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Existence of oscillatory solutions for fourth order superlinear ordinary differential equations

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1. Introduction

In this paper we consider the fourth order Emden-Fowler type equation

$$y^{(4)} = p(t)|y|^{\alpha} \operatorname{sgn} y, \tag{1}$$

where $\alpha > 1$ is a constant and $p(t)$ is a positive continuous function on $[t_0, \infty)$, $t_0 > 0$. We are concerned with oscillatory and nonoscillatory properties of proper solutions of (1). A nontrivial real-valued solution $y(t)$ of (1) is called proper if it exists on some half-line $[T_y, \infty) \subset [t_0, \infty)$. A proper solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

We denote by $\mathscr{S}$ the set of all proper solutions of (1). From the viewpoint of oscillatory and nonoscillatory properties, $\mathscr{S}$ can be decomposed into a disjoint union

$$\mathscr{S} = \mathscr{O} \cup \mathscr{N},$$

where $\mathscr{O}$ (resp. $\mathscr{N}$) is the set of all oscillatory (resp. nonoscillatory) solutions of (1). Moreover $\mathscr{N}$ can be decomposed into a disjoint union

$$\mathscr{N} = \mathscr{N}_0 \cup \mathscr{N}_2 \cup \mathscr{N}_4,$$

where $\mathscr{N}_0$, $\mathscr{N}_2$ and $\mathscr{N}_4$ denote the sets of nonoscillatory solutions $y(t)$ satisfying,

$$y(t)y'(t) < 0,\ y(t)y''(t) > 0,\ y(t)y'''(t) < 0,$$

$$y(t)y'(t) > 0,\ y(t)y''(t) > 0,\ y(t)y'''(t) < 0$$

and

$$y(t)y'(t) > 0,\ y(t)y''(t) > 0,\ y(t)y'''(t) > 0$$

respectively, for all sufficiently large t. The following results are known:

THEOREM A (Kiguradze [3]). $\mathscr{N}_0 \neq \emptyset$.

THEOREM B (Kitamura [6]). $\mathscr{N}_2 = \emptyset$ *if and only if*

$$\int_{t_0}^{\infty} t^{2+\alpha} p(t)dt = \infty. \tag{2}$$

THEOREM C (Kiguradze [2]). (i) $\mathcal{N}_4=\emptyset$ *if*

$$\liminf_{t\to\infty} t^{1+3\alpha}p(t) > 0. \tag{3}$$

(ii) $\mathcal{N}_4\neq\emptyset$ *if*

$$\int_{t_0}^{\infty} t^{3\alpha}p(t)dt < \infty.$$

THEOREM D (Kiguradze [5]). *$\mathcal{O}\neq\emptyset$ if $p(t)$ is locally absolutely continuous on $[t_0, \infty)$ and satisfies the condition* (2).

It seems to be unknown when $\mathcal{O}=\emptyset$ holds for (1). The purpose of this paper is to establish conditions under which $\mathcal{O}=\emptyset$ or $\mathcal{O}\neq\emptyset$. In Section 2 we give conditions for (1) to have no oscillatory solution. In Section 3 we prove the existence of oscillatory solutions of (1) without the above condition (2). That our results are sharp is illustrated by an example. Finally we mention the paper [7] in which conditions are presented for the nonexistence of oscillatory solutions for third order Emden-Fowler type equations. The reader is referred to the survey article of Kiguradze [4] for typical results concerning the qualitative theory of solutions of n-th order Emden-Fowler type equations.

2. Nonoscillation criteria

In this section we find conditions under which equation (1) has no oscillatory solution ($\mathcal{O}=\emptyset$).

THEOREM 1. *Let $(d/dt)p(t)\leq 0$ for $t\geq t_0$ and*

$$\int_{t_0}^{\infty} t^{1+2\alpha}p(t)\,dt < \infty. \tag{4}$$

Then every proper solution of (1) *is nonoscillatory.*

PROOF. Suppose to the contrary that there exists an oscillatory solution $y(t)$ of (1) on $[T, \infty)$, $T>t_0$. Let $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence of zeros of $y''(t)$ such that $\lim_{n\to\infty} t_n=\infty$. Choose, for each n, $s_n\in(t_n, t_{n+1})$ such that $|y''(s_n)|=\max\{|y''(t)|: t_n\leq t\leq t_{n+1}\}$. Consider the function

$$V(t) = y'''(t)y'(t) - \frac{1}{2}(y''(t))^2 - \frac{1}{1+\alpha}p(t)|y(t)|^{1+\alpha}.$$

