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Boundary value control of thermoelastic systems

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1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary S, and consider an n-dimensional linear elastic solid occupying Ω in its non-deformed state. Let us denote by $u(x, t)=\{u_i(x, t)\}_{1\leqq i\leqq n}$ the displacement vector from $x=\{x_i\}_{1\leqq i\leqq n}$ at the time t of the material particle which lies at x in the non-deformed state. If the temperature of the medium is not taken into consideration, then $u_i(x, t)$ $(1\leqq i\leqq n)$ satisfy the system of equations

$$(1.1)\quad \rho(x)(\partial^2 u_i/\partial t^2)(x, t) = \sum_{j=1}^n \partial\sigma_{ij}/\partial x_j + g_i(x, t) \quad \text{in} \quad \Omega\times(0, \infty),$$

where $\rho(x)$, $\sigma_{ij}(1\leqq i, j\leqq n)$ and $g(x, t)=\{g_i(x, t)\}_{1\leqq i\leqq n}$ denote the density, the stress tensors and the external force respectively. By Hook's law, there exists the linear dependence

$$\sigma_{ij} = \sum_{k,l=1}^n a_{ijkl}\varepsilon_{kl}(u), \quad 1\leqq i, j\leqq n$$

between the stress tensors σ_{ij} and the linearized strain tensors

$$\varepsilon_{ij}(u) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2, \quad 1\leqq i, j\leqq n.$$

Here a_{ijkl} are, in general, functions in t and x, but independent of the strain tensors. The functions a_{ijkl} are called the coefficients of elasticity.

The problem of controlling the deformation of the medium by applying traction forces $f(x, t)=\{f_i(x, t)\}_{1\leqq i\leqq n}$ on the boundary as

$$(1.2)\qquad \sum_{j=1}^n \nu_j(x)\sigma_{ij} = f_i(x, t) \quad \text{on} \quad S\times(0, \infty),$$

where $\nu(x)=\{\nu_i(x)\}_{1\leqq i\leqq n}$ is the outward unit normal vector at x on S, was considered by Clarke [1] and the author [15]. They obtained approximate controllability of the control system (1.1) with (1.2) when a_{ijkl} are independent of time t.

If the coefficients of elasticity are constants and further do not depend on the rotation of the coordinate axes, that is, if the elastic properties of the medium are the same in all directions, then the medium is said to be isotropic. In this case, a_{ijkl} are given by

$$(1.3)\qquad a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}),$$

where $\{\delta_{ij}\}$ is the Kronecker tensor and λ and μ are constants called Lame coefficients. The free energy of the deformed isotropic medium is given by

$$(1.4)\qquad F(u) = \mu\sum_{i,j=1}^{n} \{\varepsilon_{ij}(u) - (1/n)\delta_{ij} \sum_{k=1}^{n} \varepsilon_{kk}(u)\}^2 + (\lambda/2+\mu/n)(\sum_{k=1}^{n} \varepsilon_{kk}(u))^2.$$

The non-deformed state $u=0$ must be a minimal point of the free energy F when no exterior forces are applied on the medium. Hence the restrictions

$$(1.5)\qquad \mu > 0, \quad \lambda + 2\mu/n > 0$$

must be satisfied. When a_{ijkl} are given by (1.3) and the density $\rho(x)$ is equal to a constant ρ_0, the system of the equations (1.1) can be written as

$$(1.6)\qquad \rho_0(\partial^2 u/\partial t^2)(x, t) = \mu\Delta u(x, t) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u(x, t) + g(x, t) \quad \text{in} \quad \Omega \times (0, \infty),$$

where $\mathrm{grad}\,\phi=\{\partial\phi/\partial x_i\}_{1\leq i\leq n}$ for a scalar function $\phi(x)$, $\mathrm{div}\, u = \sum_{i=1}^{n} \partial u_i/\partial x_i$ for a vector function $u(x)=\{u_i(x)\}_{1\leq i\leq n}$ and $\Delta=\sum_{i=1}^{n} \partial^2/\partial x_i^2$. The author [16] considered control problems for the control system (1.6) with (1.2) and obtained the exact controllability by proving the "feedback stabilizability" along the lines of Russell [17], and also obtained the admissible controllability with boundary controls constrained in some prescribed sets.

The exact controllability for the control systems described by the wave equation was obtained by Russell [17], Graham and Russell [5] and Lagnese [9]. The admissible controllability for the control systems governed by partial differential equations has not yet been considered so much.

Under the assumption that the temperature of the medium is the same at all points and does not change during deformation, the influence of the temperature on deformation can be ignored. In reality, however, a deformation is followed by a variation in temperature and, conversely, a variation in temperature is followed by a deformation of the medium due to thermal expansion. Thus whenever one wishes to precisely describe the state of the medium, deformations produced by variations in temperature must be taken into account. It seems to be meaningful to consider the thermoelastic system as a continuation of [16], since it arises as often in mechanics as the isotropic elastodynamics (1.6).

Assume that when the medium is at rest, no exterior forces are applied to it and its absolute temperature is J_0. Let us denote by $J(x, t)$ the absolute temperature at the time t of the point $x+u(x, t)$ and by $\theta(x, t)$ the increment $J(x, t)-J_0$. Then the free energy $\tilde{F}(u)$ of the thermoelastic medium is given by

$$\tilde{F}(u) = -\alpha\theta \sum_{i=1}^{n} \varepsilon_{ii}(u) + F(u),$$

where α is a constant determined below and F is the free energy (1.4) for the non-thermoelastic medium. Therefore the stress tensors σ_{ij} in this case are given by

$$\sigma_{ij} = \partial \tilde{F}/\partial \varepsilon_{ij} = \textstyle\sum_{k,l=1}^{n} a_{ijkl}\varepsilon_{kl}(u) - \alpha\theta\delta_{ij}, \tag{1.7}$$

where a_{ijkl} are the coefficients of elasticity (1.3). Thus, when there exists a volume source of heat $q(x, t)$, $u(x, t)$ and $\theta(x, t)$ satisfy the system of thermoelastic equations

$$\begin{cases} \rho_0(\partial^2 u/\partial t^2)(x, t) = \mu\Delta u(x, t) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u(x, t) \\ \qquad\qquad\qquad - \alpha\,\mathrm{grad}\,\theta(x, t) + g(x, t) \\ (\partial\theta/\partial t)(x, t) + \beta\,\mathrm{div}\,(\partial u/\partial t)(x, t) = \kappa\Delta\theta(x, t) + q(x, t) \end{cases} \tag{1.8}$$

in $\Omega\times(0, \infty)$, where

$$\alpha = (\lambda+2\mu/n)\gamma > 0, \quad \beta = (c_p - c_v)/\gamma c_v > 0, \; \kappa = \bar{\kappa}/c_v > 0. \tag{1.9}$$

Here γ, c_p, c_v and $\bar{\kappa}$ mean the coefficient of linear heat expansion, the heat capacity at constant pressure, the heat capacity at constant volume and the heat conduction coefficient, respectively.

If we apply traction forces $f(x, t)$ on the boundary and deformations of the boundary cause forces proportional to the strain, then the boundary condition is given by

$$\{\textstyle\sum_{j=1}^{n} \nu_j\sigma_{ij}\}_{1\leqq i\leqq n} + \Gamma(x)u(x, t) = f(x, t) \quad \text{on} \quad S\times(0, \infty), \tag{1.10}$$

where $\Gamma(x)$ is an $n\times n$ symmetric positive matrix with smooth components and $\{\sum_{j=1}^{n} \nu_j\sigma_{ij}\}_{1\leqq i\leqq n}$ and $u(x, t)$ are taken as column vectors. On the other hand if the deformations of the boundary are given by $f(x, t)$, then the boundary condition is

$$u(x, t) = f(x, t) \quad \text{on} \quad S\times(0, \infty). \tag{1.11}$$

In this paper, we consider the thermoelastic system (1.8) under (1.5), (1.9) with boundary conditions (1.10) or (1.11), and

$$\theta(x, t) = 0 \quad \text{on} \quad S\times(0, \infty). \tag{1.12}$$

By (1.3), (1.7) and (1.12), we have

$$\textstyle\sum_{j=1}^{n} \nu_j\sigma_{ij} = \lambda\nu_i \sum_{j=1}^{n} \varepsilon_{jj}(u) + 2\mu \sum_{j=1}^{n} \nu_j\varepsilon_{ij}(u) \quad \text{on} \quad S.$$

Thus the boundary condition (1.10) turns into

$$\{\lambda\nu_i \textstyle\sum_{j=1}^{n} \varepsilon_{jj}(u) + 2\mu \sum_{j=1}^{n} \nu_j\varepsilon_{ij}(u)\}_{1\leqq i\leqq n} + \Gamma(x)u(x, t) = f(x, t)$$

on $S\times(0, \infty)$. The problems which we consider are whether it is possible to control the deformation of the medium, disregarding the values of temperature, by applying boundary forces or by giving boundary deformations $f(t)$ and further what sort of deformations can be controlled by controls constrained in some prescribed set; that is, exact controllability and admissible controllability for the control system (1.8) with (1.10) or (1.11), and (1.12).

As for mathematical treatment of thermoelastic systems, see e.g. Duvaut and Lions [2], Kupradze [8] and Landau and Lifshitz [10].

Throughout this paper, we denote by $H^s(\Omega)$ and $H^s(S)$ the Sobolev spaces of order s in Ω and S respectively and by $\boldsymbol{H}^s(\Omega)$ and $\boldsymbol{H}^s(S)$ the product spaces $H^s(\Omega)^n$ and $H^s(S)^n$. Further we denote by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. For an element $u(x)$ in $\boldsymbol{H}^s(\Omega)$ or $\boldsymbol{H}^s(S)$, $u_i(x)$ $(1\leq i\leq n)$ denotes the i-th component of $u(x)$. For n-dimensional vectors $x=\{x_i\}_{1\leq i\leq n}$ and $y=\{y_i\}_{1\leq i\leq n}$, we denote by $\langle x, y\rangle$ the Euclidean inner product $\sum_{i=1}^n x_i y_i$. For real numbers r and s, $\|\cdot\|_r$, $\|\cdot\|_{(r,s)}$ and $\langle\!\langle\cdot\rangle\!\rangle_r$ denote the norms of $\boldsymbol{H}^r(\Omega)$, $\boldsymbol{H}^r(\Omega)\times\boldsymbol{H}^s(\Omega)$ and $\boldsymbol{H}^r(S)$ respectively. Further for simplicity let (,) and $\|\cdot\|$ denote the usual inner product and the norm in $L^2(\Omega)$ or in $\boldsymbol{L}^2(\Omega)$. For a Banach space X and a non-negative integer k, by $\mathscr{E}_t^k(0, T; X)$ and $W^{k,1}(0, T; X)$ we denote the Banach spaces of k-times continuously differentiable X-valued functions on $[0, T]$ with the usual uniform norm

$$\sum_{j=0}^k \sup_{0<t<T} \|(d^j u/dt^j)(t)\|_X$$

and X-valued functions whose j-times weak derivatives with respect to t are in $L^1(0, T; X)$ for $0\leq j\leq k$ with the norm

$$\sum_{j=0}^k \int_0^T \|(d^j u/dt^j)(t)\|_X dt$$

respectively. For an element $u(t)$ in $\mathscr{E}_t^k(0, T; X)$ or $W^{k,1}(0, T; X)$, we put $u'=u_t=du/dt$, $u''=u_{tt}=d^2u/dt^2$ and $u^{(j)}(t)=d^j u/dt^j$ $(1\leq j\leq k)$.

2. Exact controllability

In this section we consider the exact controllability of deformations under the restrictions

$$\mu>0,\ \lambda+2\mu/n>0.$$

For simplicity, let ρ_0 be normalized as 1 and let us put

$$Au=\mu\Delta u+(\lambda+\mu)\operatorname{grad}\operatorname{div}u \qquad \text{in} \quad \Omega,$$

$$\partial u/\partial\nu_A=\{\lambda\nu_i\textstyle\sum_{j=1}^n \partial u_j/\partial x_j+\mu\sum_{j=1}^n \nu_j(\partial u_i/\partial x_j+\partial u_j/\partial x_i)\}_{1\leq i\leq n},$$

$B_D u = u$ and $B_\Gamma u = \partial u/\partial \nu_A + \Gamma(x)u$ on S. The control systems (1.6) with (1.2), and (1.8) with (1.10) or (1.11) and (1.12), are written respectively as follows:

$$[E, g, B]: \begin{cases} u_{tt} = Au + g & \text{in} \quad \Omega \times (0, \infty) \\ Bu = f & \text{on} \quad S \times (0, \infty), \end{cases}$$

$$[TE, g, q, B]: \begin{cases} u_{tt} = Au - \alpha \operatorname{grad} \theta + g & \\ \theta_t + \beta \operatorname{div} u_t - \kappa \Delta \theta = q & \text{in} \quad \Omega \times (0, \infty) \\ Bu = f & \\ \theta = 0 & \text{on} \quad S \times (0, \infty), \end{cases}$$

where $B = B_D$ or B_Γ. In the sequel we do not distinguish between column vectors ${}^t[u, v]$, ${}^t[u, v, \theta]$ and line vectors $[u, v]$, $[u, v, \theta]$, when it causes no confusions.

The problem of controlling the deformations, disregarding the values of temperature, is formulated as follows:

For a given set $[u_0, v_0, \theta_0]$ of initial state of deformation and an increment of temperature and a given final state of deformation $[u_1, v_1]$, are there at all a control $f(t)$ and a time T_0 for which there exists a solution $[u(t), \theta(t)]$ of $[TE, g, q, B]$ satisfying $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$ and $[u(T_0), u_t(T_0)] = [u_1, v_1]$?

When there exists such an $f(t)$, we say that the control $f(t)$ steers $[u_0, v_0, \theta_0]$ to $[u_1, v_1]$ or to $[u_1, v_1, \theta_1]$ $(\theta_1 = \theta(T_0))$ at T_0 and a solution $[u(t), \theta(t)]$ is often called a trajectory which connects $[u_0, v_0, \theta_0]$ and $[u_1, v_1]$ or $[u_1, v_1, \theta_1]$. We consider trajectories and controls which have the appropriate regularity naturally determined by initial and final states. Namely, when initial and final states are in $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega) \times H^m(\Omega)$ with $m \geqq 2$, or in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$ (the case $m = 1$), we take the trajectory in

$$\mathscr{E}_m[0, T] = \begin{cases} \cap_{j=0}^m \mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j}(\Omega)) \\ \times \{\cap_{j=0}^{m-2} \mathscr{E}_t^j(0, T; H^{m-j}(\Omega)) \cap \mathscr{E}_t^{m-1}(0, T; L^2(\Omega))\} \quad (m \geqq 2) \\ \cap_{j=0}^1 \mathscr{E}_t^j(0, T; \boldsymbol{H}^{1-j}(\Omega)) \times \mathscr{E}_t^0\,(0, T; L^2(\Omega)) \quad (m = 1), \end{cases}$$

which are called the trajectory spaces. Further, according to the regularity of initial and final states described above, we take the spaces $\cap_{j=0}^{m-1} \mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j-1/2}(S))$ $(m \geqq 1)$ in case $B = B_D$ and $\cap_{j=0}^{m-2} \mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j-3/2}(S))$ $(m \geqq 2)$ in case $B = B_\Gamma$ as the control spaces and denote them by $\mathscr{F}_D^m[0, T]$ and $\mathscr{F}_N^m[0, T]$ respectively.

REMARK 2.1. For the control system $[E, g, B]$, all states $[u_0, v_0]$ in $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ are controllable, that is, the space $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ is exactly controllable. See [16]. But for the control system $[TE, g, q, B]$, not all states $[u_0, v_0, \theta_0]$ in $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega) \times H^m(\Omega)$ can be controlled when $m \geqq 2$. In fact, if there exists a solution $[u(t), \theta(t)]$ in $\mathscr{E}_m[0, T]$ with the initial state $[u_0, v_0, \theta_0]$, then the following compatibility conditions (2.1) and (2.2) must be satisfied:

$$(2.1)\qquad \begin{cases} Bu_j = f^{(j)}(0), \quad 0 \leqq j \leqq m-2, \quad \text{on } S \\ \text{and further} \\ Bu_{m-1} = f^{(m-1)}(0) \quad \text{on } S \text{ when } B = B_D \end{cases}$$

and

$$(2.2)\qquad \begin{cases} \theta_j \in H^{m-j}(\Omega) \cap H_0^1(\Omega), \quad 0 \leqq j \leqq m-2, \\ \theta_{m-1} \in L^2(\Omega), \end{cases}$$

where u_j, θ_j are defined inductively as follows:

$$u_1 = v_0,\ u_j = Au_{j-2} - \alpha \operatorname{grad} \theta_{j-2} + g^{(j-2)}(0), \quad 2 \leqq j \leqq m-1,$$

$$\theta_j = -\beta \operatorname{div} u_j + \kappa \Delta \theta_{j-1} + q^{(j-1)}(0), \quad 1 \leqq j \leqq m-1.$$

It is possible to choose a control $f(t)$ so that the compatibility conditions (2.1) are satisfied, while (2.2) must be satisfied a priori, since they are the conditions on the given functions u_0, v_0, θ_0, $g(t)$ and $q(t)$.
Thus we put

$$W^m_{g,q}(\Omega) = \{[u_0, v_0, \theta_0] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega) \times H^m(\Omega) \mid [u_0, v_0, \theta_0] \text{ satisfies (2.2)}\}$$

for $m \geqq 2$ and $W^1_{g,q}(\Omega)\ (= W^1(\Omega)) = \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$.

Now we define weak solutions of $[TE, g, q, B]$.

DEFINITION 2.1. For $[u_0, v_0, \theta_0] \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$, $f(t) \in L^2(0, T; \boldsymbol{L}^2(S))$, $g(t) \in L^2(0, T; \boldsymbol{L}^2(\Omega))$ and $q(t) \in L^2(0, T; H^{-1}(\Omega))$, a function $[u(t), \theta(t)] \equiv [u(x, t), \theta(x, t)]$ is called *a weak solution of* $[TE, g, q, B_\Gamma]$ with the initial state $[u_0, v_0, \theta_0]$ if

$$(2.3)\qquad \begin{cases} u(t) \in \mathscr{E}^0_t(0, T; \boldsymbol{H}^1(\Omega)) \cap \mathscr{E}^1_t(0, T; \boldsymbol{L}^2(\Omega)), \\ \theta(t) \in \mathscr{E}^0_t(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \end{cases}$$

and

$$(2.4)\qquad -\int_0^T (u_t, \Phi_t)dt - (v_0, \Phi(0)) + \int_0^T \int_S \langle \Gamma u, \Phi \rangle S dt + \int_0^T a(u, \Phi)dt$$
$$+ \alpha \int_0^T (\operatorname{grad} \theta, \Phi)dt - \int_0^T (\theta, \Phi_t)dt - (\theta_0, \phi(0)) - \beta(\operatorname{div} u_0, \phi(0))$$
$$- \beta \int_0^T (\operatorname{div} u, \phi_t)dt + \kappa \int_0^T (\operatorname{grad} \theta, \operatorname{grad} \phi)dt$$
$$= \int_0^T (g, \Phi)dt + \int_0^T \int_S \langle f, \Phi \rangle dS dt + \int_0^T (q, \phi)dt,$$

for all smooth n-dimensional vector functions $\Phi(t) \equiv \Phi(x, t)$ and smooth scalar functions $\phi(t) \equiv \phi(x, t)$ which satisfy

$$\Phi(T) = 0, \phi(T) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \phi(t) = 0 \quad \text{on} \quad S \times (0, T).$$

Further $[u(t), \theta(t)]$ is called *a weak solution of* $[TE, g, q, B_D]$ with the initial state $[u_0, v_0, \theta_0]$ if it satisfies (2.3) and

$$\begin{aligned}(2.5) \qquad & -\int_0^T (u_t, \Phi_t)dt - (v_0, \Phi(0)) - \int_0^T (u, A\Phi)dt + \alpha \int_0^T (\operatorname{grad} \theta, \Phi)dt \\ & - \int_0^T (\theta, \phi_t)dt - (\theta_0, \phi(0)) - \beta (\operatorname{div} u_0, \phi(0)) - \beta \int_0^T (\operatorname{div} u, \phi_t)dt \\ & + \kappa \int_0^T (\operatorname{grad} \theta, \operatorname{grad} \phi)dt \\ & = \int_0^T (g, \Phi)dt - \int_0^T \int_S \langle f, (\partial\Phi/\partial\nu_A)\rangle \, dSdt + \int_0^T (q, \phi)dt,\end{aligned}$$

for all smooth functions $\Phi(t) \equiv \Phi(x, t)$ and $\phi(t) \equiv \phi(x, t)$ on $\bar{\Omega} \times [0, T]$ which satisfy

$$\Phi(t) = 0 \quad \text{on} \quad S \times (0, T), \quad \Phi(T) = 0 \quad \text{in} \quad \Omega$$

and

$$\phi(t) = 0 \quad \text{on} \quad S \times (0, T), \, \phi(T) = 0 \quad \text{in} \quad \Omega.$$

Here (q, ϕ) appearing in (2.4) and (2.5) means the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ and $a(\, , \,)$ is a bilinear form defined on $\boldsymbol{H}^1(\Omega)$ as

$$\begin{aligned} a(u, v) &= \textstyle\sum_{i,j,k,l=1}^n (a_{ijkl}\varepsilon_{kl}(u), \varepsilon_{ij}(v)) \\ &= \lambda (\operatorname{div} u, \operatorname{div} v) + \mu \textstyle\sum_{i,j=1}^n (\partial u_i/\partial x_j + \partial u_j/\partial x_i, \partial v_i/\partial x_j)\end{aligned}$$

for any $u, v \in \boldsymbol{H}^1(\Omega)$. We note that if $u \in \boldsymbol{H}^2(\Omega)$ and $v \in \boldsymbol{H}^1(\Omega)$, then

$$(Au, v) = -a(u, v) - \int_S \langle \Gamma u, v\rangle \, dS + \int_S \langle B_\Gamma u, v\rangle dS.$$

It is easily verified that, when $g(t)$, $q(t)$ and $f(t)$ are sufficiently smooth, a classical solution is a weak solution and conversely, a smooth weak solution is a classical solution. The uniqueness of the weak solution can be proved in the usual way. See e.g. Duvaut and Lions [2, p.p. 130].

