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On the behavior of potentials near a hyperplane

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1. Introduction

Let R^n $(n \geqq 2)$ be the n-dimensional Euclidean space, and set

$$R^n_+ = \{x=(x', x_n);\ x_n>0\}.$$

In this paper we investigate the behavior near the boundary ∂R^n_+ of α-potentials

$$U^f_\alpha(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy,$$

where $0<\alpha<n$ and f is a nonnegative measurable function on R^n satisfying the condition:

$$(1) \qquad \int_{R^n} f(y)^p |y_n|^\beta dy < \infty.$$

For $\gamma \geqq 1$, we say that a function u has a T_γ-limit ℓ at $\xi \in \partial R^n_+$ if

$$\lim_{x\to\xi, x\in T_\gamma(\xi,a)} u(x) = \ell$$

for any $a>0$, where

$$T_\gamma(\xi, a) = \{(x', x_n) \in R^n_+;\ |(x', 0)-\xi| < a x_n^{1/\gamma}\}.$$

If u has a T_γ-limit at ξ for any $\gamma>1$, then u is said to have a T_∞-limit at ξ. Our first aim is to prove the following result:

THEOREM 1. *Let $\alpha p>n$ and f be a nonnegative measurable function on R^n satisfying (1) with $\beta<p-1$.*

(i) *If $n-\alpha p+\beta>0$, then for each $\gamma \geqq 1$ there exists a set $E_\gamma \subset \partial R^n_+$ such that $H_{\gamma(n-\alpha p+\beta)}(E_\gamma)=0$ and U^f_α has a T_γ-limit at any $\xi \in \partial R^n_+ - E_\gamma$.*

(ii) *If $n-\alpha p+\beta=0$, then there exists a set $E \subset \partial R^n_+$ such that $B_{n/p,p}(E)=0$ and U^f_α has a T_∞-limit at any $\xi \in \partial R^n_+ - E$.*

(iii) *If $n-\alpha p+\beta<0$, then U^f_α has a limit at any $\xi \in \partial R^n_+$.*

Here H_ℓ denotes the ℓ-dimensional Hausdorff measure, and $B_{\ell,p}$ the Bessel capacity of index (ℓ, p) (cf. [5]).

As an application of (ii) of Theorem 1, we can prove a result of Cruzeiro

[4] concerning the existence of T_∞-limits of harmonic functions with gradient in $L^n(R^n_+)$.

In case $\alpha p \leqq n$, if we further restrict the set of approach, then we can obtain a similar result by replacing "T_γ-limit" by "(α, p)-fine T^*_γ-limit". To do so, we need a capacity $C_{\alpha,p}(\cdot\,;\,\cdot)$, which is a special case of the capacities of Meyers [5].

Let G be an open set in R^n. For $E \subset R^n$, define

$$C_{\alpha,p}(E;\,G) = \inf \|g\|_p^p,$$

where the infimum is taken over all nonnegative measurable functions g on R^n such that $g=0$ outside G and $U^g_\alpha(x) \geqq 1$ for every $x \in E$, and $\|\cdot\|_p$ denotes the L^p-norm in R^n. A set E in R^n is said to be (α, p)-thin at $\xi \in \partial R^n_+$ relative to T_γ if for any a, b, a' and b' with $0<a'<a<b<b'$,

$$\sum_{i=1}^{\infty} 2^{i\gamma(n-\alpha p)} C_{\alpha,p}(E_i \cap T_\gamma(\xi, a, b);\, G_i \cap T_\gamma(\xi, a', b')) < \infty, \tag{2}$$

where $E_i=\{x \in E;\, 2^{-i} \leqq |x-\xi| < 2^{-i+1}\}$, $G_i=\{x;\, 2^{-i-1}<|x-\xi|<2^{-i+2}\}$ and $T_\gamma(\xi, a, b)=\{x=(x', x_n) \in R^n_+;\, a x_n^{1/\gamma} < |\xi'-x'| < b x_n^{1/\gamma}\}$. We say that a function u has an (α, p)-fine T^*_γ-limit ℓ at ξ if there exists a set $E \subset R^n_+$ such that E is (α, p)-thin at ξ relative to T_γ and

$$\lim_{x \to \xi,\, x \in T_\gamma(\xi, a, b) - E} u(x) = \ell$$

for any a and b with $0<a<b$; u is said to have an (α, p)-fine T^*_∞-limit at ξ if it has an (α, p)-fine T^*_γ-limit at ξ for any $\gamma>1$.

Now we are ready to state our second result.

THEOREM 2. *Let $p>1$, $\alpha p \leqq n$ and $\beta < p-1$. Let f be a nonnegative measurable function on R^n satisfying* (1).

(i) *If $n-\alpha p+\beta>0$, then for each $\gamma \geqq 1$ there exists a set $E_\gamma \subset \partial R^n_+$ such that $H_{\gamma(n-\alpha p+\beta)}(E_\gamma)=0$ and U^f_α has an (α, p)-fine T^*_γ-limit at any $\xi \in \partial R^n_+ - E_\gamma$.*

(ii) *If $n-\alpha p+\beta=0$, then there exists a set $E \subset \partial R^n_+$ such that $B_{n/p,p}(E)=0$ and U^f_α has an (α, p)-fine T^*_∞-limit at any $\xi \in \partial R^n_+ - E$.*

(iii) *If $n-\alpha p+\beta<0$, then U^f_α has an (α, p)-fine T^*_∞-limit at any $\xi \in \partial R^n_+$.*

We shall also discuss the existence of T_γ-limits and (α, p)-fine T^*_γ-limits of α-Green potentials in R^n_+, and give a generalization of a result of Wu [12; Theorem 1], in which he treated only the case $n-2p+\beta>0$ $(\alpha=2)$. Since T_1-limit ((α, p)-fine T^*_1-limit) coincides with nontangential limit (nontangential (α, p)-fine limit), Theorems 2 and 3 in [10] are included in Theorems 5, 7 and 10 of the present paper.

