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Invariant measures for uniformly recurrent diffusion kernels

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In this paper we shall characterize invariant measures for a uniformly recurrent diffusion kernel T on a locally compact Hausdorff space X. Our main result is summarized as follows: Denote by $H(T)$ the cone generated by non-negative T-invariant measures and put $X_o = \mathrm{cl}(\cup_{\mu \in H(T)} \mathrm{supp}(\mu))$. Then there exists a strictly positive diffusion kernel W on X_o, uniquely determined except for the equivalence of diffusion kernels, such that $TW = W$ and $H(T)$ coincides with W-potentials.

In sections 2 and 3, we shall discuss when $H(T)$ is one dimensional and when the cone formed by non-negative invariant functions with respect to the transposed kernel of T is one dimensional.

We remark in section 4 that similar results are valid for uniformly recurrent continuous diffusion semi-groups on X.

A typical example of a uniformly recurrent diffusion kernels is an idempotent kernel on X. Applying our theorem to the idempotent kernels and using results in M. Itô [10], we see that a weakly regular diffusion kernel on X may be considered as a weakly regular Hunt diffusion kernel on some quotient space of X.

In section 6, applying our theorem to diffusion kernels of convolution type on homogeneous spaces, we represent explicitly the above diffusion kernel W. In this direction, for a locally compact abelian group G and non-negative adapted Radon measure σ on G, G. Choquet and J. Deny [4] showed that all extreme rays of the convex cone $H(\sigma)$ formed by non-negative σ-invariant measures are generated by exponentials on G. In a non-abelian case, H. Furstenberg [6] pointed out that the extreme rays of $H(\sigma)$ are generated by multiplier functions on a certain Lie group G and some particular measure σ; however a caracterization of the extreme rays is not known in the general case. But if σ is recurrent, our theorem shows that $H(\sigma)$ is generated by at most one exponential on G even if G is not commutative (see also [7]). Using our theorem, we can characterize non-negative finite order measures on locally compact Hausdorff groups, particularly, we see that non-negative idempotent measures are the normalized Haar measures (cf. [9] and [13]).

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§ 1. Basic notation and preliminaries

Let X be a locally compact Hausdorff space with a countable base. We denote by $C(X)$ the Fréchet space of finite continuous functions on X, by $C_K(X)$ the topological vector space of finite continuous functions on X with compact support, by $M(X)=C_K(X)^*$ the topological vector space of real Radon measures on X with w^*-topology and by $M_K(X)=C(X)^*$ the topological vector space of real Radon measures on X with compact support with w^*-topology. Their subsets of non-negative elements are denoted by $C^+(X)$, $C_K^+(X)$, $M^+(X)$ and $M_K^+(X)$ respectively.

A linear operator T from $M_K(X)$ into $M(X)$ is said to be a *diffusion kernel* on X if it is continuous and positive, i.e., $T\mu \in M^+(X)$ whenever $\mu \in M_K^+(X)$. The transposed kernel T^* of T is the linear continuous operator from $C_K(X)$ into $C(X)$ defined by $T^*\psi(x)=\int \psi dT\varepsilon_x$, where ε_x is the Dirace measure at $x \in X$. Then T^* is positive, i.e., for $\psi \in C_K^+(X)$, $T^*\psi \in C^+(X)$. In the sequel, for a diffusion kernel T on X, its transposed kernel is always denoted by T^*. We put

$$\mathscr{D}(T) = \{\lambda \in M(X); \int T^*\psi|d\lambda| < \infty \quad \text{for all} \quad \psi \in C_K^+(X)\},$$

$$\mathscr{D}(T^*) = \{f \in C(X); \text{the function} \int |f|dT\varepsilon_x \text{ is continuous on } X\},$$

$\mathscr{D}^+(T)=\mathscr{D}(T) \cap M^+(X)$ and $\mathscr{D}^+(T^*)=\mathscr{D}(T^*) \cap C^+(X)$. Then T (resp. T^*) can be extended to a positive linear operator from $\mathscr{D}(T)$ into $M(X)$ (resp. from $\mathscr{D}(T^*)$ into $C(X)$ by defining $T^*f(x)=\int f dT\varepsilon_x$).

Let T_j ($j=1, 2$) be a diffusion kernel on X. We say that T_1 is *equivalent* to T_2 if for any $x \in X$, there exists $c_x>0$ such that $T_1\varepsilon_x=c_xT_2\varepsilon_x$. We say that a sequence $(T_j)_{j=1}^\infty$ of diffusion kernels on X converges to a diffusion kernel T on X if $\lim_{j\to\infty} T_j\varepsilon_x=T\varepsilon_x$ for all $x \in X$. In this case, we denote by $\lim_{j\to\infty} T_j=T$.

Let T be a diffusion kernel on X and X_o be a closed subset of X. We may consider that $M(X_o) \subset M(X)$. If $T\mu \in M(X_o)$ for any $\mu \in M_K(X_o)$, then T may be regarded as a diffusion kernel on X_o.

Let T_j ($j=1, 2$) be a diffusion kernel on X. If for $\mu \in M_K(X)$, $T_2\mu \in \mathscr{D}(T_1)$ and the mapping $M_K(X) \ni \mu \to T_1(T_2\mu)$ defines a diffusion kernel on X, the resulting diffusion kernel is denoted by T_1T_2. In this case, for $\psi \in C_K(X)$, $T_1^*\psi \in \mathscr{D}(T_2^*)$ and the mapping $C_k(X) \ni \psi \to T_2^*(T_1^*\psi)$ is positive linear and $(T_1T_2)^*=T_2^*T_1^*$ holds.

In particular, for a diffusion kernal T on X and a positive integer $n \geqq 1$, we denote by T^n the diffusion kernel defined inductively by $T^{n-1}T$ provided that it is defined, where $T^1=T$. In case $T\neq 0$, T^0 means the identity operator I on $M(X)$.

A family $(V_p)_{p>0}$ of diffusion kernels on X is said to be a *resolvent* on X

if V_pV_q is defined and $V_p - V_q = (q-p)V_pV_q$ for all $p>0$, $q>0$. For a diffusion kernel V on X, if there exists a resolvent $(V_p)_{p>0}$ satisfying $\lim_{p\to 0} V_p = V$, then it is unique and by putting $V_0 = V$, we call $(V_p)_{p\geqq 0}$ the *resolvent associated with V.* In this case, we denote by $V\in \boldsymbol{R}$ and for any $p>0$, V_pV and VV_p are defined and $V - V_p = pVV_p = pV_pV$.

Let T be a diffusion kernel on X. For any $p>0$, we assume that V_p defined by

$$V_p\mu = (p+1)^{-1} \sum_{n=0}^{\infty} (p+1)^{-n} T^n\mu \qquad (\mu \in M_K(X))$$

is a diffusion kernel on X. In this case, we see easily that $(V_p)_{p>0}$ is a resolvent and call it the *resolvent generated by T.*

DEFINITION 1. (1) A resolvent $(V_p)_{p>0}$ on X is said to be *uniformly recurrent* if there exist a family $(u_p)_{p>0}$ in $C^+(X)$ and a positive number p_o satisfying the following four conditions:

(a) $u_p > 0$ on X for any $p>0$.

(b) u_p converges pointwise to 0 as $p\to 0$.

(c) For any $\psi \in C_K^+(X)$, $(u_pV_p^*\psi)_{p_o>p>0}$ forms a normal family on any compact set.

(d) For any $x\in X$, there exists $\psi \in C_K^+(X)$ such that

$$\inf_{p_o>p>0} u_p(x)V_p^*\psi(x) > 0.$$

(2) A diffusion kernel T on X is said to be *uniformly recurrent* if the resolvent generated by it exists and is uniformly recurrent.

DEFINITION 2 (cf. [10] p. 331). (1) A subset A of $X\times X$ is said to be of *transitive type* if for any (x, y), $(y, z)\in A$, $(x, z)\in A$ and $A\supset \{(x, x); x\in X\}$, and A is said to be *symmetric* if $(x, y)\in A$ implies $(y, x)\in A$.

(2) Let F be a closed set of transitive type. A function $e(x, y)>0$ on F is said to be of *exponential type* if it is continuous and for any (x, y), $(y, z)\in F$, $e(x, y)e(y, z) = e(x, z)$

If F is symmetric, $e(x, y)e(y, x) = 1$ for any $(x, y)\in F$ evidently.

For a diffusion kernel T on X, we put

$$\operatorname{supp}(T) = \operatorname{cl}\{(x, y)\in X\times X;\ x\in X,\ y\in \operatorname{supp}(T\varepsilon_x)\},$$

and call it the support of T.

DEFINITION 3 (cf. [10] p. 332). (1) A diffusion kernel T on X is said to be of *sub-exponential type* if there exists a closed set F of transitive type with $F\supset \operatorname{supp}(T)$ and a function $e(x, y)$ on F of exponential type such that for any $x\in X$, $\int e(x, y)dT\varepsilon_x(y) \leqq 1$.

(2) In the above case, if in addition the function $\int e(x, y)dT\varepsilon_x(y)$ on X is

continuous, we say that T is of *continuous sub-exponential type.*

(3) In particular, if $\int e(x, y)dT\varepsilon_x(y)\equiv 1$ on X, we say that T is of *exponential type.*

REMARK 4. Let T be a diffusion kernel on X and $e(x, y)$ be a function of exponential type on a certain closed set F of transitive type with $F\supset \mathrm{supp}(T)$. We can define the diffusion kernel eT on X by putting

$$\int \psi d(eT\mu) = \iint e(x, y)\psi(y)dT\varepsilon_x(y)d\mu(x)$$

for $\mu\in M_K(X)$ and $\psi\in C_K(X)$.

If T is of continuous sub-exponential type, then the resolvent generated by T exists, because for any $p>0$, we see

$$\textstyle\sum_{n=0}^{k}(p+1)^{-n}T^n = (1/e)\sum_{n=0}^{k}(p+1)^{-n}(eT)^n$$

for each k and the right hand side defines a diffusion kernel on X as $k\to\infty$, where e is a function of exponential type as in (1) and (2) of Definition 3.

