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On the stability of periodic solutions of the Navier-Stokes equations in a noncylindrical domain

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Introduction

In the previous paper [10] we discussed the existence and uniqueness of weak solutions of the initial-boundary value problem for the Navier-Stokes equations in a time dependent domain in R^n ($n=2, 3, 4$), and proved the existence of a periodic solution for an arbitrary external force under the assumption that the domain moves periodically and the boundary data are small enough. In this paper we shall give certain sufficient conditions for our periodic solutions to be stable in the case $n=2, 3$.

In [12] Serrin has given a condition in terms of the Reynolds number of the flow, under which there exists a unique and universally stable periodic solution. Here "universally stable" means that any other solution tends to this periodic solution as $t \to \infty$. To obtain this result, he required that for any continuous initial velocity the equation is solvable globally in t, and that there exists a certain solution which is equicontinuous in the space variables for any time t and has a sufficiently small low Reynolds number. Though these requirements seem to be rather restrictive mathematically, he offered these requirements because of their plausibility on the ground of physical intuition.

Instead of the conditions stated above, we discuss a condition under which our weak solutions including the periodic one are regular after some instant. Then, it will be shown that if our periodic solution has a sufficiently low Reynolds number, then it is stable among weak solutions with sufficiently small initial velocities.

We begin with the formulation of the problem and state the theorem on the existence of global weak solutions established in [10] in Section 1. In Section 2 we discuss the regularity of weak solutions. To do this we employ the method of evolution equation in Hilbert space which was first developed by Fujita and Kato [2], [6] in the case of time independent domains and then applied by Inoue and Wakimoto [4] to the case of time dependent domains. Using the regularity result given in Section 2, we show in Section 3 that the same condition as in [12] implies the stability of our periodic solution. Apart from this result, in the case $n=2$ we state another condition similar to the one in [5] for the universal stability and uniqueness of periodic solution.

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1. Reduction to the case of cylindrical domain

Let $\Omega(t)$, $t \in R$, be bounded domains in R^n ($n=2, 3$) with smooth boundaries and let $Q_\infty = \bigcup_{t\in R} \Omega(t) \times \{t\}$ be a noncylindrical domain in $R^{n+1} = R^n \times R$. We consider in Q_∞ the initial-boundary value problem for the Navier-Stokes equations:

$$(1.1)\qquad \begin{aligned} \partial v/\partial t - \Delta v + (v, \mathrm{grad})v &= f - \mathrm{grad}\, p, && x \in \Omega(t), && t > 0, \\ \operatorname{div} v &= 0, && x \in \Omega(t), && t \geq 0, \\ v &= \beta, && x \in \partial\Omega(t), && t > 0, \\ v(x, 0) &= v_0(x), && x \in \Omega(0). \end{aligned}$$

Here $v = \{v^j(x, t)\}_{j=1}^n$, $p = p(x, t)$ denote respectively unknown velocity and pressure while $f = \{f^j(x, t)\}_{j=1}^n$, $v_0 = \{v_0^j(x, t)\}_{j=1}^n$ denote respectively given external force and initial velocity; $\beta = \{\beta^j(x, t)\}_{j=1}^n$ is given on the boundary of Q_∞. As in [10] we impose on Q_∞ and β the following assumptions:

(A.1) There exist a cylindrical domain $\tilde{Q}_\infty = \tilde{\Omega} \times R$ and a C^∞ diffeomorphism Φ: $\bar{Q}_\infty \to \bar{\tilde{Q}}_\infty$; $(y, s) = \Phi(x, t) = (\phi^1(x, t), \dots, \phi^n(x, t), t)$ such that

$$(1.2)\qquad \det[\partial\phi^i(x, t)/\partial x^j] = J(t)^{-1} > 0 \quad \text{for} \quad (x, t) \in \bar{Q}_\infty.$$

(A.2) β is the restriction to the boundary of Q_∞ of a C^2 vector field ψ, which is divergence-free on each $\Omega(t)$ and bounded on $\bar{Q}_\infty$ together with its first and second derivatives.

We note that the condition (1.2) for the Jacobian is of no restriction; see [10]. Setting $v = \psi + u$, we obtain from (1.1),

$$(1.1)'\qquad \begin{aligned} \partial u/\partial t - \Delta u + (u, \nabla)\psi + (\psi, \nabla)u + (u, \nabla)u &= F - \nabla p, && x \in \Omega(t), && t > 0, \\ \operatorname{div} u &= 0, && x \in \Omega(t), && t \geq 0, \\ u &= 0, && x \in \partial\Omega(t), && t > 0, \\ u(x, 0) &= a(x), && x \in \Omega(0), \end{aligned}$$

where $F = f + \Delta\psi - (\psi, \nabla)\psi - \partial\psi/\partial t$ and $a(x) = v_0(x) - \psi(x, 0)$. If we regard u, a, ψ and F as vector fields and p a scalar field on $\Omega(t)$, then by setting

$$(1.3)\qquad \begin{aligned} &\tilde{u}^j(y, s) = \textstyle\sum_{k=1}^n (\partial y^j/\partial x^k) u^k(\Phi^{-1}(y, s)), \text{ and similarly for } a, \psi, F, \\ &\tilde{p}(y, s) = p(\Phi^{-1}(y, s)), \end{aligned}$$

(1.1)′ is transformed into the following problem in $\tilde{Q}_\infty$:

$$(1.1)'' \quad \begin{aligned} &\partial\tilde{u}/\partial s - L\tilde{u} + M\tilde{u} + N_1\tilde{u} + N_2\tilde{u} = \tilde{F} - \nabla_g\tilde{p}, && y\in\tilde{\Omega}, && s>0,\\ &\operatorname{div}\tilde{u} = \textstyle\sum_{j=1}^n \partial\tilde{u}^j/\partial y^j = 0, && y\in\tilde{\Omega}, && s\geq 0,\\ &\tilde{u} = 0, && y\in\partial\tilde{\Omega}, && s>0,\\ &\tilde{u}(y,0) = \tilde{a}(y), && y\in\tilde{\Omega}, \end{aligned}$$

where

$$\begin{aligned} &(L\tilde{u})^i = g^{jk}\nabla_j\nabla_k\tilde{u}^i,\ (N_2\tilde{u})^i = \tilde{u}^j\nabla_j\tilde{u}^i,\\ &(M\tilde{u})^i = (\partial y^j/\partial t)\nabla_j\tilde{u}^i + (\partial y^i\partial x^k)(\partial^2 x^k/\partial s\partial y^j)\tilde{u}^j,\\ &(N_1\tilde{u})^i = \tilde{\psi}^j\nabla_j\tilde{u}^i + \tilde{u}^j\nabla_j\tilde{\psi}^i,\ (\nabla_g\tilde{p})^i = g^{ij}\partial\tilde{p}/\partial y^j,\\ &g^{ij} = (\partial y^i/\partial x^k)(\partial y^j/\partial x^k),\ g_{ij} = (\partial x^k/\partial y^i)(\partial x^k/\partial y^j),\\ &\nabla_j\tilde{u}^i = \partial\tilde{u}^i/\partial y^j + \Gamma^i_{jk}\tilde{u}^k,\\ &\nabla_k(\nabla_j\tilde{u}^i) = \partial(\nabla_j\tilde{u}^i)/\partial y^k + \Gamma^i_{kl}(\nabla_j\tilde{u}^l) - \Gamma^l_{kj}(\nabla_l\tilde{u}^i),\\ &2\Gamma^k_{ij} = g^{kl}(\partial g_{il}/\partial y^j + \partial g_{jl}/\partial y^l - \partial g_{ij}/\partial y^l)\\ &\qquad = 2(\partial y^k/\partial x^l)(\partial^2 x^l/\partial y^i\partial y^j). \end{aligned}$$

Here and hereafter we use the summation convention. Note that ∇_j is the covariant differentiation with respect to the Riemannian connection $\{\Gamma^i_{jk}\}$ induced by the metric $g=(g_{ij})$. From the assumption (A.1) it follows that

$$(1.4) \qquad (g^{ij}) = (g_{ij})^{-1}, \quad \det(g_{ij}) = J(t)^2.$$

This implies that the divergence operator is left unchanged under the change of variables: $x\to y$. Further we note that $\partial\tilde{u}/\partial s + M\tilde{u}$ and $L\tilde{u}$ correspond respectively to $\partial u/\partial t$ and Δu under the coordinate transformation.