2. Proof of Theorem 1

For a nonnegative measurable function f on R^n, we set

$$U_\alpha^f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy.$$

LEMMA 1. *For $x^0 \in R^n$ and $c>0$, we have*

$$\lim_{x\to x^0} \int_{\{y;|x-y|>c|x^0-x|\}} |x-y|^{\alpha-n} f(y)dy = U_\alpha^f(x^0).$$

PROOF. If $U_\alpha^f(x^0)=\infty$, then Fatou's lemma gives the required equality. Assume $U_\alpha^f(x^0)<\infty$. If $|x-y|>c|x^0-x|$, then

$$|x^0-y| \leqq |x^0-x| + |x-y| < (1+c^{-1})|x-y|,$$

so that Lebesgue's dominated convergence theorem establishes the required equality.

LEMMA 2. *Let f be a nonnegative measurable function satisfying* (1) *with real numbers $p>1$ and β. If we set*

$$B_d = \left\{\xi\in \partial R_+^n;\ \limsup_{r\downarrow 0} r^{-d} \int_{B(\xi,r)} f(y)^p |y_n|^\beta dy>0\right\}, \quad d>0,$$

then $H_d(B_d)=0$, where $B(\xi, r)$ denotes the open ball with center at ξ and radius r.

LEMMA 3. *Let f be as above and define*

$$B_0 = \left\{\xi\in \partial R_+^n;\ \limsup_{r\downarrow 0} (\log r^{-1})^{p-1} \int_{B(\xi,r)} f(y)^p |y_n|^\beta dy>0\right\}.$$

Then $B_{n/p,p}(B_0)=0$.

These lemmas follow from the facts in [6; p. 165] and [5; Theorem 21].

LEMMA 4. *Let $\alpha p>n$, $\beta<p-1$, $p'=p/(p-1)$, $\xi\in\partial R_+^n$ and $x\in R_+^n$. Then there exists a positive constant C independent of x such that*

$$\left\{\int_{B(x,|\xi-x|/2)} |x-y|^{p'(\alpha-n)}|y_n|^{-\beta p'/p}dy\right\}^{1/p'} \leqq C\begin{cases} x_n^{(\alpha p-\beta-n)/p} & \text{if } n-\alpha p+\beta>0,\\ [\log(x_n^{-1}|\xi-x|+2)]^{1/p'} & \text{if } n-\alpha p+\beta=0,\\ |\xi-x|^{(\alpha p-\beta-n)/p} & \text{if } n-\alpha p+\beta<0.\end{cases}$$

PROOF. Let $\xi^*=(O, 1)$. By change of variables, we see that the left hand side is equal to

$$x_n^{\alpha-n-\beta/p+n/p'}\left\{\int_{\{z;|\xi^*-z|\leq x_n^{-1}|\xi-x|/2\}}|\xi^*-z|^{p'(\alpha-n)}|z_n|^{-\beta p'/p}dz\right\}^{1/p'},$$

which is dominated by

$$Cx_n^{(\alpha p-\beta-n)/p}\left\{\int_{B(\xi^*,1/2)}|\xi^*-z|^{p'(\alpha-n)}dz\right.$$
$$\left.+\int_{B(O,x_n^{-1}|\xi-x|/2+1)}(1+|z|)^{p'(\alpha-n)}|z_n|^{-\beta p'/p}dz\right\}^{1/p'}.$$

Evaluating these integrals by the aid of polar coordinates in R^n, we obtain the required inequalities.

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. We write $U_\alpha^f=U_1+U_2$, where

$$U_1(x)=\int_{\{y;|x-y|>|\xi-x|/2\}}|x-y|^{\alpha-n}f(y)\,dy,$$
$$U_2(x)=\int_{\{y;|x-y|\leq|\xi-x|/2\}}|x-y|^{\alpha-n}f(y)\,dy.$$

By Lemma 1, $\lim_{x\to\xi}U_1(x)=U_\alpha^f(\xi)$.

First let $n-\alpha p+\beta>0$. It suffices to prove that U_2 has T_γ-limit zero at $\xi\in\partial R_+^n-B_{\gamma(n-\alpha p+\beta)}$, since $H_{\gamma(n-\alpha p+\beta)}(B_{\gamma(n-\alpha p+\beta)})=0$ on account of Lemma 2. By Hölder's inequality and Lemma 4, we have

$$U_2(x)\leq\text{const.}\left\{x_n^{\alpha p-\beta-n}\int_{B(\xi,2|\xi-x|)}f(y)^p|y_n|^\beta dy\right\}^{1/p}.$$

Hence if $\xi\in\partial R_+^n-B_{\gamma(n-\alpha p+\beta)}$ and $x\in T_\gamma(\xi, a)\cap B(\xi, 1)$, then

$$U_2(x)\leq\text{const.}\left\{|x-\xi|^{\gamma(\alpha p-\beta-n)}\int_{B(\xi,2|\xi-x|)}f(y)^p|y_n|^\beta dy\right\}^{1/p},$$

which tends to zero as $x\to\xi$, $x\in T_\gamma(\xi, a)$. This implies that U_2 has T_γ-limit zero at $\xi\in\partial R_+^n-B_{\gamma(n-\alpha p+\beta)}$.