Since $y'''(s_n)=0$, we have $V(s_n)=-(y''(s_n))^2/2-p(s_n)|y(s_n)|^{1+\alpha}/(1+\alpha)$. On the other hand, from our assumption,

$$V'(t) = -\frac{1}{1+\alpha}p'(t)|y(t)|^{1+\alpha} \geq 0$$

for $t \geq T$. Therefore there exists $M>0$ such that $|y''(s_n)| \leq M$ for all n. From the choice of s_n, it follows that $|y''(t)| \leq M$, $t \geq t_1$. Consequently, $|y(t)| \leq Mt^2$, $t \geq T_1$, provided $T_1 \geq t_1$ is sufficiently large. This together with (4) implies that

$$\int_{T_1}^{\infty} t^3 p(t)|y(t)|^{\alpha-1} dt < \infty. \tag{5}$$

Now $y(t)$ can be considered as an oscillatory solution of the linear equation

$$z^{(4)} = p(t)|y(t)|^{\alpha-1} z, \tag{6}$$

and, as is well-known, (5) is a sufficient condition for (6) to have no oscillatory solution. This contradiction completes the proof.

THEOREM 2. *Suppose that there exist positive constants* ε *and* K *such that* $p(t)t^{(3\alpha+5+\varepsilon)/2} \geq K$ *and*

$$\frac{d}{dt}\left[p(t)t^{(3\alpha+5+\varepsilon)/2}\right] \leq 0 \tag{7}$$

for $t \geq t_0$. *Then every proper solution of* (1) *is nonoscillatory.*

The following lemma (cf. Bellman [1, p 155]) is needed in proving Theorem 2.

LEMMA 1. *Let* $y'(t)$ *be bounded on* $[T, \infty)$ *and* $y(t) \in L^2[T, \infty)$. *Then* $\lim_{t\to\infty} y(t) = 0$.

PROOF OF THEOREM 2. If $\varepsilon > \alpha-1$, then, as easily verified, $p(t)$ satisfies the assumptions of Theorem 1. Therefore it suffices to consider the case $0<\varepsilon\leq\alpha-1$. We make the change of variables

$$x = \log t, \quad w(t) = t^{-\lambda} y(t), \tag{8}$$

where $\lambda = 3/2 + \varepsilon/2(\alpha-1)$, which transforms (1) into

$$w^{(4)} + a_1 \dddot{w} + a_2 \ddot{w} + a_3 \dot{w} + a_4 w - f(x)|w|^{\alpha} \operatorname{sgn} w = 0, \tag{9}$$

where $\cdot = d/dx$, $a_1 = 4\lambda - 6$, $a_2 = 6\lambda^2 - 18\lambda + 11$, $a_3 = 4\lambda^3 - 18\lambda^2 + 22\lambda - 6$, $a_4 = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$ and $f(x) = p(t)t^{(3\alpha+5+\varepsilon)/2}$. Suppose that (1) has an oscillatory solution $y(t)$ on $[T, \infty)$, $T > t_0$. Then the function $w(x)$ defined by (8) is also an oscillatory solution of (9) on $[x_0, \infty)$, $x_0 = \log T$. Let $\{x_n\}_{n=1}^{\infty}$ be an increasing sequence of zeros of $w(x)$ such that $\lim_{n\to\infty} x_n = \infty$. Choose, for each n, $s_n \in (x_n, x_{n+1})$ such that $|w(s_n)| = \max\{|w(x)| : x_n \leq x \leq x_{n+1}\}$. Consider the function

$$F(x) = \dddot{w}(x)\dot{w}(x) - \frac{1}{2}(\ddot{w}(x))^2 + a_1\ddot{w}(x)\dot{w}(x) + \frac{1}{2}a_2(\dot{w}(x))^2$$
$$- a_1\int_{x_0}^{x}(\ddot{w}(s))^2 ds + a_3\int_{x_0}^{x}(\dot{w}(s))^2 ds + \frac{1}{2}a_4(w(x))^2$$
$$- \frac{1}{1+\alpha}f(x)|w(x)|^{1+\alpha}.$$