Let $C^\infty_{0,\sigma}(\tilde{\Omega})$ be the space of all smooth divergence-free vector fields with compact support in $\tilde{\Omega}$. Let $\tilde{H}$ and $\tilde{V}$ be respectively the closures of $C^\infty_{0,\sigma}(\tilde{\Omega})$ in $(L^2(\tilde{\Omega}))^n$ and in $(H^1(\tilde{\Omega}))^n$, where $H^1(\tilde{\Omega})$ is the usual Sobolev space. For each $t\in R$ we denote by $\tilde{H}_t$ the Hilbert space $\tilde{H}$ endowed with the inner product:

$$(1.5) \qquad \langle\tilde{u}, \tilde{v}\rangle_t = \int_{\tilde{\Omega}} g_{ij}(y,t)\tilde{u}^i(y)\tilde{v}^j(y)J(t)dy.$$

The norm in $\tilde{H}_t$ is denoted by $|\cdot|_t$. Note that the norms $|\cdot|_t$, $t\in R$, are mutually equivalent in $\tilde{H}$. Further $\tilde{V}$ is a Hilbert space with respect to any of the inner products:

$$(1.6) \qquad \langle\nabla_g\tilde{u}, \nabla_g\tilde{v}\rangle_t = \int_{\tilde{\Omega}} g_{ij}(y,t)g^{kl}(y,t)\nabla_k\tilde{u}^i(y)\nabla_l\tilde{v}^j(y)J(t)dy.$$

For $\tilde{u} \in \tilde{V}$ we set $|\nabla_g \tilde{u}|_t = \langle \nabla_g \tilde{u}, \nabla_g \tilde{u} \rangle_t^{1/2}$. The following result was proved in [10] (see [10, Th. 2.2] and its proof).

THEOREM 1.1. *Fix $T>0$ and let $\tilde{V}^*$ be the dual of $\tilde{V}$. Then for each $\tilde{a} \in \tilde{H}$ and each $\tilde{F} \in L^2(0, T; \tilde{V}^*)$ there exists a function $\tilde{u}$ belonging to $L^2(0, T; \tilde{V}) \cap L^\infty(0, T; \tilde{H})$ which satisfies (1.1)″ in the following sense:*

$$\begin{aligned} (1.7)\qquad & -\int_0^T \langle \tilde{u}(t), \tilde{v}(t) + M\tilde{v}(t) \rangle_t dt + \int_0^T \langle N_1 \tilde{u}(t) + N_2 \tilde{u}(t), \tilde{v}(t) \rangle_t dt \\ & + \int_0^T \langle \nabla_g \tilde{u}(t), \nabla_g \tilde{v}(t) \rangle_t dt = \langle \tilde{a}, \tilde{v}(0) \rangle_0 + \int_0^T \langle \tilde{F}(t), \tilde{v}(t) \rangle_t dt \end{aligned}$$

for all $\tilde{v}(t) = h(t)\tilde{w}$ such that $h \in C^1([0, T]; R)$, $h(T)=0$ and $\tilde{w} \in \tilde{V}$. Furthermore, $\tilde{u}$ satisfies the energy inequality:

$$\begin{aligned} (1.8)\qquad & |\tilde{u}(t)|_t^2 + 2\int_0^t |\nabla_g \tilde{u}(\tau)|_\tau^2 d\tau + 2\int_0^t \langle \tilde{u}^j(\tau) \nabla_j \tilde{\psi}^i(\tau), \tilde{u}^i(\tau) \rangle_\tau d\tau \\ & \leq |\tilde{a}|_0^2 + 2\int_0^t \langle \tilde{F}(\tau), \tilde{u}(\tau) \rangle_\tau d\tau, \qquad \textit{for a.e. } t \in [0, T]. \end{aligned}$$

We call such a function $\tilde{u}$ given in Theorem 1.1 a weak solution of (1.1)″ with the initial value $\tilde{a}$.

2. Regularity of weak solutions

In this section we denote functions $\tilde{u}$, $\tilde{a}$, $\tilde{F}$ *etc.* simply by u, a, F *etc.* and equip $\tilde{H}$ with the usual inner product:

$$(2.1)\qquad (u, v) = \int_{\tilde{\Omega}} u^i(y) v^i(y) dy.$$

we set $|u| = (u, u)^{1/2}$. The norm of the Sobolev space $(W^{s,r}(\tilde{\Omega}))^n$ is denoted by $|\cdot|_{s,r}$. Let P denote the orthogonal projection: $(L^2(\tilde{\Omega}))^n \to \tilde{H}$ associated with the following decomposition ([7], [8]):

$$(L^2(\tilde{\Omega}))^n = \tilde{H} \oplus \tilde{G};\ \tilde{G} = \tilde{H}^\perp = \{\operatorname{grad} q : q \in H^1(\tilde{\Omega})\}.$$

Multiplying the first equation in (1.1)″ by the matrix $g = (g_{ij})$, then applying the projection P and using the fact that Pg defines a linear isomorphism of $\tilde{H}$ ([4, Lemma 3.3]), we obtain the following evolution equation in $\tilde{H}$.

$$\begin{aligned} (2.2)\qquad & du/dt + A(t)u = \Psi(t) + E(t, u), \quad t > 0, \\ & u(0) = a, \end{aligned}$$

where

$$
(2.3)\qquad \begin{aligned} A(t) &= -(Pg)^{-1}Pg(L-M-N_1), \\ \Psi(t) &= (Pg)^{-1}PgF,\ E(t, u) = -(Pg)^{-1}PgN_2u. \end{aligned}
$$

LEMMA 2.1. *For each* $t\in[0, T]$, $T>0$ *fixed, and for* $\lambda>0$ *large enough,* $A_\lambda(t)=A(t)+\lambda$ *is a regularly accretive operator* ([14, Chap. 2]) *in* $\tilde{H}_t$ *and satisfies the estimate*:

$$
(2.4)\qquad C_\lambda^{-1}|u|_{2,2} \le |A_\lambda(t)u|_{0,2} \le C_\lambda|u|_{2,2} \qquad \text{for} \quad u\in\tilde{V}\cap(H^2(\tilde{\Omega}))^n.
$$

In particular, $A(t)$ *is a closed operator in* $\tilde{H}$ *defined on* $D(A(t))=\tilde{V}\cap(H^2(\tilde{\Omega}))^n$.