Next let $n-\alpha p+\beta=0$. Then it follows from Lemma 4 that

$$U_2(x)\leq\text{const.}\left\{[\log(x_n^{-1}|x-\xi|+2)]^{p-1}\int_{B(\xi,2|\xi-x|)}f(y)^p|y_n|^\beta dy\right\}^{1/p}.$$

If $\xi\in\partial R_+^n-B_0$ and $x\in T_\gamma(\xi, a)$, then

$$U_2(x)\leq\text{const.}\left\{[\log(|x-\xi|^{-1}+2)]^{p-1}\int_{B(\xi,2|\xi-x|)}f(y)^p|y_n|^\beta dy\right\}^{1/p},$$

and hence U_2 has T_γ-limit zero at ξ. Since γ is arbitrary, U_2 has T_∞-limit zero at $\xi \in \partial R^n_+ - B_0$. By Lemma 3, $B_{n/p,p}(B_0)=0$.

In case $n-\alpha p+\beta<0$, we obtain

$$U_2(x) \leqq \text{const.} \left\{|\xi - x|^{\alpha p-\beta-n} \int_{B(\xi,2|\xi-x|)} f(y)^p |y_n|^\beta dy\right\}^{1/p},$$

which tends to zero as $x \to \xi$. Thus Theorem 1 is established.

A function u is said to have a nontangential limit at $\xi \in \partial R^n_+$ if it has a T_1-limit at ξ. The following can be obtained with a slight modification of the above proof.

THEOREM 3. *Let $\alpha p>n$ and f be a nonnegative measurable function on R^n satisfying* (1) *with a real number β.*

(i) *If $n-\alpha p+\beta>0$, then U^f_α has a nontangential limit at any $\xi \in \partial R^n_+ - B_{n-\alpha p+\beta}$.*

(ii) *If $n-\alpha p+\beta \leqq 0$, then U^f_α has a nontangential limit at any $\xi \in \partial R^n_+$.*

3. (α, p)-fine T^*_γ-limit

For a nonnegative measurable function f on R^n, we write $U^f_\alpha = U_1+U_2+U_3$, where

$$U_1(x) = \int_{R^n - B(x,|x-\xi|/2)} |x-y|^{\alpha-n} f(y)dy,$$

$$U_2(x) = \int_{B(x,|x-\xi|/2)-B(x,x_n/2)} |x-y|^{\alpha-n} f(y)dy,$$

$$U_3(x) = \int_{B(x,x_n/2)} |x-y|^{\alpha-n} f(y)dy.$$

Lemma 1 implies that $\lim_{x\to\xi} U_1(x) = U^f_\alpha(\xi)$.

LEMMA 5. *Let $p>1$, $\beta<p-1$, $x \in R^n_+$ and $\xi \in \partial R^n_+$. Then there exists a positive constant C independent of x such that*

$$U_2(x)^p \leqq C \begin{cases} x_n^{\alpha p-\beta-n} F(x) & \text{in case } n-\alpha p+\beta>0, \\ [\log(x_n^{-1}|x-\xi|+2)]^{p-1} F(x) & \text{in case } n-\alpha p+\beta=0, \\ |x-\xi|^{\alpha p-\beta-n} F(x) & \text{in case } n-\alpha p+\beta<0, \end{cases}$$

where $F(x) = \int_{B(\xi,2|\xi-x|)} f(y)^p |y_n|^\beta dy$.

This lemma can be proved in the same way as Lemma 4 with the aid of Hölder's inequality.

LEMMA 6. *Let f be a nonnegative measurable function on R^n satisfying* (1) *with real numbers $p>1$ and β. For $\beta'>\beta$, set*

$$A_{\gamma,\beta'} = \left\{\xi\in\partial R^n_+;\ \int_{B(\xi,1)} (|y'-\xi'|^{2\gamma}+|y_n|^2)^{(\alpha p-\beta'-n)/2} f(y)^p|y_n|^{\beta'}dy = \infty\right\}.$$

Then $H_{\gamma(n-\alpha p+\beta)}(A_{\gamma,\beta'})=0$ for $\gamma\geqq 1$ and $\beta'>\beta$.

REMARK. If we set $A_\gamma=\cap_{\beta'>\beta} A_{\gamma,\beta'}$, then $H_{\gamma(n-\alpha p+\beta)}(A_\gamma)=0$.

PROOF OF LEMMA 6. If $n-\alpha p+\beta\leqq 0$, then $A_{\gamma,\beta'}$ is empty. Suppose $n-\alpha p+\beta>0$ and $H_{\gamma(n-\alpha p+\beta)}(A_{\gamma,\beta'})>0$. By [3; Theorems 1 and 3 in §II] we can find a nonnegative measure μ such that $\mu(A_{\gamma,\beta'})>0$, $\mu(R^n-A_{\gamma,\beta'})=0$ and

$$\mu(B(x,r))\leqq r^{\gamma(n-\alpha p+\beta)} \qquad \text{for every } x \text{ and } r.$$

Then, since $\int(|y'-\xi'|^{2\gamma}+|y_n|^2)^{(\alpha p-\beta'-n)/2}d\mu(\xi)\leqq\text{const.}\ |y_n|^{\beta-\beta'}$, we have

$$\begin{aligned}\infty &= \int\left\{\int(|y'-\xi'|^{2\gamma}+|y_n|^2)^{(\alpha p-\beta'-n)/2}f(y)^p|y_n|^{\beta'}dy\right\}d\mu(\xi)\\ &= \int\left\{\int(|y'-\xi'|^{2\gamma}+|y_n|^2)^{(\alpha p-\beta'-n)/2}d\mu(\xi)\right\}f(y)^p|y_n|^{\beta'}dy\\ &\leqq \text{const.}\int f(y)^p|y_n|^{\beta}dy<\infty,\end{aligned}$$

which is a contradiction. Thus the lemma is proved.

