U\phi_t dxdt \quad as \quad h \longrightarrow 0.$$

LEMMA 4.2. *For arbitrary integers γ_n $(n\geqq 1)$ satisfying $0\leqq\gamma_n\leqq\mu_n$ $(n\geqq 1)$, let $u_h^{(\gamma)}(t, x)$ be the function defined by*

$$u_h^{(\gamma)}(t_n+\theta k_{n+1}, x) = u_h^{n+1,\gamma_{n+1}}(x)$$

for all $\theta\in[0, 1)$ and $n\geqq 0$, where $\gamma=(\gamma_1, \gamma_2,\dots, \gamma_n,\dots)$. Then

(4.11) $$\max_\gamma \|u_h^{(\gamma)} - U\|_{L^\infty(G)} \longrightarrow 0 \quad as \quad h \longrightarrow 0,$$

(4.12) $$\max_\gamma \|(u_h^{(\gamma)})_x - U_x\|_{L^p(G)} \longrightarrow 0 \quad as \quad h \longrightarrow 0 \qquad (1 \leqq p < +\infty),$$

where G is any bounded subdomain of $\mathscr{H}$.

LEMMA 4.3. *Let*

(4.13) $$C_h = -\sum_n hk_{n+1}(1/\mu_{n+1}) \sum_{r=0}^{\mu_{n+1}-1} [\sum_{i=L_{n+1}+1}^{R_{n+1}-2} \{mu_i^{n+1,r}(\delta u_i^{n+1,r})(\delta\phi_i^n) + m(\delta u_i^{n+1,r})^2\phi_{i+1}^n\}],$$

(4.14) $$D_h = -\sum_n (1/\mu_{n+1}) \sum_{r=0}^{\mu_{n+1}-1} \int_{t_n}^{t_{n+1}} \int_{l_{n+1}}^{r_{n+1}} [mu_h^{(p(r))}(u_h^{(p(r))})_x\phi_x + m\{(u_h^{(p(r))})_x\}^2\phi] dxdt,$$

where

$$\delta f_i^n = (f_{i+1}^n - f_i^n)/h \quad and \quad p(r) = (p_{r1}, p_{r2},\dots, p_{rn},\dots)$$

with

$$p_{rn} = \begin{cases} r & if \quad r \leqq \mu_n, \\ \mu_n & if \quad r > \mu_n. \end{cases}$$

Then it follows that

(4.15) $$|A_h - C_h| \longrightarrow 0 \quad as \quad h \longrightarrow 0,$$

(4.16) $$|C_h - D_h| \longrightarrow 0 \quad as \quad h \longrightarrow 0,$$

(4.17) $$D_h \longrightarrow -\int \{mUU_x\phi_x + m(U_x)^2\phi\}dxdt \quad as \quad h \longrightarrow 0.$$

LEMMA 4.4. *Let*

(4.18) $$E_h = \sum_n hk_{n+1} \sum_{i=L_n}^{R_n-1} a(\delta u_i^n)^2\phi_i^n,$$

and

(4.19) $$F_h = \sum_n \int_{t_n}^{t_{n+1}} \int_{l_n}^{r_n} a((u_h)_x)^2\phi dxdt.$$

Then

$$|B_h - E_h| \longrightarrow 0 \quad as \quad h \longrightarrow 0, \tag{4.20}$$

$$|E_h - F_h| \longrightarrow 0 \quad as \quad h \longrightarrow 0, \tag{4.21}$$

$$F_h \longrightarrow \int a(U_x)^2 \phi dxdt \quad as \quad h \longrightarrow 0. \tag{4.22}$$

Following the argument by Graveleau and Jamet (see the proof of Theorem 7.2 in [5]), we obtain

THEOREM 4.2. *Assume Condition* A. *Let* $\{h\}$ *be the sequence which tends to zero, and assume that the stability condition* (3.9) *holds for each* h *and* $\tilde{k}_{n+1}$ $(n \geqq 0)$. *Then* $(u_h)^{1/(m-1)}$ *converges uniformly in any bounded subdomain of* $\mathscr{H}$ *to the unique weak solution* v *of* (1.1)–(1.2).

5. Proofs of Lemmas

5.1. Proof of Lemma 3.1

In the initial value problem (2.2), (2.3) it can be shown by the conservation law that $\int_{-\infty}^{\infty} w(t, x)dx$ is independent of t. Using this fact we can write $u_i^{n+1,0}$ as follows:

Case (i) When either $y_i(k_{n+1}) < ih$ or $z_{i1}(k_{n+1}) \leqq z_{i2}(k_{n+1}) < ih$ holds for some $i \in \{L_n, \ldots, R_n - 1\}$,

$$u_i^{n+1,0} = u_i^n + a(\delta u_i^n)^2 k_{n+1}; \tag{5.1}$$

Case (ii) When either $y_i(k_{n+1}) \geqq ih$ or $ih \leqq z_{i1}(k_{n+1}) \leqq z_{i2}(k_{n+1})$ holds for some $i \in \{L_n + 1, \ldots, R_n\}$,

$$u_i^{n+1,0} = u_i^n + a(\delta u_{i-1}^n)^2 k_{n+1}, \tag{5.2}$$

Case (iii) When $z_{i1}(k_{n+1}) < ih < z_{i2}(k_{n+1})$ holds for some $i \in \{L_n, \ldots, R_n\}$,

$$u_i^{n+1,0} = u_i^n; \tag{5.3}$$

Case (iv) At the points $(t_{n+1}, L_{n+1}h)$ and $(t_{n+1}, R_{n+1}h)$

$$u_{L_{n+1}}^{n+1,0} = \begin{cases} u_{L_n}^{n+1,0} & \text{if} \quad L_{n+1} = L_n, \\ (L_{n+1}h - l_{n+1})\delta u_{l_n}^n & \text{if} \quad L_{n+1} = L_n - 1, \end{cases} \tag{5.4}$$

where $\delta u_{l_n}^n = u_{L_n}^n/(L_n h - l_n)$, and

$$u_{R_{n+1}}^{n+1,0} = \begin{cases} u_{R_n}^{n+1,0} & \text{if} \quad R_{n+1} = R_n, \\ (R_{n+1}h - r_{n+1})\delta u_{R_n}^n & \text{if} \quad R_{n+1} = R_n + 1, \end{cases} \tag{5.5}$$

where $\delta u^n_{R_n} = -u^n_{R_n}/(r_n - R_n h)$. Now let us show (3.3). From (5.1) and (5.2) we have

$$0 < u^n_i \leqq u^{n+1,0}_i \leqq u^n_i + h\delta u^n_i = u^n_{i+1} \leqq C_0 \quad \text{(Case (i))},$$

$$0 < u^n_i \leqq u^{n+1,0}_i = u^n_{i-1} + (h + a\delta u^n_{i-1} k_{n+1})\delta u^n_{i-1} \leqq u^n_{i-1} \leqq C_0 \quad \text{(Case (ii))}$$

Here we used the inequality $\delta u^n_i \geqq 0$ (resp. $\delta u^n_{i-1} \leqq 0$) in Case (i) (resp. Case (ii)).

In Case (iii) it is obvious that (3.3) holds. Since $\delta u^n_{l_n} > 0$, $\delta u^n_{R_n} < 0$, $L_{n+1}h - l_{n+1} \leqq L_n h - l_n$ and $R_{n+1}h - r_{n+1} \geqq R_n h - r_n$, (3.3) also holds in Case (iv). Hence (3.3) can be proved. Next (3.4) can be shown by the following properties:

(5.6) $$w(t_{n+1}, x) = 0 \qquad \text{for} \quad x \in \boldsymbol{R}^1 \backslash (l_{n+1}, r_{n+1}),$$

(5.7) $$\int_{l_{n+1}}^{r_{n+1}} w(t_{n+1}, x)dx = \int_{l_n}^{r_n} w(t_n, x)dx = 0,$$

where $w(t, x)$ is the solution of the initial value problem (2.2), (2.3). (5.7) is given by the conservation law.

Let us show (3.5). It follows from (2.9) that

(5.8) $$\delta u^{n+1,0}_i = \int_{ih}^{(i+1)h} w(t_{n+1}, x)dx/h \quad (i \in \{L_{n+1}, \dots, R_{n+1} - 1\}),$$

(5.9) $$\delta u^{n+1,0}_{l_{n+1}} = u^{n+1,0}_{L_{n+1}}/(L_{n+1}h - l_{n+1}) = \int_{l_{n+1}}^{L_{n+1}h} w(t_{n+1}, x)dx/(L_{n+1}h - l_{n+1})$$

and

(5.10) $$\begin{aligned} \delta u^{n+1,0}_{R_{n+1}} &= -u^{n+1,0}_{R_{n+1}}/(r_{n+1} - R_{n+1}h) \\ &= -\int_{R_{n+1}h}^{r_{n+1}} w(t_{n+1}, x)dx/(r_{n+1} - R_{n+1}h). \end{aligned}$$

Then we have

$$|\delta u^{n+1,0}_i|, \ |\delta u^{n+1,0}_{l_{n+1}}|, \ |\delta u^{n+1,0}_{R_{n+1}}| \leqq C_1 \quad (i = L_{n+1}, \dots, R_{n+1} - 1),$$

because $\|w(t_{n+1}, \cdot)\|_\infty \leqq \|w(t_n, \cdot)\|_\infty \leqq C_1$. Hence (3.5) is proved. Since the entropy condition yields the property