PROOF. Since

$$
\langle v, w\rangle_t = (gv, w)J(t) = (v, gw)J(t) \qquad \text{for} \quad v, w\in\tilde{H}
$$

and since P is the orthogonal projection onto $\tilde{H}$, we can easily see

$$
-\langle (Pg)^{-1}PgLv, w\rangle_t = \langle \nabla_g v, \nabla_g w\rangle_t \qquad \text{for} \quad v, w\in\tilde{V}\cap(H^2(\tilde{\Omega}))^n.
$$

Therefore we have for $w\in D(A(t))$

$$
(2.5)\qquad \begin{aligned} \langle A_\lambda(t)w, w\rangle_t &= |\nabla_g w|_t^2 + \lambda|w|_t^2 + \langle Mw, w\rangle_t + \langle N_1 w, w\rangle_t \\ &\ge |\nabla_g w|_t^2 + \lambda|w|_t^2 - C(|w|_t + |\nabla_g w|_t)|w|_t \\ &\le 2^{-1}|\nabla_g w|_t^2 + (\lambda - C - C^2)|w|_t^2. \end{aligned}
$$

This and the Poincaré inequality show that for $\lambda\ge C+C^2$, $A_\lambda(t)$ is regularly accretive in $\tilde{H}_t$. On the other hand the operator L is obtained from the Laplacian Δ in the coordinates $x\in\Omega(t)$. Hence, according to the apriori estimate in [7, Chap. 3] the estimate (2.4) with $A_\lambda(t)$ replaced by $-(Pg)^{-1}PgL$ holds. Taking account of the smoothness of $g=(g_{ij})$, we can see that (2.4) holds for $A_\lambda(t)$.

In what follows we denote $A_\lambda(t)$ by $A(t)$, so that $A(t)$, $t\in[0, T]$, are invertible in $\tilde{H}$. This causes no essential trouble in solving the equation (2.2); see [9, Sect. 4]. The results in the next lemma are well known in operator theory, so we omit the proofs; see [14].

LEMMA 2.2. (i) *For each* $t\in[0, T]$, $-A(t)$ *generates in* $\tilde{H}_t$ *a holomorphic semigroup of class* C_0.

(ii) *If* ψ *is of class* C^3 *on* $\bar{\tilde{Q}}_\infty=\overline{\tilde{\Omega}\times R}$, *then* $A(t)^{-1}$ *is of class* C^2 *in* t *with respect to the norm topology of bounded operators in* $\tilde{H}$.

(iii) *There are constants* $C>0$ *and* $\theta\in(0, \pi/2)$ *such that, for each* $t\in[0, T]$,

$$
\|(d/dt)(A(t)-\mu)^{-1}\| \le C/|\mu| \qquad \textit{whenever} \quad |\arg\mu|\ge\theta,
$$

where $\|\cdot\|$ *denotes the operator norm of bounded operators in* $\tilde{H}$.

According to [14, Chap. 5, Sect. 3], Lemma 2.2 implies the following.

THEOREM 2.3. (i) *$A(t)$, $t\in[0, T]$, generates in $\tilde{H}$ a parabolic evolution operator, which we denote by $\{U(t, s); 0\leq s\leq t\leq T\}$.*

(ii) *If $s<t$, the operator function $U(t, s)$ is differentiable in t and s, and satisfies*

$$U(s, s) = I, \quad U(t, r)U(r, s) = U(t, s) \quad \text{for} \quad 0 \leq s \leq r \leq t \leq T,$$

$$\partial U(t, s)/\partial t + A(t)U(t, s) = 0 \quad \text{for} \quad s < t,$$

$$\partial U(t, s)/\partial s - U(t, s)A(s) = 0 \quad \text{for} \quad s < t,$$

$$\|A(t)U(t, s)\| \leq C(t-s)^{-1}, \quad \|U(t, s)A(s)\| \leq C(t-s)^{-1} \quad \text{for} \quad s < t.$$

Since $A(t)$ is regularly accretive in $\tilde{H}_t$, its adjoint $A(t)^*$ in $\tilde{H}_t$ is also regularly accretive (see [14]), and therefore the fractional powers $A(t)^\alpha$, $A(t)^{*\alpha}=(A(t)^\alpha)^*$, $\alpha\in R$, are defined in the usual manner. Let $B=-P\Delta$ be the Stokes operator in $\tilde{H}$ defined on $D(B)=\tilde{V}\cap(H^2(\tilde{\Omega}))^n$. It is known that B is a positive definite self-adjoint operator in $\tilde{H}$; see [2], [6]. Further, using the result of [3] and interpolartion theory, one can easily show that the space $D(B^\alpha)$ endowed with the graph norm of B^α is continuously imbedded into $(H^{2\alpha}(\tilde{\Omega}))^n$, $\alpha\in[0, 1]$. On the other hand, since $A(t)$ is regularly accretive, it follows from Kato's generalization of Heinz inequality ([14, Chap. 2]) that $D(A(t)^\alpha)=D(B^\alpha)$, $\alpha\in[0, 1]$. Also note that, by the Poincaré inequality we have

$$|A(t)v|_t\leq C|(Pg)^{-1}PgLv|_t \qquad \text{for} \quad v\in D(A(t)).$$

Then, by Heinz inequality, it also follows that

$$|A(t)^{1/2}v|_t \leq C|(-(Pg)^{-1}PgL)^{1/2}v|_t = C|\nabla_g v|_t \qquad \text{for} \quad v\in D(B^{1/2}).$$

Hence, we have

LEMMA 2.4. *$D(A(t)^\alpha)$ is continuously imbedded into $(H^{2\alpha}(\tilde{\Omega}))^n$. In particular, for $\alpha=1/2$ there is a constant $C_\Omega>0$ such that*

$$C_\Omega^{-1}|\nabla_g v|_t \leq |A(s)^{1/2}v| \leq C_\Omega|\nabla_g v|_t,$$

for any $v\in\tilde{V}$ and any $t, s\in[0, T]$.

The estimates in the lemma below are shown in [13, Sect. 1], so we omit the proof.

LEMMA 2.5. *The evolution operator given in Theorem 2.3 satisfies*

$$\|A(t)^\alpha U(t, s)^{-\beta}\| \leq C_{\alpha\beta}(t-s)^{\beta-\alpha}, \quad \text{for} \quad s < t, 0 \leq \beta < \alpha \leq 1, \tag{2.6}$$

$$\|A(t)^\alpha U(t, s)A(s)^\beta\| \leq C_{\alpha\beta}(t-s)^{-\alpha-\beta}, \quad \text{for} \quad s < t, 0 \leq \alpha, \beta \leq 1. \tag{2.7}$$

Let us now construct a solution of (2.2); we rewrite it in the integral form:

$$(2.2)' \qquad u(t) = U(t, 0)a + \int_0^t U(t, s)\{\Psi(s) + E(s, u(s))\}ds,$$

and consider the following iteration scheme:

$$(2.2)'' \qquad \begin{aligned} u_0(t) &= U(t, 0)a + \int_0^t U(t, s)\Psi(s)ds, \\ u_{m+1}(t) &= u_0(t) + \int_0^t U(t, s)E(s, u_m(s))ds, \ m = 0, 1, 2, \ldots \end{aligned}$$

We put $E(t, v, w) = -(Pg)^{-1}PgN(v, w)$, $N(v, w)^i = v^j \nabla_j w^i$, $(i=1,\ldots, n)$. For a while we restrict ourselves to the case $n=3$. The following lemma is an immediate consequence of [6, Lemma 3].