Putting

$$\omega_m(\alpha, \beta, \kappa) = \begin{cases} (\alpha\beta/\kappa)(1+\kappa^{-(m-1)}) & \text{when } m = 1, 2, 3 \\ (\alpha\beta/\kappa)(1+\kappa^{-2(m-1)}) & \text{when } m \geqq 4, \end{cases}$$

we have the following

THEOREM 2.1. *Let* $m \geqq 2$, $B = B_D$ *(resp.* $B = B_\Gamma$*),* $g(t) \in \cap_{j=0}^{m-1} \mathscr{E}_t^j(0, \infty; \boldsymbol{H}^{m-1-j}(\Omega))$, $q(t) \in \cap_{j=0}^{m-1} \mathscr{E}_t^j(0, \infty; H^{m-1-j}(\Omega))$ *and let* T_0 *be the time appearing in Theorem A stated below. If* $\omega_m(\alpha, \beta, \kappa)$ *is smaller than a positive number* d_m, *depending only on* m, A, *and* Ω, *then for the control system* $[TE, g, q, B]$ *and for any* $[u_0, v_0, \theta_0] \in W_{g,q}^m(\Omega)$ *and* $[u_1, v_1] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ *there exists a control* $f(t)$ *in* $\mathscr{F}_D^m[0, T_0]$ *(resp.* $\mathscr{F}_N^m[0, T_0]$*) which steers the initial state* $[u_0, v_0, \theta_0]$ *to the final state* $[u_1, v_1]$ *at the time* T_0.

Further if $g(t) \in \mathscr{E}_t^0(0, \infty; \boldsymbol{L}^2(\Omega))$, $q(t) \in \mathscr{E}_t^0(0, \infty; L^2(\Omega))$, *then the statement is also valid for* $m = 1$ *and* $B = B_D$ *with weak solutions as the trajectories.*

REMARK 2.2. If the speed of heat conduction is so small in comparison with the propagation speed of vibration that the thermal expansion can be ignored, then the heat conduction coefficient κ is regarded to be equal to zero. In this case, the system of equations (1.8) with an initial state $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$ turns into the equations

$$\begin{aligned}\rho_0(\partial^2 u/\partial t^2)(x, t) = {} & \mu \Delta u(x, t) + (\lambda + \mu + \alpha\beta) \operatorname{grad} \operatorname{div} u(x, t) \\ & + g(x, t) - \alpha \operatorname{grad} \theta_0(x) - \alpha\beta \operatorname{grad} \operatorname{div} u_0(x) \\ & - \alpha \int_0^t \operatorname{grad} q(x, \tau) d\tau\end{aligned}$$

and

$$\theta(x, t) = \theta_0(x) - \beta \operatorname{div} u(x, t) + \beta \operatorname{div} u_0(x) + \int_0^t q(x, \tau) d\tau.$$

Hence this control system is equal to the system $[E, g_{ad}, B]$ with Lame coefficients

$$\mu_{ad} = \mu, \ \lambda_{ad} = \lambda + \alpha\beta$$

and the external force

$$\begin{aligned}g_{ad}(x, t) = {} & g(x, t) - \alpha \operatorname{grad} \theta_0(x) - \alpha\beta \operatorname{grad} \operatorname{div} u_0(x) \\ & - \alpha \int_0^t \operatorname{grad} g(x, \tau) d\tau.\end{aligned}$$

By [16], for the control system $[E, g_{ad}, B]$, the space $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ is exactly controllable in $\mathscr{F}_D^m[0, T]$ when $B = B_D$ and in $\mathscr{F}_N^m[0, T]$ when $B = B_\Gamma$. Hence we obtain the results in Theorem 2.1 in this case, although the hypotheses are not satisfied. But for the thermoelastic system $[TE, g, q, B]$ with a small nonzero κ, we do not know whether the exact controllability holds or not.

Before the proof of Theorem 2.1, we recall some results obtained in [16]. Although the results in [16] are given for the control system $[E, 0, B_\Gamma]$ with

$\Gamma=0$, we easily obtain the same results for $[E, 0, B_D]$ and $[E, 0, B_\Gamma]$. Namely we have

THEOREM A. *Let $m \geqq 2$ and $B=B_D$ (resp. $B=B_\Gamma$). Then for the control system $[E, 0, B]$, there exists a positive time T_0 such that the space $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ is exactly controllable in $\mathscr{F}_D^m[0, T_0]$ (resp. $\mathscr{F}_N^m[0, T_0]$); namely, for any $[u_0, v_0]$ and $[u_1, v_1]$ in $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$, there exists a control $f(t)$ in $\mathscr{F}_D^m[0, T_0]$ (resp. $\mathscr{F}_N^m[0, T_0]$) for which the system has a solution $u(t)$ in $\cap_{j=0}^m \mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$ satisfying $[u(0), u_t(0)]=[u_0, v_0]$ and $[u(T_0), u_t(T_0)]=[u_1, v_1]$.*

Further if we consider weak solutions as trajectories, then the statement is also valid for $m=1$ and $[E, 0, B_D]$.

Further from the way of construction of the control in the proof of Theorem A, we see

COROLLARY B. *Under the same assumptions as in Theorem A, there exist bounded linear operators K_D from $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ to $\mathscr{F}_D^m[0, T_0]$ (resp. K_Γ from $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ to $\mathscr{F}_N^m[0, T_0]$) and L from $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ to $\cap_{j=0}^m \mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$ such that, for each $[u_1, v_1] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$, $K_D[u_1, v_1]$ (resp. $K_\Gamma[u_1, v_1]$) is the control which steers $[0, 0]$ to $[u_1, v_1]$ for the control system $[E, 0, B_D]$ (resp. $[E, 0, B_\Gamma]$) and $L[u_1, v_1]$ is the trajectory for the control $K_D[u_1, v_1]$ (resp. $K_\Gamma[u_1, v_1]$).*

In the proof of Theorem A, the following lemma, which is Lemma 2.2 in [16], plays an essential part.

LEMMA 2.1. *Let m be a nonnegative integer and B a bounded open ball in R^n which contains $\Omega \cup S$. Then there exist bounded linear operators E_m and F_m from $\boldsymbol{H}^m(\Omega)$ to $\boldsymbol{H}^m(R^n)$ which satisfy the following (1)~(4) for any $u \in \boldsymbol{H}^m(\Omega)$:*

(1) $E_m u + F_m u = u$ in Ω;

(2) $\operatorname{div} E_m u = 0$;

(3) *there exists a function ϕ in $H^{m+1}(R^n)$ such that $F_m u = \operatorname{grad} \phi$;*

(4) *the supports of $E_m u$ and $F_m u$ are contained in B.*

An outline of the proof of Lemma 2.1 was given in [16], but the boundedness of the operators E_m and F_m were not proved there. Hence we give here afresh a proof of Lemma 2.1 in detail.

PROOF OF LEMMA 2.1. Let G be a simply connected domain in R^n such that $\Omega \subset G \subset \bar{G} \subset B$. Then there exists a bounded linear operator P_m from $\boldsymbol{H}^m(\Omega)$ to $\boldsymbol{H}^m(G)$ such tht $P_m u = u$ in Ω for all $u \in \boldsymbol{H}^m(\Omega)$. (See Lions and Magenes [11, p.p. 75–76].) Since G is simply connected, $\boldsymbol{H}^m(G)$ is decomposed into the direct sum of two closed subspaces: $\boldsymbol{H}^m(G)=X^m(G)+Y^m(G)$, where

$$X^m(G) = \{\delta w \mid w \in \mathscr{H}^{m+1}(G),\ \delta w \cdot \nu = 0 \quad \text{on} \quad S\},$$
$$Y^m(G) = \{d\phi \mid \phi \in H^{m+1}(G)\}.$$

Here elements in $\boldsymbol{H}^m(G)$ and $\mathscr{H}^{m+1}(G)$ are regarded as 1-forms and 2-forms in $H^m(G)$ and $H^{m+1}(G)$ respectively. The operator d is the exterior differentiation and δ is its formal adjoint. Thus, in case $n=3$, $\delta w = \operatorname{rot} w$ and $d\phi = \operatorname{grad}\phi$. For details see [12]. Put

$$\tilde{\mathscr{H}}^{m+1}(G) = \{w \in \mathscr{H}^{m+1}(G) \mid \delta w \cdot \nu = 0 \quad \text{on} \quad S\}.$$

Then for any $u \in \boldsymbol{H}^m(\Omega)$, there exsits a 2-form w in $\tilde{\mathscr{H}}^{m+1}(G)$ and a function ϕ in $H^{m+1}(G)$ such that $P_m u = \delta w + d\phi$. We define a closed subspace N of $\tilde{\mathscr{H}}^{m+1}(G)$ as $N=\{x \in \mathscr{H}^{m+1}(G) \mid \delta w = 0\}$ and let Q_{m+1} be the orthogonal projection of $\tilde{\mathscr{H}}^{m+1}(G)$ onto the orthogonal complement $N^\perp$ of N. Let $\bar{P}_{m+1}$ be a bounded linear extension operator from $\mathscr{H}^{m+1}(G)$ to $\mathscr{H}^{m+1}(R^n)$ such that the support of $\bar{P}_{m+1}v$ is contained in B for each $v \in \mathscr{H}^{m+1}(G)$, and we define an operator E_m as

$$E_m u = \delta \bar{P}_{m+1} Q_{m+1} w \qquad \text{for} \quad P_m u = \delta w + d\phi.$$

If $P_m u = \delta w_1 + d\phi_1 = \delta w_2 + d\phi_2$ ($w_i \in \tilde{\mathscr{H}}^{m+1}(G)$, $\phi_i \in H^{m+1}(G)$, $i=1, 2$), then $Q_{m+1}w_1 = Q_{m+1}w_2$. Thus the operator E_m is well defined. It is clear that the operator E_m is linear and the support of $E_m u$ is contained in B for any $u \in \boldsymbol{H}^m(\Omega)$. We show the boundedness of E_m. Since δ is a differential operator of order 1, it is bounded from $N^\perp$ to $X^m(G)$. By the definitions of $X^m(G)$ and $N^\perp$, it is bijective. Hence, by the closed graph theorem, it is a homeomorphism from $N^\perp$ to $X^m(G)$. Thus we have, for any $u \in \boldsymbol{H}^m(\Omega)$ with $P_m u = \delta w + d\phi$,

$$\|E_m u\|_m = \|\delta \bar{P}_{m+1} Q_{m+1} w\|_m \leqq \text{const.}\ \|\bar{P}_{m+1} Q_{m+1} w\|_{m+1} \leqq \text{const.}\ \|Q_{m+1} w\|_{m+1}$$
$$\leqq \text{const.}\ \|\delta Q_{m+1} w\|_m \leqq \text{const.}\ \|P_m u\|_m \leqq \text{const.}\ \|u\|_m,$$

where const. are independent of u. Hence E_m is a bounded operator from $\boldsymbol{H}^m(\Omega)$ to $\boldsymbol{H}^m(R^n)$. Similarly we can construct a bounded operator F_m satisfying the properties (1), (3) and (4).

In order to prove Theorem 2.1, we give some lemmas.

We define closed operators $\mathscr{A}_\Gamma$ on $H^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ and $\mathscr{L}_\Gamma$ on $\dot{\boldsymbol{H}}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$ as

$$\mathscr{A}_\Gamma[u, v] = [v, Au],\ \mathscr{D}(\mathscr{A}_\Gamma) = \{[u, v] \in \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^1(\Omega) \mid B_\Gamma u = 0 \quad \text{on} \quad S\}$$

and

$$\mathscr{L}_\Gamma[u, v, \theta] = [v, Au - \alpha \operatorname{grad} \theta, -\beta \operatorname{div} v + \kappa \Delta\theta],$$

$$\mathscr{D}(\mathscr{L}_\Gamma) = \{[u, v, \theta] \in \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^1(\Omega) \times H^2(\Omega) \mid B_\Gamma u = 0 \quad \text{and} \quad \theta = 0 \quad \text{on} \quad S\}.$$

The closed operators $\mathscr{A}_D$ and $\mathscr{L}_D$, associated to the boundary conditions (1.11) and (1.12), are defined on the Hilbert spaces $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$ in the same manner.

Now we have

LEMMA 2.2. *The closed operator $\mathscr{L}_D$ (resp. $\mathscr{L}_\Gamma$) generates a C_0 semigroup on $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$ (resp. $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$).*

PROOF. We consider the bilinear form $a(\ ,\ )$ on $\boldsymbol{H}^1(\Omega)$ defined in Definition 2.1. By the assumption (1.5), we have

$$a(u, u) \geqq 2\mu \textstyle\sum_{i,j=1}^n \|\varepsilon_{ij}(u)\|^2 \qquad \text{when} \quad \lambda \geqq 0$$

and

$$a(u, u) \geqq (n\lambda + 2\mu) \textstyle\sum_{i=1}^n \|\varepsilon_{ii}(u)\|^2 + 2\mu \sum_{i \neq j}^n \|\varepsilon_{ij}(u)\|^2 \quad \text{when} \quad \lambda < 0$$

for any $u \in \boldsymbol{H}^1(\Omega)$. Hence there exists a constant $c_1 > 0$ such that the inequality

$$a(u, u) \geqq c_1 \textstyle\sum_{i,j=1}^n \|\partial u_i/\partial x_j\|^2 \tag{2.6}$$

holds for any $u \in \boldsymbol{H}^1(\Omega)$, since as is well known (see e.g. Gobert [4]) Korn's inequality

$$\textstyle\sum_{i,j=1}^n \|\partial u_i/\partial x_j\|^2 \leqq \text{const.} \{\sum_{i,j=1}^n \|\partial u_i/\partial x_j + \partial u_j/\partial x_i\|^2\}$$

holds for any $u \in \boldsymbol{H}^1(\Omega)$. Let $\mathscr{H}$ be the space $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$ with the inner product

$$(U_1, U_2)_{\mathscr{H}} = a(u_1, u_2) + (u_1, u_2) + (v_1, v_2) + (\theta_1, \theta_2) + \int_S \langle \Gamma u_1, u_2 \rangle dS$$

and the corresponding norm $\|U_1\|_{\mathscr{H}} = (U_1, U_1)_{\mathscr{H}}^{1/2}$ for $U_i = [u_i, v_i, \theta_i] \in \mathscr{H} (i = 1, 2)$. Then by the inequalities (2.6) and

$$0 \leqq \int_S \langle \Gamma u, u \rangle dS \leqq \text{const.} \langle\!\langle u \rangle\!\rangle_0^2 \leqq \text{const.} \|u\|_1^2 \quad \text{for any} \quad u \in \boldsymbol{H}^1(\Omega),$$

the norm is equivalent to the standard one in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$.

For any $U = [u, v, \theta] \in \mathscr{D}(\mathscr{L}_\Gamma)$ and positive number ξ_0, integrating by parts and applying arithmetic-geometric mean inequality, we have

$$((\mathscr{L}_\Gamma - \xi_0)U, U)_{\mathscr{H}} = a(v-\xi_0 u, u) + (v-\xi_0 u, u) + \int_S \langle \Gamma(v-\xi_0 u), u\rangle dS$$
$$+ (Au - \alpha \operatorname{grad}\theta - \xi_0 v, v) + (-\beta \operatorname{div} v + \kappa\Delta\theta - \xi_0\theta, \theta)$$
$$= -\xi_0 a(u, u) + (v, u) - \xi_0\|u\|^2 - \xi_0\int_S \langle \Gamma u, u\rangle dS - \alpha(\operatorname{grad}\theta, v)$$
$$-\xi_0\|v\|^2 + (v, \operatorname{grad}\theta) - \kappa\|\operatorname{grad}\theta\|^2 - \xi_0\|\theta\|^2$$
$$\leqq -\xi_0 a(u, u) - (\xi_0 - 1/2)\|u\|^2 - \xi_0\int_S \langle \Gamma u, u\rangle dS$$
$$-\{\xi_0 - 1/2 - (\alpha^2+\beta^2)/\kappa\}\|v\|^2 - (\kappa/2)\|\operatorname{grad}\theta\|^2 - \xi_0\|\theta\|^2,$$

since u and θ satisfy the boundary conditions $\partial u/\partial \nu_A + \Gamma(x)u = 0$ and $\theta = 0$ on S. Thus if $\xi_0 > 1/2 + (\alpha^2+\beta^2)/\kappa$, then we have

$$((\mathscr{L}_\Gamma - \xi_0)U, U)_{\mathscr{H}} \leqq 0 \qquad \text{for any} \quad U \in \mathscr{D}(\mathscr{L}_\Gamma).$$

This means that the closed operator $\mathscr{L}_\Gamma - \xi_0$ is dissipative on $\mathscr{H}$. Next we show that $\mathscr{L}_\Gamma - \xi_0$ is maximal. To show this, it is sufficient to prove that, for some $\xi \geqq \xi_0$, the operator $\mathscr{L}_\Gamma - \xi$ is surjective. Let $F = [f, g, h]$ be any element in $\mathscr{H}$ and consider the equation

$$(\mathscr{L}_\Gamma - \xi)U = F. \tag{2.7}$$

If $U = [u, v, \theta]$, then this equation is equivalent to

$$v - \xi u = f,\ Au - \alpha \operatorname{grad}\theta - \xi v = g \quad \text{and} \quad -\beta \operatorname{div} v + \kappa\Delta\theta - \xi\theta = h.$$

By substituting the first equation into the second and the third equations, we have

$$\begin{aligned} & Au - \alpha \operatorname{grad}\theta - \xi^2 u = g + \xi f \qquad \text{and} \\ & -\xi\beta \operatorname{div} u + \kappa\Delta\theta - \xi\theta = h + \beta \operatorname{div} f \qquad \text{in} \quad \Omega. \end{aligned} \tag{2.8}$$

By (2.6), there exists a constant $c_2 > 0$ such that

$$a(w, w) + \|w\|^2 \geqq c_2\|w\|_1^2 \qquad \text{for any } w \in \boldsymbol{H}^1(\Omega).$$

Let us take ξ so large that $\xi > 2$, $\alpha^2/\kappa\xi^2 < c_2/(n\beta^2\xi)$ and choose ε such that $\alpha^2/\kappa\xi^2 < \varepsilon < c_2/(n\beta^2\xi)$. We define a bilinear form B on $\boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ as

$$B([w_1, \phi_1], [w_2, \phi_2]) = a(w_1, w_2) + \alpha(\operatorname{grad}\phi_1, w_2) + \xi^2(w_1, w_2)$$
$$+ \int_S \langle \Gamma w_1, w_2\rangle dS$$
$$+ \varepsilon\{\kappa(\operatorname{grad}\phi_1, \operatorname{grad}\phi_2) + \xi\beta(\operatorname{div} w_1, \phi_2) + \xi(\phi_1, \phi_2)\}$$

for $[w_i, \phi_i] \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ $(i = 1, 2)$. Then it is easy to see that

$$B([w_1, \phi_1], [w_2, \phi_2]) \leqq \text{const.}(\|w_1\|_1 + \|\phi_1\|_1)(\|w_2\|_1 + \|\phi_2\|_1),$$

where const. depends on ξ. By Schwarz's inequality and arithmetic-geometric mean inequality, we have

$$\begin{aligned} B([w, \phi], [w, \phi]) &= a(w, w) + \alpha(\operatorname{grad}\phi, w) + \xi^2\|w\|^2 + \int_S \langle \Gamma w, w\rangle dS \\ &\quad + \varepsilon\{\kappa\|\operatorname{grad}\phi\|^2 + \xi\beta(\operatorname{div} w, \phi) + \xi\|\phi\|^2\} \\ &\geqq c_2\|w\|_1^2 - (\alpha^2/2\xi^2)\|\operatorname{grad}\phi\|^2 - (\xi^2/2)\|w\|^2 + (\xi^2-1)\|w\|^2 \\ &\quad + \int_S \langle \Gamma w, w\rangle dS + \varepsilon\{\kappa\|\operatorname{grad}\phi\|^2 - (\xi\beta^2/2)\|\operatorname{div} w\|^2 + (\xi/2)\|\phi\|^2\} \\ &\geqq (c_2/2)\|w\|_1^2 + (\alpha^2/2\xi^2)\|\operatorname{grad}\phi\|^2 + (\xi\varepsilon/2)\|\phi\|^2 \end{aligned}$$

for any $[w, \phi] \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$. Thus we can take a constant $\delta > 0$ such that $B([w, \phi], [w, \phi]) \geqq \delta(\|w\|_1^2 + \|\phi\|_1^2)$ holds for any $[w, \phi] \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$. For any given $F = [f, g, h] \in \mathscr{H}$, the functional $L_F[w, \phi] = -(g + \xi f, w) - \varepsilon(h + \beta \operatorname{div} f, \phi)$ is bounded linear on $\boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$. Hence, by Lax-Milgram's theorem, there exists a unique element $[u, \theta]$ in $\boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$ satisfying $B([u, \theta], [w, \phi]) = L_F[w, \phi]$ for any $[w, \phi] \in \boldsymbol{H}^1(\Omega) \times H_0^1(\Omega)$. In particular, taking $w \in C_0^\infty(\Omega)^n$, $\phi = 0$ and $w = 0$, $\phi \in C_0^\infty(\Omega)$, we see that the two equalities (2.8) hold in the weak sense. Further by taking $w \in C^\infty(\bar{\Omega})^n$, $\phi = 0$ and $w = 0$, $\phi \in C^\infty(\bar{\Omega})$, we see that u and θ satisfy the boundary conditions $B_\Gamma u = 0$ and $\theta = 0$ on S in the weak sense. By the general regularity theorem for the elliptic boundary value problems, u and θ belong to $\boldsymbol{H}^2(\Omega)$ and $H^2(\Omega)$ respectively (see e.g. Fichera [3]).

Putting $v = f + \xi u$, we have a solution $U = [u, v, \theta] \in \mathscr{D}(\mathscr{L}_\Gamma)$ of (2.7). Thus the closed operator $\mathscr{L}_\Gamma - \xi_0$ is maximal dissipative on $\mathscr{H}$. A closed operator with a dense domain is the generator of a contractive C_0 semigroup if and only if it is maximal dissipative (see e.g. Tanabe [22, Chap. 3]). Therefore $\mathscr{L}_\Gamma - \xi_0$ generates a C_0 semigroup in $\mathscr{H}$, and hence $\mathscr{L}_\Gamma$ is the generator of a C_0 semigroup on $\mathscr{H}$. Since the norm $\|\cdot\|_{\mathscr{H}}$ is equivalent to the standard one in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$, the closed operator $\mathscr{L}_\Gamma$ generates a C_0 semigroup on $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$.

The proof for the operator $\mathscr{L}_D$ is similar.