For a diffusion kernel T on X, we shall consider the following convex cones:

$$H(T) = \{\mu\in\mathscr{D}^+(T);\ T\mu = \mu\}$$

and

$$H(T^*) = \{f\in\mathscr{D}^+(T^*);\ T^*f = f\}.$$

An element in $H(T)$ is called a non-negative *T-invariant measure* and an element in $H(T^*)$ is called a non-negative *T^*-invariant function.*

PROPOSITION 5. *Let T be uniformly recurrent. If $\mu\in\mathscr{D}^+(T)$ satisfies $T\mu\leqq\mu$ then $\mu\in H(T)$.*

In fact, let $(V_p)_{p>0}$ be the resolvent generated by T. Then, by conditions (b) and (d) of Definition 1, $V_p\varepsilon_x$ does not converge to a measure as $p\to 0$ for each $x\in X$. Hence if $T\mu\leqq\mu$ and $T\mu\neq\mu$, then $V_p(\mu-T\mu)$ does not converge to a measure as $p\to 0$, which contradicts the fact that $V_p(\mu-T\mu)\leqq(1+p)\mu$.

From the above proposition the following corollary follows.

COROLLARY 6. *Let T be uniformly recurrent. Then $H(T)$ is a closed convex cone in the metrizable space $M^+(X)$ and is the union of its caps*[1].

In fact, we remark that $M^+(X)$ is metrizable, because X has a countable base.

1) In general, for a closed convex cone K in a locally convex space, a non-empty subset C of K is called a cap of K if C is a compact convex set and if K–C is also convex.

Since the mapping $\mathscr{D}^+(T)\ni\mu\to T\mu\in M^+(X)$ is lower semi-continuous, Proposition 5 gives the first part. For $\mu\neq 0$ in $H(T)$, we choose $f_\mu\in C^+(X)$ with $f_\mu>0$ on X and $\int f_\mu d\mu=1$. Put

$$H(T;f_\mu)=\{\lambda\in H(T);\int f_\mu d\lambda\leqq 1\}.$$

Then $H(T;f_\mu)$ is a cap of $H(T)$, which gives the second part.

We shall introduce some of the fundamental potential theoretic principles for diffusion kernels.

Let V be a diffusion kernel on X. For an open set ω in X and $\mu\in\mathscr{D}^+(T)$, we put

$$B^V(\mu;\omega)=\{\mu'\in\mathscr{D}^+(V);\ \mathrm{supp}(\mu')\subset\bar{\omega},\ V\mu'\leqq V\mu \text{ on } X \text{ and } V\mu'=V\mu \text{ in } \omega\}$$

and

$$B^V_m(\mu;\omega)=\left\{\mu'\in B^V(\mu;\omega);\ \begin{array}{l}\text{For any } \lambda\in\mathscr{D}^+(V),\ V\mu\leqq V\lambda \text{ in}\\ \omega \text{ implies } V\mu'\leqq V\lambda \text{ on } X\end{array}\right\}.$$

If $B^V(\mu;\omega)\neq\emptyset$ for any relatively compact open set ω and $\mu\in M_K^+(X)$, V is said to satisfy the *balayage principle* and we denote $V\in\boldsymbol{B}$.

We say that V (resp. V^*) satisfies the *domination principle*, $V\in\boldsymbol{D}$ (resp. $V^*\in\boldsymbol{D}$) in symbol, if for any $\mu,\lambda\in M_K^+(X)$ (resp. $f, g\in C_K^+(X)$) $V\mu\leqq V\lambda$ in a certain neighborhood of $\mathrm{supp}(\mu)$ (resp. $V^*f\leqq V^*g$ on $\mathrm{supp}(f)$) implies $V\mu\leqq V\lambda$ on X (resp. $V^*f\leqq V^*g$ on X).

For the relation between the above principles, we have the following results.

LEMMA 7 (*see* [10]). *Assume that V is a strictly positive diffusion kernel on X, i.e., $V\varepsilon_x\neq 0$ for any $x\in X$. Then*

(1) *$V\in\boldsymbol{R}\Rightarrow V\in\boldsymbol{B}\Leftrightarrow V^*\in\boldsymbol{D}\Rightarrow V+cI\in\boldsymbol{D}$ for any $c>0$.*

(2) *If $V\in\boldsymbol{B}$, then $B^V_m(\mu;\omega)\neq\emptyset$ for any relatively compact open set ω and any $\mu\in\mathscr{D}^+(V)$.*

The implication $V\in\boldsymbol{B}\Rightarrow V\in\boldsymbol{D}$ is not true in general; however we have the following

LEMMA 8. *Assume that $V\in\boldsymbol{B}$ and is strictly positive. Then for any $\mu,\lambda\in M_K^+(X)$, if $V\mu\leqq V\lambda$ in a certain neighborhood of* supp (μ) *and if there exists $\tau\in\mathscr{D}^+(V)$ such that μ is absolutely continuous with respect to $V\tau$, then $V\mu\leqq V\lambda$ on X.*

PROOF. By $V\in\boldsymbol{B}$, we may assume that $\tau\in M_K^+(X)$. Write $\mu=fV\tau$ with some non-negative function f on X. For any $n\geqq 1$, we put $g_n=\min\{f, n\}$ and $\mu_n=g_nV\tau$. Then for any $c>0$,

$$(V+cI)\mu_n \leqq (V+cI)(\lambda+cn\tau)$$

in a certain neighborhood of $\mathrm{supp}(\mu_n)$. It follows from $V+cI\in \boldsymbol{D}$ that above inequality holds on X. Letting $c\downarrow 0$ and next $n\uparrow\infty$, we have $V\mu\leqq V\lambda$ on X. This completes the proof.

§2. Determination of H(T)

We now state the main result in this section.

THEOREM 9. *Let T be a uniformly recurrent diffusion kernel on X and let $X_o=\mathrm{cl}(\cup_{\mu\in H(T)}\mathrm{supp}(\mu))$. Then there exists a strictly positive diffusion kernel W on X_o, uniquely determined except for the equivalence of diffusion kernels, satisfying the following*:

(1) $W\in \boldsymbol{B}$.

(2) $H(T)=\{W\lambda;\ \lambda\in\mathscr{D}^+(W)\}$[2].

(3) *There exists a uniquely determined funjtion $e(x, y)$ of exponential type on the closed symmetric set of transitive type generated by* $\mathrm{supp}(W)$[3] *such that, for any $x\in X_o$ and any $y\in\mathrm{supp}(W\varepsilon_x)$, $W\varepsilon_x=e(x, y)W\varepsilon_y$.*

(4) $\mathrm{exr}H(T)=\{[W\varepsilon_x];\ x\in X_o\}$, *where* $\mathrm{exr}H(T)$ *is the set of all extreme rays in $H(T)$*[4] *and $[\mu]=\{c\mu;\ c\geqq 0\}$ for any $\mu\neq 0$ in $M^+(X)$.*

It will be convenient to begin the proof with a couple of lemmas. The first is concerned with the construction of the diffusion kernel W which appears in Theorem 9.

LEMMA 10. *Let T and X_o be as in* Theorem 9. *Then there exists a strictly positive diffusion kernel W on X_o such that $W\in\boldsymbol{B}$ and $\{W\varepsilon_x;\ x\in X_o\}\subset H(T)$.*

PROOF. Let $(V_p)_{p>0}$ be the resolvent generated by T and $(u_p)_{p>0}\subset C^+(X)$ be the family as in Definition 1. First we remark that T is a diffusion kernel on X_o (note that the last part of this proof gives $H(T)\neq\{0\}$ and hence $X_o\neq\emptyset$). In fact, for any $x\in\mathrm{supp}(\mu)$ with $\mu\in H(T)$,

$$\mathrm{supp}(T\varepsilon_x)\subset\mathrm{supp}(T\mu)=\mathrm{supp}(\mu)\subset X_o,$$

which implies $\mathrm{cl}\{\cup_{x\in X_o}\mathrm{supp}(T\varepsilon_x)\}\subset X_o$ and hence $T\lambda\in M(X_o)$ for any $\lambda\in M_K(X_o)$. Similar arguments show that each V_p, $p>0$, is also a diffusion kernel on X_o and

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2) Here and hereafter, we consider that $M(X_o)\subset M(X)$.

3) This means the smallest closed symmetric set of transitive type containing supp (W).

4) A ray $[\mu]$ in $H(T)$ is a set of the form $\{c\mu;\ c\geqq 0\}$, where $0\neq\mu\in H(T)$, and we say that $[\mu]$ is an extreme ray if for any $\eta\in[\mu]$ and any ν, $\tau\in H(T)$, $\eta=c\nu+(1-c)\tau$ with $0<c<1$ implies ν, $\tau\in[\mu]$.

$(V_p)_{p>0}$ is the resolvent generated by the diffusion kernel T on X_o. Hence by restricting $(u_p)_{p>0}$ to X_o, we see that T is a uniformly recurrent diffusion kernel on X_o. By Definition 1 (c), for a countable dense subset Λ in $C_K^+(X_o)$, there exists a decreasing sequence $J=(p_n)_{n=1}^{\infty}$ of positive numbers with $\lim_{n\to\infty} p_n=0$ such that for any $\psi\in\Lambda$, $u_{p_n}V_{p_n}^*\psi$ converges uniformly on any compact set as $n\to\infty$. Let $f\in C_K^+(X_o)$ and let ω be a relatively compact open set in X_o with $\omega\supset\mathrm{supp}(f)$. We choose $\psi\in C_K^+(X_o)$ with $\psi\geqq 1$ on $\bar{\omega}$. Then for any $\varepsilon<0$, there exists $\psi_\varepsilon\in\Lambda$ with $\mathrm{supp}(\psi_\varepsilon)\subset\omega$ and $|f(x)-\psi_\varepsilon(x)|<\varepsilon\psi(x)$ on X_o. Since $(u_pV_p^*\psi)_{p\in J}$ is locally bounded, we see that $u_{p_n}V_{p_n}^*f$ converges uniformly on any compact set as $n\to\infty$. Define for $\mu\in M_K(X_o)$ and $f\in C_K(X_o)$,

$$\int f dW\mu = \lim_{n\to\infty}\int u_{p_n}V_{p_n}^* f d\mu.$$

Then W is a diffusion kernel on X_o. The strict positiveness of W follows from the condition (d) in Definition 1. Since $V_p^*\in \boldsymbol{D}$ (see, for example, [11] p. 60), we see $(u_pV_p)^*\in\boldsymbol{D}$, where $(u_pV_p)\mu=\int u_p(x)V_p\varepsilon_x d\mu(x)$ $(\mu\in M_K(X_o))$. Hence we have $W^*\in\boldsymbol{D}$ by the usual limiting process which we give here for reader's convenience. Suppose that for f, $g\in C_K^+(X_o)$, $W^*f\leqq W^*g$ on $\mathrm{supp}(f)$. Let $h\in C_K^+(X_o)$ satisfying $W^*h\geqq 1$ on $\mathrm{supp}(f)$. Since for $\psi\in C_K(X_o)$, $u_{p_n}V_{p_n}^*\psi$ converges to $W^*\psi$ uniformly on any compact set as $n\to\infty$, for any $\varepsilon>0$, there exists $N>0$ such that

$$(u_{p_n}V_{p_n})^*f\leqq(u_{p_n}V_{p_n})^*(g+\varepsilon h)$$

on $\mathrm{supp}(f)$ for any $n\geqq N$. Then $(u_pV_p)^*\in\boldsymbol{D}$ implies that the above inequality holds on X_o. Letting $n\uparrow\infty$ and $\varepsilon\downarrow 0$, we have $W^*f\leqq W^*g$ on X_o, that is, $W^*\in\boldsymbol{D}$. Hence $W\in\boldsymbol{B}$ by Lemma 7 (1).

