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On the map defined by regarding embeddings as immersions

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Introduction

Let M be a closed connected smooth manifold of dimension n and R^m the m-dimensional Euclidean space. Denote by $[M \subseteq R^m]$ the set of regular homotopy classes of immersions of M in R^m and by $[M \subset R^m]$ the set of isotopy classes of embeddings of M in R^m, and consider the commutative diagram

$$\begin{array}{ccc} [M \subset R^{m+1}] & \xrightarrow{J_{m+1}} & [M \subseteq R^{m+1}] \\ E_m \uparrow & & I_m \uparrow \\ [M \subset R^m] & \xrightarrow{J_m} & [M \subseteq R^m], \end{array}$$

where E_m and I_m are the maps induced from the natural inclusion $R^m \subset R^{m+1}$ and J_k is the one defined by regarding embeddings as immersions.

The set $[M \subseteq R^m]$ for $2m > 3n+1$ is an abelian group by taking 0 arbitrarily if it is not empty, and the map I_m is a homomorphism by taking $I_m(0)=0$; while so are the set $[M \subset R^m]$ and the maps E_m and J_m for $2m > 3(n+1)$ (see J. C. Becker [2]).

The purpose of this paper is to study the above commutative diagram when $m = 2n-1$:

$$(*) \qquad \begin{array}{ccc} [M \subset R^{2n}] & \xrightarrow{J_{2n}} & [M \subseteq R^{2n}] \\ E \uparrow & & I \uparrow \\ [M \subset R^{2n-1}] & \xrightarrow{J_{2n-1}} & [M \subseteq R^{2n-1}] \end{array} \qquad (E = E_{2n-1},\ I = I_{2n-1}),$$

(here we assume that the sets in consideration are not empty).

When $n \geqq 4$, the upper groups are determined by A. Haefliger and M. W. Hirsch [3], [5], [6] and so is the group $[M \subseteq R^{2n-1}]$ by D. R. Bausum [1, Th. 37 and Prop. 41], L. L. Larmore and E. Thomas [10, Th. 5.1] and R. D. Rigdon [11, Th. 10.4], and moreover it is proved by R. D. Rigdon [11, Th. 10.4] that I is trivial for even n and is surjective for odd n, respectively. When $n \geqq 6$, $[M \subset R^{2n-1}]$ is an abelian group and Im E is determined by R. D. Rigdon [11, Th. 11.11 and Th. 11.26]. Together with these results, we have the following

MAIN THEOREM. *Let M be a closed connected smooth manifold of dimension* *n with the i-th Stiefel-Whitney class* $w_i \in H^i(M; Z_2)$, *and let*

$$Sq^1: H^{n-1}(M; Z_2) \longrightarrow H^n(M; Z_2),$$
$$\beta_2: H^{n-2}(M; Z_2) \longrightarrow H^{n-1}(M; Z)$$

be the squaring operation and the Bockstein operator, respectively, and $H^i(M; Z[w_1])$ *be the integral cohomology twisted by* w_1. *Then in the diagram* (*) *there hold the following properties* (i)'s,..., (iv)'s, *respectively, when*

(i) n *is even and* $w_1=0$, (ii) n *is even and* $w_1\neq 0$,

(iii) n *is odd and* $w_1=0$, (iv) n *is odd and* $w_1\neq 0$.

(1) *Assume that* $n\geqq 4$. *Then*

(i) $[M\subset R^{2n}] = H^{n-1}(M; Z_2)$, $[M\subseteq R^{2n}] = Z$, $J_{2n} = 0$,

$$[M\subseteq R^{2n-1}] = \begin{cases} H^{n-1}(M; Z_2) & \text{if } n \equiv 0(4), \\ H^{n-1}(M; Z_2)+Z_2 & \text{if } n \equiv 2(4), \end{cases} \quad I = 0;$$

(ii) $[M\subset R^{2n}] = Z + \mathrm{Ker}\, Sq^1$, $[M\subseteq R^{2n}] = Z$, $J_{2n}(a, b) = 2a$,

$$[M\subseteq R^{2n-1}] = \begin{cases} H^{n-1}(M; Z_2) & \text{if } n \equiv 0(4), \\ \mathrm{Ker}\, Sq^1+Z_4 & \text{if } n \equiv 2(4), \end{cases} \quad I = 0;$$

(iii) $[M\subset R^{2n}] = H^{n-1}(M; Z)$, $[M\subseteq R^{2n}] = Z_2$, $J_{2n} = 0$,

$$[M\subseteq R^{2n-1}] = \begin{cases} H^{n-1}(M; Z)+Z_2+Z_2, & I(a, b, c)=b \quad \text{if } n\equiv 1(4), \\ H^{n-1}(M; Z)+Z_4, & I(a, b)\equiv b(2) \quad \text{if } n\equiv 3(4); \end{cases}$$

(iv) $[M\subset R^{2n}] = H^{n-1}(M; Z_2)$, $[M\subseteq R^{2n}] = Z_2$, $J_{2n} = 0$,

$[M\subseteq R^{2n-1}] = H^{n-1}(M; Z[w_1])+Z_2$, $I(a, b) = b$.