LEMMA 7. *Let f be a nonnegative measurable function on R^n satisfying* (1) *with real numbers $p>1$ and β. Let $\alpha p\leqq n$ and $\gamma\geqq 1$. Then for each $\xi\in\partial R^n_+-A_\gamma$, there exists a set $E\subset R^n_+$ such that E is (α, p)-thin at ξ relative to T_γ and*

$$\lim_{x\to\xi,x\in T_\gamma(\xi,a,b)-E} U_3(x)=0 \qquad \textit{for any } a \textit{ and } b \textit{ with } b>a>0. \tag{3}$$

PROOF. Suppose $\xi\in\partial R^n_+-A_{\gamma,\beta'}$, $\beta'>\beta$. Take a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty}a_i=\infty$ and

$$\textstyle\sum_{i=1}^\infty a_i\int_{G_i}(|y'-\xi'|^{2\gamma}+|y_n|^2)^{(\alpha p-\beta'-n)/2}f(y)^p|y_n|^{\beta'}dy<\infty,$$

where $G_i=\{x;\ 2^{-i-1}<|x-\xi|<2^{-i+2}\}$. Consider the sets

$$E_i=\{x\in B(\xi,2^{-i+1})-B(\xi,2^{-i});\ U_3(x)\geqq a_i^{-1/p}\}.$$

Let $0<a'<a<b<b'$, and find $c>0$ such that $c<1/2$ and $B(x,cx_n)\subset T_\gamma(\xi,a',b')$ whenever $x\in T_\gamma(\xi,a,b)$ and $0<x_n<1$. Set

$$U_3'(x) = \int_{B(x,x_n/2)-B(x,cx_n)} |x-y|^{\alpha-n} f(y)dy,$$

$$U_3''(x) = \int_{B(x,cx_n)} |x-y|^{\alpha-n} f(y)dy.$$

By Hölder's inequality,

$$U_3'(x) \leqq \text{const.} \left\{x_n^{\alpha p-n} \int_{B(x,x_n/2)} f(y)^p dy\right\}^{1/p}$$

$$\leqq \text{const.} \left\{\int_{B(x,x_n/2)} f(y)^p y_n^{\alpha p-n} dy\right\}^{1/p}.$$

Find $b''>0$ such that $B(x, x_n/2) \subset T_\gamma(\xi, b'')$ whenever $x \in T_\gamma(\xi, b)$ and $0<x_n<1$. Since $\sum_{i=1}^{\infty} a_i \int_{G_i \cap T_\gamma(\xi,b'')} f(y)^p y_n^{\alpha p-n} dy < \infty$, we may assume that $U_3'(x) < 2^{-1} a_i^{-1/p}$ for all $x \in E_i \cap T_\gamma(\xi, a, b)$, and hence

$$U_3''(x) \geqq 2^{-1} a_i^{-1/p} \qquad \text{for all} \quad x \in E_i \cap T_\gamma(\xi, a, b).$$

Consequently it follows from the definition of capacity $C_{\alpha,p}$ that

$$C_{\alpha,p}(E_i \cap T_\gamma(\xi, a, b); G_i \cap T_\gamma(\xi, a', b'))$$

$$\leqq 2^p a_i \int_{G_i \cap T_\gamma(\xi,a',b')} f(y)^p dy$$

$$\leqq \text{const.}\ 2^{-i\gamma(n-\alpha p)} a_i \int_{G_i \cap T_\gamma(\xi,b')} f(y)^p y_n^{\alpha p-n} dy.$$

Define $E = \cup_{i=1}^{\infty} E_i$. Then we see that E satisfies (2) and (3). Thus the lemma is established.

With the aid of Lemmas 5 and 7, we deduce the following result, which proves Theorem 2 in view of Lemmas 2, 3 and the remark after Lemma 6.

THEOREM 2'. *Let $p>1$, $\alpha p \leqq n$ and $\beta < p-1$. Let f be a nonnegative measurable function on R^n satisfying* (1).

(i) *If $n-\alpha p+\beta>0$ and $\xi \in \partial R_+^n - (A_\gamma \cup B_{\gamma(n-\alpha p+\beta)})$ for some $\gamma \geqq 1$, then U_α^f has an (α, p)-fine T_γ^*-limit $U_\alpha^f(\xi)$ at ξ.*

(ii) *If $n-\alpha p+\beta=0$ and $\xi \in \partial R_+^n - B_0$, then U_α^f has an (α, p)-fine T_∞^*-limit $U_\alpha^f(\xi)$ at ξ.*

(iii) *If $n-\alpha p+\beta<0$, then U_α^f has an (α, p)-fine T_∞^*-limit at any $\xi \in \partial R_+^n$.*

REMARK 1. In case $n-\alpha p=\beta=0$, for each $\xi \in \partial R_+^n - B_0$ one can find a set $E \subset R_+^n$ such that

$$\lim_{x\to\xi, x\in T_\gamma(\xi,a,b)-E} U_\alpha^f(x) = U_\alpha^f(\xi)$$

and

$$\lim_{r\downarrow 0}(\log r^{-1})^{p-1}C_{\alpha,p}(E\cap B(\xi, r)\cap T_\gamma(\xi, a, b); B(\xi, 2r)\cap T_\gamma(\xi, a', b'))=0$$

for any $\gamma>1$ and any a, b, a', b' with $0<a'<a<b<b'$.