Then it follows from (7) that $\dot{F}(x) = -\dot{f}(x)|w(x)|^{1+\alpha}/(1+\alpha) \geq 0$, $x \geq x_0$, so that $F(x)$ is nondecreasing. Since $\dot{w}(s_n)=0$, we have

$$F(s_n) = -\frac{1}{2}(\ddot{w}(s_n))^2 - a_1\int_{x_0}^{s_n}(\ddot{w}(s))^2 ds + a_3\int_{x_0}^{s_n}(\dot{w}(s))^2 ds$$
(10)
$$+ \frac{1}{2}a_4(w(s_n))^2 - \frac{1}{1+\alpha}f(s_n)|w(s_n)|^{1+\alpha}.$$

We wish to show that

(11) $$\lim_{x\to\infty} w(x) = 0.$$

We consider the case where $0<\varepsilon<\alpha-1$. Then we obtain $3/2<\lambda<2$ and

(12) $$a_1 > 0, \quad a_3 < 0, \quad a_4 > 0.$$

Since

$$\frac{1}{2}a_4(w(s_n))^2 - \frac{1}{1+\alpha}f(s_n)|w(s_n)|^{1+\alpha} \leq |w(s_n)|^2\left(\frac{1}{2}a_4 - \frac{K}{1+\alpha}|w(s_n)|^{\alpha-1}\right),$$

it follows from the choice of s_n that $w(x)$ is bounded. Therefore, letting $n\to\infty$ in (10) and using (12), we have

(13) $$\int_{x_0}^{\infty}(\ddot{w}(s))^2 ds < \infty, \quad \int_{x_0}^{\infty}(\dot{w}(s))^2 ds < \infty.$$

Transforming back to the original variables, we see from the boundedness of $w(t)$ that $y(t)=O(t^{\lambda})$ as $t\to\infty$, so that, by (1) and (7), $y^{(4)}(t)=O(t^{\lambda-4})$ as $t\to\infty$. Since $y(t)$ is oscillatory, we obtain $y'''(t)=O(t^{\lambda-3})$, $y''(t)=O(t^{\lambda-2})$ and $y'(t)=O(t^{\lambda-1})$ as $t\to\infty$. Since

(14) $$\dddot{w}(x) = -\lambda^3 t^{-\lambda}y(t) + (3\lambda^2-3\lambda+1)t^{1-\lambda}y'(t) + 3(1-\lambda)t^{2-\lambda}y''(t) + t^{3-\lambda}y'''(t),$$

$\dddot{w}(x)$ is bounded. Applying Lemma 1 yields

(15) $$\lim_{x\to\infty}\ddot{w}(x) = \lim_{x\to\infty}\dot{w}(x) = 0.$$

Consider the function

$$V(x) = \ddot{w}(x)\dot{w}(x) - \frac{1}{2}(\ddot{w}(x))^2 + a_1\ddot{w}(x)\dot{w}(x) + \frac{1}{2}a_2(\dot{w}(x))^2 + \frac{1}{2}a_4(w(x))^2 - \frac{1}{1+\alpha}f(x)|w(x)|^{1+\alpha}.$$

Then by (7) and (12) we see that

$$\dot{V}(x) = a_1(\ddot{w}(x))^2 - a_3(\dot{w}(x))^2 - \frac{1}{1+\alpha}\dot{f}(x)|w(x)|^{1+\alpha} \geq 0, \quad x \geq x_0.$$

It follows from (15) that $\lim_{n\to\infty} V(x)=0$, so that $\lim_{x\to\infty} V(x)=0$, which implies from (15) and the boundedness of $\ddot{w}(x)$ that

$$\lim_{x\to\infty}\left(\frac{1}{2}a_4(w(x))^2 - \frac{1}{1+\alpha}f(x)|w(x)|^{1+\alpha}\right) = 0. \tag{16}$$

