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The additive structure of $\tilde{K}(S^{4n+3}/Q_t)$

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§1. Introduction

Let t be a positive integer and let Q_t be the group of order $4t$ given by

$$Q_t = \{x, y: x^t = y^2, xyx = y\},$$

the group generated by two elements x and y with the relations $x^t = y^2$ and $xyx = y$, that is, Q_t is the subgroup of the unit sphere S^3 in the quaternion field H generated by the two elements

$$x = \exp(\pi \boldsymbol{i}/t) \quad \text{and} \quad y = \boldsymbol{j};$$

and $Q_1 = Z_4$ and Q_t for $t = 2^{m-1}$ ($m \geqq 2$) is the generalized quaternion group which is denoted by H_m in [4].

Then, Q_t acts on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action, and we have the quotient manifold

$$S^{4n+3}/Q_t \quad \text{of dimension} \quad 4n+3.$$

Some partial results on the reduced K-ring $\tilde{K}(S^{4n+3}/Q_t)$ of this manifold are obtained by [4], D. Pitt [14], T. Mormann [13] and K. Kojima. In this paper, we shall determine completely the additive structure of $\tilde{K}(S^{4n+3}/Q_t)$.

Consider the complex representations a_0, a_1 and b_1 of Q_t given by

$$\begin{cases} a_0(x) = 1, \\ a_0(y) = -1, \end{cases} \quad \begin{cases} a_1(x) = -1, \\ a_1(y) = \begin{cases} i & \text{if } t \text{ is odd,} \\ 1 & \text{if } t \text{ is even,} \end{cases} \end{cases} \quad \begin{cases} b_1(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \\ b_1(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{cases}$$

and the elements

(1.1) $\qquad \alpha_i = \xi(a_i - 1), \quad \beta_1 = \xi(b_1 - 2) \quad \text{in} \quad \tilde{K}(S^{4n+3}/Q_t) \qquad$ (cf. (3.3)),

where ξ is the natural ring homomorphism of the representation ring of Q_t to $\tilde{K}(S^{4n+3}/Q_t)$. Furthermore, consider the following subgroups of Q_t:

(1.2) $\qquad G_0 = Q_r$ generated by x^q and y, $\qquad G_1 = Z_q$ generated by x^{2r},

where $t = rq$, $r = 2^{m-1}$, $m \geqq 1$ and q is odd. Then, we have the ring homomorphisms

$$(1.3)\quad \begin{aligned} i_0^*&: \tilde{K}(S^{4n+3}/Q_t) \longrightarrow \tilde{K}(S^{4n+3}/Q_r),\\ i_1^*&: \tilde{K}(S^{4n+3}/Q_t) \longrightarrow \tilde{K}(L^{2n+1}(q)) \quad (L^{2n+1}(q)=S^{4n+3}/Z_q), \end{aligned}$$

induced from the natural projections i_k: $S^{4n+3}/G_k \to S^{4n+3}/Q_t$. Let

$$(1.4)\quad c\colon \widetilde{KO}(L_0^{2n+1}(q)) \longrightarrow \tilde{K}(L_0^{2n+1}(q)) = \tilde{K}(L^{2n+1}(q))$$

be the complexification, where $\widetilde{KO}(\ )$ is the reduced KO-ring and $L_0^{2n+1}(q)$ is the $(4n+2)$-skeleton of $L^{2n+1}(q)$.

Then, we have the following

THEOREM 1.5. (i) *The ring* $\tilde{K}(S^{4n+3}/Q_t)$ *is generated by the elements* α_1 *when* $t=1$, α_1 *and* β_1 *when* t *is odd* $\geqq 3$, α_0, α_1 *and* β_1 *when* t *is even, respectively, where* α_i *and* β_1 *are the ones in* (1.1).

(ii) *Put* $t=rq$ *where* $r=2^{m-1}$, $m\geqq 1$ *and* q *is odd. Then, the ring isomorphism*

$$\pi = \pi_0 \oplus \pi_1\colon \tilde{K}(S^{4n+3}/Q_t) \cong \tilde{K}(S^{4n+3}/Q_r) \oplus \widetilde{KO}(L_0^{2n+1}(q))$$

can be defined by

$$\pi_0 = i_0^* \quad and \quad \pi_1 = c^{-1}\circ i_1^*$$

by using i_k^* *in* (1.3) *and the monomorphism* c *in* (1.4). *Further, for the generators* α_i *and* β_1 *in* $\tilde{K}(S^{4n+3}/Q_t)$ *or* $\tilde{K}(S^{4n+3}/Q_r)$, *there hold the equalities*

$$\pi(\alpha_i) = \alpha_i, \quad \pi(\beta_1) = \begin{cases} \alpha_1^3 + 3\alpha_1^2 + 4\alpha_1 + \bar{\sigma} & \text{if } t \text{ is odd,}\\ \beta_1 + \bar{\sigma} & \text{if } t \text{ is even,} \end{cases}$$

where $\bar{\sigma}$ *is the real restriction of the stable class* $\eta-1$ *of the canonical complex line bundle* η *over* $L_0^{2n+1}(q)$ *and it generates the ring* $\widetilde{KO}(L_0^{2n+1}(q))$.

Consider the following integers $u(i)$ and elements δ_i and $\bar{\alpha}_1$ in $\tilde{K}(S^{4n+3}/Q_r)$ with $r=2^{m-1}$ $(m\geqq 2)$, where α_i and β_1 are the ones in (1.1) for $t=r$ and

$$\beta(0) = \beta_1, \quad \beta(s) = \beta(s-1)^2 + 4\beta(s-1) \quad (s\geqq 1):$$