REMARK 2. Let $p>1$, $\alpha p<n$, $\gamma>1$ and $0<a'<a<b<b'$. If E satisfies (2) and $E\subset T_\gamma(\xi, a, b)$, then there exists a nonnegative measurable function f on R^n such that

(i) $U_\alpha^f(\xi)<\infty$; (ii) $\lim_{x\to\xi, x\in E} U_\alpha^f(x)=\infty$; (iii) $\int f(y)^p|y_n|^{\alpha p-n}dy<\infty$.

For $\xi\in\partial R_+^n$ and $\zeta=(\zeta', 1)$, we set

$$t_\gamma(\xi, \zeta)=\{(\xi'+r\zeta', r^\gamma); 0<r<1\}.$$

THEOREM 4. *Let p, β and f be as in Theorem 2. Let $\gamma>1$. Then for each $\xi\in\partial R_+^n-(A_\gamma\cup B^*_{\gamma(n-\alpha p+\beta)})$ there exists a set $E\subset H=\{(\zeta', 1); \zeta'\in R^{n-1}\}$ such that E has Hausdorff dimension at most $n-\alpha p$ and*

$$\lim_{x\to\xi, x\in t_\gamma(\xi,\zeta)} U_\alpha^f(x)=U_\alpha^f(\xi) \tag{4}$$

for every $\zeta\in H-E$, where $B_d^*=B_d$ if $d\geqq 0$ and B_d^* is empty if $d<0$.

To prove this, we need the following result (cf. [2; Theorem IX, 7]).

LEMMA 8. *Let μ be a nonnegative measure on R^n such that $U_\alpha^\mu(x)=\int|x-y|^{\alpha-n}d\mu(y)\not\equiv\infty$, and $x^0\in R^n$. Then there exists a set $E\subset H$ whose Riesz capacity of order α is zero such that*

$$\lim_{r\downarrow 0} r^{n-\alpha}U_\alpha^\mu(x^0+r\zeta)=\mu(\{x^0\}) \quad \textit{for every} \quad \zeta\in H-E.$$

PROOF OF THEOREM 4. Let $\xi\in\partial R_+^n-B^*_{\gamma(n-\alpha p+\beta)}$. Then Lemmas 1 and 5 imply that

$$\lim_{x\to\xi, x\in R_+^n}\int_{R^n-B(x,x_n/2)}|x-y|^{\alpha-n}f(y)dy=U_\alpha^f(\xi).$$

Let $0<\varepsilon<\alpha$. By Hölder's inequality we derive

$$\begin{aligned}
&\int_{B(x,x_n/2)}|x-y|^{\alpha-n}f(y)dy\\
&\leqq\left\{\int_{B(x,x_n/2)}|x-y|^{(\alpha-\varepsilon)p'-n}dy\right\}^{1/p'}\left\{\int_{B(x,x_n/2)}|x-y|^{\varepsilon p-n}f(y)^p dy\right\}^{1/p}\\
&\leqq \text{const.}\left\{x_n^{(\alpha-\varepsilon)p}\int_{B(x,x_n/2)}|x-y|^{\varepsilon p-n}f(y)^p\,dy\right\}^{1/p}\\
&\leqq \text{const.}\left\{z_n^{n-\varepsilon p}\int_{B(z,cz_n)}|z-w|^{\varepsilon p-n}g(w)dw\right\}^{1/p},
\end{aligned}$$

where c is a positive constant independent of $z=(x', x_n^{1/\gamma})$ and $g(w)=f(w', w_n^\gamma)^p w_n^{\gamma(\alpha p-n)+\gamma-1}$. If $\xi\in\partial R_+^n - A_\gamma$, then $\int_{T_\gamma(\xi,a)} f(y)^p y_n^{\alpha p-n}dy<\infty$ for any $a>1$, so that $\int_{T_1(\xi,a)} g(w)dw<\infty$ for any $a>1$. By Lemma 8, we can find a set $E_\varepsilon\subset H$ whose Riesz capacity of order $n-\varepsilon p$ is zero such that

$$\lim_{x\to\xi, x\in t_\gamma(\xi,\zeta)} \int_{B(x,x_n/2)} |x-y|^{\alpha-n} f(y)dy = 0$$

for every $\zeta\in H-E_\varepsilon$. Define $E=\bigcap_{0<\varepsilon<\alpha} E_\varepsilon$. Then E has Hausdorff dimension at most $n-\alpha p$, and (4) holds for any $\zeta\in H-E$.

4. T_γ-limits of Green potentials

For a nonnegative measurable function f on R_+^n, we define

$$G_\alpha^f(x) = \int_{R_+^n} G_\alpha(x, y)f(y)dy,$$

where $G_\alpha(x, y)=|x-y|^{\alpha-n}-|\bar{x}-y|^{\alpha-n}$, $\bar{x}=(x', -x_n)$ for $x=(x', x_n)$. We first note the following property of G_α.

Lemma 9. *There exist $c_1>0$ and $c_2>0$ such that*

$$c_1 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2} \leqq G_\alpha(x, y) \leqq c_2 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2}$$

for every $x=(x', x_n)$ and $y=(y', y_n)$ in R_+^n.

Corollary. *$G_\alpha^f \not\equiv \infty$ if and only if $\int_{R_+^n} (1+|y|)^{\alpha-n-2} y_n f(y)dy<\infty$.*

For $0\leqq\delta<1$, define

$$E_\delta = \left\{\xi\in\partial R_+^n;\ \limsup_{r\downarrow 0} r^{\alpha-\delta-n-1}\int_{B(\xi,r)\cap R_+^n} y_n f(y)dy > 0\right\}.$$

Lemma 10 (cf. [10; Lemma 3]). *For $\xi\in\partial R_+^n$ and $c>0$, define*

$$G_1(x) = \int_{\{y\in R_+^n;\, |x-y|>c|x-\xi|\}} G_\alpha(x, y)f(y)dy.$$

If $G_\alpha^f\not\equiv\infty$ and $0\leqq\delta<1$, then $\lim_{x\to\xi, x\in R_+^n} x_n^{-\delta}G_1(x)=0$ if and only if $\xi\in\partial R_+^n - E_\delta$.