REMARK 2.3. If $[u(t), \theta(t)]$ is the trajectory in $\mathscr{E}_m[0, T]$ of the control system $[TE, g, q, B_D]$ (resp. $[TE, g, q, B_\Gamma]$) for a control $f(t)$, then by the usual trace theorem $f(t)$ belongs to $\mathscr{F}_D^m[0, T]$ (resp. $\mathscr{F}_N^m[0, T]$) and further the compatibility conditions (2.1) and (2.2) are satisfied at $t = 0$. Conversely, we may ask the question whether for given $f(t)$ in $\mathscr{F}_D^m[0, T]$ (resp. $\mathscr{F}_N^m[0, T]$) and an initial state $[u_0, v_0, \theta_0]$ in $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega) \times H^m(\Omega)$ satisfying the compatibility conditions (2.1) and (2.2) with $B = B_D$ (resp. $B = B_\Gamma$), there exists a trajectory $[u(t), \theta(t)]$ in $\mathscr{E}_m[0, T]$ of the control system $[TE, g, q, B_D]$ (resp. $[TE, g, q, B_\Gamma]$) with the control $f(t)$ and $[u(0), u_t(0), \theta(0)]=[u_0, v_0, \theta_0]$. For the initial boundary value problem of the wave equation

$$u_{tt}(x, t) - \Delta u(x, t) = g(x, t) \quad \text{in } \Omega \times (0, T)$$

with Dirichlet boundary condition (resp. Neumann boundary condition), that is,

$$u(x, t) = f(x, t) \text{ (resp. } (\partial u/\partial \nu)(x, t) = f(x, t)) \quad \text{on } S \times (0, T),$$

Sakamoto [20], Miyatake [13], [14] obtained the energy inequality

$$\begin{aligned}(2.9)\qquad \|u(t)\|_m^2 + \|u_t(t)\|_{m-1}^2 &\leqq \text{const.}\{\|u(0)\|_m^2 + \|u_t(0)\|_{m-1}^2 \\ &+ \textstyle\sum_{j=0}^{m-1} \int_0^t (\|g^{(j)}(\tau)\|_{m-1-j}^2 + \langle\!\langle f^{(j)}(\tau)\rangle\!\rangle_{m-j}^2) d\tau\}\end{aligned}$$

(resp.

$$\begin{aligned}(2.10)\qquad \|u(t)\|_m^2 + \|u_t(t)\|_{m-1}^2 &\leqq \text{const.}\{\|u(0)\|_m^2 + \|u_t(0)\|_{m-1}^2 \\ &+ \textstyle\sum_{j=0}^{m-1} \int_0^t (\|g^{(j)}(\tau)\|_{m-1-j}^2 + \langle\!\langle f^{(j)}(\tau)\rangle\!\rangle_{m-j-1/2}^2) d\tau\}).\end{aligned}$$

They also obtained the energy inequalities for various other boundary conditions. It seems to us that the energy inequalities (2.9) and (2.10) are best possible. Further the results of Graham and Russell [5], when the domain Ω is a unit ball, make us to conjecture that the corresponding question for the wave equation is negatively answered (see Remark 3.2 in [16]). Hence it seems to us that for the control system $[TE, g, q, B]$ the answer to the above question is negative.

In spite of Remark 2.3, for the special controls the answer to this question is affirmative.

LEMMA 2.3. *Let $m \geqq 1$ and $B=B_D$ (resp. $m \geqq 2$ and $B=B_\Gamma$). Assume that, for a control $f(t)$ in $\mathscr{F}_D^m[0, T]$ (resp. $\mathscr{F}_N^m[0, T]$), there exists a solution $v(t)$ in $\bigcap_{j=0}^m \mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j}(\Omega))$ of the control system $[E, 0, B]$ with the initial state $[v(0), v_t(0)]=[0, 0]$.*

Then there exists a solution $[u(t), \theta(t)]$ in $\mathscr{E}_m[0, T]$ of the control system $[TE, 0, 0, B]$ with the control $f(t)$ and the null initial state $[u(0), u_t(0), \theta(0)]=[0, 0, 0]$.

Further we have the energy inequality

$$\begin{aligned}(2.11)\qquad &(1/2)\{\|w^{(k+1)}(t)\|^2 + a(w^{(k)}(t), w^{(k)}(t)) + (\alpha/\beta)\|\theta^{(k)}(t)\|^2\} \\ &+ (\kappa\alpha/\beta - \delta)\int_0^t \|\operatorname{grad}\theta^{(k)}(\tau)\|^2 d\tau \\ &\leqq (\alpha^2/4\delta)\int_0^t \|v^{(k+1)}(\tau)\|^2 d\tau\end{aligned}$$

(resp.

$$(2.12)\qquad (1/2)\{\|w^{(k+1)}(t)\|^2 + a(w^{(k)}(t), w^{(k)}(t)) + (\alpha/\beta)\|\theta^{(k)}(t)\|^2 + \int_S \langle \Gamma w^{(k)}(t), w^{(k)}(t)\rangle dS\} + (\kappa\alpha/\beta-\delta)\int_0^t \|\mathrm{grad}\,\theta^{(k)}(\tau)\|^2 d\tau \leqq (\alpha^2/4\delta)\int_0^t \|v^{(k+1)}(\tau)\|^2 d\tau)$$

for $0\leqq t\leqq T$, $0\leqq k\leqq m-1$ *and a constant* δ *with* $0<\delta\leqq\kappa\alpha/\beta$, *where* $w(t)=u(t)-v(t)$.

PROOF. We prove the case $B=B_\Gamma$. Given $v(t)$, consider the equations

$$(2.13)\qquad \begin{cases} w_{tt} - Aw + \alpha\,\mathrm{grad}\,\theta = 0 \\ \theta_t + \beta\,\mathrm{div}\,w_t - \kappa\Delta\theta = -\beta\,\mathrm{div}\,v_t \end{cases} \quad \text{in} \quad \Omega\times(0, T)$$

with the homogeneous boundary conditions $B_\Gamma w=0$ and $\theta=0$ on $S\times(0, T)$ and the null initial state $[w(0), w_t(0), \theta(0)]=[0, 0, 0]$. By Lemma 2.2, the operator $\mathscr{L}_\Gamma$ generates a C_0 semigroup on $\boldsymbol{H}^1(\Omega)\times\boldsymbol{L}^2(\Omega)\times L^2(\Omega)$, which we denote by $S_\Gamma(t)$. Since $v_t(t)$ belongs to $\mathscr{E}_t^{m-1}(0, T; \boldsymbol{L}^2(\Omega))$ and satisfies the initial conditions $v^{(k)}(0)=0$ $(0\leqq k\leqq m)$, we can take $\phi_l(t)\equiv\phi_l(x, t)\in C_0^\infty(\Omega\times(0, T])^n$, $l=1, 2,\ldots$, such that $\{\phi_l(t)\}$ converges to $v_t(t)$ in $\mathscr{E}_t^{m-1}(0, T; \boldsymbol{L}^2(\Omega))$ as $l\to\infty$. Let us define $[w_l(t), \theta_l(t)]$ by

$$[w_l(t), w_l'(t), \theta_l(t)] = \int_0^t S_\Gamma(t-\tau)[0, 0, -\beta\,\mathrm{div}\,\phi_l(\tau)]d\tau.$$

Since $[0, 0, \mathrm{div}\,\phi_l(t)]\in\bigcap_{j=1}^\infty \mathscr{D}(\mathscr{L}_\Gamma^j)$ for each t, $[w_l(t), \theta_l(t)]$ is smooth in t and x and $[w_l^{(k)}(t), \theta_l^{(k)}(t)]$ $(0\leqq k\leqq m-1)$ satisfy the equation (2.13) and the boundary conditions with $B=B_\Gamma$ (see Tanabe [22]) and $[w_l^{(k)}(0), \theta_l^{(k)}(0)]=[0, 0]$. Multiply the first equation of (2.13) with $[w(\tau), \theta(\tau)]=[w_l^{(k)}(\tau), \theta_l^{(k)}(\tau)]$, by $w_l^{(k+1)}(\tau)$ and the second equation by $\theta_l^{(k)}(\tau)$ and integrate over $\Omega\times(0, t)$. Integrating by parts, we have

$$(1/2)\Big\{\|w_l^{(k+1)}(t)\|^2 + a(w_l^{(k)}(t), w_l^{(k)}(t)) + \int_S \langle \Gamma w_l^{(k)}(t), w_l^{(k)}(t)\rangle dS\Big\} - \alpha\int_0^t (\theta_l^{(k)}(\tau), \mathrm{div}\,w_l^{(k+1)}(\tau))d\tau = 0,$$

$$(1/2)\|\theta_l^{(k)}(t)\|^2 + \kappa\int_0^t \|\mathrm{grad}\,\theta_l^{(k)}(\tau)\|^2 d\tau + \beta\int_0^t (\theta_l^{(k)}(\tau), \mathrm{div}\,w_l^{(k+1)}(\tau))d\tau = \beta\int_0^t (\phi_l^{(k)}(\tau), \mathrm{grad}\,\theta_l^{(k)}(\tau))d\tau.$$

Multiplying the second equation by α/β and summing up, we have

$$(2.14)\qquad (1/2)\Big\{\|w_l^{(k+1)}(t)\|^2 + a(w_l^{(k)}(t),\ w_l^{(k)}(t)) + (\alpha/\beta)\|\theta_l^{(k)}(t)\|^2 + \int_S \langle \Gamma w_l^{(k)}(t),\ w_l^{(k)}(t)\rangle dS\Big\} + (\kappa\alpha/\beta)\int_0^t \|\mathrm{grad}\,\theta_l^{(k)}(\tau)\|^2 d\tau$$
$$= \alpha\int_0^t (\phi_l^{(k)}(\tau),\ \mathrm{grad}\,\theta_l^{(k)}(\tau))d\tau$$
$$\leqq \delta\int_0^t \|\mathrm{grad}\,\theta_l^{(k)}(\tau)\|^2 d\tau + (\alpha^2/4\delta)\int_0^t \|\phi_l^{(k)}(\tau)\|^2 d\tau$$

for any positive δ. Let us take δ with $0<\delta<\kappa\alpha/\beta$. By considering $\{w_l^{(k)}(t)-w_{l'}^{(k)}(t)\}$ and $\{\theta_l^{(k)}(t)-\theta_{l'}^{(k)}(t)\}$ in place of $\{w_l^{(k)}(t)\}$ and $\{\theta_l^{(k)}(t)\}$, the inequality (2.14) implies that the sequences $\{w_l^{(k)}(t)\}$ and $\{\theta_l^{(k)}(t)\}$ $(0\leqq k\leqq m-1)$ are Cauchy sequences in $\mathscr{E}_t^1(0,\ T;\ \boldsymbol{L}^2(\Omega))\cap\mathscr{E}_t^0(0,\ T;\ \boldsymbol{H}^1(\Omega))$ and $\mathscr{E}_t^0(0,\ T;\ L^2(\Omega))\cap L^2(0,\ T;\ H^1(\Omega))$, since $\{\phi_l(t)\}$ converges in $\mathscr{E}_t^{m-1}(0,\ T;\ L^2(\Omega))$. Here we use the fact that the norm defined by the bilinear form

$$a(u,\ v) + (u,\ v) \qquad (u,\ v\in \boldsymbol{H}^1(\Omega))$$

is equivalent to the usual one in $\boldsymbol{H}^1(\Omega)$ and the inequality

$$\|w(t)\| \leqq \|w(0)\| + \int_0^t \|w_t(\tau)\|d\tau.$$

Thus there exist $\bar{w}_k(t)$ and $\bar{\theta}_k(t)$ such that

$$w_l^{(k)}(t) \longrightarrow \bar{w}_k(t) \qquad \text{in} \quad \mathscr{E}_t^0(0,\ T;\ \boldsymbol{H}^1(\Omega))\cap\mathscr{E}_t^1(0,\ T;\ \boldsymbol{L}^2(\Omega))$$
$$\theta_l^{(k)}(t) \longrightarrow \bar{\theta}_k(t) \qquad \text{in} \quad \mathscr{E}_t^0(0,\ T;\ L^2(\Omega))\cap L^2(0,\ T;\ H^1(\Omega))$$

as $l\to\infty$ for each $0\leqq k\leqq m-1$. It is easy to see that $\bar{w}_{k+1}(t)=\bar{w}_0^{(k+1)}(t)$ and $\bar{\theta}_k(t)=\bar{\theta}_0^{(k)}(t)$ for $0\leqq k\leqq m-1$. Furthermore, $[u(t),\ \theta(t)]=[w_l(t),\ \theta_l(t)]$ satisfies (2.4) with $f(t)=0$, $g(t)=0$ and $q(t)=-\beta\,\mathrm{div}\,\phi_l(t)$. Letting $l\to\infty$, we see that $[\bar{w}_0(t),\ \bar{\theta}_0(t)]$ is a weak solution of (2.13) with boundary conditions $B_\Gamma w=0$ and $\theta=0$ on $S\times(0,\ T)$ and initial state $[0,\ 0,\ 0]$ in the sense of Definition 2.1. Since $\bar{w}_0^{(k)}(t)\in\mathscr{E}_t^0(0,\ T;\ \boldsymbol{H}^1(\Omega))\cap\mathscr{E}_t^1(0,\ T;\ \boldsymbol{L}^2(\Omega))$ and $\bar{\theta}_0^{(k)}(t)\in\mathscr{E}_t^0(0,\ T;\ L^2(\Omega))\cap L^2(0,\ T;\ H^1(\Omega))$ $(0\leqq k\leqq m-1)$, we see that, by the ellipticity of A and Δ with the boundary conditions $B_\Gamma w=0$ and $\theta=0$, the function $[\bar{w}_0(t),\ \bar{\theta}_0(t)]$ belongs to $\mathscr{E}_m[0,\ T]$ and satisfies the equation (2.13). Passage to the limit as $l\to\infty$ in (2.14) gives the energy inequality (2.12) for $[\bar{w}_0(t),\ \bar{\theta}_0(t)]$. Putting $u(t)=v(t)+\bar{w}_0(t)$ and $\theta(t)=\bar{\theta}_0(t)$, we obtain the results for the case when $B=B_\Gamma$.

The proof for the case when $B=B_D$ is similar.

LEMMA 2.4. *Let m, $g(t)$, $q(t)$ be as in Theorem 2.1 and $B=B_D$ (resp. $B=B_\Gamma$).*

Then for any $T>0$ and $[u_0, v_0, \theta_0] \in W^m_{g,q}(\Omega)$, there exists a function $f(t)$ in $\mathscr{F}^m_D[0, T]$ (resp. $\mathscr{F}^m_N[0, T]$) such that there exists a trajectory $[u(t), \theta(t)]$ in $\mathscr{E}_m[0, T]$ of the control system $[TE, g, q, B]$ with the control $f(t)$ and the initial state $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$.

PROOF. Let E be a bounded linear extension operator from $H^k(\Omega)$ to $H^k(R^n)$ $(0 \leqq k \leqq m)$ and let R be the restriction operator from $H^k(R^n)$ to $H^k(\Omega)$. Now we consider the system of equations

$$(2.15) \qquad \begin{cases} \tilde{u}_{tt} - A\tilde{u} + \alpha \operatorname{grad} E\theta = Eg & \text{in } R^n \times (0, T) \\ \theta_{tt} + \beta R \operatorname{div} \tilde{u}_t - \kappa \Delta\theta = q & \text{in } \Omega \times (0, T) \end{cases}$$

with the boundary condition $\theta = 0$ on $S \times (0, T)$ and the initial state $[\tilde{u}(0), \tilde{u}_t(0), \theta(0)] = [Eu_0, Ev_0, \theta_0]$. Here the differential operator A is thought to be defined over R^n in the same way as in Ω. Let us define the closed operator $\tilde{\mathscr{L}}$ on $\boldsymbol{H}^1(R^n) \times \boldsymbol{L}^2(R^n) \times L^2(\Omega)$ similarly to $\mathscr{L}_D$, that is,

$$\tilde{\mathscr{L}}[\tilde{u}, \tilde{v}, \theta] = [\tilde{v}, A\tilde{u} - \alpha \operatorname{grad} E\theta, -\beta R \operatorname{div} \tilde{v} + \kappa \Delta\theta],$$
$$\mathscr{D}(\tilde{\mathscr{L}}) = \boldsymbol{H}^2(R^n) \times \boldsymbol{H}^1(R^n) \times \{H^2(\Omega) \cap H^1_0(\Omega)\}.$$

Further let $\tilde{\mathscr{H}}$ be the space $\boldsymbol{H}^1(R^n) \times \boldsymbol{L}^2(R^n) \times L^2(\Omega)$ with the inner product

$$(\tilde{U}_1, \tilde{U}_2)_{\tilde{\mathscr{H}}} = \tilde{a}(\tilde{u}_1, \tilde{u}_2) + (\tilde{u}_1, \tilde{u}_2)_{R^n} + (\tilde{v}_1, \tilde{v}_2)_{R^n} + (\theta_1, \theta_2)$$

and the resulting norm $\|\tilde{U}_1\|_{\tilde{\mathscr{H}}} = (\tilde{U}_1, \tilde{U}_1)_{\tilde{\mathscr{H}}}{}^{1/2}$ for $\tilde{U}_i = [\tilde{u}_i, \tilde{v}_i, \theta_i] \in \tilde{\mathscr{H}}$ $(i=1, 2)$, where $\tilde{a}(\ ,\ )$ denotes the bilinear form on $\boldsymbol{H}^1(R^n)$ defined by A similarly to $a(\ ,\ )$ and $(\ ,\ )_{R^n}$ means the $\boldsymbol{L}^2(R^n)$-inner product. Since E is a bounded operator from $H^1(\Omega)$ to $H^1(R^n)$ and $\theta = 0$ on S, the inequalities

$$|(\operatorname{grad} E\theta, \tilde{v})_{R^n}| \leqq \text{const. } \|\theta\|_1 \|\tilde{v}\|_{R^n}$$

and

$$|(R \operatorname{div} \tilde{v}, \theta)| = |(R\tilde{v}, \operatorname{grad} \theta)| \leqq \text{const. } \|\theta\|_1 \|\tilde{v}\|_{R^n}$$

hold for any $\tilde{v} \in \boldsymbol{H}^1(R^n)$, $\theta \in H^1_0(\Omega)$. From these inequalities, it follows that $\tilde{\mathscr{L}} - \xi$ is a maximal dissipative operator for some $\xi > 0$, and hence $\tilde{\mathscr{L}}$ generates a C_0 semigroup, which we denote by $\tilde{S}(t)$. Then, by the general semigroup theory, if $\tilde{U}_0$ is in $\mathscr{D}(\tilde{\mathscr{L}})$ and $\tilde{F}(t)$ is in $\mathscr{E}^1_t(0, T; \tilde{\mathscr{H}})$, then the mild solution

$$\tilde{U}(t) = \tilde{S}(t)\tilde{U}_0 + \int_0^t \tilde{S}(t-\tau)\tilde{F}(\tau)d\tau$$

is in $\mathscr{E}^1_t(0, T; \tilde{\mathscr{H}})$ and is the strong solution of the equation $(d/dt)\tilde{U}(t) = \tilde{\mathscr{L}}\,\tilde{U}(t) + \tilde{F}(t)$. Further if $\tilde{F}(t)$ is in $\mathscr{E}^{m-1}_t(0, T; \tilde{\mathscr{H}})$ and

$$(2.16)\quad \tilde{U}_0,\ \ \tilde{U}_0 + \tilde{F}(0),\ \ \tilde{\mathscr{L}}^2\tilde{U}_0 + \tilde{\mathscr{L}}\tilde{F}(0) + \tilde{F}_t(0),\ldots,$$
$$\tilde{\mathscr{L}}^{m-2}\tilde{U}_0 + \tilde{\mathscr{L}}^{m-3}\tilde{F}(0) + \tilde{\mathscr{L}}^{m-4}\tilde{F}_t(0) + \cdots + \tilde{F}^{(m-3)}(0) \in \mathscr{D}(\tilde{\mathscr{L}}),$$

then $\tilde{U}(t)$ belongs to $\mathscr{E}_t^{m-1}(0, T; \tilde{\mathscr{H}})$. If we put $\tilde{F}(t) = [0, Eg(t), q(t)]$ and $\tilde{U}_0 = [Eu_0, Ev_0, \theta_0]$, then it is easily verified that $\tilde{F}(t)$ belongs to $\mathscr{E}_t^{m-1}(0, T; \tilde{\mathscr{H}})$ and the compatibility conditions (2.2) imply (2.16). Hence the mild solution $\tilde{U}(t)$ is in $\mathscr{E}_t^{m-1}(0, T; \tilde{\mathscr{H}})$ and if we put $\tilde{U}(t) = [\tilde{u}(t), \tilde{u}_t(t), \theta(t)]$, then $[\tilde{u}(t), \theta(t)]$ satisfies the equation (2.15). Since $[Eg(t), q(t)]$ is in $\cap_{j=0}^{m-1} \mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j-1}(R^n)) \times \cap_{j=0}^{m-1} \mathscr{E}_t^j(0, T; H^{m-j-1}(\Omega))$ when $m \geqq 2$ and in $\cap_{j=0}^{1} \mathscr{E}_t^j(0, T; \boldsymbol{H}^{1-j}(R^n)) \times \mathscr{E}_t^0(0, T; L^2(\Omega))$ when $m = 1$, $[\tilde{u}(t), \theta(t)]$ is in $\cap_{j=0}^{m} \mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j}(R^n)) \times \cap_{j=0}^{m-1} \mathscr{E}_t^j(0, T; H^{m-j}(\Omega))$ when $m \geqq 2$ and in $\cap_{j=0}^{1} \mathscr{E}_t^j(0, T; \boldsymbol{H}^{1-j}(R^n)) \times \mathscr{E}_t^0(0, T; L^2(\Omega))$ when $m=1$. Letting $u(t) = R\tilde{u}(t)$ and $f(t) = B\tilde{u}(t)$, we see that $[u(t), \theta(t)]$ belongs to $\mathscr{E}_m[0, T]$ and is a trajectory of $[TE, g, q, B]$ for the control $f(t)$ with the initial state $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$ and further, by the general trace theorem, $f(t)$ belongs to $\mathscr{F}_D^m[0, T]$ when $B = B_D$ and to $\mathscr{F}_N^m[0, T]$ when $B = B_\Gamma$. By the uniqueness of solution, $[u(t), \theta(t)]$ and $f(t)$ are the required functions.