We shall show that $\{W\varepsilon_x;\ x\in X_o\}\subset H(T)$. By the definition of V_p, $TV_p\varepsilon_x\leqq(p+1)V_p\varepsilon_x$ so that

$$T(u_p(x)V_p\varepsilon_x)=u_p(x)TV_p\varepsilon_x\leqq(p+1)u_p(x)V_p\varepsilon_x.$$

Since $\lim_{n\to\infty}u_{p_n}V_{p_n}=W$, $TW\varepsilon_x\leqq W\varepsilon_x$ and Proposition 5 gives $W\varepsilon_x\in H(T)$. This completes the proof.

LEMMA 11. *Let W be as in* Lemma 10.

(1) *Let $[\mu]\in\mathrm{exr}H(T)$. Then for any $x\in\mathrm{supp}(\mu)$, there exists $c_x>0$ such that $\mu=c_xW\varepsilon_x$. In particular,* $\mathrm{exr}H(T)\subset\{[W\varepsilon_x];\ x\in X_o\}$.

(2) *$\{c\mu;\ [\mu]\in\mathrm{exr}H(T),\ c\geqq 0\}$ is closed in $M^+(X_o)$.*

PROOF. We keep the notation as in Lemma 10 and its proof. Let $[\mu]\in\mathrm{exr}H(T)$, $x\in\mathrm{supp}(\mu)$ and let ω_x be an arbitrary relatively compact open neighborhood of x in X_o. It follows from $T\mu=\mu$ that $pV_p\mu=\mu$, and hence $pV_p\mu'_p\leqq\mu$ on X_o and $pV_p\mu'_p=\mu$ in ω_x for $\mu'_p\in B^{V_p}(\mu;\ \omega_x)$. Hence there exists a subsequence

$J_o \subset J$ and $\tau \in M^+(X_o)$ such that $pV_p\mu'_p$ converges to τ as $p \in J_o$ tends to 0. By the same discussion as in Lemma 10, we see that $\tau \in H(T)$ and the extremeness of μ leads to $\tau = \mu$ on X_o. We choose $\psi \in C_K^+(X_o)$ with $W^*\psi > 2$ on $\overline{\omega_x}$. Without loss of generality, we may assume that $u_p(y)V_p^*\psi(y) \geqq 1$ on $\overline{\omega_x}$ for each $p \in J$. Since

$$\int \psi d(pV_p\mu'_p) = \int u_p(y)V_p^*\psi(y)(p/u_p(y))d\mu'_p(y),$$

$((p/u_p)\mu'_p)_{p \in J_o}$ is vaguely bounded. Therefore denoting by λ its accumulation point, we have $\mu = W\lambda$ and $\mathrm{supp}(\lambda) \subset \overline{\omega_x}$. Rewriting $\mu = cW(\lambda/c)$ with $c = \int d\lambda$ and letting $\omega_x \downarrow \{x\}$, we have $\mu = c_x W\varepsilon_x$ with some constant $c_x > 0$.

Let $(\mu_n)_{n=1}^\infty$ be a vaguely convergent sequence in $M^+(X_o)$ with $[\mu_n] \in \mathrm{exr}H(T)$ and $\lim_{n\to\infty} \mu_n = \mu$. The closedness of $H(T)$ gives $\mu \in H(T)$. By (1), $\mu_n = c_n W\varepsilon_{x_n}$ for some $x_n \in X_o$ and $c_n > 0$. We may assume that $\mu \neq 0$. Let $H(T; f_\mu)$ be the cap defined in Corollary 6. Since $\{[\nu];\ \nu \in \mathrm{ex}H(T; f_\mu)\} \subset \mathrm{exr}H(T)$[5], where $\mathrm{ex}H(T; f_\mu)$ is the set of all extreme points in $H(T; f_\mu)$, the Choquet integral representation theorem[6] yields that there exists $[\nu] \in \mathrm{exr}H(T)$ such that $\mathrm{supp}(\mu) \supset \mathrm{supp}(\nu)$. Let $y \in \mathrm{supp}(\nu)$ and ω_y be an arbitrary relatively compact open neighborhood of y in X_o. We may assume that $W\varepsilon_{x_n}(\omega_y) > 0$ for all $n \geqq 1$. For $\varepsilon'_n \in B^W(\varepsilon_{x_n}; \omega_y)$, $[W\varepsilon_{x_n}] \in \mathrm{exr}H(T)$ gives $c_n W\varepsilon_{x_n} = c_n W\varepsilon'_n$ on X_o. Choosing $\psi \in C_K^+(X_o)$ with $W^*\psi \geqq 1$ on $\overline{\omega_y}$, we have

$$\int d(c_n\varepsilon'_n) \leqq c_n \int W^*\psi d\varepsilon'_n = \int \psi d(c_n W\varepsilon_{x_n}).$$

Since $(c_n W\varepsilon_{x_n})_{n=1}^\infty$ is vaguely bounded, so is $(c_n\varepsilon'_n)_{n=1}^\infty$ and hence we get $\mu = W\tau$ for some $\tau \in M^+(X_o)$ with $\mathrm{supp}(\tau) \subset \overline{\omega_y}$. In the same way as above, $\mu = c_y W\varepsilon_y$ for some $c_y > 0$. By (1), $\nu = c' W\varepsilon_y$ with some $c' > 0$, which gives $[\mu] = [\nu]$. Thus Lemma 11 is proved.

PROOF OF THEOREM 9. Let W be as in Lemma 10. Keep the preceding notation. Let $\mu \neq 0$ be in $H(T)$. By the Choquet integral representation theorem, we find a regular Borel measure Φ on $H(T; f_\mu)$ with $\int d\Phi \leqq 1$ carried by $\{c_x W\varepsilon_x;\ x \in X_o,\ c_x = 1/W^*f_\mu(x)\}$ such that

$$\mu = \int \lambda d\Phi(\lambda).$$

5) Let K be a closed convex cone in a locally convex space and C be its cap. Then every extreme point of C lies on an extreme ray in K (see [14] p. 88). In this case, $\mathrm{exr}H(T)$ means $\{c\mu,\ [\mu] \in \mathrm{exr}H(T),\ c \geqq 0\}$.

6) Let C be a metrizable compact convex subset of a locally convex space. Then $\mathrm{ex}C$, the set of all extreme points in C, is a G_δ-set and, for any $x \in C$, there exists a regular Borel probability measure Φ on C carried by $\mathrm{ex}C$ which represents x, i.e., for any continuous linear functional f, $f(x) = \int f(y)d\Phi(y)$ (see [14] p. 7 and p. 19).

For an exhaustion $(K_n)_{n=1}^{\infty}$ of X_o[7], we put

$$\Phi_n = \text{the restriction of } \Phi \text{ to } \{c_x W\varepsilon_x;\ x\in K_n - K_{n-1}\}$$

where $K_0=\emptyset$. Let $(\Phi_{n,m})_{m=1}^{\infty}$ be a family of Borel measures on $H(T; f_\mu)$ which are finite linear combination of the Dirac measures with positive coefficients supported by $\{c_x W\varepsilon_x;\ x\in K_n\}$ and which converges vaguely to Φ_n as m$\to\infty$. Then there exists a family $(\nu_{n,m})_{m=1}^{\infty}\subset M_K^+(X_o)$ supported by K_n such that $\int \lambda d\Phi_{n,m}(\lambda)=W\nu_{n,m}$. Hence letting $m\to\infty$, we obtain $\int \lambda d\Phi_n(\lambda)=W\nu_n$ with some $\nu_n\in M_K^+(X_o)$. Since

$$\mu = \sum_{n=1}^{\infty}\int \lambda d\Phi_n(\lambda) = \sum_{n=1}^{\infty} W\nu_n,$$

putting $\nu=\sum_{n=1}^{\infty}\nu_n$, we have $\nu\in\mathscr{D}^+(W)$ and $\mu=W\nu$, that is, $H(T)\subset\{W\nu;\ \nu\in\mathscr{D}^+(W)\}$.

The converse inculusion follows from Lemma 10 and hence W satisfies (2).

Next we shall show that W satisfies (4). Putting $X_1=\{x\in X_o;\ [W\varepsilon_x]\in \mathrm{exr}\cdot H(T)\}$, we see $X_o=\mathrm{cl}(\cup_{x\in X_1}\mathrm{supp}(W\varepsilon_x))$. In fact, by the Choquet integral representation theorem and the above proof of (2), we see that for any $\mu=\sum_{n=1}^{\infty}W\nu_n\in H(T)$ and each $n\geqq 1$, $\mathrm{supp}(W\nu_n)\subset\mathrm{cl}(\cup_{x\in X_1}\mathrm{supp}(W\varepsilon_x))$, which gives the desired assertion. Therefore for any $y\in X_o$ we can choose sequences $(y_n)_{n=1}^{\infty}\subset X_o$ and $(x_n)_{n=1}^{\infty}\subset X_1$ which satisfy $y_n\in\mathrm{supp}(W\varepsilon_{x_n})$ and y_n converges to y as $n\to\infty$. By Lemma 11, $[W\varepsilon_{y_n}]=[W\varepsilon_{x_n}]\in\mathrm{exr}H(T)$ and hence $[W\varepsilon_y]\in\mathrm{exr}H(T)$. This gives (4).