For $i=2^s+d\leqq N'=\min\{r, n\}$ with $0\leqq s<m$ and $0\leqq d<2^s$, put

$$n' = 2n+1 \quad \text{if } n \text{ is odd,} \quad = 2n \quad \text{if } n \text{ is even,}$$

$$n' = 2^s a'_s + b'_s, \quad 0 \leqq b'_s < 2^s;$$

$$u(1) = 2^{m-1+2a'_1}, \quad \delta_1 = \beta_1 \quad \text{if } i = 1;$$

$$u(i) = 2^{m-s-2+a'_s}, \quad \delta_i = \beta(s) + \textstyle\sum_{t=1}^{s} 2^{(2^t-1)(a'_s+1)}\beta(s-t)$$

$$\text{if } i = 2^s,\ 1 \leqq s < m;$$

(1.6)
$$\begin{cases} u(i)=2^{m-s-3+a(i)}, \qquad a(i)=\begin{cases} a'_{s+1}+1 & \text{for } 2d\leqq b'_{s+1}, \\ a'_{s+1} & \text{for } 2d>b'_{s+1}, \end{cases} \\ \delta_i=\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)) - 2^{a(i)-1}\beta_1^d\beta(s) \\ \qquad + \sum_{t=2}^{s+1}2^{(2^t-1)a(i)-1}\beta_1^d\beta(s+1-t) \quad \text{if } i=2^s+d\geqq 3,\ d\geqq 1; \\ \bar{\alpha}_1=\alpha_1-2\sum_{s=1}^{m-3}\beta(s)\prod_{t=s+1}^{m-3}(2+\beta(t)). \end{cases}$$

Then, the additive structure of $\tilde{K}(S^{4n+3}/Q_r)$ is given by the following theorem where $Z_k\langle x\rangle$ denotes the cyclic group of order k generated by x:

THEOREM 1.7. *Let $r=2^{m-1}$, $m\geqq 2$ and $N'=\min\{r, n\}$. Then, we have*

$$\tilde{K}(S^{4n+3}/Q_r) = Z_{2^{n+1}}\langle\alpha_0\rangle\oplus Z_{2^{n+1}}\langle\bar{\alpha}_1\rangle\oplus B^n(m), \quad B^n(m) = \sum_{i=1}^{N'} Z_{u(i)}\langle\delta_i\rangle,$$

where $B^n(m)$ is the subring of $\tilde{K}(S^{4n+3}/Q_r)$ generated by β_1, which is isomorphic to the subring of $\widetilde{KO}(L^{n'}(2^m))$ generated by $\bar{\sigma}$ by sending β_1 to $\bar{\sigma}$.

We notice that the additive structure of $\tilde{K}(S^{4n+3}/Q_1)=\tilde{K}(L^{2n+1}(4))$ is determined in [10, Th. A].

For the reduced KO-group $\widetilde{KO}(L_0^{2n+1}(q))$ (q: odd) in Theorem 1.5 (ii), it is sufficient to determine its additive structure in case when q is a power of an odd prime (cf. (6.1)).

Let p be an odd prime and $r\geqq 1$, and consider the elements

$$(1.8)\qquad \bar{\sigma}'(s) = \sum_{i=0}^{q(s)}(p^s/(2i+1))\binom{q(s)+i}{2i}\bar{\sigma}^i \quad \text{in } KO(L_0^n(p^r)) \quad (0\leqq s\leqq r),$$

where $q(s)=(p^s-1)/2$ and $\bar{\sigma}$ is the one given in Theorem 1.5 (ii). ($\bar{\sigma}'(s)$ is well defined as an integral polynomial in $\bar{\sigma}$ because the order of $\bar{\sigma}^i$ is a power of p by [9, Th. 1.1 (ii) and Prop. 2.11 (ii)].) Furthermore, consider the following integers $t(2i)$ and elements $\bar{\sigma}(s, k)$ in $\widetilde{KO}(L_0^n(p^r))$, where $0\leqq s<r$, $0\leqq k<p^s(p-1)/2$ and $i=q(s)+k+1\leqq[N/2]$ ($N=\min\{p^r-1, n\}$):

$$n-p^s+1=a_sp^s(p-1)+b_s, \quad 0\leqq b_s<p^s(p-1);$$

$$t(2i)=p^{r-s+1+\bar{a}_s}, \quad \bar{a}_s=\begin{cases} a_s+1 & \text{if } 2k+1<b_s, \\ a_s & \text{if } 2k+1\geqq b_s, \end{cases}$$

$$(1.9)\qquad \bar{\sigma}(s, k)=\begin{cases} \sum_{t=0}^{s}p^{(p^t-1)\bar{a}_s}\bar{\sigma}^{q(t)+k+1}\bar{\sigma}'(s-t)^{p^t} \\ \quad \text{if } b_s\leqq 2k+1<b_s+p^s-1 \text{ or } 2k+1<b_s-p^s(p-2)-1, \\ \bar{\sigma}^{k+1}\bar{\sigma}'(s) \quad \text{otherwise.} \end{cases}$$

Then, we have the following

THEOREM 1.10. *Let p be an odd prime and $r \geqq 1$. Then the additive structure of $\widetilde{KO}(L_0^n(p^r))$ is given by*

$$\widetilde{KO}(L_0^n(p^r)) = \sum_{i=1}^{[N/2]} Z_{t(2i)}\langle \bar{\sigma}(s, k)\rangle,$$

where $N=\min\{p^r-1, n\}$, $i=(p^s+2k+1)/2$ and $0 \leqq k < p^s(p-1)/2$.

We prepare some results on the complex representation rings $R(Q_t)$ and $R(G_k)$ for Q_t and the subgroups G_k given in (1.2) in §2. In §3, we define the elements $\alpha_i\,(i=0, 1, 2)$ and $\beta_j\,(j \in Z)$ of $\widetilde{K}(S^{4n+3}/Q_t)$ and study the homomorphism $i_k^*\colon \widetilde{K}(S^{4n+3}/Q_t) \to \widetilde{K}(S^{4n+3}/G_k)$ of (1.3) in Proposition 3.10. In §4, we first determine the order of $\widetilde{K}(S^{4n+3}/Q_t)$ by using the Atiyah-Hirzebruch spectral sequence, and prove Theorem 1.5 in Theorem 4.7 by using the known results on c in (1.4) given in [9, Prop. 2.11] and the ones obtained in §3.

In §5, we study the subring $B^n(m)$ of $\widetilde{K}(S^{4n+3}/Q_r)$ $(r=2^{m-1}, m \geqq 2)$ generated by β_1 using the ring monomorphism $f\colon B^n(m) \to \widetilde{KO}(L_0^{n\prime}(2r))$ of Lemma 5.10 and the additive structure of $\widetilde{KO}(L_0^{n\prime}(2r))$ given in [5, Th. 1.9], and prove Theorem 1.7 by showing some relations in $\widetilde{K}(S^{4n+3}/Q_r)$. Theorem 1.10 is proved in §6 by using the additive structure of $\widetilde{K}(L_0^n(p^r))$ given in [11, Th. 1.7] and the complexification $c\colon \widetilde{KO} \to \widetilde{K}$ which is monomorphic for $L_0^n(p^r)$.