$$\|w(t_{n+1}, \cdot)\|_1 \leqq \|w(t_n, \cdot)\|_1,$$

it follows

$$\begin{aligned} \|(u^{n+1,0}_h)_x\|_1 &= (L_{n+1}h - l_{n+1})|\delta u^{n+1,0}_{l_{n+1}}| + h|\delta u^{n+1,0}_{L_{n+1}}| \\ &\quad + \cdots + h|\delta u^{n+1,0}_{R_{n+1}-1}| + |r_{n+1} - R_{n+1}h| \, |\delta u^{n+1,0}_{R_{n+1}}| \\ &\leqq \int_{l_{n+1}}^{r_{n+1}} |w(t_{n+1}, x)|dx \leqq \|w(t_n, \cdot)\|_1 \leqq C_2, \end{aligned}$$

which leads to (3.6). Since $w(t_n, x)$ is piecewise constant with respect to x, the solution $w(t, x)(t_n \leqq t \leqq t_{n+1})$ consists of constant states 0, $w(t_n, l_n+0)$, $w(t_n, L_nh+0), \dots, w(t_n, R_nh+0)$, 0, which are separated by shock waves and are connected by rarefaction waves. Hence

$$V(w(t_{n+1}, \cdot)) \leqq V(w(t_n, \cdot)) \leqq C_3,$$

which implies (3.7).

Finally we show (3.8). For this end it suffices to prove that (3.8) holds in the following cases:

Case (a)	$L_{n+1} = L_n - 1$	and	$R_{n+1} = R_n + 1$;
Case (b)	$L_{n+1} = L_n - 1$	and	$R_{n+1} = R_n$;
Case (c)	$L_{n+1} = L_n$	and	$R_{n+1} = R_n + 1$;
Case (d)	$L_{n+1} = L_n$	and	$R_{n+1} = R_n$.

We now show (3.8) in Case (a). For simplicity we write $u_{\cdot}^{n+1,0}$ as $u_{\cdot}^{n+1}$ and let

$$u_{l_n}^{n+1} = \{1 - (l_n - L_{n+1}h)/h\}u_{L_{n+1}}^{n+1} + \{(l_n - L_{n+1}h)/h\}u_{L_n}^{n+1}, \tag{5.11}$$

$$u_{r_n}^{n+1} = \{(R_{n+1}h - r_n)/h\}u_{R_n}^{n+1} + \{1 - (R_{n+1}h - r_n)/h\}u_{R_{n+1}}^{n+1}. \tag{5.12}$$

Then it holds that

$$\begin{aligned}
\|(u_h^{n+1} - u_h^n)/k_{n+1}\|_1 &= [|u_{L_{n+1}}^{n+1}|(L_{n+1}h - l_{n+1}) \\
&\quad + (|u_{L_{n+1}}^{n+1}| + |u_{l_n}^{n+1}|)(l_n - L_{n+1}h) \\
&\quad + (|u_{l_n}^{n+1}| + |u_{L_n}^{n+1} - u_{L_n}^n|)(L_nh - l_n) \\
&\quad + \textstyle\sum_{i=L_n}^{R_n-1}(|u_i^{n+1} - u_i^n| + |u_{i+1}^{n+1} - u_{i+1}^n|)h \\
&\quad + (|u_{R_n}^{n+1} - u_{R_n}^n| + |u_{r_n}^{n+1}|)(r_n - R_nh) \\
&\quad + (|u_{r_n}^{n+1}| + |u_{R_{n+1}}^{n+1}|)(R_{n+1}h - r_n) \\
&\quad + |u_{R_{n+1}}^{n+1}|(r_{n+1} - R_{n+1}h)]/(2k_{n+1}) \\
&= [|u_{L_{n+1}}^{n+1}|(L_nh - l_{n+1}) + |u_{L_n}^{n+1}|(l_n - L_{n+1}h) \\
&\quad + |u_{L_n}^{n+1} - u_{L_n}^n|(L_nh - l_n) \\
&\quad + \textstyle\sum_{i=L_n}^{R_n-1}(|u_i^{n+1} - u_i^n| + |u_{i+1}^{n+1} - u_{i+1}^n|)h \\
&\quad + |u_{R_n}^{n+1} - u_{R_n}^n|(r_n - R_nh) \\
&\quad + |u_{R_n}^{n+1}|(R_{n+1}h - r_n) + |u_{R_{n+1}}^{n+1}|(r_{n+1} - R_nh)]/(2k_{n+1}) \\
&= [|u_{L_{n+1}}^{n+1}|(L_nh - l_{n+1}) + |u_{L_n}^n|(l_n - L_{n+1}h) \\
&\quad + 2\textstyle\sum_{i=L_n}^{R_n}|u_i^{n+1} - u_i^n|h \\
&\quad + |u_{R_n}^n|(R_{n+1}h - r_n) + |u_{R_{n+1}}^{n+1}|(r_{n+1} - R_nh)]/(2k_{n+1}).
\end{aligned} \tag{5.13}$$