(2) *Assume that* $n\geqq 6$. *Then*

(i) $\mathrm{Im}\, E = [M\subset R^{2n}]$,

$$\mathrm{Im}\, J_{2n-1} = \begin{cases} H^{n-1}(M; Z_2) & \text{if } n \equiv 2(4) \text{ and } w_2(\mathrm{Ker}\,\beta_2) = 0, \\ [M\subseteq R^{2n-1}] & \text{otherwise}; \end{cases}$$

(ii) $\mathrm{Im}\, E = \mathrm{Ker}\, Sq^1$,

$$\mathrm{Im}\, J_{2n-1} = \begin{cases} \mathrm{Ker}\, Sq^1+Z_2 & \text{if } n\equiv 2(4) \text{ and } w_1^2+w_2\neq 0, \\ \mathrm{Ker}\, Sq^1 & \text{otherwise}; \end{cases}$$

(iii) $\mathrm{Im}\, E = \mathrm{Im}\, \beta_2$,

$$\operatorname{Im} J_{2n-1} = \begin{cases} \operatorname{Im}\beta_2 + 0 + Z_2 & \textit{if} \quad n \equiv 1(4) \ \textit{and} \ w_2(\operatorname{Ker}\beta_2) \neq 0, \\ \operatorname{Im}\beta_2 + Z_2 & \textit{if} \quad n \equiv 3(4) \ \textit{and} \ w_2(\operatorname{Ker}\beta_2) \neq 0, \\ \operatorname{Im}\beta_2 & \textit{otherwise}; \end{cases}$$

(iv) $\operatorname{Im} E = [M \subset R^{2n}]$, $\quad \operatorname{Im} J_{2n-1} = H^{n-1}(M; Z[w_1])$.

The group $[M \subset R^{2n-1}]$ will be studied in the forthcoming paper [14].

In §1, the group structures and the filtrations on $[M \subseteq R^m]$ and $[M \subset R^m]$ are recalled according to [1], [2], [8], [11] and [13], and the methods for computing I_m, E_m and J_m are stated. The groups $[M \subseteq R^{2n}]$, $[M \subset R^{2n}]$ and $[M \subseteq R^{2n-1}]$ are restated in §2 and the results on J_{2n} and I are proved. The map J_{2n-1} is investigated in §§3–4, by using the results on the cohomology of $(\Lambda^2 M, \Delta M)$ due to L. L. Larmore [7] together with the remarks given in §5. In §5, the twisted integral cohomology groups $H^i(\Lambda^2 M, \Delta M; Z[v])$ for $i \geqq 2n-3$ ($v \in H^1(\Lambda M^2 - \Delta M; Z_2)$) are treated.

§1. Preliminaries

Let M be a closed connected smooth manifold of dimension n. Then there is a fixed point free involution on the tangent sphere bundle SM over M, which is the antipodal map on each fibre S^{n-1}. Thus, for an immersion $f: M \subseteq R^m$, we have the Z_2-equivariant map

$$\pi S(f):\ SM \xrightarrow{S(f)} R^m \times S^{m-1} \xrightarrow{\pi} S^{m-1},$$

where $S(f)$ is the Z_2-equivariant map induced from the derivation of f and π is the projection.

THEOREM (Haefliger-Hirsch [4]). *If $2m > 3n+1$, then the correspondence which associates the Z_2-equivariant homotopy class of $\pi S(f)$ with a regular homotopy class of an immersion f is a bijection between $[M \subseteq R^m]$ and the set of Z_2-equivariant homotopy classes of Z_2-equivariant maps of SM to S^{m-1}.*

On the other hand, let ΔM be the diagonal of $M \times M$. Then there is a fixed point free involution on $M \times M - \Delta M$ defined by the interchange of factors. Thus, for an embedding $f: M \subset R^m$, we have the Z_2-equivariant map

$$f':\ M \times M - \Delta M \longrightarrow S^{m-1},$$
$$f'(x, y) = (f(x) - f(y))/\|f(x) - f(y)\| \quad (x, y \in M, x \neq y).$$

THEOREM (Haefliger [3]). *If $2m > 3(n+1)$, then the correspondence which associates the Z_2-equivariant homotopy class of f' with an isotopy class of an* *embedding f is a bijection between* $[M\subset R^m]$ *and the set of* Z_2*-equivariant homotopy classes of* Z_2*-equivariant maps of* $M\times M-\Delta M$ *to* S^{m-1}.