§2. The complex representation ring $R(Q_t)$

Let t be a positive integer and let Q_t be the subgroup of order $4t$ of the unit sphere S^3 in the quaternion field H generated by the two elements

$$x = \exp(\pi \boldsymbol{i}/t) \quad \text{and} \quad y = \boldsymbol{j}.$$

Consider the complex representations a_i $(i=0, 1, 2)$ and b_j $(j \in Z)$ of Q_t given by

$$(2.1) \quad \begin{cases} a_0(x)=1, \\ a_0(y)=-1, \end{cases} \qquad \begin{cases} a_i(x)=-1, \\ a_i(y)=\begin{cases} (-1)^{i-1}\boldsymbol{i} & \text{if } t \text{ is odd}, \\ (-1)^{i-1} & \text{if } t \text{ is even}, \end{cases} \end{cases} \qquad \begin{cases} b_j(x)=\begin{pmatrix} x^j & 0 \\ 0 & x^{-j} \end{pmatrix}, \\ b_j(y)=\begin{pmatrix} 0 & (-1)^j \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Then, we see easily the following

PROPOSITION 2.2 (cf. [3, §47.15, Example 2]). *The complex representation ring $R(Q_t)$ of Q_t is a free Z-module generated by* 1, a_i $(i=0, 1, 2)$ *and* b_j $(1 \leqq j < t)$, *and the multiplicative structure is given as follows:*

$$a_0^2=1,\quad a_1^2=\begin{cases} a_0 & \text{if } t \text{ is odd,} \\ 1 & \text{if } t \text{ is even,}\end{cases}\quad a_2=a_0a_1,\quad b_0=1+a_0,\quad b_t=a_1+a_2,$$

$$b_{t+i}=b_{t-i},\quad b_{-i}=b_i,\quad b_ib_j=b_{i+j}+b_{i-j},\quad a_0b_i=b_i,\quad a_1b_i=b_{t-i}.$$

Let

(2.3) $$\alpha_i=a_i-1\ (i=0,1,2) \quad \text{and} \quad \beta_j=b_j-2\ (j\in Z)$$

be the elements in the reduced representation ring $\tilde{R}(Q_t)$. Then, we have

PROPOSITION 2.4 (cf. [4, Prop. 3.3]). *The reduced representation ring $\tilde{R}(Q_t)$ is a free Z-module generated by α_i $(i=0, 1, 2)$ and β_j $(1\leqq j<t)$, and the multiplicative structure is given as follows:*

$$\alpha_0^2=-2\alpha_0,\quad \alpha_1^2=\begin{cases} \alpha_0-2\alpha_1 & \text{if } t \text{ is odd,} \\ -2\alpha_1 & \text{if } t \text{ is even,}\end{cases}\quad \alpha_2=\alpha_0\alpha_1+\alpha_0+\alpha_1,$$

$$\beta_0=\alpha_0,\quad \beta_t=\alpha_1+\alpha_2,\quad \beta_{t+i}=\beta_{t-i},\quad \beta_{-i}=\beta_i,$$

$$\beta_i\beta_j=\beta_{i+j}+\beta_{i-j}-2(\beta_i+\beta_j),\quad \alpha_0\beta_i=-2\alpha_0,\quad \alpha_1\beta_i=\beta_{t-i}-\beta_i-2\alpha_1.$$

These show that the ring $\tilde{R}(Q_t)$ is generated by α_1 if $t=1$, α_1 and β_1 if t is odd $\geqq 3$, and α_0, α_1 and β_1 if t is even.

The following lemmas are well known:

LEMMA 2.5 (cf. [7, Ch. 13, Th. 3.1]). *$R(S^3)$ is the polynomial ring $Z[\zeta]$, where ζ is given by*

$$\zeta(z_1+jz_2)=\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \quad \text{for } z_1+jz_2\in S^3.$$

LEMMA 2.6 (cf. [1, §8]). *$R(Z_k)$ is the truncated polynomial ring $Z[\mu]/\langle\mu^k-1\rangle$, where μ is given by $z\mapsto\exp(2\pi i/k)$ for the generator z of Z_k and $\langle\mu^k-1\rangle$ means the ideal of $Z[\mu]$ generated by μ^k-1.*

Consider the following three subgroups G_k of Q_t, where

$$t=rq,\ r=2^{m-1},\ m\geqq 1 \text{ and } q \text{ is odd:}$$

(2.7)
$$G_0=Q_r \text{ generated by } x^q \text{ and } y,$$
$$G_1=Z_q \text{ generated by } x^{2r},\quad G_2=Z_{2r} \text{ generated by } x^q.$$

Then the inclusion $i_k\colon G_k\subset Q_t$ induces the ring homomorphism

(2.8) $$i_k^*\colon \tilde{R}(Q_t)\longrightarrow \tilde{R}(G_k)$$

by the restriction of representations of Q_t to G_k. By the definitions (2.1) and (2.3), Proposition 2.4 and Lemma 2.6, we see easily the following

PROPOSITION 2.9. (i) $i_0^*(\alpha_i) = \alpha_i$ $(i=0, 1, 2)$,

$$\begin{cases} i_0^*(\beta_{2i}) = \alpha_0, \quad i_0^*(\beta_{2i+1}) = \alpha_1+\alpha_2 & \text{if } t \text{ is odd,} \\ i_0^*(\beta_i) \ = \beta_i & \text{if } t \text{ is even.} \end{cases}$$

(ii) $i_1^*(\alpha_i) = 0$, $i_1^*(\beta_i) = \mu^i+\mu^{-i}-2$.

(iii) $i_2^*(\alpha_0) = 0$, $i_2^*(\alpha_i) = \mu^r-1$ $(i=1, 2)$, $i_2^*(\beta_i) = \mu^i+\mu^{-i}-2$.