By using (5.1)–(5.5) we have from (5.13)

$$\begin{aligned}
\|(u_h^{n+1}-u_h^n)/k_{n+1}\|_1 \leqq\ & [\{(L_{n+1}h-l_{n+1})(L_nh-l_{n+1}) \\
& + (L_nh-l_n)(l_n-L_{n+1}h)\}|\delta u_{L_n}^n| \\
& + 2\textstyle\sum_{i=L_n}^{R_n-1} a(\delta u_i^n)^2 k_{n+1}h \\
& + \{(r_n-R_nh)(R_{n+1}h-r_n) \\
& + (r_{n+1}-R_{n+1}h)(r_{n+1}-R_nh)\}|\delta u_{R_n}^n|]/(2k_{n+1}) \\
=\ & [\{(L_{n+1}h-l_{n+1}) + (L_nh-l_n)\}(l_n-l_{n+1})|\delta u_{L_n}^n| \\
& + 2\textstyle\sum_{i=L_n}^{R_n-1} a(\delta u_i^n)^2 k_{n+1}h \\
& + \{(r_{n+1}-R_{n+1}h) + (r_n-R_nh)\}(r_{n+1}-r_n)|\delta u_{R_n}^n|]/(2k_{n+1}) \\
\leqq\ & aC_1|\delta u_{L_n}^n|(L_nh-l_n) + aC_1 \textstyle\sum_{i=L_n}^{R_n-1} |\delta u_i^n|h \\
& + aC_1|\delta u_{R_n}^n|(r_n-R_nh) \\
\leqq\ & aC_1C_2,
\end{aligned}$$

which is the desired inequality (3.8).

Similarly (3.8) can be shown as in Cases (b), (c) and (d), and hence we omit the proofs. Thus, Lemma 3.1 is proved.

5.2. Proof of Lemma 3.2

For simplicity we write $u_\cdot^{n+1,r}$ as $u_\cdot^r$. From (2.10) it follows that

$$u_i^{r+1} = (1-2\lambda m u_i^r)u_i^r + \lambda m u_i^r u_{i+1}^r + \lambda m u_i^r u_{i-1}^r \quad (i \in \{L_{n+1}+1,\dots, R_{n+1}-1\}),$$

$$0 < u_{L_{n+1}}^{r+1} = u_{L_{n+1}}^{n+1,0} \leqq C_0$$

and

$$0 < u_{R_{n+1}}^{r+1} = u_{R_{n+1}}^{n+1,0} \leqq C_0,$$

where $\lambda = \tilde{k}_{n+1}/h^2$. Since the stability condition (3.9) gives $1-2\lambda m u_i^r \geqq 0$, we have

$$0 < u_i^{r+1} \leqq \max(u_i^r, u_{i+1}^r, u_{i-1}^r) \leqq C_0 \quad (i \in \{L_{n+1}+1,\dots, R_{n+1}-1\}).$$

Hence (3.3) holds with $u_h^{n+1,0} = u_h^{r+1}$.

Since $u_i^{r+1}=0$ for all $i \in Z\backslash\{L_{n+1},\dots, R_{n+1}\}$, we have (3.4) with $u_h^{n+1,0} = u_h^{r+1}$. From (2.10) we obtain

$$\begin{aligned}
(5.14)\qquad \delta u_i^{r+1} &= (u_{i+1}^{r+1}-u_i^{r+1})/h \\
&= \lambda m u_i^r \delta u_{i-1}^r + (1-\lambda m u_i^r - \lambda m u_{i+1}^r)\delta u_i^r + \lambda m u_{i+1}^r \delta u_{i+1}^r \\
&\qquad\qquad (i \in \{L_{n+1}+1,\dots, R_{n+1}-2\}),
\end{aligned}$$

(5.15) $$\delta u^{r+1}_{L_{n+1}} = (u^{r+1}_{L_{n+1}+1} - u^{r+1}_{L_{n+1}})/h$$
$$= (1-\lambda m u^r_{L_{n+1}+1})\delta u^r_{L_{n+1}} + \lambda m u^r_{L_{n+1}+1}\delta u^r_{L_{n+1}+1},$$

(5.16) $$\delta u^{r+1}_{l_{n+1}} = u^{r+1}_{L_{n+1}}/(L_{n+1}h - l_{n+1}) = \delta u^r_{l_{n+1}},$$

(5.17) $$\delta u^{r+1}_{R_{n+1}-1} = (u^{r+1}_{R_{n+1}} - u^{r+1}_{R_{n+1}-1})/h$$
$$= (1-\lambda m u^r_{R_{n+1}-1})\delta u^r_{R_{n+1}-1} + \lambda m u^r_{R_{n+1}-1}\delta u^r_{R_{n+1}-2}$$

and

(5.18) $$\delta u^{r+1}_{R_{n+1}} = -\, u^{r+1}_{R_{n+1}}/(r_{n+1} - R_{n+1}h) = \delta u^r_{R_{n+1}}.$$