Suppose that $\limsup_{x\to\infty}|w(x)| \geq \delta > 0$, where δ is a constant. Since $w(x)$ is oscillatory, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} z_n = \infty$ and $|w(z_n)| = N$, where $N = \min\{\delta/2, [a_4(1+\alpha)/2L]^{1/(\alpha-1)}/2\}$ and L is a positive constant such that $f(x) \leq L$, $x \geq x_0$. We have for all n,

$$\frac{1}{2}a_4(w(z_n))^2 - \frac{1}{1+\alpha}f(z_n)|w(z_n)|^{1+\alpha} \geq N^2\left(\frac{1}{2}a_4 - \frac{1}{1+\alpha}LN^{\alpha-1}\right) > 0,$$

which contradicts (16). Hence (11) is valid. It remains to consider the case where $\varepsilon = \alpha - 1$. In this case we remark that $\lambda = 2$ and

$$a_1 > 0, \quad a_3 < 0, \quad a_4 = 0.$$

Letting $n\to\infty$ in (10), we have (13). Since $f(s_n) \geq K > 0$, $w(x)$ is bounded and $y(t) = O(t^2)$ as $t\to\infty$. Similarly as above we have $y^{(4)}(t) = O(t^{-2})$ and $y'''(t) = O(t^{-1})$ as $t\to\infty$. On the other hand, since $p'(t) \leq 0$, $t \geq t_0$, by (7), the proof of Theorem 1 shows that $y''(t)$ is bounded. Thus, $y''(t) = O(1)$ and $y'(t) = O(t)$ as $t\to\infty$. Proceeding with the same argument as in the case where $0 < \varepsilon < \alpha - 1$, we conclude that (16) holds. Consequently, (11) is valid. Transforming back to the original variables, we see that $y(t) = o(t^\lambda)$, so that from (7)

$$t^4 p(t)|y(t)|^{\alpha-1} = p(t)t^{(3\alpha+5+\varepsilon)/2}o(1) = o(1) \quad \text{as} \quad t \longrightarrow \infty. \tag{17}$$

Now $y(t)$ can be considered as an oscillatory solution of the linear equation

$$z^{(4)} = p(t)|y(t)|^{\alpha-1}z. \tag{18}$$

From the Leighton-Nehari's nonoscillation theorem [8, Theorem 6.2], (17) is sufficient for (18) to have no oscillatory solution. This is a contradiction and the proof is complete.

As an example, we consider the equation

(19) $$y^{(4)} = t^{\beta}|y|^{\alpha} \operatorname{sgn} y, \quad t > 1,$$

where β is a real number and $\alpha > 1$. Theorem 2 implies that every proper solution of (19) is nonoscillatory if $\beta + (3\alpha + 5)/2 < 0$.

3. Existence of oscillatory solutions

In this section we establish conditions guaranteeing the existence of oscillatory solutions of equation (1) ($\mathcal{O} \neq \emptyset$).

THEOREM 3. *Suppose that $p(t)$ is positive and locally absolutely continuous on $[t_0, \infty)$ and let*

(20) $$\frac{d}{dt}[p(t)t^{(3\alpha+5)/2}] \geq 0$$

for $t \geq t_0$. Then equation (1) *has an oscillatory solution.*

To prove Theorem 3, the following Lemma 3 will be needed. Lemma 3 will be proved by using Lemma 2. The proof of Lemma 3 was suggested by Y. Kitamura.

LEMMA 2 (Kiguradze [5, Lemma 2.6]). *Let $p(t)$ be positive and locally absolutely continuous on $[t_0, \infty)$ and let $[t_1, t_2)$, $t_0 \leq t_1 < t_2 < \infty$, be a right maximal interval of existence for a solution $y(t)$ of* (1). *Then $y(t)$ satisfies the following inequalities in a certain left neighborhood of t_2:*

$$y^{(i)}(t)y(t) > 0 \qquad (i = 0, 1, 2, 3).$$

LEMMA 3. *Suppose that $p(t)$ is positive and locally absolutely continuous on $[t_0, \infty)$. Then for any $c \in (-\infty, +\infty)$ there exists a solution $y(t)$ of* (1) *which is defined on $[t_0, \infty)$ and satisfies the following*:

(21) $$y(t_0) = y'(t_0) = 0, \quad y''(t_0) = c;$$

(22) $$\liminf_{t\to\infty} |y'''(t)| = 0.$$

PROOF OF LEMMA 3. It suffices to assume that c is positive. Let c be fixed We denote by $y(t, d)$ the solution of (1) satisfying the initial conditions

$$y(t_0) = y'(t_0) = 0, \quad y''(t_0) = c, \quad y'''(t_0) = d.$$

It is clear that in the common interval of existence of $y(t, d_1)$ and $y(t, d_2)$