For the assertion (3), we choose $f\in\mathscr{D}^+(W^*)$ with $f>0$ on X_o and put $e(x,y)=W^*f(x)/W^*f(y)$. Then $e(x,y)$ is the desired function.

Lastly we shall show that W is uniquely determined except for the equivalence of diffusion kernels. Let W_o be another diffusion kernel on X_o which satisfies the required conditions. As is seen in the proof of (4), for $x\in X_o$, there exists a sequence $(x_n)_{n=1}^{\infty}\subset X_o$ such that $x_n\in\mathrm{supp}(W\varepsilon_{x_n})$ and x_n converges to x as $n\to\infty$. Hence the extremeness of $W\varepsilon_{x_n}$ and the fact that $W_o\in\boldsymbol{B}$ show $W_o\varepsilon_{x_n}=c_nW\varepsilon_{x_n}$ with some $c_n>0$. Letting $n\to\infty$, we have $W_o\varepsilon_x=c_xW\varepsilon_x$ with some constant $c_x>0$. This completes the proof of Theorem 9.

Similarly, we have the following

PROPOSITION 12. *Let V and W be any two diffusion kernels on X. Assume that VW is defined and satisfies $VW=W$. If W is strictly positive, $W\in\boldsymbol{B}$ and $\mathrm{cl}(\cup_{x\in X}\mathrm{supp}(W\varepsilon_x))=X$, then for any $x\in X$ and any $y\in\mathrm{supp}(V\varepsilon_x)$, there exists a constant $c_{x,y}>0$ such that $W\varepsilon_x=c_{x,y}W\varepsilon_y$.*

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7) This means that for any $n\geqq 1$, K_n is compact and is contained in the interior of K_{n+1}, and $\cup_{n=1}^{\infty}K_n=X_o$.

PROOF. For an $x \in X$, choosing $(x_n)_{n=1}^{\infty} \subset X$ and a family $(a_n)_{n=1}^{\infty}$ of positive numbers which satisfy $x \in \mathrm{cl}(\cup_{n=1}^{\infty} \mathrm{supp}(W\varepsilon_{x_n}))$ and $\sum_{n=1}^{\infty} a_n \varepsilon_{x_n} \in \mathscr{D}^+(W)$, we see that $\lambda = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$ satisfies $x \in \mathrm{supp}(W\lambda)$. Let K be a compact neighborhood of x and denote by μ the restriction of $W\lambda$ to K. For any $y \in \mathrm{supp}(V\varepsilon_x)$ and any relatively compact open neighborhood ω_y of y, we have $V(W\mu - W\mu'_y) = W\mu - W\mu'_y = 0$ in ω_y, where $\mu'_y \in B_m^W(\mu; \omega_y)$. Since $V^*f(x) > 0$ and $\int V^*fd(W\mu - W\mu'_y) = \int fd(W\mu - W\mu'_y) = 0$ for any $f \in C_k^+(X)$ with $f(y) > 0$ and $\mathrm{supp}(f) \subset \omega_y$, we have $x \notin \mathrm{supp}(W\mu - W\mu'_y)$. Hence there exists an open neighborhood U_o of x such that $W\mu - W\mu'_y = 0$ in U_o. Let U be any open neighborhood of x with $\bar{U} \subset U_o$. By the equality $W\mu'_y = \int W\varepsilon'_{x,y} d\mu(x)$ for $\mu'_y \in B_m^W(\mu; \omega_y)$ and $\varepsilon'_{x,y} \in B_m^W(\varepsilon_x; \omega_y)$ (see [10] Proposition 9), we see that

$$W(\mu_U - \mu'_{U,y}) \leqq W(\mu - \mu'_y) \quad \text{on} \quad X,$$

where μ_U denotes the restriction of μ to U and $\mu'_{U,y} \in B_m^W(\mu_U; \omega_y)$, because $W(\mu - \mu'_y) - W(\mu_U - \mu'_{U,y}) = W(\mu - \mu_U) - W(\mu'_y - \mu'_{U,y})$ and $\mu'_y - \mu'_{U,y} \in B_m^W(\mu - \mu_U; \omega_y)$. Therefore $W\mu_U = W\mu'_{U,y}$ in U_o. Since $\mathrm{supp}(\mu_U) \subset U_o$ and $\mu_U \leqq W\lambda$ on X, Lemma 8 gives $W\mu_U = W\mu'_{U,y}$ on X. By multiplying some constant and letting $\omega_y \downarrow \{y\}$ and then $U \downarrow \{x\}$, we have $W\varepsilon_x = cW\varepsilon_y$ with some constant $c > 0$, which proves the proposition.

COROLLARY 13. *Let T and X_o be as in* Theorem 9. *We consider that T is a diffusion kernel on X_o. Then $H(T)$ is one dimensional if and only if the closed symmetric set of transitive type generated by* $\mathrm{supp}(T)$ *is equal to $X_o \times X_o$.*

PROOF. The "if" part: Let W be as in Theorem 9. Put $F = \{(x, y) \in X_o \times X_o;\ W\varepsilon_x = cW\varepsilon_y$ with some constant $c > 0\}$. Then F is a closed symmetric set of transitive type and $F \supset \mathrm{supp}(T)$ by Proposition 12, and hence $F = X_o \times X_o$. This implies that $H(T)$ is one dimensional.

The "only if" part: Let F be the closed symmetric set of transitive type in $X_o \times X_o$ generated by $\mathrm{supp}(T)$. For each $x \in X_o$, we put $F_x = \{y \in X_o; (x, y) \in F\}$. Then $\mathrm{supp}(T\varepsilon_x) \subset F_x$ and hence $\mathrm{supp}(V_p\varepsilon_x) \subset F_x$ for all $p > 0$, where $(V_p)_{p>0}$ is the resolvent on X_o generated by T. This gives $\mathrm{supp}(W\varepsilon_x) \subset F_x$. Since $H(T)$ is one dimensional, we see easily that $\mathrm{supp}(W\varepsilon_x) = X_o$. Thus $F_x = X_o$ and hence $F = X_o \times X_o$. This completes the proof.

A diffusion kernel T on X is said to be *idempotent* if T^2 is defined and $T = T^2$. This is a typical example of uniformly recurrent diffusion kernels.

COROLLARY 14. *Let T be a strictly positive idempotent diffusion kernel on X. Then T is uniformly recurrent and the diffusion kernel W obtained in* Theorem 9 *is equivalent to the restriction of T to $M_K(X_o)$, where* $X_o = \mathrm{cl}(\cup_{\mu \in H(T)} \mathrm{supp}(\mu)) = \mathrm{cl}(\cup_{x \in X} \mathrm{supp}(T\varepsilon_x))$.

In fact, denoting by $(V_p)_{p>0}$ the resolvent generated by T, we see $(1+p)V_p = I+(1/p)T$, and then the required results are shown immediately.

§3. The cone H(T*)

In this section, we shall consider the conditions under which the cone $H(T^*)$ is one dimensional.

We begin with the following definition.

DEFINITION 15. Let T be a diffusion kernel on X. We say that T^* is *uniformly recurrent* if for any $\psi \in C_K^+(X)$ with $\psi \neq 0$, there exists an $x \in X$ such that

$$\sum_{n=1}^{\infty} (T^*)^n \psi(x) = \infty.$$

Here $(T^*)^n$ is defined in the following manner: For a non-negative lower semi-continuous function g on X, we define $T^*g(x) = \int g dT\varepsilon_x$. Then T^*g is also non-negative lower semi-continuous. So we can define $(T^*)^n g(x) = \int (T^*)^{n-1} g dT\varepsilon_x$ inductively ($n \geqq 2$), where $(T^*)^1 = T^*$. If the diffusion kernel T^n is defined, $(T^*)^n = (T^n)^*$ of course.

REMARK 16. Let T be uniformly recurrent on X and let X_o and W be as in Lemma 10. If $X_o = X$, that is, $\mathrm{cl}(\cup_{x \in X} \mathrm{supp}(W\varepsilon_x)) = X$, then T^* is uniformly recurrent.

In fact, if $\mathrm{cl}(\cup_{x \in X} \mathrm{supp}(W\varepsilon_x)) = X$, then given $\psi \in C_K^+(X)$ with $\psi \neq 0$, we have $W^*\psi(x) \neq 0$ for some $x \in X$. Since $u_p(x)V_p^*\psi(x)$ converges to $W^*\psi(x)$ and $u_p(x)$ converges to 0 along some decreasing sequence of p, $\lim_{p \downarrow 0} V_p^*\psi(x) = \sum_{n=0}^{\infty} \cdot (T^*)^n \psi(x) = \infty$.

Analogously to Proposition 5, we have the following

PROPOSITION 17. *Let T^* be uniformly recurrent. If $f \in \mathscr{D}^+(T^*)$ satisfies $T^*f \leqq f$, then $f \in H(T^*)$.*

As a function version of Corollary 13, we have

PROPOSITION 18. *Let T be a diffusion kerenl on X. Assume that T^* is uniformly recurrent and that the closed symmetric set of transitive type generated by* $\mathrm{supp}(T)$ *is equal to $X \times X$. Then* (1)⇔(2):

(1) *For any $x \in X$, there exists $f \in H(T^*)$ such that $f(x) > 0$.*

(2) *T is of continuous sub-exponential type.*

Furthermore, in the above case $H(T^*)$ is one dimensional.

PROOF. Suppose first that (1) is fulfilled. Then we can choose a function

g in $H(T^*)$ such that $g>0$ on X. Put $e(x, y)=g(y)/g(x)$ on $X\times X$. Then $e(x, y)$ is of exponential type and for any $x\in X$, $\int e(x, y)dT\varepsilon_x(y)=1$, which shows that T is of exponential type.