LEMMA 2.6. *There is a constant $M_0>0$ such that, for any $s, t \in [0, T]$,*

$$(2.8) \qquad |E(s, v, w)| \leq M_0 |A(t)^{1/2}v|\,|A(t)^{3/4}w|, \quad v \in D(B^{1/2}), \quad w \in D(B^{3/4}).$$

Let $a \in D(B^{1/4})$ and let $\Psi(t) \in C((0, T]; \tilde{H})$ be such that $|\Psi(t)| = o(t^{-3/4})$ as $t \to 0$. Using (2.7) and (2.8), we can show by induction on m that each step of the scheme defines u_m as an element in $C([0, T]; \tilde{H}) \cap C((0, T]; D(B^\alpha))$, $\alpha \in [1/4, 1)$. Furthermore, for $T^* \in (0, T]$ and $\alpha \in [1/4, 1)$ if we define a sequence $\{K_{\alpha,m}\}_{m=0}^\infty$ by

$$(2.9) \qquad \begin{aligned} K_{\alpha,0} = \sup_{0<t\leq T^*} t^{\alpha-1/4}|A(t)^\alpha U(t, 0)a| &+ C_{\alpha 0}B(1-\alpha, 1/4) \\ &\times \sup_{0<t\leq T^*} t^{3/4}|\Psi(t)|, \end{aligned}$$

$$(2.10) \qquad K_{\alpha,m+1} = K_{\alpha,0} + C_{\alpha 0}M_0 B(1-\alpha, 1/4)K_{1/2,m}K_{3/4,m},$$

then we have $|A(t)^\alpha u_m(t)| \leq K_{\alpha,m}t^{1/4-\alpha}$ on $(0, T]$. Here $B(p, q)$ is the beta function. If we set $k_m = \max\{K_{1/2,m}, K_{3/4,m}\}$, $C_0 = \max\{C_{1/2,0}, C_{3/4,0}\}$ and $B_0 = B(1/4, 1/4)$, then from (2.10)

$$(2.11) \qquad k_{m+1} \leq k_0 + C_0 B_0 M_0 k_m^2.$$

An elementary calculation shows that if

$$(2.12) \qquad k_0 < 1/(4C_0 B_0 M_0),$$

then, for all m

$$(2.13) \qquad k_m \leq K_0 \equiv \{1-(1-4C_0B_0M_0)^{1/2}\}/(2C_0B_0M_0) < 1/(2C_0B_0M_0).$$

From (2.6) and the assumption on a and $\Psi(t)$ we can choose $T^*>0$ so that (2.12) holds. Hence we can show the following theorem in just same way as in [4] or [6].

THEOREM 2.7. *Let* $n=3$. *For each* $a \in D(B^{1/4})$ *and* $\Psi(t) \in C((0, T]; \tilde{H})$ *such that* $|\Psi(t)| = o(t^{-3/4})$ *as* $t \to 0$, *there exist a* $T^* \in (0, T]$ *and a unique function* u *on* $[0, T^*]$ *such that*

(i) $u \in C([0, T^*]; \tilde{H}) \cap C((0, T^*]; D(B^\alpha))$ *for any* $\alpha \in [1/4, 1)$;

(ii) $|A(t)^\alpha u(t)| = o(t^{1/4-\alpha})$ *as* $t \to 0$ *for any* $\alpha \in [1/4, 1)$;

(iii) $u(t)$ *satisfies the integral equation* (2.2)$'$ *on* $[0, T^*]$.

We now show the Hölder continuity of the solution u given in the above theorem. To do so we need the following

LEMMA 2.8. *For any* $\alpha, \beta \in [0, 1]$, $\alpha < \beta$, *we have the estimate*

$$\|A(t)^\alpha \{U(t, s) - I\} A(s)^{-\beta}\| \leq C'_{\alpha\beta}(t-s)^{\beta-\alpha}. \tag{2.14}$$

PROOF. By Theorem 2.3,

$$A(t)^\alpha \{U(t, s) - I\} A(s)^{-\beta} = -\int_s^t A(t)^\alpha (d/d\sigma) U(t, \sigma) A(s)^{-\beta} d\sigma$$

$$= -\int_s^t A(t)^\alpha U(t, \sigma) A(\sigma) A(s)^{-\beta} d\sigma = -\int_s^t A(t)^\alpha U(t, \sigma) A(\sigma)^{1-\beta} A(\sigma)^\beta A(s)^{-\beta} d\sigma.$$

Since $D(A(\sigma)^\beta) = D(A(s)^\beta)$, $A(\sigma)^\beta A(s)^{-\beta}$ is bounded on $\tilde{H}$, and so the estimate (2.14) follows from (2.7).

PROPOSITION 2.9. *The solution* u *given in Theorem* 2.7 *is Hölder continuous on each* $[\varepsilon, T^*]$ $(0 < \varepsilon < T^*)$ *with values in* $D(B^\alpha)$, $\alpha \in (0, 1)$.

PROOF. Set $u(t) = u_0(t) + v(t)$ with

$$v(t) = \int_0^t U(t, s) E(s, u(s)) ds.$$

For $\alpha \in (0, 1)$ and $h > 0$ we have

$$\begin{aligned} &A(t)^\alpha \{u_0(t+h) - u_0(t)\} \\ &\quad = A(t)^\alpha \{U(t+h, t) - I\} U(t, 0) a + A(t)^\alpha \int_t^{t+h} U(t+h, s) \Psi(s) ds \\ &\qquad + A(t)^\alpha \int_0^t \{U(t+h, t) - I\} U(t, s) \Psi(s) ds \equiv I_1 + I_2 + I_3. \end{aligned}$$

Noting that $A(t)^\alpha A(t+h)^{-\alpha}$ is bounded on $\tilde{H}$, by (2.6) and (2.14) we have

$$\begin{aligned} |I_1| + |I_2| &\leq C\|A(t+h)^\alpha \{U(t+h, t) - I\} A(t)^{-1}\| \, |A(t) U(t, 0) a| \\ &\qquad + C\int_t^{t+h} \|A(t+h)^\alpha U(t+h, s)\| \, |\Psi(s)| ds \\ &\leq C h^{1-\alpha} t^{-1} + C \sup_{\varepsilon \leq t \leq T^*} |\Psi(s)| \int_t^{t+h} (t+h-s)^{-\alpha} ds \leq C h^{1-\alpha}. \end{aligned}$$

I_3 is estimated as follows: when $0<\alpha<\beta<1$,

$$|I_3| \leq C\|A(t+h)^\alpha\{U(t+h,\ t)-I\}A(t)^{-\beta}\| \int_0^t |A(t)^\beta U(t,\ s)\Psi(s)|ds$$

$$\leq Ch^{\beta-\alpha} \sup_{0<s\leq T^*} s^{3/4}|\Psi(s)|\int_0^t (t-s)^{-\beta}s^{-3/4}ds \leq Ch^{\beta-\alpha}.$$

On the other hand,

$$v(t+h) - v(t) = \int_0^t \{U(t+h,\ t)-I\}U(t,\ s)E(s,\ u(s))ds + \int_t^{t+h} U(t+h,\ s)E(s,\ u(s))ds.$$

Using Lemmas 2.6 and 2.2 we have

$$|A(t)^\alpha\{v(t+h)-v(t)\}|$$
$$\leq C_\alpha\|A(t+h)^\alpha\{U(t+h,\ t)-I\}A(t)^{-\beta}\| \int_0^t \|A(t)^\beta U(t,\ s)\|\ |E(s,\ u(s))|ds$$
$$+ C_\alpha \int_t^{t+h} \|A(t+h)^\alpha U(t+h,\ s)\|\ |E(s,\ u(s))|ds.$$

Since $|E(s,\ u(s))|\leq M_0K_0^2s^{-3/4}$ by Lemma 2.6, it follows that, when $\alpha<\beta<1$, $|A(t)^\alpha\{v(t+h)-v(t)\}|$

$$\leq C_\alpha C_{\alpha\beta}C_{\beta 0}h^{\beta-\alpha}K_0^2M_0 \int_0^t (t-s)^{-\beta}s^{-3/4}ds + C'_\alpha M_0K_0^2 \int_t^{t+h} (t+h-s)^{-\alpha}s^{-3/4}ds$$
$$\leq C'_{\alpha\beta}K_0^2M_0B(1-\beta,\ 1/4)\varepsilon^{1/4-\beta}h^{\beta-\alpha} + C'_\alpha M_0K_0^2\varepsilon^{-3/4}h^{1-\alpha},$$

which completes the proof.

From Lemma 2.6 and Proposition 2.9 we have

LEMMA 2.10. *If u is the solution of* (2.2′) *given in Theorem* 2.7, *then $E(t,\ u(t))$ is Hölder continuous on each* $[\varepsilon,\ T^*]$ $(0<\varepsilon<T^*)$.