(23) $$y^{(i)}(t, d_1) < y^{(i)}(t, d_2) \quad (i = 0, 1, 2, 3) \quad \text{if} \quad d_1 < d_2 \quad \text{and} \quad t_0 \neq t.$$

Define the sets A^+ and A^- by

$$A^+ = \{d\colon y^{(i)}(t, d) > 0 \quad (i = 0, 1, 2, 3) \quad \text{for some} \quad t > t_0\}$$

and

$$A^- = \{d\colon y^{(i)}(t, d) < 0 \quad (i = 0, 1, 2, 3) \quad \text{for some} \quad t > t_0\}.$$

From (23) and the continuity of solutions of (1) with respect to initial values, it follows that A^+ and A^- are open intervals. It is clear that $A^+ \cap A^- = \emptyset$ and $0 \in A^+$. On the other hand, there exists a positive constant ε such that $y(t, 0)$ is defined on $[t_0, t_0 + 2\varepsilon]$. Choose $d_1 < 0$ such that

$$c + \frac{\varepsilon}{3}\left[d_1 + y^\alpha(t_0+\varepsilon, 0)\int_{t_0}^{t_0+\varepsilon} p(t)dt\right] < 0.$$

We show that $d_1 \notin A^+$. Assume that $d_1 \in A_+$. By (23), then, $y(t, d_1)$ is defined on $[t_0, t_0+2\varepsilon]$. Noticing that $y(t, 0) > y(t, d_1)$ and $y(t, 0) > 0$ on $(t_0, t_0+\varepsilon]$, we have

$$\begin{aligned} y(t_0+\varepsilon, d_1) &= \frac{c}{2}\varepsilon^2 + \frac{1}{6}d_1\varepsilon^3 + \frac{1}{6}\int_{t_0}^{t_0+\varepsilon}(t_0+\varepsilon-t)^3 p(t)|y(t, d_1)|^\alpha \operatorname{sgn} y(t, d_1)dt \\ &\le \frac{c}{2}\varepsilon^2 + \frac{1}{6}d_1\varepsilon^3 + \frac{1}{6}\varepsilon^3\int_{t_0}^{t_0+\varepsilon} p(t)y^\alpha(t, 0)dt \\ &\le \frac{1}{2}\varepsilon^2\left(c + \frac{\varepsilon}{3}\left[d_1 + y^\alpha(t_0+\varepsilon, 0)\int_{t_0}^{t_0+\varepsilon} p(t)dt\right]\right) < 0. \end{aligned}$$

Similarly as above, we have

$$y^{(i)}(t_0+\varepsilon, d_1) < 0 \quad (i = 0, 1, 2, 3).$$

This implies that $d_1 \in A^-$ and that $d_1 \in A^+ \cap A^-$, which contradicts $A^+ \cap A^- = \emptyset$. Therefore $d_1 \notin A^+$. By (23) and this, the set A^+ is bounded below. Hence there exists $d_0 = \inf\{d\colon d \in A^+\}$. Since A^+ is open, $d_0 \notin A^+$. Suppose that $y(t, d_0)$ cannot be extended to $+\infty$. It follows from Lemma 2 that $d_0 \in A^-$. However, since A^- is open, A^- contains a certain neighborhood of d_0, which contradicts the definition of d_0 and $A^+ \cap A^- = \emptyset$. Therefore $y(t, d_0)$ can be extended to $+\infty$ and $d_0 \notin A^+ \cup A^-$. Suppose that $y(t, d_0)$ does not satisfy (22). Then

$$\liminf_{t\to\infty} |y'''(t, d_0)| > 0.$$

In this case there exists $t > t_0$ such that $y^{(i)}(t, d_0)y(t, d_0) > 0$ $(i = 0, 1, 2, 3)$. Hence $d_0 \in A^+ \cup A^-$. From this contradiction, we conclude that $y(t, d_0)$ is a proper solution of (1) satisfying (21) and (22).