Suppose conversely that T is of continuous sub-exponential type and let $e(x, y)$ be a continuous function on $X\times X$ of exponential type such that $\int d(eT\varepsilon_x)\leqq 1$. Then $1\in\mathscr{D}^+((eT)^*)$. Since $((eT)^*)^n\psi(x)\geqq[\inf_{z\in\mathrm{supp}(\psi)} e(x, z)](T^*)^n\psi(x)$ for any $\psi\in C_K^+(X)$ and T^* is uniformly recurrent, so is $(eT)^*$ evidently. Hence it follows from Proposition 17 that $1\in H((eT)^*)$. For a $z\in X$, we put $g(x)=e(z, x)$. Then $g>0$ on X and $g\in H(T^*)$, for

$$T^*g(x) = \int e(z, y)dT\varepsilon_x(y) = \int e(z, x)e(x, y)dT\varepsilon_x(y)$$

$$= g(x)\int e(x, y)dT\varepsilon_x(y) = g(x)(eT)^*1(x) = g(x).$$

Turning to the second assertion, we choose $g\in H(T^*)$ with $g>0$ on X and let f be an arbitrary element in $H(T^*)$. Put $F=\{(x, y)\in X\times X;\ f(x)g(y)=f(y)g(x)\}$. It suffices to show that $F=X\times X$. Since F is a closed symmetric set of transitive type, it also suffices to show that $F\supset\mathrm{supp}(T)$. Let $x\in X$ and $y\in\mathrm{supp}(T\varepsilon_x)$. If $f(x)=0$, then $f(x)=T^*f(x)$ gives $f=0$ on $\mathrm{supp}(T\varepsilon_x)$. Hence $(x, y)\in F$. If $f(x)>0$, put $h=\min\{g(x)f, f(x)g\}$. Then $h\in\mathscr{D}^+(T^*)$ and $T^*h\leqq h$, so $T^*h=h$. This implies $g(x)f=f(x)g$ on $\mathrm{supp}(T\varepsilon_x)$. Thus $F\supset\mathrm{supp}(T)$, which completes the proof.

By a similar proof, we have

PROPOSITION 19. *Let T be a diffusion kernel on X. Assume that the closed symmetric set of transitive type generated by* $\mathrm{supp}(T)$ *is equal to $X\times X$ and that there exist $g\in H(T^*)$ and $\mu\in H(T)$ with $g>0$ on X,* $\mathrm{supp}(\mu)=X$ *and* $\int gd\mu<\infty$. *Then $H(T^*)$ is one dimensional.*

In fact, for any $f\in H(T^*)$, putting $h=\min\{f, g\}$, we have $h\in\mathscr{D}^+(T^*)$, $T^*h\leqq h$ and $\int hd\mu=\int hdT\mu=\int T^*hd\mu$, which imply $h\in H(T^*)$. Thus by the same argument as above, we obtain Proposition 19.

EXAMPLE 20. Let Ω be a bounded domain in a Euclidean space with smooth boundary and let $G(x, y)$ be the Green function on Ω of a second order uniformly elliptic differential operator with bounded smooth coefficients. It is well-known that there exist $\lambda>0$ (the first eigen value) and $f_j\in C^+(\Omega)$ $(j=1, 2)$ such that

$$\int G(x, y)f_1(y)d\xi(y) = \lambda f_1(x),$$

$$\int G(y, x)f_2(y)d\xi(y) = \lambda f_2(x)$$

and $\int f_j^2(x)d\xi(x)<\infty$, where ξ denotes the Lebesgue measure (cf. for example [11] p. 105). Considering the diffusion kernel T defined by

$$T\mu = ((1/\lambda)\int G(x, y)d\mu(y))\xi \qquad (\mu \in M_K(\Omega)),$$

we see $f_1\xi \in H(T)$ and $f_2 \in H(T^*)$. By Proposition 19, $H(T^*)=\{cf_2;\ c\geqq 0\}$. This means that the eigen space for the first eigen value λ is also one dimensional even if we consider it in the space $C^+(\Omega)$.

§4. Invariant measures for uniformly recurrent continuous diffusion semi-groups

A family $(T_t)_{t\geqq 0}$ of diffusion kernels on X is said to be a *continuous diffusion semi-group* if $T_0=I$, T_tT_s is defined and $T_tT_s=T_{t+s}$ for all $t, s\geqq 0$ and for any $\mu\in M_K(X)$, $t\to T_t\mu\in M(X)$ is continuous, i.e., the mapping $t\to\int T_t^*\psi d\mu$ is continuous for any $\psi\in C_K(X)$.

We say that a family of diffusion kernels $(V_p)_{p>0}$ defined by

$$V_p\mu = \int_0^\infty e^{-pt}T_t\mu dt \qquad (\mu\in M_K(X))$$

is the *resolvent generated by* $(T_t)_{t\geqq 0}$ provided that it has a sense, and $(T_t)_{t\geqq 0}$ is uniformly recurrent if the resolvent $(V_p)_{p\geqq 0}$ is uniformly recurrent. We also say that $(T_t^*)_{t\geqq 0}$ is *uniformly recurrent* if for any $\psi\in C_K^+(X)$ with $\psi\neq 0$, there exists an $x\in X$ such that $\lim_{s\uparrow\infty}\int_0^s T_t^*\psi(x)dt=\infty$. Put

$$H((T_t)_{t\geqq 0}) = \cap_{t\geqq 0} H(T_t) \quad\text{and}\quad H((T_t^*)_{t\geqq 0}) = \cap_{t\geqq 0} H(T_t^*).$$

Using the following two lemmas, the cones $H((T_t)_{t\geqq 0})$ and $H((T_t^*)_{t\geqq 0})$ are determined by the results obtained in the previous sections.

LEMMA 21. *Let $(T_t)_{t\geqq 0}$ be a continuous diffusion semi-group on X. Assume that the resolvent $(V_p)_{p>0}$ generated by $(T_t)_{t\geqq 0}$ exists. Then for any $p>0$, $H((T_t)_{t\geqq 0})=H(pV_p)$ and $H((T_t^*)_{t\geqq 0})=H(pV_p^*)$.*

In fact, $H((T_t)_{t\geqq 0})\subset H(pV_p)$ is evident and the converse inclusion also holds by virtue of the resolvent equation and the unicity of the Laplace transformation. Similarly, $H((T_t^*)_{t\geqq 0})=H(pV_p^*)$ is obtained.

LEMMA 22. *Let $(T_t)_{t\geqq 0}$ and $(V_p)_{p>0}$ be the same as above. Then $(T_t)_{t\geqq 0}$ (resp. $(T_t^*)_{t\geqq 0}$) is uniformly recurrent if and only if pV_p (resp. pV_p^*) is uniformly recurrent for some $p>0$.*

PROOF. The resolvent $(U_q)_{q>0}$ generated by pV_p exists and

$$U_q = (q+1)^{-1}I + p(q+1)^{-2}V_{pq/(q+1)}$$

for any q with $0<q<p$. Suppose that pV_p is uniformly recurrent for some $p>0$ and let $(u_q)_{q>0}$ and q_o be as in Definition 1 corresponding to $(U_q)_{q>0}$. Put $q_1 = pq_o/(q_o+1)$ and $v_q = u_{q/(p-q)}$ $(0<q\leqq q_1)$ and $v_q = v_{q_1}$ $(q>q_1)$. Then $(v_q)_{q>0}$ and q_1 is a pair which determines the uniform recurrence of $(V_q)_{q>0}$ and hence $(T_t)_{t\geqq 0}$ is uniformly recurrent. The converse assertion is obtained similarly. We also have

$$\lim_{n\to\infty} (1/p) \sum_{k=1}^{n} (pV_p^*)^k\psi = \lim_{s\to\infty} \int_0^s T_t^*\psi dt$$

for any $\psi \in C_K^+(X)$, which implies our desired assertion and the proof is completed.

In the case of continuous diffusion semi-groups, using the above lemmas and the results in sections 2 and 3, we obtain the following

THEOREM 23. *Let $(T_t)_{t\geqq 0}$ be a uniformly recurrent continuous diffusion semi-group on X and $(V_p)_{p>0}$ be the resolvent generated by $(T_t)_{t\geqq 0}$. Then*

(1) *$H((T_t)_{t\geqq 0}) = \{Wv;\ v \in \mathscr{D}^+(W)\}$, where W is the diffusion kernel obtained in* Theorem 9 *from $T = pV_p$ with some $p>0$.*

(2) *$H((T_t)_{t\geqq 0})$ is one dimensional if and only if the closed symmetric set of transitive type generated by $\cup_{t\geqq 0}\operatorname{supp}(T_t)$ is equal to $X_o \times X_o$, where $X_o = \operatorname{cl}(\cup_{\mu\in H((T_t)_{t\geqq 0})}\operatorname{supp}(\mu))$ and we consider that $(T_t)_{t\geqq 0}$ is a continuous diffusion semi-group on X_o.*

(3) *If $(T_t^*)_{t\geqq 0}$ is uniformly recurrent and the closed symmetric set of transitive type generated by $\cup_{t\geqq 0}\operatorname{supp}(T_t)$ is equal to $X\times X$, then the following statements* (a) *and* (b) *are equivalent:*

(a) *For any $x\in X$, there exists an $f\in H((T_t^*)_{t\geqq 0})$ such that $f(x)>0$.*

(b) *$(T_t)_{t\geqq 0}$ is of continuous sub-exponential type, i.e., there exists a continuous function $e(x, y)$ on $X\times X$ of exponential type such that for any $t\geqq 0$ and any $x\in X$, $\int d(eT_t\varepsilon_x)\leqq 1$ and $1\in\mathscr{D}^+((eT_t)^*)$.*

Furthermore, in this case $H((T_t^*)_{t\geqq 0})$ is one dimensional.