§3. Some elements in $\tilde{K}(S^{4n+3}/Q_t)$

Assume that a topological group G acts freely on a topological space X. Then, the natural projection

$$p: X \longrightarrow X/G$$

defines the ring homomorphism

$$\xi: \tilde{R}(G) \longrightarrow \tilde{K}(X/G) \tag{3.1}$$

as follows (cf. [7, Ch. 12, 5.4]): For an n-dimensional representation ω of G, $\xi(\omega)$ is the complex n-plane bundle induced from the principal G-bundle $p: X \to X/G$ by the group homomorphism $\omega: G \to GL(n, C)$. Furthermore, if H is a subgroup of G, then the inclusion $i: H \subset G$ and the natural projections $p': X \to X/H$, $i: X/H \to X/G$ induce the commutative diagram

$$\begin{array}{ccc} \tilde{R}(G) & \xrightarrow{\xi} & \tilde{K}(X/G) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \tilde{R}(H) & \xrightarrow{\xi} & \tilde{K}(X/H)\,. \end{array} \tag{3.2}$$

Now, Q_t acts on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action

$$q(q_1,\dots, q_{n+1})=(qq_1,\dots, qq_{n+1}) \qquad \text{for} \quad q\in Q_t\subset S^3,\ q_i\in H.$$

Then the natural projection $S^{4n+3}\to S^{4n+3}/Q_t$ defines the ring homomorphism

$$\xi: \tilde{R}(Q_t) \longrightarrow \tilde{K}(S^{4n+3}/Q_t)$$

of (3.1), and by using the same letter, we define the elements

$$\alpha_i=\xi(\alpha_i)\ (i=0, 1, 2), \quad \beta_j=\xi(\beta_j)\ (j\in Z) \quad \text{in } \tilde{K}(S^{4n+3}/Q_t), \tag{3.3}$$

where α_i, $\beta_j \in \tilde{R}(Q_t)$ are the ones given in (2.3).

The K-ring $K(HP^n)$ of the quaternion projective space $HP^n = S^{4n+3}/S^3$ is given by

(3.4) (cf. [15, Th. 3.12]) $K(HP^n) = Z[\nu]/\langle \nu^{n+1} \rangle$,

where $\nu = \lambda - 2$ *and* λ *is the canonical complex plane bundle over* HP^n.

For the ring homomorphism $\xi\colon \tilde{R}(S^3) \to \tilde{K}(HP^n)$ of (3.1), by the definition of ζ in Lemma 2.5 and ν in (3.4), we see easily the following

LEMMA 3.5 (cf. [7, Ch. 13, Th. 3.1]). $\xi(\zeta - 2) = \nu$.

LEMMA 3.6 (cf. [4, Lemma 4.4]). $\pi^*(\nu) = \beta_1$,

where $\pi^*\colon \tilde{K}(HP^n) \to \tilde{K}(S^{4n+3}/Q_t)$ *is the homomorphism induced from the natural projection* $\pi\colon S^{4n+3}/Q_t \to HP^n$.

PROOF. We can prove the desired equality by (3.2), (2.1), (3.3) and Lemmas 2.5–6 in the same way as the proof of Lemma 4.4 in [4]. q. e. d.

The K-ring $K(L^n(k))$ of the standard lens space $L^n(k) = S^{2n+1}/Z_k$ mod k is given by

(3.7) (N. Mahammed [12]) $K(L^n(k)) = Z[\sigma]/\langle \sigma^{n+1}, (\sigma+1)^k - 1 \rangle$,

where $\sigma = \eta - 1$ *and* η *is the canonical complex line bundle over* $L^n(k)$.

For $\xi\colon \tilde{R}(Z_k) \to \tilde{K}(L^n(k))$ of (3.1), we have

LEMMA 3.8. $\xi(\mu - 1) = \eta - 1$.

PROOF. Since the first Chern class of η generates $H^2(L^n(k)) = Z_k$, we have the desired equality by the definition of η in Lemma 2.6 (cf. [1, §2 and Appendix, (3)]). q. e. d.

Let $i_k\colon S^{4n+3}/G_k \to S^{4n+3}/Q_t$ be the natural projection induced from the inclusion $i_k\colon G_k \subset Q_t$ for the subgroup G_k ($k = 0, 1, 2$) in (2.7). Then the induced homomorphism

$$i_k^*\colon \tilde{K}(S^{4n+3}/Q_t) \longrightarrow \tilde{K}(S^{4n+3}/G_k) \tag{3.9}$$

satisfies the following

PROPOSITION 3.10. *The equalities in Proposition* 2.9 *hold by replacing* α_i *and* β_j *with* α_i *and* β_j *in* (3.3) *and* μ *with* η *in* (3.7) *when* $k = 1, 2$.

PROOF. By using (3.2), Proposition 2.9, (3.3), (2.6) and Lemma 3.8, we obtain the desired equalities in each case. q. e. d.

§4. Proof of Theorem 1.5

The cohomology group of the quotient manifold $X=S^{4n+3}/Q_t$ is given as follows:

(4.1) (cf. [2, Ch. XII, §7]) $H^{4i}(X; Z)=Z_{4t}$ *if* $0<i\leqq n$,

$H^{4i+2}(X; Z)=Z_4$ (t: *odd*), $=Z_2\oplus Z_2$ (t: *even*) *if* $0\leqq i\leqq n$,

$H^{2i+1}(X; Z)=0$ *if* $0\leqq i\leqq 2n$, $H^0(X; Z)=H^{4n+3}(X; Z)=Z$.

By (4.1) and the Atiyah-Hirzebruch spectral sequence for $K(X)$, we have

LEMMA 4.2. $\#\tilde{K}(S^{4n+3}/Q_t)=2^{4n+2}t^n$,

where $\#A$ *denotes the order of a group* A.

We prepare two lemmas for the proof of Theorem 1.5. Put

$$t=rq, \text{ where } r=2^{m-1},\ m\geqq 1 \text{ and } q \text{ is an odd integer.}$$

Then, we have the following

LEMMA 4.3. *i_0^*: $\tilde{K}(S^{4n+3}/Q_t)\to\tilde{K}(S^{4n+3}/Q_r)$ is epimorphic, where i_0^* is the homomorphism in* (3.9) *for* $G_0=Q_r$.