It is obvious to see that

$$|\delta u^{r+1}_{l_{n+1}}|, \quad |\delta u^{r+1}_{R_{n+1}}| \leqq C_1.$$

Since the stability condition (3.9) gives

$$1 - \lambda m u^r_i - \lambda m u^r_{i+1} \geqq 0 \quad (i \in \{L_{n+1}+1, \dots, R_{n+1}-2\}),$$
$$1 - \lambda m u^r_{L_{n+1}+1} \geqq 0,$$
$$1 - \lambda m u^r_{R_{n+1}-1} \geqq 0,$$

it follows from (5.14), (5.15) and (5.17) that

$$|\delta u^{r+1}_i| \leqq C_1 \quad (\{i \in L_{n+1}, \dots, R_{n+1}-1\}).$$

Thus (3.5) holds with $u^{n+1,0}_h = u^{r+1}_h$. From (5.14)–(5.18) we have

$$\begin{aligned}
\|(u^{r+1}_h)_x\|_1 &= |\delta u^{r+1}_{l_{n+1}}|(L_{n+1}h - l_{n+1}) \\
&\quad + \textstyle\sum_{i=L_{n+1}}^{R_{n+1}-1} |\delta u^{r+1}_i| h + |\delta u^{r+1}_{R_{n+1}}|(r_{n+1} - R_{n+1}h) \\
&\leqq |\delta u^r_{l_{n+1}}|(L_{n+1}h - l_{n+1}) \\
&\quad + |(1-\lambda m u^r_{L_{n+1}+1})\delta u^r_{L_{n+1}} + \lambda m u^r_{L_{n+1}+1}\delta u^r_{L_{n+1}+1}| h \\
&\quad + \textstyle\sum_{i=L_{n+1}+1}^{R_{n+1}-2} |\lambda m u^r_i \delta u^r_{i-1} + (1-\lambda m u^r_i - \lambda m u^r_{i+1})\delta u^r_i + \lambda m u^r_{i+1}\delta u^r_{i+1}| h \\
&\quad + |(1-\lambda m u^r_{R_{n+1}-1})\delta u^r_{R_{n+1}-1} + \lambda m u^r_{R_{n+1}-1}\delta u^r_{R_{n+1}-2}| h \\
&\quad + |\delta u^r_{R_{n+1}}|(r_{n+1} - R_{n+1}h) \\
&= |\delta u^r_{l_{n+1}}|(L_{n+1}h - l_{n+1}) + \textstyle\sum_{i=L_{n+1}}^{R_{n+1}-1} |\delta u^r_i| h + |\delta u^r_{R_{n+1}}|(r_{n+1} - R_{n+1}h) \\
&= \|(u^r_h)_x\|_1 \leqq C_2,
\end{aligned}$$

which is the desired inequality (3.6) with $u^{n+1,0}_h = u^{r+1}_h$. We now estimate $V((u^{r+1}_h)_x)$. From (5.14)–(5.18) it follows that

(5.19) $$|\delta u^{r+1}_{l_{n+1}} - \delta u^{r+1}_{L_{n+1}}| = |\delta u^r_{l_{n+1}} - \delta u^r_{L_{n+1}} + \lambda m u^r_{L_{n+1}+1}(\delta u^r_{L_{n+1}} - \delta u^r_{L_{n+1}+1})|,$$

$$(5.20)\quad |\delta u_{L_{n+1}}^{r+1} - \delta u_{L_{n+1}+1}^{r+1}| = |(1-2\lambda m u_{L_{n+1}+1}^r)(\delta u_{L_{n+1}}^r - \delta u_{L_{n+1}+1}^r) + \lambda m u_{L_{n+1}+2}^r(\delta u_{L_{n+1}+1}^r - \delta u_{L_{n+1}+2}^r)|,$$

$$(5.21)\quad |\delta u_i^{r+1} - \delta u_{i+1}^{r+1}| = |\lambda m u_i^r(\delta u_{i-1}^r - \delta u_i^r) + (1-2\lambda m u_{i+1}^r)(\delta u_i^r - \delta u_{i+1}^r) + \lambda m u_{i+2}^r(\delta u_{i+1}^r - \delta u_{i+2}^r)| \quad (i \in \{L_{n+1}+1, \dots, R_{n+1}-3\}),$$

$$(5.22)\quad |\delta u_{R_{n+1}-2}^{r+1} - \delta u_{R_{n+1}-1}^{r+1}| = |\lambda m u_{R_{n+1}-2}^n(\delta u_{R_{n+1}-3}^r - \delta u_{R_{n+1}-2}^r) + (1-2\lambda m u_{R_{n+1}-1}^r)(\delta u_{R_{n+1}-2}^r - \delta u_{R_{n+1}-1}^r)|,$$

$$(5.23)\quad |\delta u_{R_{n+1}-1}^{r+1} - \delta u_{R_{n+1}}^{r+1}| = |\delta u_{R_{n+1}-1}^r - \delta u_{R_{n+1}}^r + \lambda m u_{R_{n+1}-1}^r(\delta u_{R_{n+1}-2}^r - \delta u_{R_{n+1}-1}^r)|.$$