§5. Weakly regular Hunt diffusion kernels

We say that a diffusion kernel V is a *weakly regular Hunt diffusion kernel* on X if

(H.1) there exists a continuous diffusion semi-group $(T_t)_{t\geqq 0}$ on X such that $V = \int_0^\infty T_t dt$, i.e., for $\mu\in M_K(X)$ and $\psi\in C_K(X)$, $\int \psi dV\mu = \int_0^\infty dt \int \psi dT_t\mu$;

(H.2) $B_m^V(\mu;\ \omega)\neq\emptyset$ for any open set ω and any $\mu\in M_K^+(X)$;

(H.3) for any $\psi\in C_K^+(X)$, the greatest V-subharmonic minorant of $V^*\psi$ on X vanishes and V has the lower regularization property: Let Ω be an open set and

u be a real-valued Borel function satisfying $|u(x)| \leqq \int f dV\varepsilon_x$ with some $f \in C_K^+(X)$. If for any $x \in \Omega$, any compact set $K \subset \Omega$ and any $\varepsilon'_{x,CK} \in B^V(\varepsilon_x; CK)$, $u(x) \geqq \int u d\varepsilon'_{x,CK}$, then $\underline{u}(x) = \liminf_{y \to x} u(y)$ is V-superharmonic in Ω.

Here a real-valued Borel function u on X is said to be *V-superharmonic* in Ω if u is lower semi-continuous in Ω and for any $x \in \Omega$ and any compact set K with $K \subset \Omega$, $u(x) \geqq \int u d\varepsilon'_{x,CK}$, where $\varepsilon'_{x,CK} \in B_m^V(\varepsilon_x; CK)$. Also if $-u$ is V-superharmonic in Ω, then u is said to be *V-subharmonic* in Ω.

Weakly regular Hunt diffusion kernels are characterized by M. Itô in [10]. That is, a diffsion kernel V on X is a weakly regular Hunt diffusion kernel if and only if

(D.1) V, $V^* \in \boldsymbol{D}$;

(D.2) V is non-degenerate, i.e., for any $x, y \in X$ with $x \neq y$, $V\varepsilon_x \neq 0$ and $V\varepsilon_y$ is not proportional to $V\varepsilon_x$;

(D.3) for any closed set F in X and any $\psi \in C_K^+(X)$, the V-reduced function $H_F^{V^*\psi}$ of $V^*\psi$ on F is upper semi-continuous and $H_\infty^{V^*\psi}(x) = \inf\{H_{CK}^{V^*\psi};\ K\text{: compact}\} = 0$ on X.

Here, V-reduced functions are defined as follows: For a diffusion kernel V on X, a real-valued lower semi-continuous function u on X is said to be *V-supermedian* if for any $x \in X$ and $\lambda \in M_K^+(X)$ with $V\lambda \leqq V\varepsilon_x$, $\int u d\lambda \leqq u(x)$. We denote by $S^+(V)$ the totality of non-negative V-supermedian functions. For a subset A in X and a function $g \geqq 0$ on A, we define the *V-reduced function of g on A* by

$$H_A^g(x) = \inf\{u(x);\ u \in S^+(V),\ u \geqq g \quad \text{on} \quad A\} \quad \text{on} \quad X$$

whenever $\{u \in S^+(V);\ u \geqq g \text{ on } A\} \neq \emptyset$.

A strictly positive diffusion kernel satisfying (D.1) and (D.3) is called a *weakly regular diffusion kernel.*

In this section, we shall show that under some additional assumption, the condition (D.2) in M. Itô's result can be removed in the following sense: A weakly regular diffusion kernel V on X may be regard as a weakly regular Hunt diffusion kernel on some quotient space of X.

We begin with the following lemmas.

LEMMA 24. *Let V be a weakly regular diffusion kernel on X and let $h \in C^+(X)$ with $h > 0$ on X. For $e(x, y) = h(y)/h(x)$, the diffusion kernel eV (see* Remark 4) *is also weakly regular.*

In fact, the strict positiveness of eV is clear. Since V, $V^* \in \boldsymbol{D}$ if and only if $eV, (eV)^* \in \boldsymbol{D}$, we see that eV satisfies (D.1). By the definition, the equality

$$H_F^{(eV)^*\psi}(x) = (1/h(x)) H_F^{V^*(\psi h)}(x)$$

holds on X for any $\psi \in C_K^+(X)$ and any closed set F in X, which shows that eV satisfies (D.3).

For a diffusion kernel V on X and an $x \in X$, we put

$$F_x = \{y \in X;\ V\varepsilon_x = cV\varepsilon_y \quad \text{with some} \quad c > 0\}.$$

LEMMA 25. *Let V be a weakly regular diffusion kernel on X. Then:*

(1) *For any compact set K, $\bigcup_{x\in K} F_x$ is also compact.*

(2) *For any closed set F, $\bigcup_{x\in F} F_x$ is also closed.*

(3) *The relation $x \sim y$ on X defined by $x \in F_y$ is an equivalence relation and the quotient space $X/\sim$ is a locally compact Hausdorff space with a countable base.*

PROOF. Choosing $f \in \mathscr{D}^+(V^*)$ with $f>0$ on X, we put $e(x, y) = V^*f(y)/V^*f(x)$ on $X \times X$. Then $F_x = \{y \in X;\ eV\varepsilon_x = eV\varepsilon_y\}$ for any $x \in X$. Let $(\omega_n)_{n=1}^{\infty}$ be an open exhaustion of X[8]. First we show that for any compact set K in X, $\bigcup_{x\in X} F_x$ is relatively compact. Choose $\psi \in C_K^+(X)$ with $(eV)^*\psi(x) \geqq 1$ on K. By the proof of Lemma 13 in [10] and Lemma 24, $H_{C\omega_n}^{(eV)^*\psi}$ converges to 0 uniformly on any compact set as $n \to \infty$. Hence there exists an $N>0$ such that $H_{C\omega_N}^{(eV)^*\psi}(x) \leqq 1/2$ on K. It follows from the definition of eV-reduced functions that

$$H_{C\omega_N}^{(eV)^*\psi} = (eV)^*\psi \quad \text{on} \quad C\omega_N.$$

On the other hand from $eV\varepsilon_x = eV\varepsilon_y$ $(y \in F_x)$ it follows that

$$H_{C\omega_N}^{(eV)^*\psi}(y) = H_{C\omega_N}^{(eV)^*\psi}(x).$$

The above two equalities and $(eV)^*\psi(y) = (eV)^*\psi(x)$ for $y \in F_x$ imply that $\bigcup_{x\in K} F_x \subset \omega_N$, that is, $\bigcup_{x\in K} F_x$ is relatively compact.

Next we shall show (2). If (2) is valid, (1) holds by the above proof. Let $(y_n)_{n=1}^{\infty}$ be a convergent sequence in $\bigcup_{x\in F} F_x$ with $\lim_{n\to\infty} y_n = y$. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in F with $eV\varepsilon_{y_n} = eV\varepsilon_{x_n}$. By the above proof, $\bigcup_{n=1}^{\infty} F_{y_n}$ is relatively compact and contains $(x_n)_{n=1}^{\infty}$ so that we may assume that $(x_n)_{n=1}^{\infty}$ converges to some $x \in F$ as $n \to \infty$. This implies $y \in F_x$ and hence $\bigcup_{x\in F} F_x$ is closed.

Clearly $x \sim y$ is an equivalent relation. By (2) the natural projection π from X onto $X/\sim$ is a closed mapping and by (1), F_x is compact for any $x \in X$. Hence a theorem in [12] (Theorem 20, p. 148) gives (3). This completes the proof.

PROPOSITION 26. *Let V be a weakly regular diffusion kernel on X. As-*

8) This means that for any $n \geqq 1$, ω_n is relatively compact open and $\bar{\omega}_n \subset \omega_{n+1}$ and that $\bigcup_{n=1}^{\infty} \omega_n = X$.

sume that there exists an accumulation point W of $(pV_p)_{p\geqq 0}$ as $p\uparrow\infty$, where $(V_p)_{p\geqq 0}$ is the resolvent associated with V. Then:

(1) *W is strictly positive idempotent and $VW=WV=V$.*

(2) $\mathrm{cl}(\cup_{x\in X}\mathrm{supp}(W\varepsilon_x))=X$.

(3) *For any $x\in X$, $F_x\supset\mathrm{supp}(W\varepsilon_x)$.*

(4) *For any $\mu\in M_K^+(X)$, $W\mu\in M_K^+(X)$.*

(5) *For any $\psi\in C_K^+(X)$, $W^*\psi\in C_K^+(X)$.*

PROOF. First we remark that $V\in\boldsymbol{R}$ (see Proposition 15 and Remark 16 in [10]). By our assumption, there exists an increasing sequence $(p_n)_{n=1}^{\infty}$ such that $\lim_{p_n\uparrow\infty}p_nV_{p_n}=W$. Note that $\lim_{p_n\uparrow\infty}V_{p_n}=0$. Since $pVV_p\varepsilon_x=V\varepsilon_x-V_p\varepsilon_x\leqq V\varepsilon_x$ for any $x\in X$, (D.3) gives $VW\varepsilon_x=V\varepsilon_x$ (use Lemma 13 in [10]) and hence V_pW is also defined and $V_pW=V_p$ for all $p\geqq 0$. By the Fatou lemma and strict positiveness of V, W is strictly positive and idempotent. Furthermore we see $V_pW=WV_p=V_p$ for any $p\geqq 0$, which shows (1).

For any $x\in X$, $V\varepsilon_x\neq 0$ and $V\in\boldsymbol{D}$ imply $x\in\mathrm{supp}(V\varepsilon_x)$. Therefore

$$\mathrm{cl}(\cup_{x\in X}\mathrm{supp}(W\varepsilon_x))\supset\mathrm{cl}(\cup_{x\in X}\mathrm{supp}(WV\varepsilon_x)) = \cup_{x\in X}\mathrm{supp}(V\varepsilon_x)=X,$$

which shows (2).

The assertion (3) follows from Proposition 12, and (4) and (5) follow from (3) and Lemma 25 immediately. Thus Proposition 26 is shown.