PROOF. By Proposition 3.10, $i_0^*(\alpha_i)=\alpha_i$ $(i=0, 1)$ and $i_0^*(\beta_1)=\beta_1$ hold. On the other hand, the ring $\tilde{K}(S^{4n+3}/Q_r)$ is generated by α_0, α_1 and β_1 by [4, Th. 1.1]. Thus, we have the desired result. q.e.d.

Consider the homomorphism

$$(4.4)\qquad \xi\colon \tilde{R}(Q_t)\longrightarrow\tilde{K}(S^{4n+3}/Q_t)$$

of (3.1) for the natural projection $S^{4n+3}\to S^{4n+3}/Q_t$, and set

$$R=\operatorname{Im}\xi.$$

Then, concerning with the homomorphism

$$i_1^*\colon \tilde{K}(S^{4n+3}/Q_t)\longrightarrow\tilde{K}(L^{2n+1}(q))\quad (L^{2n+1}(q)=S^{4n+3}/Z_q)$$

in (3.9) for $G_1=Z_q$, we have the following

LEMMA 4.5. $i_1^*(R)=\operatorname{Im}(c\colon \widetilde{KO}(L_0^{2n+1}(q))\to\tilde{K}(L_0^{2n+1}(q))=\tilde{K}(L^{2n+1}(q)))$,

where c is the complexification and $L_0^k(q)$ is the $2k$-skeleton of $L^k(q)$.

PROOF. By (3.3) and Proposition 3.10, we have the equalities

$$i_1^*(\beta_i)=\eta^i+\eta^{-i}-2=c(r(\eta^i-1)),\quad i_1^*(\alpha_j)=0\ (j=0, 1, 2);$$

while the ring $\widetilde{KO}(L_0^{2n+1}(q))$ is generated by $r(\eta^i-1)$ $(i\geqq 1)$, where $r\colon \tilde{K}\to\widetilde{KO}$ is the real restriction and is epimorphic for $L_0^{2n+1}(q)$ (q: odd), (cf. [9, Prop. 2.11]). Therefore, we obtain the desired result by the first half of Proposition 2.4.

q. e. d.

Now, we consider the ring homomorphism

$$(4.6)\qquad \pi=\pi_0\oplus\pi_1\colon R(=\operatorname{Im}\xi)\longrightarrow \tilde{K}(S^{4n+3}/Q_r)\oplus\widetilde{KO}(L_0^{2n+1}(q))$$
$$\text{given by } \pi_0 = i_0^*|R \text{ and } \pi_1 = c^{-1}\circ(i_1^*|R),$$

where i_0^* is the one in Lemma 4.3 and π_1 is defined by the above lemma since the complexification c in that place is monomorphic for odd q (cf. [9, Prop. 2.11]).

THEOREM 4.7. (i) *ξ in* (4.4) *is an epimorphism and $R=\tilde{K}(S^{4n+3}/Q_t)$.*

(ii) *Let $t=rq$, $r=2^{m-1}$, $m\geqq 1$ and q is odd. Then π in* (4.6) *is a ring isomorphism*

$$\pi=\pi_0\oplus\pi_1\colon \tilde{K}(S^{4n+3}/Q_t)\cong \tilde{K}(S^{4n+3}/Q_r)\oplus\widetilde{KO}(L_0^{2n+1}(q)).$$

PROOF. In (4.6), π_0 is epimorphic by (3.3) and the proof of Lemma 4.3, and so is π_1 by Lemma 4.5. On the other hand, by Lemma 4.2 and [9, Prop. 2.11],

$$\#\tilde{K}(S^{4n+3}/Q_r) = 2^{(m+3)n+2} \quad\text{and}\quad \#\widetilde{KO}(L_0^{2n+1}(q)) = q^n.$$

Therefore π in (4.6) is also epimorphic since q is odd, and we see the theorem because $\#R\leqq\#\tilde{K}(S^{4n+3}/Q_t)=2^{(m+3)n+2}q^n$ by Lemma 4.2. q. e. d.

REMARK 4.8. By the definition of π in (4.6), Proposition 3.10 and the proof of Lemma 4.5, *we have the following equalities for π in the above theorem*:

$$\pi(\alpha_i) = \alpha_i\ (i=0,1,2),$$
$$\begin{cases} \pi(\beta_{2i}) = \alpha_0+r(\eta^{2i}-1), \\ \pi(\beta_{2i+1}) = \alpha_1+\alpha_2+r(\eta^{2i+1}-1) & \text{if } t \text{ is odd}, \end{cases}$$
$$\pi(\beta_i) = \beta_i+r(\eta^i-1) \qquad \text{if } t \text{ is even}.$$

REMARK 4.9. By (3.3) and Theorem 4.7 (i), *the relations in Proposition* 2.4 *hold in $\tilde{K}(S^{4n+3}/Q_t)$ and so the ring $\tilde{K}(S^{4n+3}/Q_t)$ is generated by α_1 if $t=1$, α_1 and β_1 if t is odd$\geqq 3$, and α_0, α_1 and β_1 if t is even.*

Combining Theorem 4.7 (ii) with the above remarks, we complete the proof of Theorem 1.5.

§5. The group $\tilde{K}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$

In this section, we shall determine the additive structure of $\tilde{K}(S^{4n+3}/Q_r)$ for $r=2^{m-1}$ with $m\geqq 2$ by giving an additive base. In case $m=1$, $\tilde{K}(S^{4n+3}/Q_1)=\tilde{K}(L^{2n+1}(4))$ and its additive structure is given in [10, Th. A]. The results in case $m=2$ is given in [4, Th. 1.2]. For $m=3$, T. Mormann [13] and Kazuyoshi Kojima have determined its additive structure.

Let $m\geqq 2$ and, in addition to the elements α_i and β_j in $\tilde{K}(S^{4n+3}/Q_r)$ of (3.3), define $\beta(s)$ in $\tilde{K}(S^{4n+3}/Q_r)(r=2^{m-1})$ inductively as follows:

(5.1) $$\beta(0) = \beta_1, \quad \beta(s) = \beta(s-1)^2 + 4\beta(s-1) \quad (s\geqq 1).$$

Then, we have the relations in $\tilde{K}(S^{4n+3}/Q_r)$ given by the following lemmas.