Remark. If $G_\alpha^f\not\equiv\infty$, then $H_{n-\alpha+\delta+1}(E_\delta)=0$. If in addition $\int_{R_+^n} f(y)^p y_n^\beta dy<\infty$ with $p>1$ and $\beta<2p-1$, then $H_{n-\alpha p+\beta+\delta p}(E_\delta)=0$ (see [10; Corollary to Lemma 5]).

The following result can be proved in the same way as Lemma 4.

LEMMA 11. *Let $\alpha p>n$ and $\xi \in \partial R_+^n$. Then*

$$\left\{\int_{\{y\in R_+^n;|x-y|<|\xi-x|/2\}} G_\alpha(x, y)^{p'} y_n^{-\beta p'/p} dy\right\}^{1/p'}$$

$$\leqq \text{const.}\begin{cases} x_n^{(\alpha p-\beta-n)/p} & \text{if} \quad n-\alpha p+\beta+p>0, \\ x_n[\log(x_n^{-1}|\xi-x|+2)]^{1/p'} & \text{if} \quad n-\alpha p+\beta+p=0, \\ x_n|\xi-x|^{(\alpha p-\beta-p-n)/p} & \text{if} \quad n-\alpha p+\beta+p<0. \end{cases}$$

By Lemmas 10 and 11 we can establish the following theorems.

THEOREM 5. *Let $\alpha p>n$, $0\leqq\delta<1$ and f be a nonnegative measurable function on R_+^n such that $G_\alpha^f \not\equiv \infty$ and*

$$\int_{R_+^n} f(y)^p y_n^\beta dy < \infty, \quad \beta < 2p-1. \tag{5}$$

(i) *If $n-\alpha p+\beta+\delta p>0$ and $\gamma\geqq 1$, then $x_n^{-\delta}G_\alpha^f(x)$ has T_γ-limit zero at any $\xi \in \partial R_+^n-(E_\delta \cup B_{\gamma(n-\alpha p+\beta+\delta p)})$.*

(ii) *If $n-\alpha p+\beta+\delta p\leqq 0$, then $x_n^{-\delta}G_\alpha^f(x)$ has limit zero at any $\xi\in\partial R_+^n$.*

THEOREM 6. *Let $\alpha p>n$ and f be as above. Set*

$$G(\xi) = 2(n-\alpha)\int_{R_+^n} |\xi-y|^{\alpha-n-2} y_n f(y)dy, \quad \xi\in\partial R_+^n.$$

(i) *If $n-\alpha p+\beta+p>0$ and $\gamma\geqq 1$, then $x_n^{-1}G_\alpha^f(x)$ has a T_γ-limit $G(\xi)$ at any $\xi\in\partial R_+^n-B_{\gamma(n-\alpha p+\beta+p)}$.*

(ii) *If $n-\alpha p+\beta+p=0$, then $x_n^{-1}G_\alpha^f(x)$ has a T_∞-limit $G(\xi)$ at any $\xi\in \partial R_+^n-B_0$.*

(iii) *If $n-\alpha p+\beta+p<0$, then $\lim_{x\to\xi, x\in R_+^n} x_n^{-1}G_\alpha^f(x)=G(\xi)$ for any $\xi\in\partial R_+^n$.*

As to T_γ^*-limits of Green potentials, we have the next result.

THEOREM 7. *Let $p>1$, $0\leqq\delta<1$, $\alpha p\leqq n$ and f be a nonnegative measurable function on R_+^n satisfying (5) with $\beta<2p-1$ such that $G_\alpha^f\not\equiv\infty$.*

(i) *If $n-\alpha p+\beta+\delta p>0$ and $\gamma\geqq 1$, then $x_n^{-\delta}G_\alpha^f(x)$ has (α, p)-fine T_γ^*-limit zero at any $\xi\in\partial R_+^n-(E_\delta \cup A_{\gamma,\delta}^* \cup B_{\gamma(n-\alpha p+\beta+\delta p)})$.*

(ii) *If $n-\alpha p+\beta+\delta p\leqq 0$, then $x_n^{-\delta}G_\alpha^f(x)$ has (α, p)-fine T_∞^*-limit zero at any $\xi\in\partial R_+^n$.*

Here $A_{\gamma,\delta}^*=\bigcap_{\beta'>\beta+\delta p} A_{\gamma,\beta'}$. Note that $H_{\gamma(n-\alpha p+\beta+\delta p)}(E_\delta\cap A_{\gamma,\delta}^*)=0$ in the case of (i) of Theorem 7.