The main result in this section is the following

THEOREM 27. *Let V be a weakly regular diffusion kernel on X. Assume that there exists an accumulation point W of $(pV_p)_{p\geqq 0}$ as $p\uparrow\infty$, where $(V_p)_{p\geqq 0}$ is the resolvent associated with V. Let $x\sim y$ be the equivalence relation defined by $V\varepsilon_x=c_{x,y}V\varepsilon_y$ with some $c_{x,y}>0$. Then there exist a weakly regular Hunt diffusion kernel $\tilde{V}$ on the quotient space $X/\sim$ and a function $e(x,y)$ on $X\times X$ of exponential type with $e(x,y)=c_{x,y}$ such that, for any $f\in C_K(X/\sim)$ and any $x\in X$,*

$$\int fd\tilde{V}\varepsilon_{\pi(x)}=\int f(\pi(y))dVeW\varepsilon_x(y),$$

where π is the natural projection of X onto $X/\sim$.

PROOF. By Lemma 25, $X/\sim$ is locally compact. Let $e(x,y)$ be the same as in the proof of Lemma 25. Then $eV\varepsilon_x=eV\varepsilon_y$ for any $x\in X$ and any $y\in F_x$. Hence $e(x,y)=c_{x,y}$. The equality $eV\varepsilon_x=eV\varepsilon_y$ and the resolvent equation give $eV_p\varepsilon_x=eV_p\varepsilon_y$ for any $p>0$ so that $eW\varepsilon_x=eW\varepsilon_y$. Therefore, for $\tilde{x}\in X/\sim$, $VeW\varepsilon_x$ is independent of $x\in\tilde{x}$. For any $f\in C_K(X/\sim)$, we define

$$\int fd\tilde{V}\varepsilon_{\pi(x)}=\int f(\pi(y))dVeW\varepsilon_x(y).$$

Then $\tilde{V}$ is a diffusion kernel on $X/\sim$. Let $\tilde{V}_p$ be a diffusion kernel defined by $\int fd\tilde{V}_p\varepsilon_{\pi(x)}=\int f(\pi(y))dV_peW\varepsilon_x(y)$ for all $f\in C_K(X/\sim)$. Then $(\tilde{V}_p)_{p\geqq 0}$ is the resolvent associated with $\tilde{V}$, which implies $\tilde{V}^*\in \boldsymbol{D}$. We shall show $\tilde{V}\in \boldsymbol{D}$. Let $\tilde{\lambda}_j\in M_K^+(X/\sim)$ $(j=1, 2)$ and suppose that $\tilde{V}\tilde{\lambda}_1\leqq\tilde{V}\tilde{\lambda}_2$ in a certain relatively compact open neighborhood $\tilde{\omega}$ of $\operatorname{supp}(\tilde{\lambda}_1)$. Let $\lambda_j\in M_K^+(X)$ satisfy $\int\psi d\lambda_j=\int\overline{(eW)^*\psi}\cdot(\pi(x))d\tilde{\lambda}_j(\pi(x))$ $(j=1, 2)$ for any $\psi\in C_K(X)$, where $\overline{(eW)^*\psi}(\pi(x))=(eW)^*\psi(x)=\int\psi deW\varepsilon_x$. Then by Proposition 26, $\operatorname{supp}(\lambda_j)\subset\{x;\ \pi(x)\in\operatorname{supp}(\tilde{\lambda}_j)\}$ and $V\lambda_1\leqq V\lambda_2$ in $\{x;\ \pi(x)\in\tilde{\omega}\}$. Hence $V\in\boldsymbol{D}$ gives $V\lambda_1\leqq V\lambda_2$ on X. This implies $\tilde{V}\tilde{\lambda}_1\leqq\tilde{V}\tilde{\lambda}_2$ on $X/\sim$, that is, $\tilde{V}\in\boldsymbol{D}$.

Clearly $\tilde{V}$ is non-degenerate. Since V satisfies (D.3), Lemma 13 in [10] shows that $\tilde{V}$ also satisfies (D.3). Therefore by [10], $\tilde{V}$ is a weakly retular Hunt diffusion kernel on $X/\sim$, which completes the proof.

§6. Diffusion kernels of convolution type

Let G be a locally compact Hausdorff group with a countable base and let X be a homogeneous space of G. Then X may be identified with the (left) coset space G/H, where $H=\{g\in G;\ gx_o=x_o\}$ for some $x_o\in X$. Denote by π_H the natural projection of G onto $X=G/H$.

In the sequel, we always assume that H is compact.

For any $\nu\in M^+(G)$ and $\mu\in M^+(X)$, the convolution $\nu*\mu$ may be defined by

$$\int_X\psi(y)d\nu*\mu(y)=\int_X\int_G\psi(gx)d\nu(g)d\mu(x)$$

provided that the right hand side is finite for any $\psi\in C_K^+(X)$. The convolution of two measures ν, $\tau\in M^+(G)$ is defined similarly and we use the same notation $\nu*\tau$.

Denoting by ξ_H the normalized Haar measure on H, we see that the mapping η_H^*: $C_K(G)\to C_K(X)$ determined by

$$\eta_H^*(f)(\pi_H(g))=\int f(gh)d\xi_H(h)\qquad (f\in C_K(G))$$

is well-defined and clearly it is continuous, positive, linear and surjective (cf. [2] and [3]), so that the transposed mapping η_H: $M(X)\to M(G)$ defined by

$$\int\psi d\eta_H(\mu)=\int\eta_H^*(\psi)d\mu\qquad (\mu\in M(X)\text{ and }\psi\in C_K(X))$$

is continuous, positive and injective. Then for $\nu\in M^+(G)$ and $\mu\in M^+(X)$, $\nu*\eta_H(\mu)=\eta_H(\nu*\mu)$ and $\eta_H(\varepsilon_{\pi_H(g)})=\varepsilon_g*\xi_H$ for any $g\in G$.

DEFINITION 28. Let $X=G/H$ be a homogeneous space of G.

(1) A diffusion kernel T on X is said to be of *convolution type* if for any $x\in X$,

$$T\varepsilon_x = \eta_H(T\varepsilon_{\pi_H(e)})*\varepsilon_x,$$

where e denotes the neutral element of G. We say that $\eta_H(T\varepsilon_{\pi_H(e)})\in M^+(G)$ is the *representing measure* of T, which is denoted by σ_T in the sequel. Then $T\mu=\sigma_T*\mu$ for any $\mu\in\mathscr{D}(T)$ and $\sigma_T*\varepsilon_h=\sigma_T$ for any $h\in H$.

(2) A continuous diffusion semi-group $(T_t)_{t\geqq 0}$ on X (resp. a resolvent $(V_p)_{p>0}$ on X) is said to be of *convolution type* if each element is of convolution type.

REMARK 29. (1) Let T be of convolution type and let $\sigma\in M^+(G)$ satisfy $T\varepsilon_x=\sigma*\varepsilon_x$ for any $x\in X$. Then $\sigma*\xi_H=\sigma_T$.

(2) Let $(T_t)_{t\geqq 0}$ be a continuous diffusion semi-group on X of convolution type. Then $\sigma_{T_0}=\xi_H$, $\sigma_{T_t}*\sigma_{T_s}=\sigma_{T_{t+s}}$ for any t, $s\geqq 0$ and the mapping $t\to\sigma_{T_t}$ is vaguely continuous.

(3) Let $(V_p)_{p>0}$ be a resolvent on X of convolution type. Then $(\sigma_{V_p})_{p>0}$ satisfies the resolvent equation, i.e.,

$$\sigma_{V_p} - \sigma_{V_q} = (q-p)\sigma_{V_p}*\sigma_{V_q}$$

for any p, $q>0$.

In fact, $\sigma_T=\eta_H(T\varepsilon_{\pi_H(e)})=\eta_H(\sigma*\varepsilon_{\pi_H(e)})=\sigma*\eta_H(\varepsilon_{\pi_H(e)})=\sigma*\xi_H$, which implies (1). The assertions (2) and (3) follow from (1) easily.

From the manner of the construction of diffusion semi-groups and resolvents (see [10] p. 317 and p. 323-325) we obtain

PROPOSITION 30. (1) *Let V be a Hunt diffusion kernel on X, i.e., there exists a continuous diffusion semi-group $(T_t)_{t\geqq 0}$ on X such that $V=\int_0^\infty T_t dt$. If V is of convolution type, then so is $(T_t)_{t\geqq 0}$.*

(2) *If $V\in\boldsymbol{R}$ and is of convolution type, then the resolvent associated with V is also of convolution type.*

DEFINITION 31. We say that $\sigma\in M^+(G)$ is *recurrent* if for some $f\in C_K^+(G)$,

$$\sum_{n=1}^\infty \iint\cdots\int f(g_1g_2\cdots g_n)d\sigma(g_1)d\sigma(g_2)\cdots d\sigma(g_n) = \infty$$

and that σ is *sub-exponential* if there exists an exponential E[9] on the closed subgroup generated by supp(σ) such that $\int E(g)d\sigma(g)\leqq 1$.

9) A positive continuous function E on a topological group Γ is called an exponential on Γ if it satisfies $E(gg')=E(g)E(g')$ for any g, $g'\in\Gamma$.

For the existence of recurrent and sub-exponential measures, we refer to [1] and [7].

Using Theorem 9, we see the following theorem, which is the main result in this section.

THEOREM 32. *Let T be a diffusion kernel on $X=G/H$ of convolution type. Then T is uniformly recurrent if and only if its representing measure σ_T is recurrent and sub-exponential. In this case, there exist a closed uni-modular subgroup Γ in G and an exponential E on Γ such that*

(1) $\mathrm{cl}(\cup_{n=0}^{\infty} \mathrm{supp}((\sigma_T)^n))$[10] $=\Gamma$,

(2) $\int E(g)d\sigma_T(g)=1$,

(3) $H(T)=\{(1/E)\xi_\Gamma*\nu;\ \nu\in\mathscr{D}^+(T)\}$,

where ξ_Γ denotes a Haar measure on Γ.