Let $PM=SM/Z_2$ and $M^*=(M\times M-\Delta M)/Z_2$ be the tangent projective bundle over M and the reduced symmetric product of M, respectively. Moreover, let

$$\eta: PM \longrightarrow P^\infty \quad \text{and} \quad \xi: M^* \longrightarrow P^\infty$$

be the classifying maps of the double coverings $SM\to PM$ and $M\times M-\Delta M\to M^*$, respectively. Now, $S^\infty\to P^\infty$ is the universal double covering and $S^\infty\times_{Z_2}S^{m-1}\to P^\infty$ is homotopically equivalent to the natural inclusion $P^{m-1}\subset P^\infty$. Therefore the above theorems are restated as follows, where

$$[X, P^{m-1}; \alpha] = [X, S^\infty\times_{Z_2}S^{m-1}; \alpha] \qquad \text{for} \quad \alpha: X \longrightarrow P^\infty$$

denotes the homotopy sets of liftings of α to $S^\infty\times_{Z_2}S^{m-1}$:

THEOREM 1.1. *There exist bijections*

$$A: [M\subseteq R^m] \cong [PM, P^{m-1}; \eta] \qquad \text{if} \quad 2m > 3n+1,$$

$$B: [M\subset R^m] \cong [M^*, P^{m-1}; \xi] \qquad \text{if} \quad 2m > 3(n+1).$$

Each set of the right hand sides has the structure of an abelian group by [2] if it is not empty, which induces those of $[M\subseteq R^m]$ and $[M\subset R^m]$.

Now PM is a manifold of dimension $2n-1$ and M^* has the homotopy type of a CW-complex of dimension less than $2n$.

PROPOSITION 1.2 (Bausum [1, Prop. 5 and Prop. 6], Larmore-Rigdon [8, Prop. 4.1], Yasui [13, Prop. 1.1]). *Assume that X has the homotopy type of a CW-complex of dimension less than* $2n$ $(n\geqq 4)$. *Then for a map* $\alpha: X\to P^\infty$, *there exist decreasing filtrations*

$$[X, P^{2n-1}; \alpha] = G_0(\alpha) \supset G_1(\alpha) = 0, \quad G_0(\alpha) = H^{2n-1}(X; Z);$$

$$[X, P^{2n-2}; \alpha] = F_0(\alpha) \supset F_1(\alpha) \supset F_2(\alpha) = 0,$$

$$F_0(\alpha)/F_1(\alpha) = H^{2n-2}(X; Z[v]),$$

$$F_1(\alpha) = \mathrm{Coker}\,(\Theta: H^{2n-3}(X; Z[v]) \longrightarrow H^{2n-1}(X; Z_2)),$$

where $H^i(X; Z[v])$ *is the integral cohomology of X twisted by* $v=\alpha^*u$ ($u\in H^1(P^\infty; Z_2)$ *is the generator) and*

$$\Theta = Sq^2\tilde{\rho}_2 + \binom{2n-1}{2}v^2\tilde{\rho}_2$$

($\tilde{\rho}_2: H^i(X; Z[v])\to H^i(X; Z_2)$ *is the reduction* mod 2).

By the definitions of the maps I_m, E_m and J_m in the introduction and the bijections A and B in Theorem 1.1, we have the commutative diagram

$$\begin{array}{ccccccc}
[M\subset R^{m+1}] & \xleftarrow{E_m} & [M\subset R^m] & \xrightarrow{J_m} & [M\subseteq R^m] & \xrightarrow{I_m} & [M\subseteq R^{m+1}] \\
B\downarrow\cong & & B\downarrow\cong & & A\downarrow\cong & & A\downarrow\cong \\
[M^*, P^m; \xi] & \xleftarrow{i_\#} & [M^*, P^{m-1}; \xi] & \xrightarrow{j^\#} & [PM, P^{m-1}; \eta] & \xrightarrow{i_\#} & [PM, P^m; \eta]
\end{array}$$

for $2m>3(n+1)$ (cf. [8], [11]), where $i\colon P^{m-1}\subset P^m$ is the natural inclusion and

$$j\colon PM \longrightarrow M^* \text{ is the embedding with } \xi j = \eta$$

induced from the Z_2-equivariant map $j\colon SM\to M\times M-\Delta M$ defined by $j(u)=(\exp(u), \exp(-u))$.