Using (5.19)–(5.23), we have

$$\begin{aligned} V((u_h^{r+1})_x) &= |\delta u_{l_{n+1}}^{r+1}| + |\delta u_{l_{n+1}}^{r+1} - \delta u_{L_{n+1}}^{r+1}| \\ &\quad + \textstyle\sum_{i=L_{n+1}}^{R_{n+1}-1} |\delta u_i^{r+1} - \delta u_{i+1}^{r+1}| + |\delta u_{R_{n+1}}^{r+1}| \\ &\leqq |\delta u_{l_{n+1}}^r| + |\delta u_{l_{n+1}}^r - \delta u_{L_{n+1}}^r| \\ &\quad + \textstyle\sum_{i=L_{n+1}}^{R_{n+1}-1} |\delta u_i^r - \delta u_{i+1}^r| + |\delta u_{R_{n+1}}^r| \\ &= V((u_h^r)_x) \leqq C_3, \end{aligned}$$

and (3.7) holds with $u_h^{n+1,0} = u_h^{r+1}$. Finally we show the estimate (3.10). From (2.10) we have

$$\begin{aligned} \|(u_h^{r+1} - u_h^r)/\tilde{k}_{n+1}\|_1 &= \textstyle\sum_{i=L_{n+1}+1}^{R_{n+1}-1} |(u_i^{r+1} - u_i^r)/\tilde{k}_{n+1}| h \\ &= \textstyle\sum_{i=L_{n+1}+1}^{R_{n+1}-1} |m u_i^r(\delta u_i^r - \delta u_{i-1}^r)| \\ &\leqq m C_0 V((u_h^r)_x) \leqq m C_0 C_3, \end{aligned}$$

which completes the proof.

5.3. Proof of Lemma 4.1

It is clear that (4.9) holds. For all $(t, x) \in \mathscr{H}$ we have

$$\begin{aligned} |\tilde{u}_h(t, x) - u_h(t, x)| &\leqq \sup_n \{\max(u_{L_n}^n, u_{R_n}^n)\} \\ &= \sup_n \{\max(|\delta u_{l_n}^n|(L_n h - l_n), |\delta u_{R_n}^n|(r_n - R_n h))\} \\ &\leqq C_1 h, \end{aligned}$$

which immediately yields (4.8). Moreover, (4.10) follows from (4.3) and (4.8).

5.4. Proof of Lemma 4.2

Lemma 4.2 can be shown by the argument similar to Graveleau and Jamet's (see the proof of Lemma 6.3 in [5]).

5.5. Proof of Lemma 4.3

From (4.6) and (2.10) we have

$$\begin{aligned}
A_h &= \textstyle\sum_n \sum_{i=L_{n+1}+1}^{R_{n+1}-1} k_{n+1}\phi_i^n(1/\mu_{n+1})\{\sum_{r=0}^{\mu_{n+1}-1} mu_i^{n+1,r}(\delta u_i^{n+1,r}-\delta u_{i-1}^{n+1,r})\} \\
&= \textstyle\sum_n k_{n+1}(1/\mu_{n+1}) \sum_{r=0}^{\mu_{n+1}-1}\{-\phi_{L_{n+1}+1}^n mu_{L_{n+1}+1}^{n+1,r}\delta u_{L_{n+1}}^{n+1,r} \\
&\qquad + \textstyle\sum_{i=L_{n+1}+1}^{R_{n+1}-2}(\phi_i^n mu_i^{n+1,r}-\phi_{i+1}^n mu_{i+1}^{n+1,r})\delta u_i^{n+1,r} \\
&\qquad + \phi_{R_{n+1}-1}^n mu_{R_{n+1}-1}^{n+1,r}\delta u_{R_{n+1}-1}^{n+1,r}\} \\
&= C_h + \textstyle\sum_n k_{n+1}(1/\mu_{n+1}) \sum_{r=0}^{\mu_{n+1}-1}\{-\phi_{L_{n+1}+1}^n mu_{L_{n+1}+1}^{n+1,r}\delta u_{L_{n+1}}^{n+1,r} \\
&\qquad + \phi_{R_{n+1}-1}^n mu_{R_{n+1}-1}^{n+1,r}\delta u_{R_{n+1}-1}^{n+1,r}\}.
\end{aligned}$$

Then it holds that

$$(5.24)\qquad |A_n - C_h| \leqq \|\phi\|_\infty 4mhC_1^2K,$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathscr{X})}$ and K is a positive constant such that

$$(5.25)\qquad \phi(t, x) = 0 \qquad \text{for all } (t, x)\in[K, \infty)\times \boldsymbol{R}^1.$$

Hence (4.15) follows from (5.24). Let

$$D'_h = -\textstyle\sum_n (1/\mu_{n+1}) \sum_{r=0}^{\mu_{n+1}-1}\displaystyle\int_{t_n}^{t_{n+1}}\int_{(L_{n+1}+1)h}^{(R_{n+1}-1)h}[mu_h^{(p(r))}(u_h^{(p(r))})_x\phi_x + m\{(u_h^{(p(r))})_x\}^2\phi]\,dxdt.$$