PROOF. Suppose that T is uniformly recurrent. We keep the same notation as used in Theorem 9 and its proof. Since the resolvent $(V_p)_{p>0}$ generated by T is of convolution type, there exists a sequence $J=(p_n)_{n=1}^{\infty}$ of positive numbers with $\lim_{n\to\infty}p_n=0$ such that $(u_p(x)\sigma_{V_p}*\varepsilon_x)_{p\in J}$ $(x\in X)$ converges vaguely to an element in $H(T)$ as $n\to\infty$. Put $\rho_W=W\varepsilon_{\pi_H(e)}$. Then, for any $x\in X$, $\rho_W*\varepsilon_x=c_xW\varepsilon_x$, where $c_x=\lim_{n\to\infty}u_{p_n}(\pi_H(e))/u_{p_n}(x)>0$. Since $\lim_{n\to\infty}u_{p_n}(\pi_H(e))=0$, we have

$$\sum_{n=1}^{\infty}\iint\cdots\int f(g_1g_2\cdots g_n)d\sigma_T(g_1)d\sigma_T(g_2)\cdots d\sigma_T(g_n)$$
$$=\lim_{p\downarrow 0,p\in J}\int f(g)d\sigma_{V_p}(g)=\infty$$

for any $f\in C_K^+(G)$ with $\mathrm{supp}(f)\cap\mathrm{supp}(\rho_W)\neq\emptyset$, which implies that σ_T is recurrent. Put $\Gamma=\mathrm{supp}(\rho_W)$. Then for every $p>0$, $\Gamma\subset\mathrm{supp}(\sigma_{V_p})=\mathrm{cl}(\cup_{n=0}^{\infty}\mathrm{supp}((\sigma_T)^n))$. Since $V_pT=TV_p$, we have $\sigma_T*\rho_W=\rho_W*\rho_T=\rho_W$. By Theorem 9, we see that for any $g\in\mathrm{supp}(\sigma_T)$, $\rho_W=c_g\rho_W*\varepsilon_g=c'_g\varepsilon_g*\rho_W$ with some $c_g>0$ and $c'_g>0$, because $\mathrm{cl}(\cup_{g\in G}\mathrm{supp}(\rho_W*\varepsilon_g))=\mathrm{cl}(\cup_{g\in G}\mathrm{supp}(\varepsilon_g*\rho_W))=G$. Hence $\Gamma=\mathrm{cl}(\cup_{n=0}^{\infty}\mathrm{supp}\cdot((\sigma_T)^n))$, Γ is a closed subgroup and for every $g\in\Gamma$, $\rho_W=c_g\rho_W*\varepsilon_g$ with some $c_g>0$. Then putting $E(g)=1/c_g$, we see easily that E is an exponential on Γ and $E\rho_W$ is invariant with respect to the right translation on Γ, that is, $E\rho_W$ is a right Haar measure on Γ, which we denote by ξ_Γ. Since for any $\mu\in\mathscr{D}^+(W)$, $W\mu=\rho_W*\bar{\mu}$, where $\bar{\mu}=\int c_x\varepsilon_x d\mu(x)$, we have $H(T)=\{(1/E)\xi_\Gamma*\nu;\ \nu\in\mathscr{D}^+(W)\}$ so that (3) is shown.

The equalities

$$(1/E)\xi_\Gamma*\sigma_T=\sigma_T*(1/E)\xi_\Gamma=(1/E)\xi_\Gamma$$

10) In this case, $(\sigma_T)^n=(\sigma_T)^{n-1}*\sigma_T$ is defined inductively for $n\geqq 1$, where $(\sigma_T)^0=\varepsilon_e$.

show that

$$\int E(g)d\sigma_T(g) = 1 \quad \text{and} \quad \int \Delta_\Gamma(g^{-1})E(g)d\sigma_T(g) = 1,$$

where Δ_Γ denotes the modular function of Γ, which imply the assertions (1) and (2). In fact, by simple calculation and by Corollary 13, we see $(\Delta_\Gamma/E)\xi_\Gamma \in \{\mu \in M^+(\Gamma);\ \sigma_T*\mu=\mu\} = \{\mu \in M^+(\Gamma);\ \mu*\sigma_T=\mu\} = \{c(1/E)\xi_\Gamma;\ c \geqq 0\}$, which shows $\Delta_\Gamma \equiv 1$.

Finally we shall show that if σ_T is recurrent and sub-exponential then T is uniformly recurrent. Let E be the exponential on the closed subgroup Γ generated by $\mathrm{supp}(\sigma_T)$ which satisfies $\int E(g)d\sigma_T(g) \leqq 1$. Since σ_T is recurrent, $E\sigma_T$ is also recurrent so that $\int E(g)d\sigma_T(g)=1$. Hence $(1/E)\xi_\Gamma \in H(T)$, where ξ_Γ is a left Haar measure on Γ. Put $\mu=(1/E)\xi_\Gamma$. It follows from $\sigma_T*\mu=\mu$ that the resolvent $(V_p)_{p>0}$ generated by T exists and $\sigma_{V_p}=\sum_{n=0}^\infty (1+p)^{-n}(\sigma_T)^n \in M(G)$ and $p\sigma_{V_p}*\mu=\mu$. Let ω be a relatively compact open set in G with $e \in \omega$. Since $\sigma_{V_p} \in \boldsymbol{B}$, we choose $\mu'_p \in B^{\sigma_{V_p}}(p\mu;\ \omega)$. Put $u_p(x) \equiv \int d\mu'_p$ on X. Then $(u_p)_{p>0}$ is a family defining the uniform recurrence of T. In fact, since σ_T is recurrent, we see $\lim_{p\downarrow 0}\int d\mu'_p=0$, and since $\sigma_{V_p}*\mu'_p=\mu$ in ω, $(u_p(x)\sigma_{V_p})_{p>0}$ is vaguely bounded and its vague accumulation points as $p\to 0$ are non-zero. Hence $(u_p)_{p>0}$ satisfies all the conditions in Definition 1, that is, T is uniformly recurrent. This completes the proof.

COROLLARY 33 (cf. [7]). *Let $\sigma \in M^+(G)$ be recurrent. Then σ is sub-exponential if and only if $H(\sigma) \neq \{0\}$, where σ may be considered as a diffusion kernel on G of convolution type. In this case, we obtain*:

(1) *$\Gamma = \mathrm{cl}(\cup_{n=0}^\infty \mathrm{supp}(\sigma^n))$ is a uni-modular closed subgroup in G.*

(2) *There exists a uniquely determined exponential E on Γ such that $\int E(g)d\sigma(g)=1$.*

(3) *$H(\sigma)=\{(1/E)\xi_\Gamma*\nu;\ \nu \in \mathscr{D}^+(\sigma)\}$, where ξ_Γ is a Haar measure on Γ.*

We say that $\sigma \in M^+(G)$ is of *finite order* if for some $n \geqq 2$, σ^n is defined and $\sigma^n=\sigma$. In particular, we say that σ is *idempotent* if $\sigma^2=\sigma$.

Generalizing a characterization of above measures on abelian groups (cf. for example [9] and [13]), we obtain

PROPOSITION 34. *Let $0 \neq \sigma \in M^+(G)$.*

(1) *If σ is idempotent, then* $\mathrm{supp}(\sigma)$ *is a compact subgroup in G and σ is the normalized Haar measure.*

(2). *If $\sigma^n=\sigma$ for some $n \geqq 3$, then there exist a compact subgroup K in G and an element $g \in G$ such that $g^{n-1} \in K$ and $\sigma=\xi_K*\varepsilon_g=\varepsilon_g*\xi_K$, where ξ_K is the normalized Haar measure on K.*

PROOF. (1): Clearly σ is recurrent. By Corollary 33, $\sigma=cE\xi_\Gamma$, where $\Gamma=\mathrm{supp}(\sigma) \cup \{e\}$, E is some exponential on Γ and $c>0$. Thus $e \in \mathrm{supp}(\sigma)$.

By $\sigma*\sigma=\sigma$, Γ is compact and $E\equiv1$, which imply that σ is the normalized Haar measure on a compact group.

(2): Put $K=\mathrm{supp}(\sigma^{n-1})$. It follows from (1) and $\sigma^{n-1}*\sigma^{n-1}=\sigma^{n-1}$ that $\sigma^{n-1}=\xi_K$. This gives $\sigma=\xi_K*\sigma=\sigma*\xi_K$ and $\int d\sigma=1$. For any $g_o\in\mathrm{supp}(\sigma)$, we put $g=(g_o^{-1})^{n-2}$. Then $g^{-1}g', g'g^{-1}\in K$ for all $g'\in\mathrm{supp}(\sigma)$, because

$$g_1g_2\cdots g_{n-1}\in K \qquad \text{for any} \quad g_j\in\mathrm{supp}(\sigma) \quad (j=1, 2,\dots, n-1).$$

Hence $\sigma=\xi_K*\varepsilon_g=\varepsilon_g*\xi_K$. Clearly $g^{n-1}\in K$. This completes the proof.

Finally we shall describe the results corresponding to Theorem 27 for the diffusion kernels of convolution type. This generalizes the result that any diffusion kernel of convolution type with associated resolvent on a locally compact abelian group is the canonical prolongation of a Hunt diffusion kernel on a quotient group by some compact subgroup (see [8] and [13]).

THEOREM 35. *Let V be a weakly regular diffusion kernel on $X=G/H$ of convolution type. Then there exist a uniquely determined compact subgroup K with $K\supset H$ and a uniquely determined weakly regular Hunt diffusion kernel $\tilde{V}$ on G/K of convolution type such that V is the canonical prolongation of $\tilde{V}$ on G/H, that is, $\eta_H(V\varepsilon_{\pi_H(g)})=\eta_K(\tilde{V}\varepsilon_{\pi_K(g)})$ for every $g\in G$.*

PROOF. By Proposition 30, the resolvent $(V_p)_{p\geq0}$ associated with V is of convolution type. Since $(p\sigma_{V_p})_{p>0}$ is vaguely bounded, there exists an accumulation point $\sigma\in M^+(G)$ as $p\uparrow\infty$. Then by Theorem 27 we see that $\sigma*\sigma=\sigma$ and $\sigma_V*\sigma=\sigma_V$. By Proposition 34, there exists a compact subgroup K in G such that $\sigma=\xi_K$. Since $\sigma_{V_p}*\xi_H=\sigma_{V_p}$ for any $p>0$, $\sigma*\xi_H=\sigma$, which implies $K\subset H$. Using Theorem 27 again, we see that the required assertion is valid.

COROLLARY 36. *Let V be a weakly regular diffusion kernel on $X=G/H$ of convolution type. If X is a symmetric space, that is, H is a maximal compact subgroup of G, then V is a weakly regular Hunt diffusion kerenel on X.*

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