This lemma implies the following theorem.

THEOREM 2.11. *If in addition to the assumption in Theorem* 2.7 *$\Psi(t)$ is Hölder continuous on each* $[\varepsilon,\ T]$ $(0<\varepsilon<T)$, *then the solution u of* (2.2)′ *given in Theorem* 2.7 *satisfies*:

(i) $u\in C((0,\ T^*];\ D(B))$, $u'\in C((0,\ T^*];\ \tilde{H})$;

(ii) *u is a solution of the evolution equation* (2.2) *on* $(0,\ T^*]$.

When the initial value belongs to $D(B^{1/2})$, we have

PROPOSITION 2.12. *Let a belong to* $D(B^{1/2})$ *and* $\Psi(t)$ *be as in Theorem* 2.11. *Then the solution u given in Theorem* 2.11 *belongs to* $C([0, T^*]; D(B^{1/2}))$, *and hence u belongs to* $L^4(0, T^*; L^6(\tilde{\Omega}))$.

PROOF. We have only to show the continuity at $t=0$ from the right. By (2.2)′ we have

$$A(t)^{1/2}u(t) - A(0)^{1/2}a = \{A(t)^{1/2}U(t, 0)A(0)^{-1/2} - I\}A(0)^{1/2}a + \int_0^t A(t)^{1/2}U(t, s)\{\Psi(s)+E(s, u(s))\}ds.$$

Since, according to Lemmas 2.5 and 2.6, the integrand in the right hand side is summable on $(0, T^*)$, and since $A(t)^{1/2}U(t, 0)A(0)^{-1/2}$ converges to I as $t\to 0$, $A(t)^{1/2}u(t)$ tends to $A(0)^{1/2}a$ in $\tilde{H}$ as $t\to 0$. This shows the continuity of $u(t)$ at $t=0$ in $D(B^{1/2})$. The last assertion follows at once from the Sobolev imbedding theorem.

Now, applying the argument of the uniqueness theorem in [7, Chap. 6, Sect. 2] to our case with slight modification, we can show the following:

COROLLARY 2.13. *Let* $n=3$ *and let the external force* $\Psi(t)$ *be as in Theorem* 2.11. *If the initial value a belongs to* $D(B^{1/2})$, *then the weak solution u given in Theorem* 1.1 *is continuous with values in* $D(B^{1/2})$ *on the interval* $[0, T^*]$, *where* $T^*>0$ *is the constant given in Theorem* 2.7.

Let us proceed to the case $n=2$, in which Lemma 2.6 is replaced by

LEMMA 2.6′. *There is a constant* $M_0'>0$ *such that, for any* $s, t\in[0, T]$,

$$\text{(2.8)}' \quad |A(s)^{-1/4}E(s, v, w)| \leq M_0'|A(t)^{1/4}v|\,|A(t)^{1/2}w|, \; v\in D(B^{1/4}),\; w\in D(B^{1/2}).$$

This is also an immediate consequence of [6, Lemma 3′]. In the same way as before, we can show the following:

THEOREM 2.14. *Let* $n=2$. *Assume that* $\Psi(t)\in C((0, T]; \tilde{H})$ *satisfies* $|A(t)^{-1/4}\Psi(t)|=o(t^{-3/4})$ *as* $t\to 0$ *and is Hölder continuous on each* $[\varepsilon, T](0<\varepsilon<T)$. *Then for an arbitrary* $a\in\tilde{H}$, *there exist a* $T'\in(0, T]$ *and a unique function u on* $[0, T']$ *such that*

(i) $u\in C([0, T']: \tilde{H})\cap C((0, T']; D(B)),\ u'\in C((0, T']; \tilde{H})$;

(ii) *u is a solution of the evolution equation* (2.2) *on* $(0, T']$, *where* T' *depends on a and* $\Psi(t)$.

When $n=2$, we can show that our solutions of (2.2) exist on the whole interval $[0, T]$ under the following assumption:

(A.3) $F\in L^2(0, T; \tilde{H})$; $\Psi(t)\in C((0, T]; \tilde{H})$ is Hölder continuous on each $[\varepsilon, T]$ and $|A(t)^{-1/4}\Psi(t)|=o(t^{-3/4})$ as $t\to 0$; ψ is of class C^3 on $\overline{\tilde{\Omega}\times R}$.

LEMMA 2.15. *If* $u \in C((0, T']; D(B))$ *and* $u' \in C((0, T']; \tilde{H})$, *then*

$$(d/dt)|\nabla_g u|_t^2 = -2\langle u', Lu\rangle_t + R(t),$$

where $|R(t)| \leq C|\nabla_g u(t)|_t^2$ *for some constant* C *independent of* u.

PROOF. First we assume $u \in C^1((0, T']; D(B))$. Then

$$(d/dt)|\nabla_g u|_t^2 = \int_{\tilde{\Omega}} (g_{jk}g^{lm})' \nabla_l u^j \nabla_m u^k J(t)\,dy + \int_{\tilde{\Omega}} g_{jk}g^{lm}\nabla_l u^j \nabla_m u^k J'(t)\,dy$$
$$+ 2\int_{\tilde{\Omega}} g_{jk}g^{lm}(\nabla_l u^j)'\nabla_m u^k J(t)\,dy \equiv I_1 + I_2 + I_3.$$

Obviously $I_1 + I_2 = 0(|\nabla_g u|_t^2)$. To calculate I_3 we proceed as follows: Let ∇_t be the covariant differentiation in the direction of t with respect to the Riemannian connection on $\tilde{\Omega} \times R$ induced by the diffeomorphism Φ. Since this is a flat connection, we have

$$(\nabla_l u^j)' = \nabla_t \nabla_l u^j + 0(|\nabla_g u|) = \nabla_l \nabla_t u^j + 0(|\nabla_g u|)$$
$$= \nabla_l(\partial u^j/\partial t) + 0(|\nabla_g u|).$$

Hence

$$I_3 = 2\int_{\tilde{\Omega}} g_{jk}g^{lm}\nabla_l(\partial u^j/\partial t)\nabla_m u^k J(t)\,dy + 0(|\nabla_g u|_t^2)$$
$$= 2\int_{\tilde{\Omega}} \nabla_l(g_{jk}g^{lm}(\partial u^j/\partial t)\nabla_m u^k)J(t)\,dy$$
$$- 2\int_{\tilde{\Omega}} g_{jk}(\partial u^j/\partial t)g^{lm}\nabla_l\nabla_m u^k J(t)\,dy + 0(|\nabla_g u|_t^2).$$

Since $u' \in C((0, T']; D(B))$, an integration by parts shows that the first term in the right hand side vanishes. Consequently,

$$I_3 = -2\int_{\tilde{\Omega}} g_{ik}(\partial u^j/\partial t)g^{lm}\nabla_l\nabla_m u^k J(t)\,dy + 0(|\nabla_g u|_t^2)$$
$$= -2\langle u', Lu\rangle_t + 0(|\nabla_g u|_t^2).$$

Thus we have proved the assertion when $u \in C^1((0, T']; D(B))$. For general u we have only to regularize it in t after defining $u=0$ outside $(0, T')$ and then to take limit. This completes the proof.

THEOREM 2.16. *Under the assumption* (A.3) *the solution* u *of* (2.2) *given in Theorem* 2.14 *can be continued to the whole interval* $[0, T]$ *as an element in* $C([0, T]; \tilde{H}) \cap C^1((0, T]; \tilde{H}) \cap C((0, T]; D(B))$.

PROOF. Without loss of generality we may assume $a = u(0) \in D(B^{1/2})$.