Proof of Theorem 3. We define the constants K, L and the function $P(t)$ by

$$K = 27\alpha^2, \quad L = K^{1/(\alpha-1)}P(t_0)^{-1/(\alpha-1)}, \quad P(t) = p(t)t^{(3\alpha+5)/2}.$$

We choose c so that

(24) $$c^2 > 130t_0^{-1}(1+3K)^2L^2.$$

For this c, Lemma 3 guarantees that there exists a proper solution $y(t)$ of (1) satisfying (21) and (22). In the following we shall prove that $y(t)$ is oscillatory. Assume to the contrary that $y(t)$ is nonoscillatory. We may assume that $y(t)>0$ for all sufficiently large t. From (21) and (22), there exists $T>t_0$ such that

(25) $$y'(t) > 0, \quad y''(t) > 0, \quad y'''(t) < 0$$

for $t \geq T$, i.e., $y(t) \in \mathcal{N}_2$. By (22) and (25), we have

(26) $$0 \leq y''(\infty) = \lim_{t\to\infty} y''(t) < \infty, \quad y'''(\infty) = \lim_{t\to\infty} y'''(t) = 0.$$

Hence integrating (1), yields for $t \geq T$,

(27) $$y(t) = y(T) + y'(T)(t-T) + \frac{1}{2} y''(\infty)(t-T)^2 + \int_T^t \int_T^{t_1} \int_{t_2}^\infty \int_{t_3}^\infty p(t_4)y^\alpha(t_4)dt_4dt_3dt_2dt_1.$$

Using (20) and (25), we have

$$\begin{aligned} y(t) &\geq \left(\int_T^t \int_T^s d\tau ds\right)\left(\int_t^\infty \int_s^\infty p(\tau)y^\alpha(\tau)\, d\tau ds\right) \\ &= \frac{1}{2}(t-T)^2 \int_t^\infty (s-t)p(s)y^\alpha(s)ds \\ &\geq \frac{1}{2}(t-T)^2 y^\alpha(t)P(t)\int_t^\infty (s-t)s^{-(3\alpha+5)/2}ds. \end{aligned}$$

Therefore, we have

$$y(t) \geq K^{-1}y^\alpha(t)P(t)t^{-3(\alpha-1)/2}$$

for all sufficiently large t, say, $t \geq T_1 > T$. Consequently,

(28) $$y(t) \leq K^{1/(\alpha-1)}P(t)^{-1/(\alpha-1)}t^{3/2}, \quad t \geq T_1.$$

By (1) and (28), we obtain

$$y^{(4)}(t) \leq K^{\alpha/(\alpha-1)}P(t)^{-1/(\alpha-1)}t^{-5/2}, \quad t \geq T_1.$$

Integrating the above over $[t, \infty)$, $t \geq T_1$, and using (20) and (26), we obtain

$$-y'''(t) \leq \frac{2}{3} K^{\alpha/(\alpha-1)}P(t)^{-1/(\alpha-1)}t^{-3/2}, \quad t \geq T_1.$$

It follows from (20), (26), (27) and (28) that $y''(\infty)=0$. Hence we have, as above,

$$y''(t) \le \frac{4}{3} K^{\alpha/(\alpha-1)} P(t)^{-1/(\alpha-1)} t^{-1/2}$$

and

$$y'(t) \le c_0 + \frac{8}{3} K^{\alpha/(\alpha-1)} P(t_0)^{-1/(\alpha-1)} t^{1/2}$$

for $t \ge T_1$, where $c_0 = y'(T_1)$. From (20), we have the following estimates

$$\begin{aligned} &y(t) \le Lt^{3/2}, \\ &y'(t) \le c_0 + \frac{8}{3} KLt^{1/2}, \\ &y''(t) \le \frac{4}{3} KLt^{-1/2}, \\ &-y'''(t) \le \frac{2}{3} KLt^{-3/2} \end{aligned} \tag{29}$$

for $t \ge T_1$. We make the change of variables $x = \log t$, $w(x) = t^{-3/2} y(t)$, which transforms (1) into

$$w^{(4)} - \frac{5}{2}\ddot{w} + \frac{9}{16} w - f(x)|w|^{\alpha} \operatorname{sgn} w = 0, \tag{30}$$