LEMMA 2.5. *Given* $[u_1, v_1]$ *in* $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$, *let* $w(t)$ *be the function stated in Lemma* 2.3 *with* $B = B_D$, $f(t) = K_D[u_1, v_1](t)$ *and* $v(t) = L[u_1, v_1](t)$, *where* K_D *and* L *are the bounded linear operators stated in Corollary B. If* $\alpha\beta/\kappa \leqq 1$, *then there exists a constant* $c_m > 0$ *depending only on* m, A *and* Ω *such that*

$$(2.17)\quad \|w(T_0)\|_m^2 + \|w_t(T_0)\|_{m-1}^2 + (\alpha/\beta)\|\theta(T_0)\|_m^2$$
$$\leqq c_m \omega_m(\alpha, \beta, \kappa) T_0 \|\!|L|\!\|^2 \|[u_1, v_1]\|_{(m,m-1)}^2.$$

PROOF. Putting $\delta = \kappa\alpha/\beta$ and $t = T_0$ in (2.11), we have

$$(2.18)\quad (1/2)\{\|w^{(k+1)}(T_0)\|^2 + a(w^{(k)}(T_0), w^{(k)}(T_0)) + (\alpha/\beta)\|\theta^{(k)}(T_0)\|^2\}$$
$$\leqq (\alpha\beta/4\kappa) \int_0^{T_0} \|v^{(k+1)}(t)\|^2 dt.$$

By the well known Poincaré's inequality

$$\|u\|^2 \leqq \text{const.} \textstyle\sum_{i,j=1}^n \|\partial u_i/\partial x_j\|^2 \qquad (u \in \boldsymbol{H}_0^1(\Omega))$$

and (2.6), (2.17) imply

$$\|[w^{(k)}(T_0), w^{(k+1)}(T_0)]\|_{(1,0)}^2 + (\alpha/\beta)\|\theta^{(k)}(T_0)\|^2$$
$$\leqq c_0(\alpha\beta/\kappa) T_0 \sup_{0<t<T_0} \|v^{(k+1)}(t)\|^2 \qquad \text{for} \quad 0 \leqq k \leqq m-1,$$

where c_0 is a constant depending only on m, A and Ω. Since L is a bounded linear operator from $\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$ to $\cap_{j=0}^m \mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$,

$$\sup_{0<t<T_0}\|v^{(k+1)}(t)\|^2 \leqq |||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2$$

for any $[u_1, v_1]\in\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$, where $|||L|||$ denotes the operator norm of L. Thus we have

$$(2.19)\qquad \|[w^{(k)}(T_0), w^{(k+1)}(T_0)]\|_{(1,0)}^2 + (\alpha/\beta)\|\theta^{(k)}(T_0)\|^2 \\ \leqq c_0(\alpha\beta/\kappa)T_0|||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2 \qquad \text{for}\quad 0\leqq k\leqq m-1.$$

Now, we show, by induction on j,

$$(2.20)\quad \|w^{(m-j)}(T_0)\|_j^2 + \|w^{(m-j+1)}(T_0)\|_{j-1}^2 + (\alpha/\beta)\|\theta^{(m-j)}(T_0)\|_{j-1}^2 \\ \leqq c_m(\alpha\beta/\kappa)(1+\kappa^{-2(j-1)})T_0|||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2$$

for $1\leqq j\leqq m$ if $\alpha\beta/\kappa\leqq 1$, where c_m is a constant depending only on m, A and Ω. The inequality (2.19) for $k=m-1$ implies (2.20) for $j=1$. For simplicity let us put

$$e_j = (\alpha\beta/\kappa)(1+\kappa^{-2(j-1)})T_0|||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2.$$

Let us assume that (2.20) holds for $1\leqq j\leqq l\,(1\leqq l\leqq m-1)$. By the regularity theorem for elliptic systems, if

$$Aw = g \text{ in } \Omega,\quad w=0 \text{ on } S \quad\text{and}\quad \Delta\theta = h \text{ in } \Omega,\quad \theta = 0 \text{ on } S,$$

then

$$\|w\|_{k+2}\leqq C_m\|g\|_k \qquad\text{and}\qquad \|\theta\|_{k+2}\leqq C_m\|h\|_k$$

for $g\in\boldsymbol{H}^k(\Omega)$ and $h\in H^k(\Omega)$ ($k=-1, 0, 1,\dots, m-2$). Here and hereafter C_m denote constants depending only on m, A and Ω. Noting that $w(t)$ and $\theta(t)$ satisfy the equation (2.13) and the homogeneous boundary conditions, we have

$$(2.21)\qquad \|w^{(m-l-1)}(T_0)\|_{l+1}^2 \leqq C_m\|Aw^{(m-l-1)}(T_0)\|_{l-1}^2 \\ = C_m\|w^{(m-l+1)}(T_0)+\alpha\,\text{grad}\,\theta^{(m-l-1)}(T_0)\|_{l-1}^2 \\ \leqq C_m\{\|w^{(m-l+1)}(T_0)\|_{l-1}^2 + \alpha^2\|\theta^{(m-l-1)}(T_0)\|_l^2\},$$

and

$$(2.22)\qquad \alpha^2\|\theta^{(m-l-1)}(T_0)\|_l^2 \leqq C_m\alpha^2\|\Delta\theta^{(m-l-1)}(T_0)\|_{l-2}^2 \\ = C_m(\alpha^2/\kappa^2)\|\kappa\Delta\theta^{(m-l-1)}(T_0)\|_{l-2}^2 \\ \leqq C_m(\alpha^2/\kappa^2)\{\|\theta^{(m-l)}(T_0)\|_{l-1}^2 + \beta^2\|\text{div}\,w^{(m-l)}(T_0)\|_{l-1}^2 \\ + \beta^2\|\text{div}\,v^{(m-l)}(T_0)\|_{l-2}^2\}.$$

Note that the inequalities

$$\kappa^{-k}e_l \leqq 2e_{l+1}, \qquad 0 \leqq k \leqq 2,$$

hold. Hence, by (2.20) for $1 \leqq j \leqq l$, we have

$$\|w^{(m-l+1)}(T_0)\|_{l-1}^2 \leqq c_m e_l \leqq 2c_m e_{l+1}$$

and

$$\begin{aligned}&(\alpha^2/\kappa^2)\{\|\theta^{(m-l)}(T_0)\|_{l-1}^2 + \beta^2\|\operatorname{div} w^{(m-l)}(T_0)\|_{l-1}^2\\ &\qquad\qquad + \beta^2\|\operatorname{div} v^{(m-l)}(T_0)\|_{l-2}^2\}\\ &\leqq c_m(\alpha^2/\kappa^2)(\beta/\alpha + n\beta^2)e_l + n(\alpha^2\beta^2/\kappa^2)\|v^{(m-l)}(T_0)\|_{l-1}^2\\ &\leqq C_m e_{l+1}.\end{aligned}$$

In the last inequality we have used the assumption that $\alpha\beta/\kappa \leqq 1$ and the inequalities

$$\begin{aligned}&\|v^{(m-l)}(T_0)\|_{l-1}^2 \leqq \left\|\int_0^{T_0} v^{(m-l+1)}(t)\,dt\right\|_{l-1}^2\\ &\leqq T_0^2 \sup_{0<t<T_0}\|v^{(m-l+1)}(t)\|_{l-1}^2 \leqq T_0 |||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2\end{aligned}$$

when $T_0 \leqq 1$, and

$$\begin{aligned}&\|v^{(m-l)}(T_0)\|_{l-1}^2 \leq \|v^{(m-l)}(T_0)\|_l^2 \leqq |||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2\\ &\leqq T_0 |||L|||^2\|[u_1, v_1]\|_{(m,m-1)}^2\end{aligned}$$

when $T_0 > 1$. Thus, by (2.21) and (2.22), we have

$$\|w^{(m-l-1)}(T_0)\|_{l+1}^2 \leqq C_m e_{l+1}. \tag{2.23}$$

Further, by the assumption of induction, (2.22) and the above estimate, we have

$$\|w^{(m-l)}(T_0)\|_l^2 \leqq c_m e_l \leqq 2c_m e_{l+1} \quad \text{and} \quad (\alpha/\beta)\|\theta^{(m-l-1)}(T_0)\|_l^2 \leqq C_m e_{l+1}. \tag{2.24}$$

By (2.23) and (2.24), the inequality (2.20) holds for $j = l+1$. Hence (2.20) is valid for each $1 \leqq j \leqq m$. Taking $j = m$, we have (2.17) for $m \geqq 4$. Calculating directly, we have (2.17) when $m = 1$, 2 and 3.

Now, we give the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Let R be the linear operator on $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ which maps $[u_1, v_1]$ to $[w(T_0), w_t(T_0)] = [u(T_0) - u_1, u_t(T_0) - v_1]$, where $w(t)$ and $u(t)$ are the functions stated in Lemmas 2.3 and 2.5. Then the inequality (2.17) means that

(2.25) $$|||R|||^2 \leqq c_m \omega_m(\alpha, \beta, \kappa) T_0 |||L|||^2$$

if $\alpha\beta/\kappa \leqq 1$, where $|||R|||$ is the operator norm of R. Since $[u(t), \theta(t)]$ is the solution obtained in Lemma 2.3, the null state is steered to $[u_1, v_1]+R[u_1, v_1]$ at T_0 by the control $f(t)=K_D[u_1, v_1](t)$ in $\mathscr{F}_D^m[0, T_0]$ for the control system $[TE, 0, 0, B_D]$. If we put $d_m=(c_m T_0 |||L|||^2)^{-1}$ for c_m in (2.25), then

(2.26) $$\omega_m(\alpha, \beta, \kappa) < d_m$$

implies $|||R|||<1$. Hence if (2.26) is satisfied, then the operator $I+R$ is onto, so that for any $[u, v]$ in $\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$, there exist a control $f(t)\in\mathscr{F}_D^m[0, T_0]$ and an increment of temperature $\theta\in H^m(\Omega)\cap H_0^1(\Omega)$ such that $f(t)$ steers $[0, 0, 0]$ to $[u, v, \theta]$ for the control system $[TE, 0, 0, B_D]$.

By Lemma 2.4, for any $[u_0, v_0, \theta_0]\in W_{g,q}^m(\Omega)$, there exists a function $f_1(t)$ in $\mathscr{F}_D^m[0, T_0]$ which steers $[u_0, v_0, \theta_0]$ to some state $[u_1, v_1, \theta_1]$ at T_0 for the control system $[TE, g, q, B_D]$. For any $[u, v]$, let us take a control $f_2(t)$ in $\mathscr{F}_D^m[0, T_0]$ and θ such that $f_2(t)$ steers $[0, 0, 0]$ to $[u-u_1, v-v_1, \theta]$ at T_0 for the control system $[TE, 0, 0, B_D]$. Putting $f(t)=f_1(t)+f_2(t)$, we see that the control $f(t)$ steers $[u_0, v_0, \theta_0]$ to $[u, v, \theta+\theta_1]$ at T_0 for the given control system. This completes the proof of the theorem for $[TE, g, q, B_D]$.

The above controllability implies that, given $[u_0, v_0, \theta_0]$ in $W_{g,q}^m(\Omega)$ and $[u_1, v_1]$ in $\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$, there exists a solution $[u(t), \theta(t)]$ of (1.8) in $\mathscr{E}_m[0, T_0]$ satisfying (1.12), $[u(0), u_t(0), \theta(0)]=[u_0, v_0, \theta_0]$ and $[u(T_0), u_t(T_0)]=[u_1, v_1]$. If we put $f(t)=B_\Gamma u(t)$ on S for this u, then, by the general trace theorem and the uniqueness of solution, we see that $f(t)$ is in $\mathscr{F}_N^m[0, T_0]$ and steers $[u_0, v_0, \theta_0]$ to $[u_1, v_1, \theta_1]$ for the control system $[TE, g, q, B_\Gamma]$.

COROLLARY 2.1. *Let m, T_0 be as in Theorem 2.1. If $\omega_m(\alpha, \beta, \kappa)<d_m$, then there exist bounded linear operators $\bar{K}_D$ (resp. $\bar{K}_\Gamma$) and $\bar{L}$ from $\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$ to $\mathscr{F}_D^m[0, T_0]$ (resp. $\mathscr{F}_N^m[0, T_0]$) and to $\mathscr{E}_m[0, T_0]$ such that, for $[u, v]\in\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$, $\bar{K}_D[u, v]$ (resp. $\bar{K}_\Gamma[u, v]$) is the control which steers the null state $[0, 0, 0]$ to $[u, v]$ at T_0 and $\bar{L}[u, v]$ is the trajectory $[u(t), \theta(t)]$ for the control $\bar{K}_D[u, v]$ (resp. $\bar{K}_\Gamma[u, v]$) for the control system $[TE, 0, 0, B_D]$ (resp. $[TE, 0, 0, B_\Gamma]$).*

PROOF. Let M be the operator which maps $v\in\bigcap_{j=0}^m\mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$ to $[v+w, \theta]\in\mathscr{E}_m[0, T_0]$, where w is the solution of (2.13) with homogeneous boundary conditions and null initial state. It can be shown as in the above proof that M is a bounded operator. Then, $\bar{K}_D=K_D(I+R)^{-1}$, $\bar{L}=ML(I+R)^{-1}$ and $\bar{K}_\Gamma=B_\Gamma\bar{L}$ are the required operators.

In Theorem 2.1, we assume that $\omega_m(\alpha, \beta, \kappa)$ is sufficiently small. But if $g(t)=0$ and $q(t)=0$, then the subspace $\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$ is controlled by controls in $\mathscr{F}_D^m[0, T_0]$ when $B=B_D$ and in $\mathscr{F}_N^m[0, T_0]$ when $B=B_\Gamma$ without this assumption, which is seen in the following theorem. Here, for nonnegative integer k, $\tilde{X}^k(\Omega)=\{u\in \boldsymbol{H}^k(\Omega)\mid F_k u=0\}$.

THEOREM 2.2. *Let $m\geqq 2$. Then, there exists a constant $C>0$ such that, for any $[u_0, v_0]$ and $[u_1, v_1]$ in $\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$, there exist a control $f(t)$ in $\mathscr{F}_D^m[0, T_0]$ (resp. $\mathscr{F}_N^m[0, T_0]$) which steers $[u_0, v_0, 0]$ to $[u_1, v_1]$ for the control system $[TE, 0, 0, B_D]$ (resp. $[TE, 0, 0, B_\Gamma]$) at T_0 and a trajectory $[u(t), 0]$ for the control $f(t)$ satisfying the inequalities*

$$(2.27)\quad \sup_{0<t<T_0}\langle\!\langle f(t)\rangle\!\rangle_{m-1/2}\leqq C(\|[u_0, v_0]\|_{(m,m-1)}+\|[u_1, v_1]\|_{(m,m-1)})$$

$$(\text{resp.}\quad \sup_{0<t<T_0}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}\leqq C(\|[u_0, v_0]\|_{(m,m-1)}+\|[u_1, v_1]\|_{(m,m-1)})),$$

$$(2.28)\quad \|[u(t), 0]\|_{\mathscr{E}_m[0,T_0]}\leqq C(\|[u_0, v_0]\|_{(m,m-1)}+\|[u_1, v_1]\|_{(m,m-1)}).$$

PROOF. The proof is similar to that of Theorem 2.1 in [16], but slightly different. We consider the operator E_k constructed in Lemma 2.1. Then, for any $u\in \boldsymbol{H}^k(\Omega)$, there exists a 2-form p in $\mathscr{H}^{k+1}(R^n)$ such that $E_k u=\delta p$. Hence for given $[u_0, v_0]$ and $[u_1, v_1]$ in $\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$, there exist 2-forms p_i in $\mathscr{H}^{m+1}(R^n)$ and q_i in $\mathscr{H}^m(R^n)$ such that $E_m u_i=\delta p_i$ and $E_{m-1}v_i=\delta q_i$ $(i=0, 1)$. Since the restrictions of p_i and q_i to Ω belong to $\mathscr{H}^{m+1}(\Omega)$ and $\mathscr{H}^m(\Omega)$ respectively, by Russell [17], there exists a 2-form solution $p(t)$ in $\cap_{j=0}^{m+1}\mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m+1-j}(\Omega))$ of the wave equation

$$(2.29)\qquad p_{tt}(t)-\mu\Delta p(t)=0\quad\text{in }\Omega\times(0, T_0)$$

with the initial state $[p(0), p_t(0)]=[p_0, q_0]|_\Omega$ ($\equiv$the restriction of $[p_0, q_0]$ to Ω) and the final state $[p(T_0), p_t(T_0)]=[p_1, q_1]|_\Omega$. Then $u(t)=\delta p(t)$ also satisfies (2.29) and $[u(0), u_t(0)]=[\delta p_0, \delta q_0]|_\Omega=[u_0, v_0]$, $[u(T_0), u_t(T_0)]=[u_1, v_1]$, since $F_m u_i=0$ and $F_{m-1}v_i=0$ $(i=0, 1)$. Further $\operatorname{div}u(t)=-\delta u(t)=-\delta^2 p(t)=0$, since $\delta^2=0$. Thus $u(t)$ satisfies the equation

$$(2.30)\qquad u_{tt}(t)-Au(t)=0\quad\text{in }\Omega\times(0, T_0).$$

If we put $\theta(t)\equiv 0$, then $[u(t), \theta(t)]\in\mathscr{E}_m[0, T_0]$ satisfies the equation (1.8) with $g(t)=0$, $q(t)=0$ and $[u(0), u_t(0), \theta(0)]=[u_0, v_0, 0]$, $[u(T_0), u_t(T_0), \theta(T_0)]=[u_1, v_1, 0]$. Let us take $f(t)=B_D u(t)$ on S (resp. $f(t)=B_\Gamma u(t)$ on S). Then the control $f(t)$ steers $[u_0, v_0, 0]$ to $[u_1, v_1, 0]$ at T_0.

Next we show the existence of a control and the corresponding trajectory which satisfy the estimates (2.27) and (2.28). Let us denote by $S^m(0, T_0)$ the space of all $u(t)$ in $\cap_{j=0}^m\mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$ which satisfy the equation (2.30), $\operatorname{div}u(t)=0$ for all $t\in[0, T_0]$, $[u(0), u_t(0)]=[0, 0]$ and $[u(T_0), u_t(T_0)]\in\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$. Noting that $\tilde{X}^k(\Omega)$ is a closed subspace in $\boldsymbol{H}^k(\Omega)$, we easily see that $S^m(0, T_0)$ is a closed subspace in $\cap_{j=0}^m\mathscr{E}_t^j(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$. Hence it is a Banach space. Let F be the operator which maps $u(t)$ to $[u(T_0), u_t(T_0)]$. Then it is easy to see that F is a bounded linear operator from $S^m(0, T_0)$ to $\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$. We have shown that, for any $[u_1, v_1]\in\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$, there exists $u(t)\in S^m(0, T_0)$ such that $[u(T_0), u_t(T_0)]=[u_1, v_1]$. Hence the operator F is surjective. Therefore, by the open mapping theorem, F is an open map. This means that there exists a constant $C>0$ such that, for any $[u_1, v_1]\in\tilde{X}^m(\Omega)\times\tilde{X}^{m-1}(\Omega)$, there exists $u(t)\in S^m(0, T_0)$ satisfying $[u(T_0), u_t(T_0)]$ $(=F(u(t)))=[u_1, v_1]$ and

$$\textstyle\sum_{j=0}^{m}\sup_{0<t<T_0}\|u^{(j)}(t)\|_{m-j}\leqq C\|[u_1, v_1]\|_{(m,m-1)}.$$

Putting $f_1(t)=B_Du(t)$ (resp. $f_1(t)=B_\Gamma u(t)$), we have the estimates (2.27) and (2.28) when $[u_0, v_0]=[0, 0]$. Similarly we have a control $f_2(t)$ and the corresponding trajectory $\tilde{u}(t)$ such that $[\tilde{u}(0), \tilde{u}_t(0)]=[u_0, v_0]$ and $[\tilde{u}(T_0), \tilde{u}_t(T_0)]=[0, 0]$ which satisfy the estimates (2.27) and (2.28) when $[u_1, v_1]=[0, 0]$. By putting $f(t)=f_1(t)+f_2(t)$, we have the required results.

REMARK 2.4. It often appears, in mechanics, the case when the boundary condition (1.12) of the increment of temperature is replaced by

$$\gamma(x)(\partial\theta/\partial\nu)(x, t)+(1-\gamma(x))\theta(x, t)=0 \qquad \text{on} \quad S\times(0, \infty) \tag{2.31}$$

in the control system $[TE, g, q, B_D]$, where $\gamma(x)$ is a smooth function on S with $0\leqq\gamma(x)\leqq 1$. We denote this control system by $[TE, g, q]_\gamma$. But we do not obtain the exact controllability for this control system. We can show that the closed operator $\mathscr{L}_\gamma$ defined similarly to $\mathscr{L}_D$ for the boundary conditions (1.11) and (2.31), generates a C_0 semigroup on the space $\boldsymbol{H}_0^1(\Omega)\times\boldsymbol{L}^2(\Omega)\times L^2(\Omega)$. In the proof of the estimate of the resolvent, we use the Hilbert space $V_\gamma(\Omega)$ which is defined as the completion of the space of all θ each of which belongs to $C^\infty(\Omega)$ and satisfies $\theta(x)=0$ on the set $\{x\in S\,|\,\gamma(x)=0\}$ and $\|\theta\|_{V_\gamma}<\infty$ with $\|\cdot\|_{V_\gamma}$-norm defined by

$$\|\theta\|_{V_\gamma}^2=\|\theta\|_1^2+\int_{S,\gamma(x)\neq 0}\gamma^{-1}(x)(1-\gamma(x))\,|\theta(x)|^2dS;$$

cf. Inoue [6], Kaji [7].

But, for this system, we do not obtain the energy inequality of the type (2.11) appearing in Lemma 2.3. Put $S_0=\{x\in S\,|\,\gamma(x)=0\}$ and assume $\lambda+\mu=0$. By Russell [18], for the control system $[E, g, B_D]$ when $\lambda+\mu=0$, that is, the control system governed by the wave equation, the space $V^m(\Omega)\times V^{m-1}(\Omega)$ is exactly controllable at some time T by controls $f(t)$ in $\cap_{j=0}^{m-1}\mathscr{E}_t^j(0, T; \boldsymbol{H}^{m-j-1/2}(S))$ with $\operatorname{supp}f(t)\subset S_0\times[0, T]$ under the assumption that (Ω, S_0) is a "star-complemented" region. Here $V^m(\Omega)$ is the closure in $\boldsymbol{H}^m(\Omega)$ of the space of all $u\in C^\infty(\Omega)^n$ vanishing in a neighborhood of $S-S_0$. For the definition and examples of "star-complemented" regions and for the proof of exact controllability, see Russell [18], [19]. For controls $f(t)$ with $\operatorname{supp}f(t)\subset S_0\times[0, T]$ and the solutions $v(t)$

of the control system $[E, 0, B_D]$ for $f(t)$ with $[v(0), v_t(0)]=[0, 0]$, we can obtain the existence of the solutions $[u(t), \theta(t)]$ of the control system $[TE, 0, 0]_\gamma$ with $[u(0), u_t(0), \theta(0)]=[0, 0, 0]$ for this $f(t)$ and the energy inequality of the type (2.11). Further Lemma 2.4 holds with controls $f(t)$ satisfying $\operatorname{supp} f(t) \subset S_0 \times [0, T]$ if we consider the first equation of (1.8) in $R^n\text{-}\Omega^*$ with the boundary condition $u=0$ on $\partial\Omega^*$ and the second equation in Ω with $\theta=0$ on S instead of (2.15), where Ω^* is a star-shaped domain with $\partial\Omega^* \supset S_0$ and $\Omega^* \cap \Omega = \emptyset$. Thus, in this case, we obtain the exact controllability of the space

$$W^m_{g,q}(\Omega) = \{[u, v, \theta] \in V^m(\Omega) \times V^{m-1}(\Omega) \times H^m(\Omega) | \\ [u, v, \theta] \text{ satisfies the compatibility conditions}\}$$

for the control system $[TE, g, q]_\gamma$ in the same way.

In Theorem 2.1 we have shown that, when the Lame coefficients and a domain are fixed, the control system $[TE, g, q, B]$ is exactly controllable if $\omega_m(\alpha, \beta, \kappa)$ is sufficiently small. Now let the Lame coefficients λ and μ and the coefficients α, β and κ are fixed. If the domain Ω is small, then the time T_0, at which the control system $[E, g, B]$ is exactly controllable, is also small. But since the operator norm of the extension operator from Ω to R^n is not necessarily small, $|||L|||^2$ appearing in (2.25), and hence the operator norm $|||R|||$ may not be small. Thus we do not know whether the control system is exactly controllable or not under the assumption that the domain Ω is small.