Since $\langle v, w\rangle_t=(gv, w)J(t)$, we obtain from (2.2–3)

$$\langle u'+Mu, u\rangle_t - \langle Lu, u\rangle_t + \langle N_1u+N_2u, u\rangle_t = \langle F, u\rangle_t, \; t\in(0, T].$$

Note that

$$\langle N_1u+N_2u, u\rangle_t = \langle N(u, \psi) + N(\psi, u)+N(u, u), u\rangle_t.$$

As shown in [10], $2\langle u'+Mu, u\rangle_t=(d/dt)|u|_t^2$, $-\langle Lu, u\rangle_t=|\nabla_g u|_t^2$ and $\langle N(v, w), w\rangle=0$ for $v, w\in\tilde{V}$. Therefore,

$$(d/dt)|u|_t^2+2|\nabla_g u|_t^2 = 2\langle F, u\rangle_t - 2\langle N(u, \psi), u\rangle_t. \tag{2.15}$$

The second term in the right hand side is estimated as follows:

$$\begin{aligned} |\langle N(u, \psi), u\rangle_t| &= |((\bar{u}, \nabla)\bar{\psi}, \bar{u})_{\Omega(t)}| \\ &\le |\bar{u}|^2_{L^4(\Omega(t))}|\nabla\bar{\psi}|_{L^2(\Omega(t))} \\ &\le 2^{1/2}|\bar{u}|_{L^2(\Omega(t))}|\nabla\bar{\psi}|_{L^2(\Omega(t))}|\nabla\bar{u}|_{L^2(\Omega(t))} \\ &= 2^{1/2}|u|_t|\nabla_g\psi|_t|\nabla_g u|_t, \end{aligned} \tag{2.16}$$

where $\bar{u}=\Phi(\cdot, t)_*^{-1}u$ and $\bar{\psi}=\Phi(\cdot, t)_*^{-1}\psi$. Here we have used the estimate: $|u|_{L^4}\le 2^{1/4}(|u|_{L^2}|\nabla u|_{L^2})^{1/2}$ for $u\in(C_0^1(R^2))^2$; see [7] or [8]. Thus by Schwarz's inequality, we have

$$(d/dt)|u|_t^2 + |\nabla_g u|_t^2 \le |F|_t^2 + (1+2|\nabla_g\psi|_t^2)|u|_t^2, \; t\in(0, T'].$$

From this it follows easily that, for $t\in(0, T']$,

$$\begin{aligned} &|u|_t^2 \le (|a|_0^2 + \int_0^T |F|_s^2 ds)\exp\int_0^T (1+2|\nabla_g\psi|_s^2)ds. \\ &\int_0^t |\nabla_g u|_s^2 ds \le |a|_0^2 + \int_0^T |F|_s^2 ds \\ &\quad + \int_0^T (1+2|\nabla_g\psi|_s^2)ds(|a|_0^2+ \int_0^T |F|_s^2 ds)\exp\int_0^T (1+2|\nabla_g\psi|_s^2)ds. \end{aligned} \tag{2.17}$$

To estimate $|\nabla_g u|_t$ we return to the equation (1.1)″. Using the orthogonal decomposition: $(L^2(\tilde{\Omega}))^2=\tilde{H}_t\oplus G_t$, $G_t=\{\nabla_g q;\, q\in H^1(\tilde{\Omega})\}$ with respect to the inner product (1.5), and the associated projection $P_t\colon (L^2(\tilde{\Omega}))^2\to\tilde{H}_t$, we obtain from (1.1)″

$$\langle u', P_tLu\rangle_t + \langle Mu, P_tLu\rangle_t - |P_tLu|_t^2 + \langle N_1u+N_2u, P_tLu\rangle_t = \langle F, P_tLu\rangle_t.$$

Since $\langle u', P_tLu\rangle_t=\langle u', Lu\rangle_t$, it follows from Lemma 2.15 that

$$\begin{aligned} &(d/dt)|\nabla_g u|_t^2 + 2|P_tLu|_t^2 \\ &\quad \le 2|F|_t|P_tLu|_t + C_1|\nabla_g u|_t|P_tLu|_t + C_2|\nabla_g u|_t^2 + C_3|u|_t^{1/2}|\nabla_g u|_t|P_tLu|_t^{3/2}, \end{aligned}$$

so that, by Young's inequality and (2.17),

$$(d/dt)|\nabla_g u|_t^2 \leq C_4|F|_t^2 + C_5(1+|\nabla_g u|_t^2)|\nabla_g u|_t^2.$$

This implies that

$$|\nabla_g u(t)|_t^2 \leq C_6 \exp C_5 \int_0^t (1+|\nabla_g u(s)|_s^2)ds, \quad t \in (0, T']. \tag{2.18}$$

The estimates (2.17–18) bring us the bound of $|\nabla_g u(t)|_t$ on $(0, T']$ independent of T'. So the assertion follows by a standard argument of evolution equations. This completes the proof.

COROLLARY 2.17. *Let $n=2$, and assume* (A.3). *Then the weak solution of* (1.1)″ *given in Theorem* 1.1 *belongs to* $C([0, T]; \tilde{H}) \cap C^1((0, T]; \tilde{H}) \cap C((0, T]; D(B))$.

PROOF. By (2.15) and (2.17) the solution of (2.2) is a weak solution of (1.1)″. Since the weak solution of (1.1)″ is unique for $n=2$ (see [10, Theorem 2.8]). Theorem 2.16 implies the result.

REMARK 2.18. Bock [1] showed the global existence of a unique strong solution of (1.1)″ by the Faedo-Galerkin method. However, the proof in [1] is complicated compared with ours. Moreover, our assumptions on the data are weaker than those of [1].

3. On stability of periodic solutions

In this section for each vector field v on $\Omega(t)$ $\tilde{v}$ will always mean a vector field on $\tilde{\Omega}$ obtained by the transformation (1.3), and conversely. Note that, for each $t \in R$, $|\tilde{v}|_t = \|v\|_t$ if $v \in (L^2(\Omega(t)))^n$ and $|\nabla_g \tilde{w}|_t = \|\nabla w\|_t$ if $w \in (H^1(\Omega(t)))^n$, where $\|\cdot\|_t$ and $\|\nabla \cdot\|_t$ denote the usual norm in $(L^2(\Omega(t)))^n$ and $(H^1(\Omega(t)))^n$ respectively. Let us assume that $n=2$ or 3, and that the diffeomorphism $\Phi(x, t)$ and the boundary data $\psi(x, t)$ are periodic with period $T>0$. The next lemma is due to Serrin [11].

LEMMA 3.1. *Put $d=\max_{0\leq t\leq T} d(t)$, where $d(t)$ is the diameter of $\Omega(t)$. Then, for each $v \in C_{0,\sigma}^{\infty}(\Omega(t))$ we have*

$$\kappa d^{-2}\|v\|_t^2 \leq \|\nabla v\|_s^2. \tag{3.1}$$

Here κ is equal to $(1+2^{1/2})\pi^2$ when $n=2$ and is equal to $(3+13^{1/2})\pi^2/2$ when $n=3$.

In [10, Sect. 3] we proved

THEOREM 3.2. *Let Φ and $\psi(x, t)$ be as above. Then there exists a constant*

$C_1>0$ such that, if $K=\max_{0\le t\le T}\|\nabla\psi(t)\|_t<C_1$, there exists for each $\tilde{F}\in L^2(0, T;\tilde{H})$ a weak solution $\tilde{u}$ of (1.1)″on $[0, T]$ such that $\tilde{u}(0)=\tilde{u}(T)$ in $\tilde{H}$. The constant C_1 is equal to $d^{-1}(\kappa/8)^{1/2}$ when $n=2$, and is equal to $\kappa^{1/4}d^{-1/2}/4$ when $n=3$.