LEMMA 5.2. $$\beta_{2^s} = \beta(s) + (-1)^{2^{s-1}}\alpha_0 \quad (s\geqq 1).$$

PROOF. By noticing Remark 4.9, we can show $\alpha_0\beta(1)=-4\alpha_0$, $\alpha_0\beta(s)=0$ $(s\geqq 2)$ and the equality in the lemma inductively using the relations in Proposition 2.4. q.e.d.

LEMMA 5.3. $\beta_{r-1} - \beta_1 = \sum_{s=1}^{m-2}\{(2+\beta_1)\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t))\}$.

PROOF. In $R(Q_r)$, the relation $b_{2i-1}=b_ib_{i-1}-b_1$ for $i=2^{s-1}$ $(s\geqq 1)$ holds by Proposition 2.2, and so we have

$$b_{r-1}=b_1\{b_2\prod_{t=2}^{m-2}b_{2^t}-\sum_{s=1}^{m-2}\prod_{t=s+1}^{m-2}b_{2^t}\}=b_1+\sum_{s=1}^{m-2}b_1(b_{2^s}-2)\prod_{t=s+1}^{m-2}b_{2^t}.$$

Therefore, by (2.3), Lemma 5.2 and the relation $(2+\beta_1)\alpha_0=0$ in Proposition 2.4, we have

$$\begin{aligned}\beta_{r-1}-\beta_1 &= \sum_{s=1}^{m-2}(2+\beta_1)(\beta(s)+(-1)^{2^{s-1}}\alpha_0)\prod_{t=s+1}^{m-2}(\beta(t)+\alpha_0+2)\\ &= \sum_{s=1}^{m-2}(2+\beta_1)\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t)).\end{aligned}$$
q.e.d.

LEMMA 5.4 $$(2+\beta_1)\alpha_0 = 0,\quad (2+\beta_1)\alpha_1 = \beta_{r-1} - \beta_1,$$
$$(2+\beta_1)\beta(m-1) = 2(\beta_{r-1}-\beta_1),\quad \beta_1^{n+1} = 0.$$

PROOF. The first two follow from Proposition 2.4 and Remark 4.9. The third one is shown as follows:

$$\begin{aligned}(2+\beta_1)\beta(m-1) &= (2+\beta_1)(\beta_r-(-1)^{r/2}\alpha_0) \quad &&\text{(by Lemma 5.2)}\\ &= (2+\beta_1)\beta_r = 2(\beta_{r-1}-\beta_1) &&\text{(by Proposition 2.4).}\end{aligned}$$

The last one follows from (3.4) and Lemma 3.6. q.e.d.

LEMMA 5.5. *Let $P(x)$ be a polynomial in x with*

$$P(x) = ax + \textit{higher terms, where } a \textit{ is a positive integer,}$$

and $B(n, P)$ $(n \geqq 0)$ *be the ring generated by* x *with the two relations* $x^{n+1}=0$ *and* $P(x)=0$. *Then,* $\# B(n, P)=a^n$.

PROOF. We can prove the equality inductively by noticing that $B(0, P)=0$ and by showing that

$$(*) \qquad \mathrm{Ker}(p_n\colon B(n, P) \longrightarrow B(n-1, P)) = Z_a\langle x^n\rangle$$

for the natural ring epimorphism p_n given by $p_n(x)=x$.

If $p_n(y)=0$ for $y \in B(n, P)$, then $y=Q_1(x)x^n+Q_2(x)P(x)$ for some polynomials Q_i by definition, which shows that $y=kx^n$ in $B(n, P)$ for some $k \in Z$. On the other hand, $ax^n=P(x)x^{n-1}=0$ in $B(n, P)$ by definition. Conversely, if $kx^n=0$ $(k\in Z)$ in $B(n, P)$, then $kx^n=R_1(x)x^{n+1}+R_2(x)P(x)=ak'x^n$ for some polynomials R_i and some $k' \in Z$, which shows that $k\equiv 0 \mod a$. Thus we see $(*)$. q.e.d.

LEMMA 5.6. *Let* $B^n(m)$ *be the subring of* $\tilde{K}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ *generated by* β_1. *Then*

$$\# B^n(m) \leqq (4r)^n.$$

PROOF. Since $\beta(s)=2^{2s}\beta_1+$higher terms by (5.1), we see that the polynomial $P'(\beta_1)$ in β_1 given by the right hand side in Lemma 5.3 is $2^m(2^{m-2}-1)\beta_1+$higher terms. Consider the polynomial $P(\beta_1)$ in β_1 given by

$$P(\beta_1) = (2+\beta_1)\beta(m-1) - 2P'(\beta_1) = 4r\beta_1 + \text{higher terms.}$$

Then, by the definitions of $B(n, P)$ and $B^n(m)$, the equality in Lemma 5.3 and the last two ones in Lemma 5.4 show that a ring epimorphism $B(n, P)\to B^n(m)$ is defined by sending the generator x to β_1. Thus we see the lemma by the above lemma. q.e.d.

For a given integer n, put

$$(5.7) \qquad n' = 2n+1 \quad \text{if } n \text{ is odd,} \qquad = 2n \quad \text{if } n \text{ is even,}$$

and consider the ring monomorphism

$$c'\colon \widetilde{KO}(L^{n'}(2r)) \longrightarrow \tilde{K}(L^{2n+1}(2r)) \qquad (r=2^{m-1},\ m\geqq 2)$$

given by $c'=c_3$ if n is odd, $=c_0$ if n is even, where $c_3=c$ and c_0 are the ones defined in [5, Prop. 5.3] by modifying the complexification c. Furthermore, consider the ring homomorphism

$$i_2^*\colon \tilde{K}(S^{4n+3}/Q_r) \longrightarrow \tilde{K}(L^{2n+1}(2r)) \quad \text{in (3.9)}.$$

Then, by [5, Proof of Cor. 5.16] and Proposition 3.10, we have

(5.8) $$c'(\bar{\sigma}) = \eta + \eta^{-1} - 2 = i_2^*(\beta_1),$$

where $\bar{\sigma}$ is the real restriction of $\sigma=\eta-1$ in (3.7). Therefore, we can define the ring epimorphism

(5.9) $$f = c'^{-1}\circ i_2^* \colon B^n(m) \longrightarrow R^{n'}(m) \quad \text{with} \quad f(\beta_1) = \bar{\sigma},$$

where $B^n(m)$ is the subring of $\tilde{K}(S^{4n+3}/Q_r)$ generated by β_1 and $R^{n'}(m)$ is the one of $\widetilde{KO}(L^{n'}(2r))$ generated by $\bar{\sigma}$.