Here we consider the case where Ω is shrinked in similar figures. For a domain Ω and a real number r with $0<r<1$, let us put

$$\Omega_r = r\Omega \ (\equiv \{rx \,|\, x \in \Omega\}), \quad S_r = \partial\Omega_r \ (\equiv \text{the boundary of } \Omega_r),$$

and consider the control system (1.8) in Ω_r with (1.10) or (1.11) and (1.12) on S_r. By putting $y=r^{-1}x$ and $\tau=r^{-1}t$, the equation (1.8) in Ω_r with $g(t)=0$, $q(t)=0$ and $\rho_0=1$ is reduced to the equation

$$\begin{cases} \tilde{u}_{\tau\tau}(\tau) - A_y\tilde{u}(\tau) + \alpha r \operatorname{grad}_y \tilde{\theta}(\tau) = 0 \\ \tilde{\theta}_\tau(\tau) + \beta r^{-1} \operatorname{div}_y \tilde{u}_\tau(\tau) - \kappa r^{-1} \Delta_y \tilde{\theta}(\tau) = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, \infty),$$

where $\tilde{u}(y, \tau)=u(ry, r\tau)$, $\tilde{\theta}(y, \tau)=\theta(ry, r\tau)$ and A_y, grad_y, div_y and Δ_y mean that A, grad, div and Δ are taken with respect to y respectively. By Theorem 2.1, if the inequality

$$\omega_m(\alpha r, \beta r^{-1}, \kappa r^{-1}) = \begin{cases} r(\alpha\beta/\kappa)\{1 + (r/\kappa)^{m-1}\} < d_m & \text{when } 1 \leqq m \leqq 3 \\ r(\alpha\beta/\kappa)\{1 + (r/\kappa)^{2(m-1)}\} < d_m & \text{when } m \geqq 4 \end{cases}$$

holds, then each initial state in $W^m_{g,q}(\Omega)$ is controllable to any state in $\boldsymbol{H}^m(\Omega) \times$

$\boldsymbol{H}^{m-1}(\Omega)$ at $\tau = T_0$. Since $\omega_m(\alpha r, \beta r^{-1}, \kappa r^{-1}) \to 0$ as $r \to 0$, we have the following

COROLLARY 2.2. *Let λ, μ satisfying (1.5), α, β and κ be given and let Ω be a bounded domain with smooth boundary. Then there exists $r_m > 0$ such that, if $0 < r < r_m$, then for the control system $[TE, g, q, B_D]$ (resp. $[TE, g, q, B_\Gamma]$) in Ω_r with the boundary S_r, each initial state in $W^m_{g,q}(\Omega_r)$ is steered to any state in $\boldsymbol{H}^m(\Omega_r) \times \boldsymbol{H}^{m-1}(\Omega_r)$ at rT_0 by a control $f(t)$ in $\mathscr{F}^m_D[0, rT_0]$ (resp. $\mathscr{F}^m_N[0, rT_0]$). Here T_0 is the time stated in Theorem A for the domain Ω.*

3. Admissible controllability with constrained controls

As is stated in the introduction, we now introduce a constraint set and consider what sort of deformations can be controlled by controls in this constraint set.

For a subset G in $\boldsymbol{L}^2(S)$ and an integer $m \geqq 1$ we define the constraint set of controls $\mathscr{F}^m_D(G)$ as

$$\mathscr{F}^m_D(G) = \{f(t) \in \cap_{j=0}^{m-1} \mathscr{E}^j_t(0, \infty; \boldsymbol{H}^{m-j-1/2}(S)) \,|\, f(t) \in G \quad \text{for all } t \in [0, \infty)\},$$

and, for an integer $m \geqq 2$, we define $\mathscr{F}^m_N(G)$ as

$$\mathscr{F}^m_N(G) = \{f(t) \in \cap_{j=0}^{m-2} \mathscr{E}^j_t(0, \infty; \boldsymbol{H}^{m-j-3/2}(S)) \,|\, f(t) \in G \quad \text{for all } t \in [0, \infty)\}.$$

DEFINITION 3.1. (1) For the control system $[TE, g, q, B_\Gamma]$ a state $[u_0, v_0, \theta_0] \in W^m_{g,q}(\Omega)$ is said to be *admissibly controllable* to a state $[u_1, v_1] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ in the constraint set $\mathscr{F}^m_N(G)$, if there exist a positive time T and a control $f(t)$ in $\mathscr{F}^m_N(G)$ such that $f(t)$ steers $[u_0, v_0, \theta_0]$ to $[u_1, v_1]$ at the time T, i.e., there exists a solution $[u(t), \theta(t)]$ in $\mathscr{E}_m[0, T]$ of the control system $[TE, g, q, B_\Gamma]$ for the control $f(t)$ which satisfies $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$ and $[u(T), u_t(T)] = [u_1, v_1]$.
(2) A subset D of $W^m_{g,q}(\Omega)$ is said to be *admissibly controllable* to a subset R of $\boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ in the constraint set $\mathscr{F}^m_N(G)$, if any $[u_0, v_0, \theta_0]$ in D is admissibly controllable to any $[u_1, v_1]$ in R in the constraint set $\mathscr{F}^m_N(G)$.

The admissible controllability in the constraint set $\mathscr{F}^m_D(G)$ for the control system $[TE, g, q, B_D]$ is defined similarly.

REMARK 3.1. As is in the case of the isotropic elastodynamic system $[E, g, B]$, if an initial state $[u_0, v_0, \theta_0]$ is steered to a final state $[u_1, v_1, \theta_1]$ by a control $f(t)$ at a time T for the control system $[TE, g, q, B]$, then u_0 and u_1 have to satisfy the compatibility conditions

$$Bu_0 = f(0) \quad \text{and} \quad Bu_1 = f(T) \quad \text{on} \quad S.$$

Thus if a subset D is admissibly controllable in $\mathscr{F}^m_D(G)$ (resp. $\mathscr{F}^m_N(G)$), then D is contained in the set $\{[u, v, \theta] \in W^m_{g,q}(\Omega) \mid Bu \in G\}$, where $B = B_D$ (resp. $B = B_\Gamma$).

Now we put

$$\mathscr{M}^m_D(G) = \{[u, v, \theta] \in W^m_{g,q}(\Omega) \mid B_D u \in G\}$$

and

$$\mathscr{M}^m_\Gamma(G) = \{[u, v, \theta] \in W^m_{g,q}(\Omega) \mid B_\Gamma u \in G\}.$$

First we consider the control system $[TE, g, q, B_\Gamma]$. We begin with the following

LEMMA 3.1. *Let $m=2$ or 3, G be an open and connected subset of $\boldsymbol{H}^{m-3/2}(S)$ containing 0. Then, for any $[u_0, v_0, \theta_0]$ in $\mathscr{M}^m_\Gamma(G)$, there exist an element $[u_1, v_1, \theta_1]$ in $\mathscr{D}(\mathscr{L}^{m-1}_\Gamma)$ and a control $f(t)$ in $\mathscr{F}^m_N(G)$ such that $f(t)$ steers $[u_0, v_0, \theta_0]$ to $[u_1, v_1, \theta_1]$ at the time T_0 given in Theorem A, for the control system $[TE, 0, 0, B_\Gamma]$.*

PROOF. By Lemma 2.4, for $[u_0, v_0, \theta_0] \in W^m_{0,0}(\Omega)$, there exist a function $\bar{f}(t)$ in $\mathscr{F}^m_N[0, T_0]$ and a solution $[\bar{u}(t), \bar{\theta}(t)] \in \mathscr{E}_m[0, T_0]$ of the control system $[TE, 0, 0, B_\Gamma]$ for the control $\bar{f}(t)$ with the initial state $[\bar{u}(0), \bar{u}_t(0), \bar{\theta}(0)] = [u_0, v_0, \theta_0]$. Then, by Theorem A, there exists a function $\tilde{f}(t)$ in $\mathscr{F}^m_N[0, T_0]$ for which there exists a solution $v(t)$ in $\cap^m_{j=0} \mathscr{E}^j_t(0, T_0; \boldsymbol{H}^{m-j}(\Omega))$ of the control system $[E, 0, B_\Gamma]$ for the control $\tilde{f}(t)$ with $[v(0), v(0)] = [0, 0]$ and $[v(T_0), v_t(T_0)] = -[\bar{u}(T_0), \bar{u}_t(T_0)]$. For this control $\tilde{f}(t)$, by Lemma 2.3, there exists a solution $[\tilde{u}(t), \tilde{\theta}(t)]$ in $\mathscr{E}_m[0, T_0]$ of the control system $[TE, 0, 0, B_\Gamma]$ with the initial state $[\tilde{u}(0), \tilde{u}_t(0), \tilde{\theta}(0)] = [0, 0, 0]$. Since G is open and connected and since $\bar{f}(0) + \tilde{f}(0) = B_\Gamma \bar{u}(0) + B_\Gamma v(0) = B_\Gamma u_0 \in G$ and $\bar{f}(T_0) + \tilde{f}(T_0) = B_\Gamma \bar{u}(T_0) + B_\Gamma v(T_0) = 0 \in G$, there exists a function $\hat{f}(t)$ in $[C^\infty(S \times [0, T_0])]^n$ which satisfies

$$\hat{f}^{(j)}(0) = \hat{f}^{(j)}(T_0) = 0, \quad 0 \leqq j \leqq m,$$

and $\bar{f}(t) + \tilde{f}(t) + \hat{f}(t) \in G$ for all $t \in [0, T_0]$. Since $\hat{f}(t)$ is smooth, there exists a solution $[\hat{u}(t), \hat{\theta}(t)]$ in $\mathscr{E}_m[0, T_0]$ of the control system $[TE, 0, 0, B_\Gamma]$ for the control $f(t)$ with the null initial state. Putting $u(t) = \bar{u}(t) + \tilde{u}(t) + \hat{u}(t)$, $\theta(t) = \bar{\theta}(t) + \tilde{\theta}(t) + \hat{\theta}(t)$ and $f(t) = \bar{f}(t) + \tilde{f}(t) + \hat{f}(t)$, we see that $[u(t), \theta(t)]$ is the trajectory of the control system $[TE, 0, 0, B_\Gamma]$ for the control $f(t)$ with the initial state $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$. For this $[u(t), \theta(t)]$, we have

$$B_\Gamma u(T_0) = f(T_0) = 0 \qquad \text{when} \quad m = 2, 3,$$

$$B_\Gamma u_t(T_0) = f_t(T_0) = \bar{f}_t(T_0) + \tilde{f}_t(T_0) = B_\Gamma \bar{u}_t(T_0) + B_\Gamma v_t(T_0) = 0$$

when $m=3$ and

$$\theta(T_0) = 0 \qquad \text{on} \quad S \quad \text{when} \quad m = 2, 3,$$

$$-\beta \operatorname{div} u_t(T_0) + \kappa \Delta \theta(T_0) = \theta_t(T_0) = 0 \qquad \text{on} \quad S \quad \text{when} \quad m = 3,$$

since $\theta(t)=0$ on S. This means that $[u(T_0), u_t(T_0), \theta(T_0)]$ belongs to $\mathscr{D}(\mathscr{L}_T^{m-1})$. Thus $[u_0, v_0, \theta_0]$ is steered to $[u(T_0), u_t(T_0), \theta(T_0)]$ in $\mathscr{D}(\mathscr{L}_T^{m-1})$ by the control $f(t)=\bar{f}(t)+\tilde{f}(t)+\hat{f}(t)$ in the constraint set $\mathscr{F}_N^m(G)$.

We define the energy semi-norm E_m on $\mathscr{D}(\mathscr{L}_T^m)$ as

$$E_0[u, v, \theta] = a(u, u) + \|v\|^2 + \int_S \langle \Gamma u, u\rangle dS + (\alpha/\beta)\|\theta\|^2$$

$$E_m[u, v, \theta] = \textstyle\sum_{j=0}^m E_0[\mathscr{L}_T^j[u, v, \theta]]$$

and a finite dimensional subspace $\mathscr{R}_\Gamma$ of $\boldsymbol{H}^m(\Omega)$ as

$$\mathscr{R}_\Gamma = \{\phi \in C^\infty(\bar{\Omega})^n \mid a(\phi, \phi) + \int_S \langle \Gamma\phi, \phi\rangle dS = 0\}.$$

Then we have

LEMMA 3.2. *Let $m=2$ or 3. Then there exists a constant $\hat{\omega}_m (0<\hat{\omega}_m \leqq d_m/4)$, depending only on m, A, Γ and Ω, for which the following holds:*

If $\omega_m(\alpha, \beta, \kappa)$ is smaller than $\hat{\omega}_m$, then for any $\varepsilon, \eta>0$ and $[u_0, v_0, \theta_0] \in \mathscr{D}(\mathscr{L}_T^{m-1})$, there exist a positive time T, a control $f(t)$ in $\mathscr{F}_N^m[0, T]$ with $\sup_{0<t<T} \langle\!\langle f(t)\rangle\!\rangle_{m-3/2}<\eta$ and $[u_1, v_1, \theta_1] \in \mathscr{D}(\mathscr{L}_T^{m-1})$ such that $E_{m-1}[u_1, v_1, \theta_1]<\varepsilon$ and $f(t)$ steers $[u_0, v_0, \theta_0]$ to $[u_1, v_1, \theta_1]$ at T for the control system $[TE, 0, 0, B_\Gamma]$.

PROOF. First we define a semi-norm $\|\cdot\|_e$ on $\mathscr{D}(\mathscr{A}_\Gamma)$ as

$$\|[u, v]\|_e = \{a(u, u) + \|v\|^2 + \int_S \langle \Gamma u, u\rangle dS + a(v, v) + \|Au\|^2 + \int_S \langle \Gamma v, v\rangle dS\}^{1/2}.$$

By Duvaut and Lions [2, Théorème 3.4], the bilinear form

$$a(u, v) + \int_S \langle \Gamma u, v\rangle dS \qquad \text{for} \quad u, v \in \boldsymbol{H}^1(\Omega)$$

induces canonically a norm on $\boldsymbol{H}^1(\Omega)/\mathscr{R}_\Gamma$, which is equivalent to the standard quotient norm of $\boldsymbol{H}^1(\Omega)/\mathscr{R}_\Gamma$. Namely, there exists a constant $\gamma_0>0$ such that the inequality

$$\gamma_0^{-1}\{a(u, u) + \int_S \langle \Gamma u, u\rangle dS\} \leqq \inf_{\phi\in\mathscr{R}_\Gamma} \|u + \phi\|_1^2 \leqq \gamma_0\{a(u, u) + \int_S \langle \Gamma u, u\rangle dS\} \tag{3.1}$$

holds for any $u \in \boldsymbol{H}^1(\Omega)$. Since $\mathscr{R}_\Gamma \times \{0\} \subset \mathscr{D}(\mathscr{A}_\Gamma)$ and $A\phi=0$ for $\phi \in \mathscr{R}_\Gamma$, it follows that $\|\cdot\|_e$ induces cannonically a norm on $\mathscr{D}(\mathscr{A}_\Gamma)/(\mathscr{R}_\Gamma \times \{0\})$ which is equivalent to the standard quotient norm of $\mathscr{D}(\mathscr{A}_\Gamma)/(\mathscr{R}_\Gamma \times \{0\})$. Here $\mathscr{D}(\mathscr{A}_\Gamma)$ is thought to be a Banach space endowed with the graph norm. Namely, there exists a constant $\gamma_1 > 0$ such that the inequality

$$\gamma_1^{-1}\|[u, v]\|_e \leqq \inf_{\phi\in\mathscr{R}_\Gamma} \|[u + \phi, v]\|_{(2,1)} \leqq \gamma_1\|[u, v]\|_e$$

holds for any $[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma)$, since the graph norm of $\mathscr{D}(\mathscr{A}_\Gamma)$ is equivalent to $\|\cdot\|_{(2,1)}$.

By Corollary B, if $\omega_m(\alpha, \beta, \kappa) < d_m$, then there exists a positive constant δ_0 such that, for any $[u, v] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ with $\|[u, v]\|_{(m,m-1)} \leqq \delta_0$, there exists a control $f(t)$ in $\mathscr{F}_N^m[0, T_0]$ with $\sup_{0<t<T_0} \langle\!\langle f(t) \rangle\!\rangle_{m-3/2} < \eta$ which steers $[0, 0, 0]$ to $[u, v]$ at T_0 for the control system $[TE, 0, 0, B_\Gamma]$. Now let $m=2$ and let $\varepsilon < \min\{1/3, \gamma_1^{-1}\delta_0\}$. Given $[u_0, v_0, \theta_0] \in \mathscr{D}(\mathscr{L}_\Gamma)$, Let $[\tilde{u}(t), \tilde{\theta}(t)]$ be the trajectory in $\mathscr{E}_2[0, \infty)$ of the control system $[TE, 0, 0, B_\Gamma]$ for the null control $f(t) \equiv 0$ with the initial state $[\tilde{u}(0), \tilde{u}_t(0), \tilde{\theta}(0)] = [u_0, v_0, \theta_0]$. Then we have the following energy equality in the same way as in the proof of (2.14):

$$\begin{aligned}(3.2)\qquad &(1/2)\{\|\tilde{u}^{(j+1)}(T)\|^2 + a(\tilde{u}^{(j)}(T), \tilde{u}^{(j)}(T)) \\ &\qquad + \int_S \langle \Gamma\tilde{u}^{(j)}(T), \tilde{u}^{(j)}(T)\rangle dS + (\alpha/\beta)\|\tilde{\theta}^{(j)}(T)\|^2\} \\ &\qquad + (\kappa\alpha/\beta)\int_0^T \|\mathrm{grad}\, \tilde{\theta}^{(j)}(t)\|^2 dt \\ &= (1/2)\{\|\tilde{u}^{(j+1)}(0)\|^2 + a(\tilde{u}^{(j)}(0), \tilde{u}^{(j)}(0)) \\ &\qquad + \int_S \langle \Gamma\tilde{u}^{(j)}(0), \tilde{u}^{(j)}(0)\rangle dS + (\alpha/\beta)\|\tilde{\theta}^{(j)}(0)\|^2\},\end{aligned}$$

$j=0, 1$, for any $T>0$. Thus we have

$$\begin{aligned}&(\kappa\alpha/\beta)\left\{\int_0^\infty \|\mathrm{grad}\, \tilde{\theta}(t)\|^2 dt + \int_0^\infty \|\mathrm{grad}\, \tilde{\theta}_t(t)\|^2 dt\right\} \\ &\leqq (1/2)E_1[u_0, v_0, \theta_0].\end{aligned}$$

By Poincaré's inequality, we have

$$\int_0^\infty \{\|\tilde{\theta}(t)\|_1^2 + \|\tilde{\theta}_t(t)\|_1^2\} dt \leqq \text{const.}\, E_1[u_0, v_0, \theta_0].$$

Now assume that $\zeta > 0$ satisfies

$$2\alpha^2\zeta^2 + 2(\alpha/\beta)\zeta^2 < \varepsilon/3.$$

Since $\tilde{\theta}(t) \in \mathscr{E}_t^0(0, \infty; H^2(\Omega)) \cap \mathscr{E}_t^1(0, \infty; L^2(\Omega))$, there exists a time $T_1 \geqq T_0$ such that

(3.3) $\|\tilde{\theta}(T_1)\| < \zeta, \quad \|\tilde{\theta}_t(T_1)\| < \zeta \quad \text{and} \quad \|\operatorname{grad}\tilde{\theta}(T_1)\| < \zeta.$

Put $[\tilde{u}(T_1), \tilde{u}_t(T_1), \tilde{\theta}(T_1)] = [\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1]$. Then by (3.2), we have

(3.4) $$E_1[\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1] \leqq E_1[u_0, v_0, \theta_0].$$

If

$$\|[\tilde{u}_1, \tilde{v}_1]\|_e^2 \leqq \varepsilon/3,$$

then we have

$$\begin{aligned} E_1[\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1] &= a(\tilde{u}_1, \tilde{u}_1) + \|\tilde{v}_1\|^2 + \int_S \langle \Gamma\tilde{u}_1, \tilde{u}_1\rangle dS + (\alpha/\beta)\|\tilde{\theta}_1\|^2 \\ &\quad + a(\tilde{v}_1, \tilde{v}_1) + \|A\tilde{u}_1 - \alpha \operatorname{grad}\tilde{\theta}_1\|^2 + \int_S \langle \Gamma\tilde{v}_1, \tilde{v}_1\rangle dS \\ &\quad + (\alpha/\beta)\| - \beta \operatorname{div}\tilde{v}_1 + \kappa\Delta\tilde{\theta}_1\|^2 \\ &\leqq 2\|[\tilde{u}_1, \tilde{v}_1]\|_e^2 + 2\alpha^2\|\operatorname{grad}\tilde{\theta}_1\|^2 + (\alpha/\beta)\{\|\tilde{\theta}_1\|^2 + \|\tilde{\theta}_t(T_1)\|^2\} \\ &\leqq 2\varepsilon/3 + 2\alpha^2\zeta^2 + 2(\alpha/\beta)\zeta^2 < \varepsilon. \end{aligned}$$

Next we consider the case when $\|[\tilde{u}_1, \tilde{v}_1]\|_e^2 > \varepsilon/3$. Put $\|[\tilde{u}_1, \tilde{v}_1]\|_e = r$. Since $0<\varepsilon<1/3$, $r>\varepsilon$. Let $0<\delta<\varepsilon$. For $(\delta/r)[\tilde{u}_1, \tilde{v}_1]$, there exists a function ϕ_0 in $\mathscr{R}_\Gamma$ satisfying

(3.5) $$\|[\delta\tilde{u}_1/r + \phi_0, \delta\tilde{v}_1/r]\|_{(2,1)} \leqq \gamma_1(\delta/r)\|[\tilde{u}_1, \tilde{v}_1]\|_e = \gamma_1\delta.$$