REMARK 3.3. The initial value of the periodic solution $\tilde{u}$ given in the proof of [10, Theorem 3.1] satisfies

$$|\tilde{u}(0)|_0^2\le(1-\exp(-C_2T))^{-1}\kappa^{-1}d^2\int_0^T\exp(C_2(t-T))|\tilde{F}(t)|_t^2dt. \tag{3.2}$$

Here $C_2=(1-C_1^{-1}K)\kappa d^{-2}$ and C_1 is the constant introduced in Theorem 3.2.

The results below (Theorems 3.6 and 3.7) are analogous to the one in [12]. Before proceeding, we show that, when $n=3$, our weak solutions with sufficiently small initial values belong to $C([T,\infty); D(B))$, imposing on the forcing term the following assumptions:

(A.4) $\tilde{\Psi}(t)\in C((0,\infty);\tilde{H})$ is periodic with period T and is Hölder continuous on any finite subinterval in $(0,\infty)$.

(A.5) $\int_0^T|\tilde{F}(t)|_t^2dt<2^{-3/2}T^{1/2}(1-\exp(-C_2T))C_2/(8C_0B_0M_0C_{1/2}C_\Omega)^2$;

$$\sup_{0\le t\le T}|\tilde{\Psi}(t)|\le(2T)^{-3/4}/(8C_0^2B_0^2M_0).$$

Here we put $C_{1/2}=\max\{C_{1/2,1/2}, C_{3/4,1/2}\}$; the constants $C_{1/2,1/2}$, $C_{3/4,1/2}$, C_0, B_0, M_0 and C_Ω are those introduced in Section 2.

Setting $\rho=(1-(\exp(-C_2T))^{-1}\kappa^{-1}d^2\int_0^T|\tilde{F}(t)|_t^2dt$, we obtain

PROPOSITION 3.4. *Let $n=3$. Under the assumptions* (A. 4–5), *any weak solution $\tilde{v}$ of* (1.1)″ *whose initial value satisfies $|\tilde{v}(0)|_0^2\le\exp(-C_2T)\rho$ belongs to $C([T,\infty); D(B))$ and satisfies the evolution equation* (2.2) *on* $[T,\infty)$.

PROOF. First we note that, for a.e. $t\in[0, T]$,

$$|\tilde{v}(t)|_t^2+C_2\kappa^{-1}d^2\int_0^t|\nabla_g\tilde{v}(\tau)|_\tau^2d\tau\le|\tilde{v}(0)|_0^2+\kappa^{-1}d^2\int_0^t|\tilde{F}(\tau)|_\tau^2d\tau. \tag{3.3}$$

This follows from (1.8), by Schwarz's inequality and the estimate:

$$|\langle N(\tilde{v},\tilde{\psi}),\tilde{v}\rangle_t|\le 2\kappa^{-1/4}d^{1/2}K|\nabla_g\tilde{v}|_t^2=(2C_1)^{-1}K|\nabla_g\tilde{v}|_t^2 \tag{3.4}$$

which is proved in the same way as (2.16) by making use of the estimate: $|v|_{L^4}\le 2^{1/2}|v|_{L^2}^{1/4}|\nabla v|_{L^2}^{3/4}$ for $v\in(C_0^1(R^3))^3$. Then, by (A.5) and the assumption on the initial value $\tilde{v}(0)$. one can easily see that there is a point $t_0\in(0, T)$ such that $|\tilde{v}(t_0)|_{t_0}^2\le\rho$ and

(3.5) $$|\nabla_g \tilde{v}(t_0)|_{t_0} < (2T)^{-1/4}/(8C_0B_0M_0C_{1/2}C_\Omega).$$

By (A.5) we have

(3.6) $$\sup_{0<t\leq 2T}\{t^{\alpha-1/4}|A(t+t_0)^\alpha U(t+t_0,\ t_0)\tilde{v}(t_0)| + C_0B_0t^{3/4}|\tilde{\psi}(t+t_0)|\} < 1/(4C_0B_0M_0).$$

for $\alpha=1/2$, $3/4$. Therefore, by Theorem 2.7 and Corollary 2.13, $\tilde{v}$ belongs to $C((t_0,\ t_0+2T];\ D(B))$ and satisfies (2.2) on $(t_0,\ t_0+2T]$. Further, as we have deduced (2.15), we can deduce

(3.7) $$(d/dt)|\tilde{v}|_t^2+2|\nabla_g\tilde{v}|_t^2 = 2\langle\tilde{F},\ \tilde{v}\rangle_t - 2\langle N(\tilde{v},\ \tilde{\psi}),\ \tilde{v}\rangle_t,\ t_0 < t < t_0 + 2T.$$

By (3.1) and (3.4), it follows from (3.7) that

$$\exp(C_2T)|\tilde{v}(t_0+T)|_{t_0+T}^2 \leq |\tilde{v}(t_0)|_{t_0}^2 + \kappa^{-1}d^2\exp(C_2T)\int_0^T |\tilde{F}(t)|_t^2 dt.$$

Then, from $|\tilde{v}(t_0)|_{t_0}^2\leq\rho$, it follows that $|\tilde{v}(t_0+T)|_{t_0+T}^2\leq\rho$. Also (3.7) gives

$$C_2\kappa^{-1}d^2\int_{t_0+T}^{t_0+2T} |\nabla_g\tilde{v}(t)|_t^2 dt \leq |\tilde{v}(t_0 + T)|_{t_0+T}^2 + \kappa^{-1}d^2\int_0^T |\tilde{F}(t)|_t^2 dt.$$

Applying the same argument as above, we see that there is a $t_1\in(t_0+T,\ t_0+2T)$ such that (3.5) holds with t_0 replaced by t_1, and then $\tilde{v}$ belongs to $C([t_1,\ t_1+2T];\ D(B))$. Successive use of this argument yields the result.

PROPOSITION 3.5. *Let $n=3$. Under the assumptions* (A.4–5), *the periodic solution $\tilde{u}$ given in Theorem* 3.2 *belongs to* $C([0,\ \infty);\ D(B))$ *and satisfies* (2.2) *on* $[0,\ \infty)$.

PROOF. This is proved in just the same way as Proposition 3.4, by noting periodicity of $\tilde{u}$, (3.2) and (3.3) in which $\tilde{v}$ is replaced by $\tilde{u}$. So we omit the details.

From this proposition and the Sobolev imbedding theorem, the periodic solution $\tilde{u}$, given in Theorem 3.2 is continuous on $\overline{\tilde{\Omega}}\times[0,\ T]$.

THEOREM 3.6. *Let $\tilde{u}$ be the periodic solution given in Theorem* 3.2 *in the case $n=3$ and let V be the maximum of $|(\psi+u)(x,\ t)|$ on $\cup_{0\leq t\leq T}\overline{\Omega(t)}\times\{t\}$. Assume* (A.4–5) *and*

(3.8) $$Vd < \kappa^{1/2}.$$

Let $\tilde{v}$ be any weak solution of (1.1)″ *such that $|\tilde{v}(0)|_0^2\leq\exp(-C_2T)\rho$. Then for any $\varepsilon>0$ there is a $T_1>T$ such that*

$$|\tilde{u}(t)-\tilde{v}(t)|_t \leq \varepsilon \qquad \text{for} \quad t \geq T_1.$$