PROOF OF THEOREM 7. Write $G_\alpha^f(x)=G_1(x)+G_2(x)+G_3(x)$, where

$$G_1(x) = \int_{\{y\in R_+^n;|x-y|>|\xi-x|/2\}} G_\alpha(x, y)f(y)dy,$$

$$G_2(x) = \int_{\{y\in R_+^n;x_n/2<|x-y|\leqq|\xi-x|/2\}} G_\alpha(x, y)f(y)dy,$$

$$G_3(x) = \int_{B(x,x_n/2)} G_\alpha(x, y)f(y)dy.$$

First note that $\lim_{x\to\xi,x\in R_+^n} x_n^{-\delta}G_1(x)=0$ if $\xi\in\partial R_+^n-E_\delta$ according to Lemma 10. In what follows we shall prove only the case $n-\alpha p+\beta+\delta p>0$, because the remaining case can be proved similarly. Assume $n-\alpha p+\beta+\delta p>0$. Then Hölder's inequality yields

$$\begin{aligned} x_n^{-\delta}G_2(x) &\leqq c_2x_n^{1-\delta}\left\{\int_{B(x,|\xi-x|/2)-B(x,x_n/2)} |x-y|^{p'(\alpha-n-2)}|y_n|^{p'(1-\beta/p)}dy\right\}^{1/p'} \\ &\times\left\{\int_{B(\xi,2|\xi-x|)\cap R_+^n} f(y)^p y_n^\beta dy\right\}^{1/p} \\ &\leqq \text{const.}\left\{x_n^{\alpha p-\beta-\delta p-n}\int_{B(\xi,2|x-\xi|)\cap R_+^n} f(y)^p y_n^\beta dy\right\}^{1/p}. \end{aligned}$$

If $\xi\in\partial R_+^n-B_{\gamma(n-\alpha p+\beta+\delta p)}$ and $x\in T_\gamma(\xi, a)\cap B(\xi, 1)$, then

$$\begin{aligned} x_n^{-\delta}G_2(x) &\leqq \text{const.}\left\{|x-\xi|^{\gamma(\alpha p-\beta-\delta p-n)}\int_{B(\xi,2|x-\xi|)\cap R_+^n} f(y)^p y_n^\beta dy\right\}^{1/p} \\ &\longrightarrow 0 \text{ as } x\longrightarrow\xi,\ x\in T_\gamma(\xi, a). \end{aligned}$$

Since $x_n^{-\delta}G_3(x)\leqq c_2\int_{B(x,x_n/2)}|x-y|^{\alpha-n}f(y)(y_n/2)^{-\delta}dy$ on account of Lemma 9, it follows from Lemma 7 that $x_n^{-\delta}G_3(x)$ has (α, p)-fine T_γ^*-limit zero at $\xi\in\partial R_+^n-A_{\gamma,\delta}^*$. By these facts $x_n^{-\delta}G_\alpha^f(x)$ has (α, p)-fine T_γ^*-limit zero at $\xi\in\partial R_+^n-E_\delta-A_{\gamma,\delta}^*-B_{\gamma(n-\alpha p+\beta+\delta p)}$.

In a similar manner we can establish the following result.

THEOREM 8. *Let α, β, p and f be as in Theorem 7.*

(i) *If $n-\alpha p+\beta+p>0$ and $\gamma\geqq 1$, then $x_n^{-1}G_\alpha^f(x)$ has an (α, p)-fine T_γ^*-limit $G(\xi)$ at any $\xi\in\partial R_+^n-(A_{\gamma,1}^*\cup B_{\gamma(n-\alpha p+\beta+p)})$.*

(ii) *If $n-\alpha p+\beta+p\leqq 0$, then $x_n^{-1}G_\alpha^f(x)$ has an (α, p)-fine T_∞^*-limit $G(\xi)$ at any $\xi\in\partial R_+^n-B_{\gamma(n-\alpha p+\beta+p)}^*$.*

In a way similar to the proof of Theorem 4, the existence of limits along t_γ of Green potentials can be proved.

THEOREM 9 (cf. Wu [12; Theorem 1]). *Let α, β, δ, p and f be as in Theorem 7.*

(i) *If $n-\alpha p+\beta+\delta p>0$ and $\gamma>1$, then for each $\xi\in\partial R^n_+-(E_\delta \cup A^*_{\gamma,\delta}\cup B_{\gamma(n-\alpha p+\beta+\delta p)})$ there exists a set $E\subset H$ such that E has Hausdorff dimension at most $n-\alpha p$ and*

$$\lim_{x\to\xi, x\in t_\gamma(\xi,\zeta)} x_n^{-\delta} G^f_\alpha(x) = 0 \qquad \text{for every} \quad \zeta\in H-E. \tag{6}$$

(ii) *If $n-\alpha p+\beta+\delta p\leqq 0$, then for each $\xi\in\partial R^n_+$ there exists a set $E\subset H$ such that E has Hausdorff dimension at most $n-\alpha p$ and (6) holds.*

As to nontangential limits we have the following results.

THEOREM 10. *Let $0\leqq\delta<1$ and f be a nonnegative measurable function on R^n_+ such that $G^f_\alpha\not\equiv\infty$ and $\int_{R^n_+} f(y)^p y_n^\beta dy<\infty$ for some real numbers $p>1$ and β.*

(i) *If $\beta+\delta p\geqq\alpha p-n>0$, then $x_n^{-\delta}G^f_\alpha(x)$ has nontangential limit zero at any $\xi\in\partial R^n_+-(E_\delta\cup B^{**}_{n-\alpha p+\beta+\delta p})$, where $B^{**}_d=B_d$ when $d>0$ and B^{**}_d is empty when $d\leqq 0$.*

(ii) *If $\alpha p\leqq n$ and $n-\alpha p+\beta+\delta p\geqq 0$, then for each $\xi\in\partial R^n_+-(E_\delta\cup A^*_{1,\delta})$ there exists a set $E\subset R^n_+$ such that E is (α, p)-thin at ξ (relative to T_1) and*

$$\lim_{x\to\xi, x\in T_1(\xi,a)-E} x_n^{-\delta}G^f_\alpha(x) = 0 \qquad \text{for any } a>0.$$

Similar results can be obtained in case $\delta=1$.