Lemma 5.10. *f is a ring isomorphism, $\#B^n(m)=(4r)^n$ and $f(\beta(s))=\bar{\sigma}(s)$, where $\bar{\sigma}(s)\in\widetilde{KO}(L^{n'}(2r))$ is the element defined in* [5, (1.6)] *by $\bar{\sigma}(0)=\bar{\sigma}$ and $\bar{\sigma}(s)=\bar{\sigma}(s-1)^2+4\bar{\sigma}(s-1)$ $(s\geqq 1)$.*

Proof. We notice that $\#R^{n'}(m)=(\#\widetilde{KO}(L^{n'}(2r)))/2=(4r)^n$ by [5, (1.4), Th. 1.9 and Cor. 4.12]. Thus f is isomorphic by Lemma 5.6. Since $f(\beta_1)=\bar{\sigma}$, we see the desired equality by (5.1) and the definition of $\sigma(s)$. q. e. d.

Lemma 5.11. $\quad 2^{n+1}\beta(m-2)=0$ *in* $\tilde{K}(S^{4n+3}/Q_r)$ $(r=2^{m-1}\geqq 4)$.

Proof. $2^{n+1}\bar{\sigma}(m-2)=0$ in $\widetilde{KO}(L^{n'}(2r))$ for $r=2^{m-1}\geqq 4$ by [5, Lemma 6.9(i)]. Thus, the desired result follows from Lemma 5.10. q. e. d.

Lemma 5.12. *The following relations hold in $\tilde{K}(S^{4n+3}/Q_r)$ $(r=2^{m-1}\geqq 2)$:*

(i) $2^{n+1}\alpha_0 = 0.$

(ii) $2^{n+1}\alpha_1 = 2^{n+2}\{\sum_{s=1}^{m-3}\beta(s)\prod_{t=s+1}^{m-3}(2+\beta(t))\}.$

Proof. (i) follows from the relations $\alpha_0\beta_1=-2\alpha_0$ and $\beta_1^{n+1}=0$ in Lemma 5.4.

(ii)
$$\begin{aligned} 0 = \alpha_1\beta_1^{n+1} &= \beta_1^n(\beta_{r-1}-\beta_1)-2\alpha_1\beta_1^n \\ &= (\textstyle\sum_{i=0}^n(-1)^i 2^i\beta_1^{n-i})(\beta_{r-1}-\beta_1)+(-1)^{n+1}2^{n+1}\alpha_1 \\ &= (-1)^n 2^{n+1}\textstyle\sum_{s=1}^{m-2}\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t)) + (-1)^{n+1}2^{n+1}\alpha_1 \\ &= (-1)^n 2^{n+2}\textstyle\sum_{s=1}^{m-3}\beta(s)\prod_{t=s+1}^{m-3}(2+\beta(t)) + (-1)^{n+1}2^{n+1}\alpha_1, \end{aligned}$$

by Lemmas 5.3–4 and 5.11. q. e. d.

Let $u(i)$, $\bar{\alpha}_1$ and δ_i be the integers and the elements in $\tilde{K}(S^{4n+3}/Q_r)$ $(r=2^{m-1}\geqq 2)$ defined in (1.6). Then, we have the following

Lemma 5.13. (i) $2^{n+1}\bar{\alpha}_1 = 0.$

(ii) *The subring $B^n(m)$ in Lemma 5.6 is given by*

$$B^n(m) = \textstyle\sum_{i=1}^{N'} Z_{u(i)}\langle\delta_i\rangle \quad (N' = \min\{r, n\}).$$

PROOF. (i) follows from the definition of $\bar{\alpha}_1$ in (1.6) and Lemma 5.12 (ii).

(ii) By the additive structure of $\widetilde{KO}(L^{n'}(2r))$ given in [5, Th. 1.9], where $2\kappa = \bar{\sigma}(m-1)$ for the stable class κ of the non trivial real line bundle over $L^{n'}(2r)$, and by the definition (1.6) and Lemma 5.10, we see immediately that

(5.14) $$R^{n'}(m) = \sum_{i=1}^{N'} Z_{u(i)}\langle\bar{\sigma}_i\rangle \quad \text{and} \quad f(\delta_i) = \bar{\sigma}_i$$

for the isomorphism $f\colon B^n(m) \cong R^{n'}(m)$ in (5.9). Thus (ii) holds. q.e.d.

We are ready to prove Theorem 1.7.

PROOF OF THEOREM 1.7. The group $\tilde{K}(S^{4n+3}/Q_r)$ is generated additively by α_0, $\bar{\alpha}_1$ and $B^n(m)$ in Lemma 5.6 by Remark 4.9 and Lemmas 5.2–4. On the other hand, $2^{n+1}2^{n+1}(\# B^n(m)) = 2^{2n+2}(4r)^n = \#\tilde{K}(S^{4n+3}/Q_r)$ by Lemmas 4.10 and 4.2. These together with Lemmas 5.12 (i) and 5.13 complete the proof of Theorem 1.7. q.e.d.

§6. An additive base of $\widetilde{KO}(L_0^n(q))$ for odd q

In this section, we give an explicit additive base of the group $\widetilde{KO}(L_0^n(q))$ for odd q, where $L_0^n(q)$ is the $2n$-skeleton of the standard lens space $L^n(q) = S^{2n+1}/Z_q \bmod q$. For this purpose, it is sufficient to study the case $q = p^r$ (p: odd prime, $r \geqq 1$), because the following fact is known (cf. [6, Prop. 2.2]):

(6.1) *Let $q = \prod p^{\nu_p(q)}$ be the prime power decomposition of q and*

$$\pi_p\colon L_0^n(p^{\nu_p(q)}) \longrightarrow L_0^n(q)$$

be the natural projection. Then we have the isomorphism

$$\oplus\, \pi_p^*\colon \widetilde{KO}(L_0^n(q)) \cong \oplus_{p|q} \widetilde{KO}(L_0^n(p^{\nu_p(q)})) \quad \text{with } \pi_p^*(\bar{\sigma}) = \bar{\sigma},$$

where $\bar{\sigma}$ is the real restriction of $\sigma = \eta - 1$ in (3.7).