Put $[\bar{u}, \bar{v}] = [\delta\tilde{u}_1/r + \phi_0, \delta\tilde{v}_1/r]$. Then, since

$$\|[\bar{u}, \bar{v}]\|_{(2,1)} \leqq \gamma_1\delta < \gamma_1\varepsilon < \delta_0,$$

the null state $[0, 0, 0]$ can be steered to $-[\bar{u}, \bar{v}]$ at T_0 by a control $f_0(t)$ in $\mathscr{F}_N^2[0, T_0]$ with $\sup_{0<t<T_0} \langle\!\langle f_0(t)\rangle\!\rangle_{1/2} < \eta$. Further, by Lemma 2.5 and the proof of Theorem 2.1, we can take $f_0(t)$ so that

$$(\alpha/\beta)\|\bar{\theta}^{(2-j)}(T_0)\|_{j-1}^2 \leqq c_2\omega_2(\alpha, \beta, \kappa)T_0 |||L|||^2 \|[u_1, v_1]\|_{(2,1)}^2$$

with $[u_1, v_1] = (I+R)^{-1}[\bar{u}, \bar{v}]$. Hence,

(3.6) $$\begin{aligned} (\alpha/\beta)\|\bar{\theta}^{(2-j)}(T_0)\|_{j-1}^2 &\leqq c_2\omega_2(\alpha, \beta, \kappa)T_0 |||L|||^2 |||(I+R)^{-1}|||^2 \|[\bar{u}, \bar{v}]\|_{(2,1)}^2 \\ &\leqq c_2\omega_2(\alpha, \beta, \kappa)T_0 |||L|||^2 (1 - |||R|||)^{-2} \|[\bar{u}, \bar{v}]\|_{(2,1)}^2 \\ &\leqq a_2^2(1-a_2)^{-2}\|[\bar{u}, \bar{v}]\|_{(2,1)}^2 \end{aligned}$$

for $j=1, 2$ where $a_2 = \{\omega_2(\alpha, \beta, \kappa)d_2^{-1}\}^{1/2}$ $(\leqq 1/2)$. Let us put $f_1(t)=0$ on $[0, T_1 - T_0]$, $f_1(t) = f_0(t + T_0 - T_1)$ on $[T_1 - T_0, T_1]$. Then $f_1(t)$ steers $[0, 0, 0]$ to $-[\bar{u}, \bar{v}, \bar{\theta}]$ at T_1 with some $\bar{\theta} \in H^2(\Omega) \cap H_0^1(\Omega)$. If we denote by $[\bar{u}(t), \bar{\theta}(t)]$ the corresponding trajectory, then we have, by (3.6)

$$(3.7)\quad (\alpha/\beta)\{\|\bar{\theta}(T_1)\|_1^2 + \|\bar{\theta}_t(T_1)\|^2\} \leqq a_2^2(1-a_2)^{-2}\|[\bar{u}, \bar{v}]\|_{(2,1)}^2 \leqq \omega_2(\alpha, \beta, \kappa)\gamma_1^2\delta^2$$

provided that $\omega_2(\alpha, \beta, \kappa) \leqq d_2/4$. It is easy to see that the control $f_1(t)$ steers $[u_0, v_0, \theta_0]$ to $[\tilde{u}_1-\bar{u}, \tilde{v}_1-\bar{v}, \hat{\theta}_1-\bar{\theta}]$ at T_1 for the control system $[TE, 0, 0, B_\Gamma]$. Since $\bar{u}=\delta\tilde{u}_1/r+\phi_0$ and $\bar{v}=\delta\tilde{v}_1/r$, we have

$$\begin{aligned}
&E_1[\tilde{u}_1 - \bar{u}, \tilde{v}_1 - \bar{v}, \hat{\theta}_1 - \bar{\theta}] \\
&= a(\tilde{u}_1-\bar{u}, \tilde{u}_1-\bar{u}) + \|\tilde{v}_1 - \bar{v}\|^2 + \int_S \langle \Gamma(\tilde{u}_1-\bar{u}), \tilde{u}_1-\bar{u}\rangle dS \\
&\quad + (\alpha/\beta)\|\hat{\theta}_1 - \bar{\theta}\|^2 + a(\tilde{v}_1-\bar{v}, \tilde{v}_1-\bar{v}) + \|A(\tilde{u}_1-\bar{u}) - \alpha\,\mathrm{grad}\,(\hat{\theta}_1-\bar{\theta})\|^2 \\
&\quad + \int_S \langle \Gamma(\tilde{v}_1-\bar{v}), \tilde{v}_1 - \bar{v}\rangle dS + (\alpha/\beta)\|\hat{\theta}_t(T_1) - \bar{\theta}_t(T_1)\|^2 \\
&= a(\tilde{u}_1, \tilde{u}_1) + \|\tilde{v}_1\|^2 + \int_S \langle \Gamma\tilde{u}_1, \tilde{u}_1\rangle dS + (\alpha/\beta)\|\hat{\theta}_1\|^2 + a(\tilde{v}_1, \tilde{v}_1) \\
&\quad + \|A\tilde{u}_1 - \alpha\,\mathrm{grad}\,\hat{\theta}_1\|^2 + \int_S \langle \Gamma\tilde{v}_1, \tilde{v}_1\rangle dS + (\alpha/\beta)\|\hat{\theta}_t(T_1)\|^2 \\
&\quad - (2\delta/r-\delta^2/r^2)\{a(\tilde{u}_1, \tilde{u}_1) + \|\tilde{v}_1\|^2 + \int_S \langle \Gamma\tilde{u}_1, \tilde{u}_1\rangle dS \\
&\qquad\qquad + a(\tilde{v}_1, \tilde{v}_1) + \|A\tilde{u}_1\|^2 + \int_S \langle \Gamma\tilde{v}_1, \tilde{v}_1\rangle dS\} \\
&\quad + 2\alpha(1-\delta/r)(A\tilde{u}_1, \mathrm{grad}\,\bar{\theta}) + (\alpha/\beta)\|\bar{\theta}\|^2 + \alpha^2\|\,\mathrm{grad}\,\bar{\theta}\|^2 \\
&\quad + (\alpha/\beta)\|\bar{\theta}_t(T_1)\|^2 + 2(\delta\alpha/r)(A\tilde{u}_1, \mathrm{grad}\,\tilde{\theta}_1) - 2\alpha^2(\mathrm{grad}\,\tilde{\theta}_1, \mathrm{grad}\,\bar{\theta}) \\
&\quad - (2\alpha/\beta)\{(\tilde{\theta}_1, \bar{\theta}) + (\tilde{\theta}_t(T_1), \theta_t(T_1))\} \\
&= E_1[\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1] - (2\delta/r-\delta^2/r^2)\|\,[\tilde{u}_1, \tilde{v}_1]\|_e^2 \\
&\qquad + 2\alpha(1-\delta/r)(A\tilde{u}_1, \mathrm{grad}\,\bar{\theta}) + H_1 + H_2.
\end{aligned}$$

Here

$$\begin{aligned}
H_1 &= (\alpha/\beta)\|\bar{\theta}\|^2 + \alpha^2\|\,\mathrm{grad}\,\bar{\theta}\|^2 + (\alpha/\beta)\|\bar{\theta}_t(T_1)\|^2, \\
H_2 &= 2(\delta\alpha/r)(A\tilde{u}_1, \mathrm{grad}\,\tilde{\theta}_1) - 2\alpha^2(\mathrm{grad}\,\tilde{\theta}_1, \mathrm{grad}\,\bar{\theta}) \\
&\quad - (2\alpha/\beta)\{(\tilde{\theta}_1, \bar{\theta}) + (\tilde{\theta}_t(T_1), \bar{\theta}_t(T_1))\}.
\end{aligned}$$

By the well-known L^2-estimate, (3.5) and (3.7), we have

$$\begin{aligned}
(3.8)\quad \alpha^2\|\,\mathrm{grad}\,\bar{\theta}\|^2 &\leqq C\alpha^2\|\Delta\bar{\theta}\|^2 \leqq C(\alpha^2/\kappa^2)\{\|\bar{\theta}_t(T_1)\|^2 + \beta^2\|\mathrm{div}\,\bar{u}_t(T_1)\|^2\} \\
&\leqq C(\alpha^2/\kappa^2)\{(\beta/\alpha)\omega_2(\alpha, \beta, \kappa)\gamma_1^2\delta^2 + n\beta^2\|\bar{v}\|_1^2\} \\
&\leqq C(1+n)(\alpha^2\beta^2/\kappa^2)(1+1/\kappa)^2\delta^2 = C(1+n)\omega_2(\alpha, \beta, \kappa)^2\delta^2
\end{aligned}$$

with a constant $C>0$ depending only on Ω. Then by (3.7) and (3.8), we have

$$|H_1| \leqq \{C(1+n) + \gamma_1^2\}\omega_2(\alpha, \beta, \kappa)\delta^2$$

and noting that $\|A\tilde{u}_1\| \leqq \|[\tilde{u}_1, \tilde{v}_1]\|_e = r$, by (3.3), (3.7) and (3.8), we have

$$\begin{aligned}(3.9) \quad |H_2| &\leqq 2\zeta\{\alpha\delta + C^{1/2}(1+n)^{1/2}\alpha\omega_2(\alpha, \beta, \kappa)\delta + 2(\alpha/\beta)^{1/2}\omega_2(\alpha, \beta, \kappa)^{1/2}\gamma_1\delta\} \\ &\leqq 2\zeta r\{\alpha + C^{1/2}(1+n)^{1/2}\alpha + 2(\alpha/\beta)^{1/2}\gamma_1\} \\ &= 2\zeta r J(\alpha, \beta),\end{aligned}$$

where

$$J(\alpha, \beta) = \alpha + C^{1/2}(1+n)^{1/2}\alpha + 2(\alpha/\beta)^{1/2}\gamma_1.$$

Since

$$\begin{aligned} r^2 &= \|[\tilde{u}_1, \tilde{v}_1]\|_e^2 \\ &= a(\tilde{u}_1, \tilde{u}_1) + \|\tilde{v}_1\|^2 + \int_S \langle \Gamma\tilde{u}_1, \tilde{u}_1\rangle dS \\ &\qquad + a(\tilde{v}_1, \tilde{v}_1) + \|A\tilde{u}_1\|^2 + \int_S \langle \Gamma\tilde{v}_1, \tilde{v}_1\rangle dS \\ &\leqq a(\tilde{u}_1, \tilde{u}_1) + \|\tilde{v}_1\|^2 + \int_S \langle \Gamma\tilde{u}_1, \tilde{u}_1\rangle dS + a(\tilde{v}_1, \tilde{v}_1) \\ &\quad + 2\|A\tilde{u}_1 - \alpha \operatorname{grad}\tilde{\theta}_1\|^2 + \int_S \langle \Gamma\tilde{v}_1, \tilde{v}_1\rangle dS + 2\alpha^2\|\operatorname{grad}\tilde{\theta}_1\|^2 \\ &\leqq 2E_1[u_1, v_1, \theta_1] + 2\alpha^2\|\operatorname{grad}\tilde{\theta}_1\|^2 \\ &\leqq 2E_1[u_0, v_0, \theta_0] + 2\alpha^2\|\operatorname{grad}\tilde{\theta}_1\|^2, \end{aligned}$$

we have by (3.9)

$$\begin{aligned}|H_2| &\leqq 2\zeta J(\alpha, \beta)\{2E_1[u_0, v_0, \theta_0] + 2\alpha^2\|\operatorname{grad}\tilde{\theta}_1\|^2\}^{1/2} \\ &\leqq 4\zeta J(\alpha, \beta)\{E_1[u_0, v_0, \theta_0] + \alpha^2\zeta^2\}^{1/2}.\end{aligned}$$

Further we have by (3.8)

$$\begin{aligned}&|2(1-\delta/r)(A\tilde{u}_1, \alpha \operatorname{grad}\tilde{\theta})| \leqq 2(1-\delta/r)\|[\tilde{u}_1, \tilde{v}_1]\|_e\|\alpha \operatorname{grad}\tilde{\theta}\| \\ &\leqq 2(1-\delta/r)rC^{1/2}(1+n)^{1/2}\omega_2(\alpha, \beta, \kappa)\delta \\ &= 2C^{1/2}(1+n)^{1/2}\omega_2(\alpha, \beta, \kappa)(r\delta - \delta^2).\end{aligned}$$

Hence, if $\{C(1+n)+\gamma_1^2\}\omega_2(\alpha, \beta, \kappa) < 1/2$ and $C^{1/2}(1+n)^{1/2}\omega_2(\alpha, \beta, \kappa) < 1/4$, then we have

$$
\begin{aligned}
&E_1[\tilde{u}_1 - \bar{u}, \tilde{v}_1 - \bar{v}, \tilde{\theta}_1 - \bar{\theta}] \\
&\leqq E_1[\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1] - (2r\delta - \delta^2) + \{C(1+n) + \gamma_1^2\}\omega_2(\alpha, \beta, \kappa)\delta^2 \\
&\quad + C^{1/2}(1+n)^{1/2}\omega_2(\alpha, \beta, \kappa)(r\delta - \delta^2) \\
&\quad + 4\zeta J(\alpha, \beta)\{E_1[u_0, v_0, \theta_0] + \alpha^2\zeta^2\}^{1/2} \\
&\leqq E_1[\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1] - r\delta/2 + 4\zeta J(\alpha, \beta)\{E_1[u_0, v_0, \theta_0] + \alpha^2\zeta^2\}^{1/2}.
\end{aligned}
$$

Therefore, if $\delta = \varepsilon/2$ and $\zeta > 0$ has been so chosen that

$$4\zeta J(\alpha, \beta)\{E_1[u_0, v_0, \theta_0] + \alpha^2\zeta^2\}^{1/2} < \varepsilon^2/8,$$

then

$$
\begin{aligned}
E_1[\tilde{u}_1 - \bar{u}, \tilde{v}_1 - \bar{v}, \tilde{\theta}_1 - \bar{\theta}] &\leqq E_1[\tilde{u}_1, \tilde{v}_1, \tilde{\theta}_1] - \varepsilon^2/8 \\
&\leqq E_1[u_0, v_0, \theta_0] - \varepsilon^2/8.
\end{aligned}
$$

Further, clearly, $[\tilde{u}_1 - \bar{u}, \tilde{v}_1 - \bar{v}, \tilde{\theta}_1 - \bar{\theta}]$ belongs to $\mathscr{D}(\mathscr{L}_\Gamma)$. Let us put $\hat{\omega}_2 = \min\{d_2/4, (1/2)\{C(1+n)+\gamma_1^2\}^{-1}, (1/4)C^{-1/2}(1+n)^{-1/2}\}$ and assume $\omega_2(\alpha, \beta, \kappa) < \hat{\omega}_2$. Then we have seen that, under the assumption $\|[\tilde{u}_1, \tilde{v}_1]\|_e^2 < \varepsilon/3$, we can take $[u_1, v_1, \theta_1]$ in $\mathscr{D}(\mathscr{L}_\Gamma)$ such that

$$E_1[u_1, v_1, \theta_1] \leqq E_1[u_0, v_0, \theta_0] - \varepsilon^2/8$$

and $[u_0, v_0, \theta_0]$ can be steered to $[u_1, v_1, \theta_1]$ at T_1 by some control $f_1(t)$ in $\mathscr{F}_N^2[0, T_1]$ with $\sup_{0<t<T_1} \langle\!\langle f_1(t) \rangle\!\rangle_{1/2} < \eta$.

Next we start with the initial state $[u_1, v_1, \theta_1]$ at $t = T_1$. In the sequel we say that a control $f(t)$ steers a state $[u, v, \theta]$ at $t = T$ to a state $[\tilde{u}, \tilde{v}]$ or $[\tilde{u}, \tilde{v}, \tilde{\theta}]$ at $t = \tilde{T}$ $(> T)$ and $[u(t), \theta(t)]$ is a trajectory which connects $[u, v, \theta]$ at T and $[\tilde{u}, \tilde{v}, \tilde{\theta}]$ at $\tilde{T}$ for the control system $[TE, g, q, B_\Gamma]$, if there exists a solution $[u(t), \theta(t)]$ of $[TE, g, q, B_\Gamma]$ for the control $f(t)$ such that $[u(T), u_t(T), \theta(T)] = [u, v, \theta]$, $[u(\tilde{T}), u_t(\tilde{T}), \theta(\tilde{T})] = [\tilde{u}, \tilde{v}, \tilde{\theta}]$. Further we define $\mathscr{F}_N^m[T, \tilde{T}]$ and $\mathscr{E}_m[T, \tilde{T}]$ similarly to $\mathscr{F}_N^m[0, T]$ and $\mathscr{E}_m[0, T]$. Let $[\hat{u}(t), \hat{\theta}(t)]$ denote the trajectory in $\mathscr{E}_2[T_1, \infty)$ of the control system $[TE, 0, 0, B_\Gamma]$ for the null control $f(t) = 0$ with the initial state $[\hat{u}(T_1), \hat{u}_t(T_1), \hat{\theta}(T_1)] = [u_1, v_1, \theta_1]$. Then the equality (3.2) for $\hat{u}(t)$ and T_1 in the place of $\tilde{u}$ and 0 holds, and hence there exists a time $T_2 > T_1$ such that (3.3) holds for $\hat{\theta}(T_2)$, $\hat{\theta}_t(T_2)$ and $\operatorname{grad}\hat{\theta}(T_2)$. Put $[\tilde{u}_2, \tilde{v}_2] = [\hat{u}(T_2), \hat{u}_t(T_2)]$. Then, by the same arguments as above, we see that, if $\|[\tilde{u}_2, \tilde{v}_2]\|_e^2 > \varepsilon/3$, then there exist $[u_2, v_2, \theta_2]$ in $\mathscr{D}(\mathscr{L}_\Gamma)$ satisfying

$$E_1[u_2, v_2, \theta_2] \leqq E_1[u_1, v_1, \theta_1] - \varepsilon^2/8.$$

and a control $f_2(t)$ in $\mathscr{F}_N^2[T_1, T_2]$ with $\sup_{T_1<t<T_2} \langle\!\langle f_2(t) \rangle\!\rangle_{1/2} < \eta$ which steers $[u_1, v_1, \theta_1]$ at T_1 to $[u_2, v_2, \theta_2]$ at T_2 for the control system $[TE, 0, 0, B_\Gamma]$. Repeating this procedure, there is an integer $N \geqq 1$ such that we can take $[\tilde{u}_n, \tilde{v}_n] \in$

$H^2(\Omega) \times H^1(\Omega)$, $[u_n, v_n, \theta_n] \in \mathscr{D}(\mathscr{L}_\Gamma)$, $T_n > T_{n-1}$ and $f_n(t) \in \mathscr{F}_N^2[T_{n-1}, T_n]$ with $\sup_{T_{n-1}<t<T_n} \langle\!\langle f_n(t) \rangle\!\rangle_{1/2} < \eta$, for which the null control steers $[u_{n-1}, v_{n-1}, \theta_{n-1}]$ at T_{n-1} to $[\tilde{u}_n, \tilde{v}_n]$ at T_n, $f_n(t)$ steers $[u_{n-1}, v_{n-1}, \theta_{n-1}]$ at T_{n-1} to $[u_n, v_n, \theta_n]$ at T_n and the inequalities

$$E_1[u_n, v_n, \theta_n] \leqq E_1[u_{n-1}, v_{n-1}, \theta_{n-1}] - \varepsilon^2/8$$

hold for $1 \leqq n \leqq N$, and either the inequality

$$\|[\tilde{u}_N, \tilde{v}_N]\|_e^2 \leqq \varepsilon/3 \quad \text{or} \quad E_1[u_N, v_N, \theta_N] < \varepsilon$$

holds.

Putting $f(t) = f_n(t)$, $T_{n-1} \leqq t \leqq T_n$, $n = 1, 2, \ldots, N-1$, and

$$f(t) = \begin{cases} 0 & \text{when} \quad \|[u_N, v_N]\|_e^2 < \varepsilon/3 \\ f_N(t) & \text{when} \quad E_1[u_N, v_N, \theta_N] < \varepsilon, \end{cases}$$

$T_{N-1} \leqq t \leqq T_N$, we obtain the required control in the case of $m = 2$.

In case $m = 3$, we consider the energy $E_2[u, v, \theta]$ and the semi-norm

$$\begin{aligned} \|[u, v]\|_{\tilde{e}} = \{ &\|[u, v]\|_e^2 + \|Av\|^2 \\ &+ \int_S \langle \Gamma(Au - (\alpha\beta/\kappa) \operatorname{grad} [\Delta^{-1} (\operatorname{div} v)]), Au - (\alpha\beta/\kappa) \operatorname{grad} [\Delta^{-1} (\operatorname{div} v)] \rangle dS \\ &+ a(Au - (\alpha\beta/\kappa) \operatorname{grad} [\Delta^{-1} (\operatorname{div} v)], Au - (\alpha\beta/\kappa) \operatorname{grad} [\Delta^{-1} (\operatorname{div} v)])\}^{1/2} \end{aligned}$$

instead of the energy $E_1[u, v, \theta]$ and the semi-norm $\|[u, v]\|_e$ respectively. Here we denote by Δ^{-1} Green's operator related to Dirichlet's homogeneous problem for Δ, i.e., if $f \in L^2(\Omega)$, then $\Delta^{-1} f$ is defined as the solution u of the problem

$$u \in H_0^1(\Omega) \cap H^2(\Omega), \quad \Delta u = f \quad \text{in } \Omega.$$

It is well known that Δ^{-1} is a bounded linear operator from $L^2(\Omega)$ to $H_0^1(\Omega) \cap H^2(\Omega)$ and from $H^1(\Omega)$ to $H_0^1(\Omega) \cap H^3(\Omega)$. If $[u_0, v_0, \theta_0]$ belongs to $\mathscr{D}(\mathscr{L}_\Gamma^2)$, then for $\zeta > 0$, we can take a positive time T such that, in addition to (3.3), the inequalities

$$\|\operatorname{grad} \theta_t(T)\| < \zeta, \quad \|\theta_{tt}(T)\| < \zeta$$

and

$$\|\operatorname{grad} \Delta^{-1}\theta_t(T)\|_1 \leqq \|\Delta^{-1}\theta_t(T)\|_2 \leqq \text{const.}\, \|\theta_t(T)\| < \zeta$$

hold for the trajectory $[u(t), \theta(t)]$ of the control system $[TE, 0, 0, B_\Gamma]$ for the null control $f(t) = 0$ with the initial state $[u(0), u_t(0), \theta(0)] = [u_0, v_0, \theta_0]$. Since

$$\begin{aligned} Au(T) - \alpha \operatorname{grad} \theta(T) = Au(T) &- (\alpha\beta/\kappa) \operatorname{grad} [\Delta^{-1} (\operatorname{div} u_t(T))] \\ &- (\alpha/\kappa) \operatorname{grad} [\Delta^{-1}(\theta_t(T))], \end{aligned}$$

if $\alpha\beta/\kappa$ is small, then the semi-norm $\|\cdot\|_{\tilde{e}}$ induces canonically the norm on $\mathscr{D}(\mathscr{A}_{\Gamma}^{2})/(\mathscr{R}_{\Gamma}\times\{0\})$ which is equivalent to the standard quotient norm of $\mathscr{D}(\mathscr{A}_{\Gamma}^{2})/(\mathscr{R}_{\Gamma}\times\{0\})$; and the norm of $\mathscr{D}(\mathscr{A}_{\Gamma}^{2})$ is equivalent to $\|\cdot\|_{(3,2)}$. Thus we can prove the result for the case $m=3$ by the same procedures as in the case of $m=2$, although estimates are much more complicated. We omit the details of the proof.

LEMMA 3.3. *For any $\eta>0$ there exists a constant $\delta>0$ satisfying the following*:

If $T>nT_0$ for a positive integer n, then for any $\phi\in\mathscr{R}_{\Gamma}$ with $\|\phi\|_m\leqq n\delta$, there exists a control $f(t)\in\mathscr{F}_N^m[0, T]$ with $\sup_{0<t<T}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}<\eta$ which steers $[0, 0, 0]$ to $[\phi, 0, 0]$ at the time T for the control system $[TE, 0, 0, B_{\Gamma}]$.