Then

$$\begin{aligned}
(5.26)\qquad |C_h - D'_h| \leqq{}& \{mC_1^2\|\phi_x\|_\infty h + mC_0C_1(\|\phi_{xx}\|_\infty h + \|\phi_{xt}\|_\infty\sqrt{h}) \\
&+ mC_1^2(\|\phi_x\|_\infty h + \|\phi_t\|_\infty\sqrt{h})\} \\
&\times \textstyle\sum_n\{(R_{n+1}-1)h - (L_{n+1}+1)h\}k_{n+1},
\end{aligned}$$

$$\begin{aligned}
(5.27)\qquad |D_h - D'_h| \leqq{}& (mC_0C_1\|\phi_x\|_\infty + mC_1^2\|\phi\|_\infty) \\
&\times \textstyle\sum_n\{(L_{n+1}+1)h - l_{n+1} + r_{n+1} - (R_{n+1}-1)h\}k_{n+1}.
\end{aligned}$$

Since

$$|l_{n+1} - a_1| \leqq aC_1K,\quad |r_{n+1} - a_2| \leqq aC_1K,$$
$$|L_{n+1}h - l_{n+1}| \leqq h,\quad |r_{n+1} - R_{n+1}h| \leqq h$$

for all $n \geqq 0$ such that $t_{n+1} \leqq K$, it follows from (5.26) and (5.27) that

$$|C_h - D'_h|,\ |D_h - D'_h| \longrightarrow 0 \quad as \quad h \longrightarrow 0.$$

Hence (4.16) holds. (4.17) can be easily shown by Lemma 4.2.

5.6. Proof of Lemma 4.4

Let S_1^n, S_2^n and S_3^n be the sets of integers i for which (5.1), (5.2) and (5.3) hold, respectively. Putting

$$B_h^n = \sum_i hk_{n+1}\phi_i^n(u_i^{n+1,0}-u_i^n)/k_{n+1},$$

and

$$E_h^n = hk_{n+1}\sum_{i=L_n}^{R_n-1} a(\delta u_i^n)^2\phi_i^n,$$

we estimate $|B_h^n-E_h^n|$ in Cases (a), (b), (c) and (d) which are introduced in Section 5.1. In Case (a), by (5.1)–(5.5) we have

$$\begin{aligned}(5.28)\qquad B_h^n = hk_{n+1}\{&\phi_{L_{n+1}}^n(L_{n+1}h-l_{n+1})\delta u_{L_n}^n/k_{n+1}\\ &+\phi_{R_{n+1}}^n(r_{n+1}-R_{n+1}h)\delta u_{R_n}^n/k_{n+1}\\ &+\textstyle\sum_{i\in S_1^n} a\phi_i^n(\delta u_i^n)^2+\sum_{i\in S_2^n} a\phi_i^n(\delta u_{i-1}^n)^2\}.\end{aligned}$$

Let S_0^n be the set of integers i satisfying $i\in S_2^n$ and $i+1\in S_1^n\cup S_3^n$ or satisfying $i\in S_3^n$ and $i+1\in S_1^n$. Then

$$\begin{aligned}(5.29)\qquad \textstyle\sum_{i=L_n}^{R_n-1}\phi_i^n(\delta u_i^n)^2 &= \textstyle\sum_{i\in S_1^n}\phi_i^n(\delta u_i^n)^2+\sum_{i\in S_2^n}\phi_{i-1}^n(\delta u_{i-1}^n)^2\\ &\quad+\textstyle\sum_{i\in S_0^n}\phi_i^n(\delta u_i^n)^2.\end{aligned}$$

By (5.28) and (5.29) we have

$$\begin{aligned}(5.30)\quad |B_h^n-E_h^n| &\leqq \|\phi\|_\infty\{(L_{n+1}h-l_{n+1})+(r_{n+1}-R_{n+1}h)\}C_1h\\ &\quad+hk_{n+1}\{\textstyle\sum_{i\in S_2^n} a|\phi_i^n-\phi_{i-1}^n|(\delta u_{i-1}^n)^2+\sum_{i\in S_0^n} a|\phi_i^n|(\delta u_i^n)^2\}\\ &\leqq 2a\|\phi\|_\infty C_1^2hk_{n+1}+a\|\phi_x\|_\infty C_1C_2hk_{n+1}\\ &\quad+a\|\phi\|_\infty\{\textstyle\sum_{i\in S_0^n}(\delta u_i^n)^2\}hk_{n+1}.\end{aligned}$$

Since

$$\textstyle\sum_{i\in S_0^n}(\delta u_i^n)^2\leqq C_1V((u_h^n)_x)\leqq C_1C_3,$$

it follows from (5.30) that

$$(5.31)\qquad |B_h^n-E_h^n|\leqq a(2C_1^2\|\phi\|_\infty+C_1C_2\|\phi_x\|_\infty+C_1C_3\|\phi\|_\infty)hk_{n+1}.$$