where $\cdot = d/dx$ and $f(x) = P(t)$. Since

$$\begin{aligned} w(x) &= t^{-3/2} y(t), \\ \dot{w}(x) &= -\frac{3}{2} t^{-3/2} y(t) + t^{-1/2} y'(t), \\ \ddot{w}(x) &= \frac{9}{4} t^{-3/2} y(t) - 2t^{-1/2} y'(t) + t^{1/2} y''(t), \\ \dddot{w}(x) &= -\frac{27}{8} t^{-3/2} y(t) + \frac{13}{4} t^{-1/2} y'(t) - \frac{3}{2} t^{1/2} y''(t) + t^{3/2} y'''(t) \end{aligned} \tag{31}$$

using (28) and (29), we obtain the following estimates:

$$\begin{aligned} &f(x)|w(x)|^{1+\alpha} \le KL^2, \\ &|w(x)| \le L, \\ &|\dot{w}(x)| \le \frac{3}{2} L + \frac{8}{3} KL + c_0 \exp(-x/2), \\ &|\ddot{w}(x)| \le \frac{9}{4} L + \frac{20}{3} KL + 2c_0 \exp(-x/2), \\ &|\dddot{w}(x)| \le \frac{27}{8} L + \frac{34}{3} KL + \frac{13}{4} c_0 \exp(-x/2) \end{aligned} \tag{32}$$

for $x \geq x_1$, $x_1 = \log T_1$. Consider the function

$$(33) \qquad F(x) = \ddot{w}(x)\dot{w}(x) - \frac{1}{2}(\ddot{w}(x))^2 - \frac{5}{4}(\dot{w}(x))^2 + \frac{9}{32}(w(x))^2 - \frac{1}{1+\alpha} f(x)|w(x)|^{1+\alpha}.$$

Then by (20),

$$\dot{F}(x) = -\frac{1}{1+\alpha}\dot{f}(x)|w(x)|^{1+\alpha} \leq 0, \quad x \geq x_0,\ x_0 = \log t_0.$$

Hence by (32), we see that $F(x)$ is bounded and that

$$(34) \qquad \lim_{x\to\infty} F(x) \leq F(x_0).$$

From (21), (31), (32) and (34), it follows that

$$\begin{aligned} F(x_0) &= -\frac{1}{2}t_0c^2 \geq \lim_{x\to\infty} F(x) \\ &\geq -\left(\frac{27}{8}L + \frac{34}{3}KL\right)\left(\frac{3}{2}L + \frac{8}{3}KL\right) - \frac{1}{2}\left(\frac{9}{4}L + \frac{20}{3}KL\right)^2 \\ &\qquad - \frac{5}{4}\left(\frac{3}{2}L + \frac{8}{3}KL\right)^2 - \frac{9}{32}L^2 - \frac{1}{1+\alpha}KL^2 \\ &> -65(1+3K)^2L^2, \end{aligned}$$

which contradicts (24). From this contradiction, we conclude that $y(t)$ is an oscillatory solution of (1). This completes the proof.

The proof of Theorem 3 shows that, under the hypotheses of Theorem 3, every proper solution $y(t)$ of (1) such that $y(t_0) = y'(t_0) = 0$ and $|y''(t_0)|$ is sufficiently large is oscillatory.

THEOREM 4. *Let $p(t)$ be a positive continuous function on $[t_0, \infty)$, $t_0 > 0$. Suppose that there exists a positive constant ε such that*

$$(35) \qquad \frac{d}{dt}\left[p(t)t^{(3\alpha+5-\varepsilon)/2}\right] \geq 0$$

for $t \geq t_0$. Then every proper solution $y(t)$ of (1) such that $y(t_0) = y'(t_0) = 0$ is oscillatory.

PROOF. We may assume that the number ε in (35) satisfies

$$(36) \qquad \varepsilon < \alpha - 1,$$

since if (35) holds for some $\varepsilon > 0$, then it also does for all smaller $\varepsilon > 0$. Suppose to the contrary that there exists a nonoscillatory solution $y(t)$ of (1) such that $y(t_0)=y'(t_0)=0$. Without loss of generality we may assume that $y(t)>0$ for all sufficiently large t. It is easy to see that (35) implies (3). Hence it follows from Theorem C that $y''(t_0)\neq 0$ and $y(t)\notin\mathcal{N}_4$. Since $y(t_0)=y'(t_0)=0$, $y(t)\notin\mathcal{N}_0$. Consequently we conclude that $y(t)\in\mathcal{N}_2$, so that (25) holds. If $y'''(\infty)<0$, then $y(t)\to-\infty$ as $t\to\infty$, which contradicts the assumption that $y(t)>0$ for all large t. Hence (26) holds and $y(t)$ satisfies the integral equation (27). From (25), (26), (27) and (35), it follows that