PROOF. It is easy to see that any element ϕ in $\mathscr{R}_{\Gamma}$ is represented as $\phi=\delta p$ with some 2-form p in $\mathscr{H}^{m+1}(\Omega)$. Hence, in the same way as in the proof of Theorem 2.2, we see that there exists a control $f(t)$ in $\mathscr{F}_N^m[0, T_0]$ which steers $[0, 0, 0]$ to $[\phi, 0, 0]$ at the time T_0 and, further, satisfies the inequality $\sup_{0<t<T_0}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}\leqq\text{const.}\,\|\phi\|_m$. Thus there exists $\delta>0$ such that, for any ϕ_0 in $\mathscr{R}_{\Gamma}$ with $\|\phi_0\|_m\leqq\delta$, the state $[0, 0, 0]$ is steered to $[\phi_0, 0, 0]$ at T_0 by a control $f_0(t)$ in $\mathscr{F}_N^m[0, T_0]$ with $\sup_{0<t<T_0}\langle\!\langle f_0(t)\rangle\!\rangle_{m-3/2}<\eta$. Noting that $[u(t), \theta(t)]=[\phi_0, 0]$, $0\leqq t\leqq T_0$, is a trajectory of the control system $[TE, 0, 0, B_{\Gamma}]$ for the null control $f(t)=0$, we see that the state $[\phi_0, 0, 0]$ is steered to $[2\phi_0, 0, 0]$ at T_0 by the control $f_0(t)$. In this way, we see that the null state is steered to $[n\phi_0, 0, 0]$ at nT_0 by the control $f(t)=f_0(t-kT_0)$, $kT_0\leqq t\leqq(k+1)T_0$, $k=0, 1,\dots, n-1$. For any $T\geqq nT_0$, put $f(t)=0$, $0\leqq t\leqq T-nT_0$, $f(t)=f(t-T+T_0)$, $T-nT_0\leqq t\leqq T$. Then, clearly this control $f(t)$ belongs to $\mathscr{F}_N^m[0, T]$, satisfies $\sup_{0<t<T}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}<\eta$ and steers the null state to $[\phi, 0, 0]$ at T for any $\phi\in\mathscr{R}_{\Gamma}$ with $\|\phi\|_m\leqq n\delta$.

LEMMA 3.4. *Given $\eta>0$ there exist $\varepsilon>0$ and $\bar{T}>0$ such that, for any $u(t)\in\mathscr{E}_t^0(0, \infty; \boldsymbol{H}^m(\Omega))\cap\mathscr{E}_t^1(0, \infty; \boldsymbol{H}^{m-1}(\Omega))$ with $\sup_{0<t<\infty}\|u_t(t)\|<\varepsilon$ and any $T>\bar{T}$, there exists a control $f(t)$ in $\mathscr{F}_N^m[0, T]$ with $\sup_{0<t<T}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}<\eta$ which steers $[0, 0, 0]$ to $[-Pu(T), 0, 0]$ at the time T for the control system $[TE, 0, 0, B_{\Gamma}]$. Here P is the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ to $\mathscr{R}_{\Gamma}$.*

PROOF. Let $r=\sup_{0<t<\infty}\|u_t(t)\|$. Then we have

$$\|Pu(T)\|\leqq\|u(T)\|\leqq\|u(0)\|+\int_0^{(n+1)T_0}\|u_t(\tau)\|d\tau\leqq\|u(0)\|+(n+1)rT_0.$$

for any $T\in[nT_0, (n+1)T_0]$. Since the subspace $\mathscr{R}_{\Gamma}$ is finite dimensional, there exists a constant γ_2 such that

$$\|\phi\|_m\leqq\gamma_2\|\phi\|\qquad\text{for all}\quad\phi\in\mathscr{R}_{\Gamma}.$$

Hence

$$\|Pu(T)\|_m\leqq\gamma_2\|u(0)\|+nr\gamma_2T_0\qquad\text{for all}\quad T\in[nT_0, (n+1)T_0].$$

For given $\eta>0$, let us take $\delta>0$ stated in Lemma 2.3 and put $\varepsilon=\delta/2\gamma_2 T_0$. Then, if $r<\varepsilon$, then choosing n_0 so large that

$$\gamma_2\|u(0)\| + (n_0+1)r\gamma_2 T_0 < n_0\delta$$

we have

$$\|Pu(T)\|_m < n\delta \qquad \text{for any} \quad T\in[nT_0, (n+1)T_0],\ n \geqq n_0. \tag{3.10}$$

Put $\bar{T}=n_0T_0$ and let $T>\bar{T}$. Then taking an integer n so that $nT_0\leqq T<(n+1)T_0$, by Lemma 3.3, we see that the null state can be steered to $-[Pu(T), 0, 0]$ at T by a control $f(t)$ in $\mathscr{F}_N^m[0, T]$ with $\sup_{0<t<T}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}<\eta$ for the control system $[TE, 0, 0, B_\Gamma]$, since (3.10) holds.

LEMMA 3.5. *Let $m=2$ or 3. Then there exist positive constants $\bar{\omega}_m$ and $\bar{d}_m$, depending only on m, A, Γ and Ω, such that if $\omega_m(\alpha, \beta, \kappa)<\bar{\omega}_m$, then the following is satisfied.*

For any $\eta>0$ and $[u_1, v_1]$ in $\mathscr{D}(\mathscr{A}_\Gamma^{m-1})$ with $\|[u_1, v_1]\|_{(m,m-1)}\leqq\bar{d}_m\eta/\omega_m(\alpha, \beta, \kappa)$, there exist a positive time T and a control $f(t)$ in $\mathscr{F}_N^m[0, T]$ with $\sup_{0<t<T}\langle\!\langle f(t)\rangle\!\rangle_{m-3/2}<\eta$ which steers $[0, 0, 0]$ to $[u_1, v_1]$ at T for the control system $[TE, 0, 0, B_\Gamma]$.

PROOF. We define a semi-norms $\|\cdot\|_{e_m}$ on $\mathscr{D}(\mathscr{A}_\Gamma^m)$ as

$$\|[u, v]\|_{e_0} = \{a(u, u) + \|v\|^2 + \int_S \langle\Gamma u, u\rangle dS\}^{1/2}$$

$$\text{for} \quad [u, v]\in \boldsymbol{H}^1(\Omega)\times \boldsymbol{L}^2(\Omega),$$

$$\|[u, v]\|_{e_m} = \{\textstyle\sum_{j=0}^m \|\mathscr{A}_\Gamma^j[u, v]\|_{e_0}^2\}^{1/2} \qquad \text{for} \quad [u, v]\in\mathscr{D}(\mathscr{A}_\Gamma^m).$$

Then $\{\|[u, v]\|_{e_m}^2+\|u\|^2\}^{1/2}$ defines a norm on $\mathscr{D}(\mathscr{A}_\Gamma^m)$ which is equivalent to the graph norm. Let Z_m be the orthogonal complement of $\mathscr{R}_\Gamma\times\{0\}$ in the space $\mathscr{D}(\mathscr{A}_\Gamma^m)$ with respect to the inner product defined by the norm $\{\|[u, v]\|_{e_m}^2+\|u\|^2\}^{1/2}$ and Q_m be the orthogonal projection from $\mathscr{D}(\mathscr{A}_\Gamma^m)$ to Z_m with respect to this inner product. By (3.1), the inequality

$$\|u\|^2 \leqq \gamma_0\{a(u, u) + \int_S \langle\Gamma u, u\rangle dS\} \tag{3.11}$$

holds if $u\in\boldsymbol{H}^1(\Omega)$ and $(u, \phi)=0$ for any $\phi\in\mathscr{R}_\Gamma$. Since $(u, \phi)=0$ for any $[u, v]\in Z_{m-1}$ and $\phi\in\mathscr{R}_\Gamma$ and the norm $\{\|[u, v]\|_{e_{m-1}}^2+\|u\|^2\}^{1/2}$ is equivalent to $\|\cdot\|_{(m,m-1)}$ by (2.6), (3.11) implies that $\|\cdot\|_{e_{m-1}}$ is a norm on Z_{m-1} which is equivalent to the standard norm induced by $\|\cdot\|_{(m,m-1)}$. Hence there exists a constant $\gamma_m>0$ satisfying

$$\|[u, v]\|_{(m,m-1)} \leqq \gamma_m\|[u, v]\|_{e_{m-1}} \qquad \text{for any} \quad [u, v]\in Z_{m-1}. \tag{3.12}$$

Let $V(t)$ be the semigroup generated by $\mathscr{A}_\Gamma$. Then the energy equality

$$\|V(t)[u, v]\|_{e_{m-1}} = \|[u, v]\|_{e_{m-1}}$$

holds for any $[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1})$ and $t \geqq 0$; cf. [15]. For any $[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1})$ and positive integer n, we put $[u_0, v_0] = [u, -v]$,

$$[u_{k+1}, v_{k+1}] = V(T_0)[u_k, v_k] - (1/(n-k))Q_{m-1}V(T_0)[u_k, v_k], \quad 0 \leqq k \leqq n-1,$$

(3.13) $\tilde{f}_k(t) = -(1/(n-k))K_\Gamma[Q_{m-1}V(T_0)[u_k, v_k]](t), \quad 0 \leqq t \leqq T_0,$

and $\tilde{f}(t) = \tilde{f}_k(t - kT_0)$, $kT_0 \leqq t \leqq (k+1)T_0$, $0 \leqq k \leqq n-1$, where K_Γ is the bounded linear operator stated in Corollary B. Let $\tilde{v}(t)$ be the trajectory of the control system $[E, 0, B_\Gamma]$ for the control $\tilde{f}(t)$ with the initial state $[u_0, v_0]$. Then we easily see that

$$\tilde{v}(t) = \tilde{P}V(t - kT_0)[u_k, v_k] - (1/(n-k))L[Q_{m-1}V(T_0)[u_k, v_k]](t)$$

on $[kT_0, (k+1)T_0]$, $0 \leqq k \leqq n-1$, where $\tilde{P}[u, v] = u$ for $[u, v] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega)$ and L is the bounded linear operator stated in Corollary B. Since $Q_{m-1}^2 = Q_{m-1}$ and

$$\|Q_{m-1}[u, v]\|_{e_{m-1}} = \|[u, v]\|_{e_{m-1}} \qquad \text{for any} \quad [u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1}),$$

we have

$$\begin{aligned}
\|[u_k, v_k]\|_{e_{m-1}} &= \|V(T_0)[u_{k-1}, v_{k-1}] - (1/(n-k+1))Q_{m-1}V(T_0)[u_{k-1}, v_{k-1}]\|_{e_{m-1}} \\
&= (1 - 1/(n-k+1))\|Q_{m-1}V(T_0)[u_{k-1}, v_{k-1}]\|_{e_{m-1}} \\
&= (1 - 1/(n-k+1))\|V(T_0)[u_{k-1}, v_{k-1}]\|_{e_{m-1}} \\
&= (1 - 1/(n-k+1))\|[u_{k-1}, v_{k-1}]\|_{e_{m-1}} = \cdots \\
&= (1 - k/n)\|[u_0, v_0]\|_{e_{m-1}}, \quad 1 \leqq k \leqq n.
\end{aligned}$$

Hence, if $t \in [kT_0, (k+1)T_0]$ $(0 \leqq k \leqq n-1)$, then

$$\begin{aligned}
(3.14) \qquad \langle\!\langle \tilde{f}(t) \rangle\!\rangle_{m-3/2} &\leqq (1/(n-k))|||K_\Gamma|||\,\|Q_{m-1}V(T_0)[u_k, v_k]\|_{(m,m-1)} \\
&\leqq (\gamma_m/(n-k))|||K_\Gamma|||\,\|Q_{m-1}V(T_0)[u_k, v_k]\|_{e_{m-1}} \\
&\leqq c_m(1/n)\|[u_0, v_0]\|_{(m,m-1)}
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{v}^{(j+1)}(t)\| &\leqq \|V(t - kT_0)[u_k, v_k]\|_{e_{m-1}} \\
&\quad + (1/(n-k))|||L|||\,\|Q_{m-1}V(T_0)[u_k, v_k]\|_{(m,m-1)} \\
&\leqq (1 - k/n)\|[u_0, v_0]\|_{e_{m-1}} + \gamma_m(1/n)|||L|||\,\|[u_0, v_0]\|_{e_{m-1}} \\
&\leqq c_m\{(n-k+1)/n\}\|[u_0, v_0]\|_{e_{m-1}}, \quad 0 \leqq j \leqq m-1,
\end{aligned}$$

where c_m is a constant depending only on m, A, Γ and Ω. Put

$$v(t) = \tilde{v}(nT_0 - t) \quad \text{and} \quad f(t) = \tilde{f}(nT_0 - t). \tag{3.15}$$

Since

$$\|[v(0), v_t(0)]\|_{e_{m-1}} = \|[\tilde{v}(nT_0), -\tilde{v}_t(nT_0)]\|_{e_{m-1}} = 0,$$

$v(0) \in \mathscr{R}_\Gamma$ and $v_t(0) = 0$. As in the proof of Lemma 2.3, we see that there exists a solution $[u(t), \theta(t)]$ in $\mathscr{E}_m[0, nT_0]$ of the control system $[TE, 0, 0, B_\Gamma]$ for the control $f(t)$ with the initial state $[u(0), u_t(0), \theta(0)] = [v(0), 0, 0]$, and $w(t) = u(t) - v(t)$ satisfies the inequalities

$$\begin{aligned}
&(1/2)\{\|w^{(j+1)}(nT_0)\|^2 + a(w^{(j)}(nT_0), w^{(j)}(nT_0)) \\
&\quad + \int_S \langle \Gamma w^{(j)}(nT_0), w^{(j)}(nT_0)\rangle dS + (\alpha/\beta)\|\theta^{(j)}(nT_0)\|^2\} \\
&\leqq (\alpha\beta/4\kappa)\int_0^{nT_0} \|v^{(j+1)}(t)\|^2 dt \\
&\leqq (\alpha\beta/4\kappa)c_m^2 T_0 \textstyle\sum_{k=0}^{n-1}\{(k+2)/n\}^2 \|[u_0, v_0]\|_{e_{m-1}}^2 \\
&\leqq (\alpha\beta/4\kappa)c_m^2 (n+1)T_0 \|[u_0, v_0]\|_{(m,m-1)}^2, \quad 0 \leqq j \leqq m-1.
\end{aligned}$$

In a way similar to the proof of Lemma 2.5, we have

$$\begin{aligned}
\|Q_{m-1}[w(nT_0), w_t(nT_0)]\|_{e_{m-1}}^2 &= \|[w(nT_0), w_t(nT_0)]\|_{e_{m-1}}^2 \\
&\leqq c_m \omega_m(\alpha, \beta, \kappa) nT_0 \|[u_0, v_0]\|_{(m,m-1)}^2
\end{aligned} \tag{3.16}$$

with a constant c_m depending only on m, A, Γ and Ω. Clearly the operator which maps $[u, v] = [u_0, -v_0]$ to $v(t)$ is linear. Thus the operator R_n which maps $[u, v]$ to $[w(nT_0), w_t(nT_0)]$ is linear. Further if $[u_0, -v_0] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1})$, then $[w(nT_0), w_t(nT_0)]$ belongs to $\mathscr{D}(\mathscr{A}_\Gamma^{m-1})$. Putting $\tilde{R}_n = Q_{m-1}R_n$, we see that $\tilde{R}_n$ is a linear operator on Z_{m-1}. Further the inequalities (3.12), (3.16) and the equality $\|[u, v]\|_{(m,m-1)} = \|[u_0, v_0]\|_{(m,m-1)}$ imply the inequality

$$|||\tilde{R}_n|||^2 \leqq \bar{\gamma}_m \omega_m(\alpha, \beta, \kappa) nT_0 \tag{3.17}$$

with a constant $\bar{\gamma}_m$ depending only on m, A, Γ and Ω. As in the proof of Theorem 2.1, $(I + \tilde{R}_n)[u, v] = Q_{m-1}[u(nT_0), u_t(nT_0)]$, where $[u(t), \theta(t)]$ is the trajectory of the control system $[TE, 0, 0, B_\Gamma]$ for the control $f(t)$, which is given by (3.13) and (3.15), with the initial state $[u(0), u_t(0), \theta(0)] = [v(0), 0, 0]$. Put

$$\tilde{B}(r) = \{[u, v] \in Z_{m-1} \mid \|[u, v]\|_{(m,m-1)} < r\}$$

for a positive number r. Then, by (3.14), if $[u, v] \in \tilde{B}(n\eta/(2c_m))$, then $\langle\!\langle f(t) \rangle\!\rangle_{m-3/2} < \eta/2$ on $[0, nT_0]$. Now assume that

(3.18) $$8\omega_m(\alpha, \beta, \kappa)\bar{\gamma}_m T_0 < 1,$$

and take a positive integer n_0 such that

(3.19) $$\{4\omega_m(\alpha, \beta, \kappa)\bar{\gamma}_m T_0\}^{-1} - 1 < n_0 \leqq \{4\omega_m(\alpha, \beta, \kappa)\bar{\gamma}_m T_0\}^{-1}.$$

Then, by (3.17),

$$|||\tilde{R}_{n_0}||| \leqq \{\bar{\gamma}_m \omega_m(\alpha, \beta, \kappa) n_0 T_0\}^{1/2} \leqq 1/2.$$

Hence $(I+\tilde{R}_{n_0})^{-1}$ exists and

$$|||(I+\tilde{R}_{n_0})^{-1}||| \leqq (1-|||\tilde{R}_{n_0}|||)^{-1} \leqq 2.$$

Therefore

$$(I+\tilde{R}_{n_0})\tilde{B}(n_0\eta/(2c_m)) \supset \tilde{B}(n_0\eta/(4c_m)).$$

By (3.18) and (3.19),

$$\begin{aligned} n_0\eta/4c_m &\geqq (\eta/4c_m)\{4\omega_m(\alpha, \beta, \kappa)\bar{\gamma}_m T_0\}^{-1} - 1 \\ &\geqq \eta/\{32 c_m \omega_m(\alpha, \beta, \kappa)\bar{\gamma}_m T_0\}. \end{aligned}$$

Putting

$$\bar{\omega}_m = 1/(8\bar{\gamma}_m T_0) \quad \text{and} \quad \hat{d}_m = 1/(32 c_m \bar{\gamma}_m T_0),$$

we have shown that, for any $[\bar{u}, \bar{v}] \in Z_{m-1}$ with $\|[\bar{u}, \bar{v}]\|_{(m,m-1)} \leqq \hat{d}_m\eta/\omega_m(\alpha, \beta, \kappa)$, there exist some elements $[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1})$ with $Q_{m-1}[u, v] = [\bar{u}, \bar{v}]$, $\phi_0 \in \mathscr{R}_\Gamma$, $\theta \in H^m(\Omega) \cap H_0^1(\Omega)$ and a control $\bar{f}(t) \in \mathscr{F}_N^m[0, n_0 T_0]$ with $\sup_{0<t<n_0T_0} \langle\langle \bar{f}(t) \rangle\rangle_{m-3/2} < \eta/2$ such that $\bar{f}(t)$ steers $[\phi_0, 0, 0]$ to $[u, v, \theta]$ at $n_0 T_0$ for the control system $[TE, 0, 0, B_\Gamma]$. Note that the inequalities

$$\begin{aligned} \|Q_{m-1}[u, v]\|_{(m,m-1)} &\leqq \gamma_m \|Q_{m-1}[u, v]\|_{e_{m-1}} \\ &\leqq c'_m \gamma_m \|[u, v]\|_{(m,m-1)} \end{aligned}$$

hold for any $[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1})$ with a positive constant c'_m depending only on m, A and Ω. Putting $\bar{d}_m = \hat{d}_m/(c'_m\gamma_m)$, we see that, for any $[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1})$ with $\|[u, v]\|_{(m,m-1)} \leqq \bar{d}_m\eta/\omega_m(\alpha, \beta, \kappa)$, there exist functions ϕ_0, $\phi_1 \in \mathscr{R}_\Gamma$, $\theta \in H^m(\Omega) \cap H_0^1(\Omega)$ and a control $\bar{f}(t) \in \mathscr{F}_N^m[0, n_0 T_0]$ with $\sup_{0<t<n_0T_0} \langle\langle \bar{f}(t) \rangle\rangle_{m-3/2} < \eta/2$ such that $\bar{f}(t)$ steers $[\phi_0, 0, 0]$ to $[u+\phi_1, v, \theta]$. Subtracting ϕ_0 from the trajectory, we see that $\bar{f}(t)$ steers $[0, 0, 0]$ to $[u+\phi, v, \theta]$ for some $\phi \in \mathscr{R}_\Gamma$.

By Lemma 3.3, the null state is steered to $[-\phi, 0, 0]$ at a time $T(\geqq n_0 T_0)$ by a control $\hat{f}(t)$ in $\mathscr{F}_N^m[0, T]$ with $\sup_{0<t<T} \langle\langle \hat{f}(t) \rangle\rangle_{m-3/2} < \eta/2$. Putting $f(t) = \hat{f}(t)$, $0 \leqq t \leqq T - n_0 T_0$, $f(t) = \bar{f}(t - T + n_0 T_0) + \hat{f}(t)$, $T - n_0 T_0 \leqq t \leqq T$, we easily see that the control $f(t)$ steers $[0, 0, 0]$ to $[u, v]$ at T and satisfies $\sup_{0<t<T} \langle\langle f(t) \rangle\rangle_{m-3/2} < \eta$. Thus we obtain the result.

Put $\tilde{\omega}_m = \min\{\hat{\omega}_m, \bar{\omega}_m\}$, where $\hat{\omega}_m$, $\bar{\omega}_m$ are constants given in Lemmas 3.2 and 3.5 respectively. Let

$$B_m(r) = \{[u, v] \in \mathscr{D}(\mathscr{A}_\Gamma^{m-1}) \mid \|[u, v]\|_{(m,m-1)} < r\} \qquad \text{for} \quad r > 0.$$

PROPOSITION 3.1. *Let $m=2$ or 3, G be an open and connected subset of $\boldsymbol{H}^{m-3/2}(S)$ containing 0 and $g(t) \in \bigcap_{j=0}^{m-1} \mathscr{E}_t^j(0, \infty; H^{m-j-1}(\Omega)) \cap W^{m-1,1}(0, \infty; L^2(\Omega))$, $q(t) \in \bigcap_{j=0}^{m-1} \mathscr{E}_t^j(0, \infty; H^{m-j-1}(\Omega)) \cap W^{m-1,1}(0, \infty; L^2(\Omega))$. Further let us assume that in case $m=2$,*

$$\|g(t)\| + \|q(t)\| \longrightarrow 0 \quad as \quad t \longrightarrow \infty,$$

and in case $m=3$, $q(t) \in H_0^1(\Omega)$ for all $t>0$ and

$$\|g_t(t)\| + \|q_t(t)\| + \|g(t)\|_1 + \|q(t)\|_2 \longrightarrow 0 \quad as \quad t \longrightarrow \infty.$$

If $\omega_m(\alpha, \beta, \kappa) < \tilde{\omega}_m$, then, for the control system $[TE, g, q, B_\Gamma]$, the subset $\mathscr{M}_\Gamma^m(G)$ is admissibly controllable to the set $B_m(\bar{d}_m\eta/2\omega_m(\alpha, \beta, \kappa))$ in the constraint set $\mathscr{F}_N^m(G)$.