5. Further results and remarks

Let D be a special Lipschitz domain as defined in Stein [11; Chap. VI]. Then similar results can be shown to hold for U^f_α with a nonnegative measurable function f on R^n such that

$$\int_{R^n} f(y)^p d(y)^\beta dy < \infty, \quad p>1, \beta<p-1, \tag{7}$$

if we replace $T_\gamma(\xi, a)$ by $\{x\in D;\ |x-\xi|<ad(x)^{1/\gamma}\}$. Here $d(y)$ denotes the distance from y to the boundary ∂D.

Let m be a positive integer and u be an (m, p)-quasi continuous function (see [7]) such that

$$\sum_{|\lambda|=m}\int_D |D^\lambda u(x)|^p d(x)^\beta dx < \infty,$$

where $D^\lambda=(\partial/\partial x_1)^{\lambda_1}\cdots(\partial/\partial x_n)^{\lambda_n}$ for a multi-index $\lambda=(\lambda_1,\dots,\lambda_n)$ with length

$|\lambda|=\lambda_1+\cdots+\lambda_n$. If $p>1$ and $\beta<p-1$, then for each bounded open set G we can find functions $f_{\lambda,G}$ satisfying

$$\int_G |f_{\lambda,G}(y)|^p d(y)^\beta dy < \infty$$

such that

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} f_{\lambda,G}(y)dy$$

holds for $x \in G \cap D$ except for a set with $C_{m,p}$ capacity zero, where a_λ are constants (cf. [7]). Thus one can discuss the boundary behavior of u by similar methods as above; one need take into account the following exceptional sets:

$$\left\{x \in G \cap \partial D;\ \int |x-y|^{m-n}|f_{\lambda,G}(y)|dy = \infty\right\},$$

which has $B_{m-\beta/p,p}$ capacity zero as will be shown in the Appendix.

For Green potentials in D, we refer to Aikawa [1], in which finely non-tangential limits of Green potentials are discussed.

6. Appendix

Here we show that $B_{\alpha-\beta/p,p}(\{x \in \partial D;\ U_\alpha^f(x)=\infty\})=0$ if f is a nonnegative measurable function on R^n satisfying (7). Set $A=\{x \in \partial D;\ U_\alpha^f(x)=\infty\}$. If $\beta \leqq 0$, then A is included in

$$A' = \left\{x \in \partial D;\ \int_{B(x,1)} |x-y|^{\alpha-\beta/p-n}[f(y)d(y)^{\beta/p}]dy = \infty\right\}.$$

Since $B_{\alpha-\beta/p,p}(A')=0$ by assumption (7), we have $B_{\alpha-\beta/p,p}(A)=0$. If $\beta \geqq \alpha p-1$, then $B_{\alpha-\beta/p,p}(\partial D)=0$, so that $B_{\alpha-\beta/p,p}(A)=0$. Now assume that $0<\beta<[\min(\alpha, 1)]p-1$. By considering a Lipschitz transformation of D to R_+^n locally, we may assume further that D is the half space R_+^n.

Let g_α denote the Bessel kernel of order α (see [5]), and note

$$A = \left\{x \in \partial R_+^n;\ \int g_\alpha(x-y)f(y)dy = \infty\right\}.$$

We see that the function $G(\xi)=\int g_\alpha(\xi-y)f(y)dy$, $\xi \in \partial R_+^n$, belongs to the Lipschitz space $\Lambda_{\alpha-(\beta+1)/p}^{p,p}(\partial R_+^n)$ (cf. [11; Chap. VI, §4.3]). Let u be the Poisson integral of G with respect to R_+^n. By the fact in [11; p. 152] we have

$$\sum_{|\lambda|=m} \int_{R_+^n} |D^\lambda u(x)|^p x_n^{p(m-\alpha)+\beta} dx < \infty,$$

where m is a positive integer greater than $\alpha-(\beta+1)/p$. By [9; Theorem 2] we can find a set $B\subset\partial R_+^n$ such that u has a finite nontangential limit at any $\xi\in\partial R_+^n-B$ and $B_{\alpha-\beta/p,p}(B)=0$. Since $\lim_{x\to\xi,x\in R_+^n}u(x)=\infty$ for any $\xi\in A$, it follows that $A\subset B$, so that $B_{\alpha-\beta/p,p}(A)=0$.

References

[1] H. Aikawa, On the minimal thinness in a Lipschitz domain, preprint.
[2] M. Brelot, On topologies and boundaries in potential theory, Lecture Notes in Math. **175**, Springer, Berlin, 1971.
[3] L. Carleson, Selected problems on exceptional sets, Van Nostrand, Princeton, 1967.
[4] A. B. Cruzeiro, Convergence au bord pour les fonctions harmoniques dans R^d de la classe de Sobolev W_1^d, C. R. Acad. Sci. Paris **294** (1982), 71–74.
[5] N. G. Meyers, A theory of capacities for potentials in Lebesgue classes, Math. Scand. **26** (1970), 255–292.
[6] N. G. Meyers, Continuity properties of potentials, Duke Math. J. **42** (1975), 157–166.
[7] Y. Mizuta, Integral representations of Beppo Levi functions of higher order, Hiroshima Math. J. **4** (1974), 375–396.
[8] Y. Mizuta, On the radial limits of potentials and angular limits of harmonic functions, Hiroshima Math. J. **8** (1978), 415–437.
[9] Y. Mizuta, Existence of various boundary limits of Beppo Levi functions of higher order, Hiroshima Math. J. **9** (1979), 717–745.
[10] Y. Mizuta, Boundary limits of Green potentials of order α, Hiroshima Math. J. **11** (1981), 111–123.
[11] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.
[12] J.-M. G. Wu, L^p-densities and boundary behaviors of Green potentials, Indiana Univ. Math. J. **28** (1979), 895–911.

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