In the rest of this section, let p be an odd prime and $r \geqq 1$.

To study the group $\widetilde{KO}(L_0^n(p^r))$, consider the elements

(6.2) $\sigma = \eta - 1 = \sigma(0)$, $\sigma(s) = \eta^{p^s} - 1 = (1+\sigma)^{p^s} - 1$ $(0 \leqq s \leqq r)$, $\sigma(r) = 0$,

in $\tilde{K}(L_0^n(p^r))$, where η is the one in (3.7). Further, consider the elements

$$\bar{\sigma}'(s) \in KO(L_0^n(p^r)) \quad \text{and} \quad \bar{\sigma}(s, k) \in \widetilde{KO}(L_0^n(p^r))$$

defined in (1.8–9). Then, we have the following three lemmas.

LEMMA 6.3. *For the complexification $c\colon KO(L_0^n(p^r)) \to K(L_0^n(p^r))$, the following equalities hold:*

(i) $c\bar{\sigma} = \sigma^2/(1+\sigma)$,

(ii) $c\bar{\sigma}'(s) = \sigma(s)/\sigma(1+\sigma)^{q(s)}$,

(iii) $c(\bar{\sigma}'(s-t)^{p^t}\bar{\sigma}^{q(t)+k+1}) = \sigma(s-t)^{p^t}\sigma^{2k+1}/(1+\sigma)^{q(s)+k+1}$,

where σ and $\sigma(s)$ are the elements in (6.2) *and* $q(s)=(p^s-1)/2$.

PROOF. (i) is proved in [9, (2.12)].

(ii) By (i) and (1.8), we see that

$$\begin{aligned}(c\bar{\sigma}'(s))(1+\sigma)^{q(s)} &= \textstyle\sum_{i=0}^{q(s)} (p^s/(2i+1)) \sum_{k=0}^{q(s)-i} \binom{q(s)+i}{2i}\binom{q(s)-i}{k}\sigma^{2i+k} \\ &= \textstyle\sum_{j=0}^{2q(s)} \{\sum_{i=0}^{j}(p^s/(2i+1))\binom{q(s)+i}{2i}\binom{q(s)-i}{j-2i}\}\sigma^j \\ &= \textstyle\sum_{j=0}^{2q(s)} (p^s/(j+1)) \sum_{i=0}^{j} \binom{q(s)+i}{j}\binom{j+1}{2i+1}\sigma^j \\ &= \textstyle\sum_{j=0}^{2q(s)} (p^s/(j+1))\binom{2q(s)}{j}\sigma^j \text{ (by [8, Lemma (3.7)])} \\ &= \textstyle\sum_{j=0}^{p^s-1} \binom{p^s}{j+1}\sigma^j = ((1+\sigma)^{p^s}-1)/\sigma = \sigma(s)/\sigma.\end{aligned}$$

This implies (ii).

(iii) follows immediately from (i), (ii) and the definition $q(s)=(p^s-1)/2$. q.e.d.

LEMMA 6.4. *For the elements $\bar{\sigma}(s, k)$ in* (1.9), *we have*

$$c\bar{\sigma}(s, k) = \sigma(s, 2k+1)/(1+\sigma)^{q(s)+k+1},$$

where $\sigma(s, d) \in \tilde{K}(L_0^n(p^r))$ is the element defined in [11, (1.6)].

PROOF. By Lemma 6.3 (iii) and the definition of $\sigma(s, d)$ in [11, (1.6)], we see easily the desired equality. q.e.d.

LEMMA 6.5. (i) $\bar{\sigma}'(s)=\sum_{j=0}^{q(s)} k_j\bar{\sigma}^j$ *with* $k_{q(s)}=1$, *and* $\bar{\sigma}\bar{\sigma}'(r)=0$ *in* $\widetilde{KO}(L_0^n(p^r))$.

(ii) *For* $0\leqq s<r$, $0\leqq k<p^s(p-1)/2$ *and* $i=q(s)+k+1\leqq[N/2]$ *with* $N=\min\{p^r-1, n\}$, *and the integer $t(2i)$ defined in* (1.6), *we have*

$$\bar{\sigma}(s, k) = \textstyle\sum_{j=1}^{i} l_j\bar{\sigma}^j \text{ with } l_i \equiv 1 \bmod p, \text{ and } \quad t(2i)\bar{\sigma}(s, k) = 0 \quad \text{in } \widetilde{KO}(L_0^n(p^r)).$$

PROOF. We see the first half of (i) by (1.8), and it implies that of (ii) by (1.9) since $\bar{a}_s$ in (1.9) is positive by definition. We have $c(\bar{\sigma}\bar{\sigma}'(r))=\sigma\sigma(r)/(1+\sigma)^{q(r)+1}=0$ by Lemma 6.3 (i), (ii) and (6.2), which implies $\bar{\sigma}\bar{\sigma}'(r)=0$ since c in Lemma 6.3 is monomorphic. Since $t(2i)\sigma(s, 2k+1)=0$ in $\tilde{K}(L_0^n(p^r))$ by [11, Th. 1.7], Lemma 6.4 implies the second half of (ii). q.e.d.

Now, we are ready to prove Theorem 1.10.

PROOF OF THEOREM 1.10. By [9, Prop. 2.11 (i)], we have the following

(6.6) *The ring* $\widetilde{KO}(L_0^n(p^r))$ *is generated by* $\bar{\sigma}$ *satisfying* $\bar{\sigma}^{[n/2]+1}=0$, *and* $\#\widetilde{KO}(L_0^n(p^r))=p^{r[n/2]}$.

This and Lemma 6.5 imply that $\widetilde{KO}(L_0^n(p^r))$ is generated additively by $\bar{\sigma}(s, k)$ in (1.6) and is $\sum_{i=1}^{[N/2]} Z_{t(2i)}\bar{\sigma}(s, k)$ $(i=q(s)+k+1)$, because we have $\prod_{i=1}^{[N/2]} t(2i)=p^{r[n/2]}$ by a routine calculation. Thus, we complete the proof of Theorem 1.10.
q. e. d.

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