Similary we can obtain (5.31) in Cases (b), (c) and (d). Hence

$$|B_h-E_h|\leqq Ka(2C_1^2\|\phi\|_\infty+C_1C_2\|\phi_x\|_\infty+C_1C_3\|\phi\|_\infty)h,$$

where K is a positive constant satisfying (5.25). Thus (4.20) follows. From (4.18) and (4.19) we have

$$|E_h - F_h| \leqq \sum_n \int_{t_n}^{t_{n+1}} \left\{ \int_{l_n}^{L_n h} a((u_h)_x)^2 |\phi| dx + \int_{R_n h}^{r_n} a((u_h)_x)^2 |\phi| dx \right\} dt$$

$$+ \sum_n \sum_{i=L_n}^{R_n - 1} \int_{t_n}^{t_{n+1}} \int_{ih}^{(i+1)h} a((u_h)_x)^2 |\phi_i^n - \phi| dx dt$$

$$\leqq K\{2aC_1^2 \|\phi\|_\infty h + aC_1 C_2 (\|\phi_x\|_\infty h + \|\phi_t\|_\infty \sqrt{h})\},$$

which yields (4.21). (4.22) can be shown by using (4.4), and the proof is complete.

6. Numerical results.

In this section, we show some numerical results. Figs. 4 and 5 show that, when an initial value $v^0(x)$ takes Barenblatt and Pattle's solution at $t=0$, a numerical solution of our scheme gives good profiles to the exact solution (1.3) as well as interface curves (1.4). Moreover, we exemplify an initial function of the form

$$\frac{m}{m-1}(v^0)^{m-1} = \begin{cases} (1-\theta)\sin^2 x + \theta \sin^4 x & \text{for} \quad x \in (-\pi, 0), \\ 0 & \text{for} \quad x \notin (-\pi, 0) \end{cases} \tag{6.1}$$

to calculate the waiting time. The numerical interface curves of solutions to the problem (1.1), (1.2) when $\theta=0$, 1/2 and 1 are drawn in Fig. 6. The detail of numerical waiting time will be reported elsewhere.

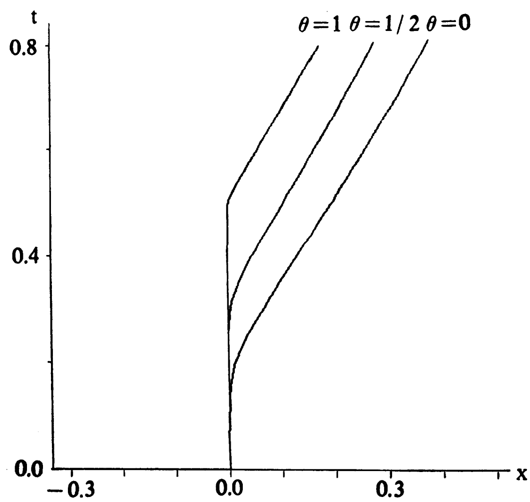

Fig. 6. Numerical right interface curves of our scheme with $h=0.03125$ subject to the initial condition (6.1) when $m=2$.

References

[1] D. G. Aronson, Regularity properties of flows through porous media, SIAM J. Appl. Math. **17** (1969), 461–467.

[2] D. G. Aronson, L. A. Caffarelli and S. Kamin, How an initially stationary interface begins to move in porous medium flow, Univ. of Minnesota Mathematics Report 81–113 (1981).

[3] G. I. Barenblatt, On some unsteady motions of a liquid and a gas in a porous medium, Prinkl. Mat. Meh. **16** (1952), 67–78.

[4] J. Descloux, On the equation of Boussinesq, Topics in Numerical Analysis III, ed. by J. H. Miller, (1977), 81–102.

[5] J. L. Graveleau and P. Jamet, A finite difference approach to some degenerate nonlinear parabolic equations, SIAM J. Appl. Math. **20** (1971), 199–223.

[6] B. F. Knerr, The porous medium equation in one dimension, Trans. Amer. Math. Soc. **234** (1977), 381–415.

[7] Y. Konishi, Une méthode de résolution d'une équation d'evolution nonlinéaire dégénérée, J. Fac. Sci, Univ. of Tokyo. **19** (1972), 243–255.

[8] M. Muskat, The flow of homogeneous fluids through porous media, McGraw-Hill, New York, (1937).

[9] O. A. Oleinik, A. S. Kalashnikov and Chzou Yui-Lin, The Cauchy problem and boundary value problems for equations of the type of nonstationary filteration, Izv. Akad. Nauk SSSR Ser. Math. **22** (1958), 667–704.

[10] R. E. Pattle, Diffusion from an instantaneous point source with a concentration-dependent coefficient. Quart. J. Mech. Appl. Math. **12** (1959), 407–409.

Osaka Institute of Technology
Osaka, Japan
and
Department of Mathematics,
Faculty of Science,
Hiroshima University