$$\begin{aligned} y(t) &\geq \left(\int_T^t\int_T^s d\tau ds\right)\left(\int_t^\infty\int_s^\infty p(\tau)y^\alpha(\tau)d\tau ds\right) \\ &\geq \frac{1}{2}(t-T)^2 y^\alpha(t)P(t)\int_t^\infty (s-t)s^{-(3\alpha+5-\varepsilon)/2}ds, \end{aligned}$$

where $P(t)=p(t)t^{(3\alpha+5-\varepsilon)/2}$. From the above there exist $T_1>T$ and $K>0$ such that

$$y(t) \leq KP(t)^{-1/(\alpha-1)}t^{3/2-\varepsilon/2(\alpha-1)},\ t\geq T_1. \tag{37}$$

By (1), we obtain

$$y^{(4)}(t) \leq K^\alpha P(t)^{-1/(\alpha-1)}t^{-5/2-\varepsilon/2(\alpha-1)},\ t\geq T_1.$$

By an argument similar to that employed in the proof of Theorem 4 we have, using (35) and (36),

$$\begin{aligned} &y^{(4)}(t) = 0(t^{-5/2-\varepsilon/2(\alpha-1)}), \quad y'''(t) = 0(t^{-3/2-\varepsilon/2(\alpha-1)}), \\ &y''(t) = 0(t^{-1/2-\varepsilon/2(\alpha-1)}), \quad y'(t) = 0(t^{1/2-\varepsilon/2(\alpha-1)}), \\ &y(t) = 0(t^{3/2-\varepsilon/2(\alpha-1)}), \end{aligned} \tag{38}$$

as $t\to\infty$. We make the change of variables $x=\log t$, $w(x)=t^{-3/2}y(t)$, which transforms (1) into (30) with $f(x)=P(t)t^{\varepsilon/2}$. Since from (37)

$$f(x)|w(x)|^{1+\alpha} \leq K^{1+\alpha}P(t)^{-2/(\alpha-1)}t^{-\varepsilon/(\alpha-1)}, \quad t\geq T_1,$$

we have, by (31) and (38),

$$\begin{aligned} &w(x) = o(1), \quad \dot{w}(x) = o(1), \quad \ddot{w}(x) = o(1), \quad \dddot{w}(x) = o(1) \\ &f(x)|w(x)|^{1+\alpha} = o(1), \end{aligned} \tag{39}$$

as $x\to\infty$. Consider the function $F(x)$ defined by (33). From (35) we obtain $\dot{F}(x)=-\dot{f}(x)|w(x)|^{1+\alpha}/(1+\alpha)\leq 0$ and hence $\lim_{x\to\infty}F(x)\leq F(x_0)$, $x_0=\log t_0$. It follows from $y(t_0)=y'(t_0)=0$, $y''(t_0)\neq 0$ and (39) that

$$\lim_{x\to\infty}F(x) = 0 \leq F(x_0) = -\frac{1}{2}t_0(y''(t_0))^2 < 0,$$

which is a contradiction. This completes the proof.

EXAMPLE. Consider the equation

$$y^{(4)} = t^{\beta}|y|^{\alpha} \operatorname{sgn} y, \quad t > 1, \tag{40}$$

where β is a real number and $\alpha > 1$. Theorem 3 implies that (40) has an oscillatory solution if $\beta + (3\alpha + 5)/2 \geq 0$. Combining this with Theorem 2, we see that (40) has an oscillatory solution if and only if $\beta + (3\alpha + 5)/2 \geq 0$. The classes of solutions of (40) mentioned in the introduction can be characterized as follows:

$\mathcal{N}_0 \neq \emptyset$;

$\mathcal{N}_2 \neq \emptyset$ if and only if $\alpha + \beta + 3 \leq 0$;

$\mathcal{N}_4 \neq \emptyset$ if and only if $3\alpha + \beta + 1 < 0$;

$\mathcal{O} \neq \emptyset$ if and only if $\beta + (3\alpha + 5)/2 \geq 0$.

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