Here η is a positive constant such that the η-neighborhood of the origin in $\boldsymbol{H}^{m-3/2}(S)$ is contained in G and $\bar{d}_m$ is the constant stated in Lemma 3.5.

PROOF. Noting $[0, g(t), q(t)] \in \bigcap_{j=0}^{m-1} \mathscr{E}_t^j(0, \infty; \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega))$ and $[0, g(0), q(0)] \in \mathscr{D}(\mathscr{L}_\Gamma)$ when $m=3$, by the general semigroup theory, we see that the null control $f(t)=0$ steers the null state to a state in $\mathscr{D}(\mathscr{L}_\Gamma^{m-1})$ at the time T_0 for the control system $[TE, g, q, B_\Gamma]$. By Lemma 3.1, any state $[u_0, v_0, \theta_0]$ in $\mathscr{M}_\Gamma^m(G)$ is steered to some state in $\mathscr{D}(\mathscr{L}_\Gamma^{m-1})$ by a control $f_0(t)$ in $\mathscr{F}_N^m(G)$ at the time T_0 for the control system $[TE, 0, 0, B_\Gamma]$. Hence we easily see that $f_0(t)$ steers $[u_0, v_0, \theta_0]$ to some state $[u_1, v_1, \theta_1]$ in $\mathscr{D}(\mathscr{L}_\Gamma^{m-1})$ at T_0 for the control system $[TE, g, q, B_\Gamma]$.

We give the proof for $m=2$. By the assumptions on $g(t)$ and $q(t)$, for any $\varepsilon>0$ we can take T_1 ($\geqq T_0$) such that

$$\|g(t)\| + (\alpha/\beta)^{1/2}\|q(t)\| < \varepsilon \qquad \text{for} \quad t \geqq T_1 \tag{3.20}$$

and

$$\int_{T_1}^{\infty} \{\|g^{(j)}(t)\| + (\alpha/\beta)^{1/2}\|q^{(j)}(t)\|\}dt < \varepsilon, \quad j = 0, 1. \tag{3.21}$$

Put $f_1(t)=f_0(t)$, $0 \leqq t \leqq T_0$, $f_1(t)=0$, $T_0 \leqq t \leqq T_1$, and let $[u_0, v_0, \theta_0]$ be steered to $[u_2, v_2, \theta_2]$ at T_1 by $f_1(t)$. Since $[u_2, v_2, \theta_2]$ is in $\mathscr{D}(\mathscr{L}_\Gamma)$, we see, by Lemma 3.2, that for any $\varepsilon>0$, $[u_2, v_2, \theta_2]$ at T_1 is steered to some $[u_3, v_3, \theta_3] \in \mathscr{D}(\mathscr{L}_\Gamma)$ with $E_1[u_3, v_3, \theta_3] < \varepsilon^2$ at T_2 ($\geqq T_1$) by a control $f_2(t)$ in $\mathscr{F}_N^2(G)$ for the control system $[TE, 0, 0, B_\Gamma]$. Let $[\bar{u}(t), \bar{\theta}(t)]$ ($t \geqq T_1$) be the trajectory for the null control $f(t) \equiv 0$ with initial state $[\bar{u}(T_1), \bar{u}_t(T_1), \bar{\theta}(T_1)] = [0, 0, 0]$ for the control system

$[TE, g, q, B_\Gamma]$ and $S_\Gamma(t)$ be the semigroup generated by $\mathscr{L}_\Gamma$. Putting $V(t)=[\bar{u}(t), \bar{u}_t(t), \bar{\theta}(t)]$ and $F(t)=[0, g(t), q(t)]$, we have

$$V(t) = \int_{T_1}^{t} S_\Gamma(t-\tau)F(\tau)d\tau.$$

Since $F(t)\in \mathscr{E}_t^1(T_1, \infty; \boldsymbol{H}^1(\Omega)\times \boldsymbol{L}^2(\Omega)\times L^2(\Omega))$, $V(t)$ belongs to $\mathscr{D}(\mathscr{L}_\Gamma)$ for each $t\in[T_1, \infty)$ and satisfies the equality

$$\mathscr{L}_\Gamma V(t) = -F(t) + S_\Gamma(t-T_1)F(T_1) + \int_{T_1}^{t} S_\Gamma(t-\tau)F_t(\tau)d\tau.$$

The equality (3.4) shows that $E_0[S_\Gamma(t)[u, v, \theta]]\leqq E_0[u, v, \theta]$ for any $t\geqq 0$ and $[u, v, \theta]\in \boldsymbol{H}^1(\Omega)\times \boldsymbol{L}^2(\Omega)\times L^2(\Omega)$. Since $E_0^{1/2}$ is a semi-norm on $\boldsymbol{H}^1(\Omega)\times \boldsymbol{L}^2(\Omega)\times L^2(\Omega)$, we have

$$\begin{aligned}
(3.22)\quad & E_1^{1/2}[u(t), u_t(t), \theta(t)] \\
&\leqq E_0^{1/2}\left[\int_{T_1}^{t} S_\Gamma(t-\tau)F(\tau)d\tau\right] \\
&\quad + E_0^{1/2}\left[-F(t) + S_\Gamma(t-T_1)F(T_1) + \int_{T_1}^{t} S_\Gamma(t-\tau)F_t(\tau)d\tau\right] \\
&\leqq \int_{T_1}^{t} E_0^{1/2}[F(\tau)]d\tau + E_0^{1/2}[F(t)] + E_0^{1/2}[F(T_1)] + \int_{T_1}^{t} E_0^{1/2}[F_t(\tau)]d\tau \\
&\leqq \int_{T_1}^{t} (\|g(\tau)\| + (\alpha/\beta)^{1/2}\|q(\tau)\|)d\tau + \|g(t)\| + (\alpha/\beta)^{1/2}\|q(t)\| \\
&\quad + \|g(T_1)\| + (\alpha/\beta)^{1/2}\|q(T_1)\| + \int_{T_1}^{t} (\|g_t(\tau)\| + (\alpha/\beta)^{1/2}\|q_t(\tau)\|)d\tau \\
&\leqq 4\varepsilon
\end{aligned}$$

for any $t\geqq T_1$, where the last inequality follows from (3.20) and (3.21). Put $f_3(t)=f_2(t)$, $T_1\leqq t\leqq T_2$, $f_3(t)=0$, $t\geqq T_2$, and let $[\hat{u}(t), \hat{\theta}(t)]$ be the trajectory for the control $f_3(t)$ with the initial state $[\hat{u}(T_1), \hat{u}_t(T_1), \hat{\theta}(T_1)]=[u_2, v_2, \theta_2]$ for the control system $[TE, g, q, B_\Gamma]$. Then $[\hat{u}(t), \hat{u}_t(t), \hat{\theta}(t)]=S_\Gamma(t-T_1)[u_3, v_3, \theta_3]+[\bar{u}(t), \bar{u}_t(t), \bar{\theta}(t)]$ for $t\geqq T_2$. Hence, by the inequalities (3.4), (3.22) and $E_1^{1/2}[u_3, v_3, \theta_3]<\varepsilon$, the inequality

$$E_1^{1/2}[\hat{u}(t), \hat{u}_t(t), \hat{\theta}(t)] \leqq 5\varepsilon$$

holds for any $t\geqq T_2$. By Lemma 3.4, we can take $\varepsilon>0$ so small that there exists a control $f_4(t)\in \mathscr{F}_N^2[T_2, T_3]$ with $\sup_{T_2<t<T_3} \langle\!\langle f_4(t)\rangle\!\rangle_{1/2}<\eta/4$ which steers the null state at T_2 to $[-Pu(T_3), 0, 0]$ at some time T_3 for the control system $[TE, 0, 0, B_\Gamma]$, since

$$\|\hat{u}_t(t)\| \leqq E_1^{1/2}[\hat{u}(t), \hat{u}_t(t), \hat{\theta}(t)] \leqq 5\varepsilon \qquad \text{for any} \quad t \geqq T_2.$$

We can take T_3 as large as we wish. Let us take T_3 so large that $T_3 \geqq T_2 + T_0$. This implies that the state $[\hat{u}(T_2), \hat{u}_t(T_2), \hat{\theta}(T_2)]$ at T_2 is steered to $[u_4, v_4, \theta_4] = [\hat{u}(T_3) - P\hat{u}(T_3), \hat{u}_t(T_3), \hat{\theta}(T_3)]$ at T_3 by the control $f_4(t)$ for the control system $[TE, g, q, B_\Gamma]$. Since P is the orthogonal projection from $L^2(\Omega)$ to $\mathscr{R}_\Gamma$, we easily see that $[u_4, v_4] \in Z_1$. Thus, by the inequality (3.12),

$$\|[u_4, v_4]\|_{(2,1)}^2 \leqq \gamma_1^2 \|[u_4, v_4]\|_e^2 = \gamma_1^2 \|[\hat{u}(T_3), \hat{u}_t(T_3)]\|_e^2$$

Noting the inequalities

$$\begin{aligned} &\|\operatorname{grad} \hat{\theta}(T_3)\|^2 \leqq \text{const.} \|\kappa \Delta \hat{\theta}(T_3)\|^2 \\ &\leqq \text{const.} \{\|\kappa \Delta \hat{\theta}(T_3) - \beta \operatorname{div} \hat{u}_t(T_3)\|^2 + \beta^2 \|\operatorname{div} \hat{u}_t(T_3)\|^2\} \\ &\leqq \text{const.}\, E_1[\hat{u}(T_3), \hat{u}_t(T_3), \hat{\theta}(T_3)], \end{aligned}$$

we have

$$\begin{aligned} \|[u_4, v_4]\|_{(2,1)}^2 &\leqq \gamma_1^2 \|[\hat{u}(T_3), \hat{u}_t(T_3)]\|_e^2 \\ &\leqq 2\gamma_1^2 \{E_1[\hat{u}(T_3), \hat{u}_t(T_3), \hat{\theta}(T_3)] + \alpha^2 \|\operatorname{grad} \hat{\theta}(T_3)\|^2\} \\ &\leqq \text{const.}\, E_1[\hat{u}(T_3), \hat{u}_t(T_3), \hat{\theta}(T_3)] \leqq \text{const.}\, \varepsilon^2, \end{aligned}$$

where const. may depend on α, β and κ. By Corollary 2.1, if ε is sufficiently small then there exist a control $f_5(t) \in \mathscr{F}_N^2[0, T_0]$ with $\sup_{0<t<T_0} \langle\!\langle f_5(t) \rangle\!\rangle_{1/2} < \eta/4$ and $\tilde{\theta}_4 \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $f_5(t)$ steers the null state to $[u_4, v_4, \tilde{\theta}_4]$ at T_0 for the control system $[TE, 0, 0, B_\Gamma]$. By Lemma 3.5, for any $[u, v] \in B_2(\bar{d}_2\eta/2\omega_2(\alpha, \beta, \kappa))$, there exist a positive time T_4 ($\geqq T_0$), a control $f_6(t) \in \mathscr{F}_N^2[0, T_4]$ with $\sup_{0<t<T_4} \langle\!\langle f_6(t) \rangle\!\rangle_{1/2} < \eta/2$ and $\theta \in H^2(\Omega) \cap H_0^1(\Omega)$ such that the control $f_6(t)$ steers the null state to the state $[u, v, \theta]$ at T_4 for the control system $[TE, 0, 0, B_\Gamma]$.

Taking T_3 greater than $T_2 + T_4$ and putting

$$f(t) = \begin{cases} f_1(t), & 0 \leqq t \leqq T_1, \\ f_2(t), & T_1 \leqq t \leqq T_2, \\ f_4(t), & T_2 \leqq t \leqq T_3 - T_4, \\ f_4(t) + f_6(t - T_3 + T_4), & T_3 - T_4 \leqq t \leqq T_3 - T_0, \\ f_4(t) - f_5(t - T_3 + T_0) + f_6(t - T_3 + T_4), & T_3 - T_0 \leqq t \leqq T_3, \end{cases}$$

we see that the control $f(t)$ belongs to $\mathscr{F}_N^2(G)$ and steers $[u_0, v_0, \theta_0]$ to $[u, v]$ at T_3. This completes the proof for the case $m = 2$.

Noting that

$$\mathscr{L}_\Gamma^2 \left(\int_{T_1}^t S_\Gamma(t-\tau) F(\tau) d\tau \right)$$

$$= \mathscr{L}_\Gamma(-F(t) + S_\Gamma(t)F(T_1) + \int_{T_1}^{t} S_\Gamma(t-\tau)F_t(\tau)d\tau)$$

$$= - \mathscr{L}_\Gamma F(t) + S_\Gamma(t)\mathscr{L}_\Gamma F(T_1) - F_t(t) + S_\Gamma(t)F_t(T_1) + \int_{T_1}^{t} S_\Gamma(t-\tau)F_{tt}(\tau)d\tau$$

holds under the assumptions on $g(t)$ and $q(t)$, we can similarly prove the case $m=3$.

As in [21], we give the definition of holdable states.

DEFINITION 3.2. An element u in $\boldsymbol{H}^m(\Omega)$ is said to be *a holdable state* for time independent external forces $g \in \boldsymbol{H}^{m-2}(\Omega)$, $q \in H^{m-2}(\Omega)$ and a constraint G, if u satisfies the following

$$- Au + \alpha \operatorname{grad} \theta = g \quad \text{in} \quad \Omega \quad \text{and} \quad B_\Gamma u \in G$$

for the solution θ of

$$- \kappa \Delta \theta = q \quad \text{in} \quad \Omega \quad \text{and} \quad \theta = 0 \quad \text{on} \quad S.$$

Now we easily come to the main theorem.

THEOREM 3.1. *Let $m=2$ or 3, G be an open and connected subset of $\boldsymbol{H}^{m-3/2}(S)$, $g_0 \in \boldsymbol{H}^{m-2}(\Omega)$ and $q_0 \in H^{m-2}(\Omega)$. Further assume that the functions $g(t)-g_0$ and $q(t)-q_0$ satisfy the assumptions in Proposition 3.1 in place of $g(t)$ and $q(t)$ respectively.*

If $\omega_m(\alpha, \beta, \kappa,) < \tilde{\omega}_*$, then, for any holdable state u_0 for g_0, q_0 and G, the set $\mathscr{M}_\Gamma^m(G)$ is admissibly controllable to the set $[u_0, 0] + B_m(\bar{d}_m \eta / 2\omega_m(\alpha, \beta, \kappa))$ in the constraint set $\mathscr{F}_N^m(G)$.

Here η is a positive constant such that the η-neighborhood of $B_\Gamma u_0$ in $\boldsymbol{H}^{m-3/2}(S)$ is contained in G.

Especially the set $\mathscr{M}_\Gamma^m(G)$ is admissibly controllable to any holdable state in the constraint set $\mathscr{F}_N^m(G)$.

PROOF. Applying Proposition 3.1 to the control system $[TE, g-g_0, q-q_0, B_\Gamma]$ and to the open set $G - B_\Gamma u_0$, and then adding $[u_0, 0, \theta_0]$ to the trajectory, we obtain the results.

For the control system $[TE, g, q, B_D]$, we have similar results in the same way. We define holdable states for the control system $[TE, g, q, B_D]$ similarly to Definition 3.2.

Then we obtain

THEOREM 3.2. *Let $m=1$ or 2, G be an open and connected subset of $\boldsymbol{H}^{m-1/2}(S)$ and the hypotheses on $g(t)$ and $q(t)$ in Theorem 2.1 be satisfied.*

Further assume that there exist functions $g_0 \in \boldsymbol{H}^{m-2}(\Omega)$ *and* $q_0 \in H^{m-2}(\Omega)$ *such that, in case* $m=1$,

$$g(t) - g_0 \in L^1(0, \infty; \boldsymbol{L}^2(\Omega)), \quad q(t) - q_0 \in L^1(0, \infty; L^2(\Omega))$$

and, in case $m=2$, *in addition to the above,*

$$g(t) \longrightarrow g_0 \quad in \ \boldsymbol{L}^2(\Omega), \quad q(t) \longrightarrow q_0 \quad in \ L^2(\Omega) \quad as \ t \longrightarrow \infty,$$
$$g_t(t) \in L^1(0, \infty; \boldsymbol{L}^2(\Omega)), \quad q_t(t) \in L^1(0, \infty; L^2(\Omega)).$$

Then, if $\omega_m(\alpha, \beta, \kappa) < \tilde{\omega}_m$, *for any holdable state* u_0 *for* g_0, q_0 *and* G, *the set* $\mathscr{M}^m_D(G)$ *is admissibly controllable to the set* $[u_0, 0] + B_m(\bar{d}_m\eta/2\omega_m(\alpha, \beta, \kappa))$ *in the constraint set* $\mathscr{F}^m_D(G)$.

Here η *is a positive constant such that the* η*-neighborhood of* $B_D u_0$ *in* $\boldsymbol{H}^{m-1/2}(S)$ *is contained in* G.

REMARK 3.2. For the control system $[E, g, B_\Gamma]$ (resp. $[E, g, B_D]$), we obtain the results corresponding to Proposition 3.1 under the simpler assumptions

$$\|g^{(j)}(t)\| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \quad 0 \leqq j \leqq m-1. \tag{3.23}$$

In fact, for any $\varepsilon > 0$ there exists T_1 such that $\|g^{(j)}(t)\| < \varepsilon$, $0 \leqq j \leqq m-1$, for all $t \geqq T_1$. Hence the difference between the values of trajectories at $T_0 + T$ for the external force $g(t)$ and the zero external force with the same initial state at $t = T$ $(\geqq T_1)$ is estimated by $c_m \varepsilon T_0$ with a constant c_m depending only on m, A and Ω. For any $\eta > 0$, taking ε so small that the null state is steered to any state with the norm estimated by $c_m \varepsilon T_0$ at T_0 by a control $f(t)$ with $\sup_{0<t<T_0} \langle\!\langle f(t) \rangle\!\rangle_{m-3/2} < \eta$ (resp. $\sup_{0<t<T_0} \langle\!\langle f(t) \rangle\!\rangle_{m-1/2} < \eta$), we see that it is sufficient to consider the case when $g(t) = 0$. Thus we obtain the results in the same way as in the proof of Proposition 3.1.

Noting that the control system $[E, 0, B_\Gamma]$ (resp. $[E, 0, B_D]$) is invariant under time reversal, we easily see that the set $M^m_\Gamma(G)$ (resp. $M^m_D(G)$) is admissibly controllable in the constraint set $\mathscr{F}^m_N(G)$ (resp. $\mathscr{F}^m_D(G)$) under the assumptions (3.23) for any open and connected subset G in $\boldsymbol{H}^{m-3/2}(S)$ (resp. $\boldsymbol{H}^{m-1/2}(S)$) containing 0. Here

$$M^m_\Gamma(G) = \{[u, v] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega) \mid B_\Gamma u \in G\}$$

and

$$M^m_D(G) = \{[u, v] \in \boldsymbol{H}^m(\Omega) \times \boldsymbol{H}^{m-1}(\Omega) \mid B_D u \in G\}.$$

REMARK 3.3. For the control system $[TE, g, q, B_D]$, there always exists a holdable state for any $g \in \boldsymbol{H}^{m-2}(\Omega)$, $q \in H^{m-2}(\Omega)$ and G $(\neq \emptyset)$. But, for the control system $[TE, g, q, B_\Gamma]$, there does not in general exist a holdable state. In [16], the case where the controls are constrained so that small forces are exercised by means of pushing the boundary was considered. There, we considered the constraint set $\mathscr{F}_N^m(G_{\eta,\gamma})$ with

$$G_{\eta,\gamma} = \left\{ h \in \boldsymbol{H}^{m-3/2}(S) \,\middle|\, \begin{array}{l} |h(x)| = \langle h(x), h(x)\rangle^{1/2} < \eta \\ \langle \nu(x), h(x)\rangle < -(1+\gamma^2)^{-1/2}|h(x)| \end{array} \right\}$$

for constants $\eta > 0$ and γ, $0 < \gamma < 1$, and showed that, if the absolute values of

$$\int_\Omega g(x)dx, \quad \int_\Omega \{x_i g_j(x) - x_j g_i(x)\}dx \qquad (1 \leqq i < j \leqq n) \tag{3.24}$$

are small in comparison with η, then there exists a holdable state u for g and $G_{\eta,\gamma}$, which is defined as a solution of

$$-Au = g \quad \text{in } \Omega, \quad \partial u/\partial \nu_A \in G_{\eta,\gamma}.$$

There also exists a holdable state for the control system $[TE, g, q, B_\Gamma]$ for g, q and $G_{\eta,\gamma}$ under the same assumptions on g. In fact, it is easy to see that a necessary and sufficient condition for the existence of the solution of the boundary value problem

$$-Aw = g - \alpha \operatorname{grad}\theta \quad \text{in } \Omega, \quad B_\Gamma w = h \text{ on } S \tag{3.25}$$

for given functions g, θ and h, is that the equality

$$\int_\Omega \langle g - \alpha \operatorname{grad}\theta, \phi\rangle dx = -\int_S \langle h, \phi\rangle dS \tag{3.26}$$

holds for any function $\phi \in \mathscr{R}_\Gamma$. By noting that $\mathscr{R}_\Gamma \subset \{\phi \in C^\infty(\Omega)^n \mid a(\phi, \phi) = 0\}$, we see, in the same way as in [16], that if the absolute values of (3.24) are small, then there exists a function h in $G_{\eta,\gamma}$ satisfying the equality

$$\int_\Omega \langle g, \phi\rangle dx = -\int_S \langle h, \phi\rangle dS \qquad \text{for any} \quad \phi \in \mathscr{R}_\Gamma.$$

For any given function q, there exists a solution of the equation

$$-\kappa\Delta\theta = q \quad \text{in } \Omega, \quad \theta = 0 \text{ on } S.$$

For this solution θ and for any $\phi \in \mathscr{R}_\Gamma$, we have

$$\int_\Omega \langle -\alpha \operatorname{grad}\theta, \phi\rangle dx = -\alpha\int_S \langle \theta\nu, \phi\rangle dS + \alpha\int_\Omega \langle \theta, \operatorname{div}\phi\rangle dx = 0,$$

since $\theta = 0$ on S and $\operatorname{div}\phi = 0$ for any $\phi \in \mathscr{R}_\Gamma$. Hence h satisfies (3.26) for any $\phi \in \mathscr{R}_\Gamma$. Thus, for the functions g, q and h, there exists a solution of (3.25), Therefore, for any g such that the absolute values of (3.24) are small, for any $q \in H^{m-2}(\Omega)$ and for any constraint $G_{\eta,\gamma}$, a holdable state exists.

By Sobolev's imbedding theorem, we see that the subset $G_{\eta,\gamma}$ is open and connected in $\boldsymbol{H}^{3/2}(S)$ when $n=3$. Thus we can apply Theorem 3.1 for $m=3$ and $n=3$, with the constraint set $\mathscr{F}^3_N(G_{\eta,\gamma})$.

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