PROOF. First note that for $t>T$ $\tilde{u}$ satisfies

$$\langle \tilde{u}'(t) + M\tilde{u}(t), \tilde{\phi}\rangle_t + \langle \nabla_g \tilde{u}(t), \nabla_g \tilde{\phi}\rangle_t + \langle (N_1\tilde{u} + N_2\tilde{u})(t), \tilde{\phi}\rangle_t = \langle \tilde{F}(t), \tilde{\phi}\rangle_t$$

for any $\tilde{\phi} \in \tilde{V}$, and $\tilde{v}$ also satisfies the corresponding equation. Hence $\tilde{w} = \tilde{u} - \tilde{v}$ satisfies

$$\langle \tilde{w}'(t) + M\tilde{w}(t), \tilde{\phi}\rangle_t + \langle \nabla_g \tilde{w}(t), \nabla_g \tilde{\phi}\rangle_t + \langle N_1\tilde{w}(t), \tilde{\phi}\rangle_t + \langle (N_2\tilde{u} - N_2\tilde{v})(t), \tilde{\phi}\rangle_t = 0.$$

Substituting $\tilde{w}(t)$ for $\tilde{\phi}$, we obtain

$$\langle (\tilde{w}' + M\tilde{w})(t), \tilde{w}(t)\rangle_t + |\nabla_g \tilde{w}(t)|_t^2 = -\langle N(\tilde{w}, \tilde{w})(t), (\tilde{u} + \tilde{\psi})(t)\rangle_t,$$

where we have used the fact that $\langle N(\tilde{u}, \tilde{v}), \tilde{w}\rangle_t = -\langle N(\tilde{u}, \tilde{w}), \tilde{v}\rangle_t$, which is valid for $\tilde{u}, \tilde{v}, \tilde{w} \in \tilde{V}$. Since $(d/dt)|\tilde{w}|_t^2 = 2\langle \tilde{w}' + M\tilde{w}, \tilde{w}\rangle_t$, by returning to $\Omega(t)$, we have

$$(d/dt)\|w\|_t^2 + 2\|\nabla w\|_t^2 = 2((w, \nabla)w, \psi + u)_t.$$

Here $(\cdot, \cdot)_t$ denotes the usual inner product in $(L^2(\Omega(t)))^n$. Then, by Schwarz's inequality

$$\begin{aligned}(d/dt)\|w\|_t^2 + 2\|\nabla w\|_t^2 &\leq Vd\kappa^{-1/2}\|\nabla w\|_t^2 + \kappa^{-1/2}(Vd)^{-1}\int_{\Omega(t)} |w|^2|\psi + u|^2 dx \\ &\leq Vd\kappa^{-1/2}\|\nabla w\|_t^2 + \kappa^{1/2}Vd^{-1}\|w\|_t^2.\end{aligned}$$

From (3.1) and (3.8) we have

$$(d/dt)\|w\|_t^2 + 2\kappa d^{-2}(1 - Vd\kappa^{-1/2})\|w\|_t^2 \leq 0.$$

Hence, putting $C' = 2\kappa d^{-1}(1 - Vd\kappa^{-1/2}) > 0$, we can easily deduce

$$\|w(t)\|_t^2 \leq e^{C'(T-t)}\|w(T)\|_T^2 \qquad \text{for} \quad t > T.$$

From this the assertion follows immediately.

Next we state our results in the case $n=2$, in which the assumptions on the data can be relaxed compared with the case $n=3$.

THEOREM 3.7. *Let $n=2$. Under the assumption* (A.4), *suppose the periodic solution $\tilde{u}$ given in Theorem* 3.2 *satisfies* (3.8). *Then the periodic solution of* (1.1)″ *is unique and any other weak solution of* (1.1)″ *tends to $\tilde{u}$ as $t \to \infty$ in $\tilde{H}$.*

PROOF. Let $\tilde{v}$ be any weak solution of (1.1)″. According to Corollary 2.17 both $\tilde{u}$ and $\tilde{v}$ belong to $C([0, \infty); \tilde{H}) \cap C^1((0, \infty); \tilde{H}) \cap C((0, \infty); D(B))$. In just the same way as in the proof of Theorem 3.6, setting $\tilde{w} = \tilde{u} - \tilde{v}$, we have $(C' = 2\kappa d^{-2}(1 - Vd\kappa^{-1/2}) > 0)$

$$\|w(t)\|_t^2 \leq e^{-C't}\|w(0)\|_0^2 \qquad \text{for} \quad t > 0.$$

From this the second assertion follows immediately. Suppose now that $\tilde{v}$ is also periodic with period T. Then by the above inequality $\|w(0)\|_0 = \|w(T)\|_T = 0$. Since our weak solution is unique when $n=2$, the first assertion holds.

REMARK 3.8. In this paper we have treated the Navier-Stokes flow with unit kinematic viscosity, so the condition (3.8) is the same as the one Serrin has given in [12] in terms of the Reynolds number of the flow.

Finally we give another condition in the case $n=2$ for our periodic solution to be unique and stable. Note that here the forcing term is required only to belong to $L^2(0, T; \tilde{H})$.

THEOREM 3.9. *Let $n=2$ and assume that the periodic solution $\tilde{u}$ given in Theorem 3.2 satisfies*

$$\int_0^T \|\nabla(\psi+u)(t)\|_t^2 dt < T\kappa d^{-2}/2. \tag{3.9}$$

Then the periodic solution of (1.1)″ is unique and any other weak solution of (1.1)″ defined on $[0, \infty)$ tends to $\tilde{u}$ as $t \to \infty$ in $\tilde{H}$.

PROOF. Let $\tilde{v}$ be any weak solution of (1.1)″ defined on $[0, \infty)$. In this case also $\tilde{w} = \tilde{u} - \tilde{v}$ satisfies

$$\begin{aligned}(d/dt)\|w\|_t^2 + 2\|\nabla w\|_t^2 &= 2((w, \nabla)w, \psi+u)_t \\ &= -2((w, \nabla)(\psi+u), w)_t,\end{aligned}$$

see [10, Theorem 2.8]. Then, by (2.16), we have

$$(d/dt)\|w\|_t^2 + 2\|\nabla w\|_t^2 \leq 2^{3/2}\|\nabla(\psi+u)\|_t \|\nabla w\|_t \|w\|_t.$$

Applying Schwarz's inequality and (3.1) to the right hand side, we obtain

$$(d/dt)\|w\|_t^2 \leq \zeta(t)\|w\|_t^2, \quad \zeta(t) = 2\|\nabla(\psi+u)(t)\|_t^2 - \kappa d^{-2}.$$

Hence

$$\|w(t)\|_t^2 \leq \|w(0)\|_0^2 \exp\left(\int_0^t \zeta(\tau)d\tau\right), \qquad \text{for} \quad t > 0. \tag{3.10}$$

If $\tilde{v}$ is periodic with period T, then

$$\|w(0)\|_0^2 = \|w(T)\|_T^2 \leq \|w(0)\|_0^2 \exp\left(\int_0^T \zeta(t)dt\right).$$

Since (3.9) implies $\delta = \exp\left(\int_0^T \zeta(t)dt\right) < 1$, $\|w(0)\|_0$ vanishes. This shows the uniqueness of the periodic solution $\tilde{u}$. To show the stability we note that, by periodicity of ψ and u, $\exp\left(\int_0^{kT}\zeta(t)dt\right)=\delta^k$ for any integer $k>0$. So (3.10) gives

$$\|w(t)\|_t^2 \leq \|w(0)\|_0^2\delta^k \qquad \text{if} \quad t\in[kT,\ (k+1)T].$$

By letting $k\to\infty$, the result follows.

REMARK 3.10. From the proof of Proposition 3.4 we obtain

$$\int_0^T \|\nabla u(t)\|_t^2 dt \leq C_2^{-1}(\kappa d^{-2}\|u(0)\|_0^2 + \int_0^T \|F(t)\|_t^2\, dt).$$

On account of this and (3.2), (3.9) holds when the forcing term and the boundary data with their derivatives are sufficiently small.

REMARK 3.11. The last theorem is similar to the one in [5], where Kaniel and Shinbrot, in the case of time independent domains, obtained a periodic solution which is locally stable under the assumption that the given external force is periodic and small enough.

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