Proposition 1.3 (Larmore-Rigdon [8, Prop. 5.1 and Prop. 6.1]). *Let (X, α) represent (PM, η) or (M^*, ξ), and consider the filtrations of $[X, P^{m-1}; \alpha]$ for $m=2n-1, 2n$ given in Proposition 1.2. Then*

(1) *$i_\#\colon [X, P^{2n-2}; \alpha]\to[X, P^{2n-1}; \alpha]$ preserves the filtrations and the induced homomorphism*

$$i_\#\colon F_0(\alpha)/F_1(\alpha) = H^{2n-2}(X; Z[v]) \longrightarrow G_0(\alpha) = H^{2n-1}(X; Z)$$

is just the multiplication by $V=\tilde{\beta}_2(1)\in H^1(X; Z[v])$ ($\tilde{\beta}_2\colon H^i(X; Z_2)\to H^{i+1}(X; Z[v])$ is the twisted Bockstein operator);

(2) *$j^\#\colon [M^*, P^{m-1}; \xi]\to[PM, P^{m-1}; \eta]$ preserves the filtrations and $j^\#\colon G_0(\xi)\to G_0(\eta)$ and $j^\#\colon F_i(\xi)/F_{i+1}(\xi)\to F_i(\eta)/F_{i+1}(\eta)$ are j^* on the cohomology groups and moreover $j^\#$ for $m=2n-1$ induces the map*

$$j_0^\#\colon \operatorname{Ker}(j^\#\colon F_0(\xi)/F_1(\xi) \longrightarrow F_0(\eta)/F_1(\eta)) \longrightarrow \operatorname{Coker}(j^\#\colon F_1(\xi) \longrightarrow F_1(\eta)),$$

which is equal to the functional operation

$$\Theta_j\colon \operatorname{Ker} j^*(\subset H^{2n-2}(M^*; Z[v])) \longrightarrow H^{2n-1}(PM; Z_2)/(\operatorname{Im}\Theta+\operatorname{Im} j^*)$$

given by $\delta^{-1}\Theta i^{*-1}$ in the commutative diagram

$$\begin{array}{ccccccc}
\cdots\xrightarrow{j^*} H^{2n-3}(PM; Z[j^*v]) & \xrightarrow{\delta} & H^{2n-2}(M^*, PM; Z[v]) & \xrightarrow{i^*} & H^{2n-2}(M^*; Z[v]) & \xrightarrow{j^*} & \cdots \\
\Theta\downarrow & & \Theta\downarrow & & \Theta\downarrow & & \\
\cdots\xrightarrow{j^*} H^{2n-1}(PM; Z_2) & \xrightarrow{\delta} & H^{2n}(M^*, PM; Z_2) & \xrightarrow{i^*} & H^{2n}(M^*; Z_2)(=0) & &
\end{array}$$

of the exact sequences of the pair (M^*, PM), where $v=\xi^*u$ and $i\colon M^*\subset(M^*,PM)$.

Furthermore, let $\Lambda^2M=(M\times M)/Z_2$ be the 2-fold symmetric product of M, the set of unordered pairs of M. Then $\Lambda^2M-\Delta M=M^*$ and $PM=j(PM)$ bounds a tubular neighborhood N of ΔM in Λ^2M, and the natural inclusions

$$(M^*, PM) \subset (\Lambda^2 M, N) \supset (\Lambda^2 M, \Delta M)$$

induce isomorphisms of cohomology groups (cf. [8, §5]). Thus we have the following

LEMMA 1.4. *The cohomology exact sequence of* (M^*, PM) *with any coefficients (e.g., the one in the diagram in Proposition* 1.3) *can be replaced by the exact sequence*

$$\cdots \longrightarrow H^{i-1}(M^*) \xrightarrow{j^*} H^{i-1}(PM) \xrightarrow{\delta} H^i(\Lambda^2 M, \Delta M)$$
$$\xrightarrow{i^*} H^i(M^*) \xrightarrow{j^*} H^i(PM) \longrightarrow \cdots .$$

Our study is based on these results. Moreover the cohomology of $(\Lambda^2 M, \Delta M)$ is investigated by L. L. Larmore[7]. The notations Λx and $\Delta(x, y)$ and the results stated in [7, pp. 908–915] are freely quoted hereafter. We also use the following lemma and the results remarked in §5.

LEMMA 1.5. (1) $\tilde{\rho}_r(\Lambda x) = \Lambda(\rho_r x)$ *and* $\tilde{\rho}_r(\Delta(x, y)) = \Delta(\rho_r x, \rho_r y)$ *for* $x, y \in H^*(M; Z_s)$, *where* $r \mid s$, $s \leqq \infty$ *and* ρ_r, $\tilde{\rho}_r$ *are the reductions* mod r.

(2) $\Delta(x, y) = \Lambda x \Lambda y + \Lambda(xy)$ *for* $x, y \in H^*(M; Z_2)$.

(3) $\delta(v^i x) = v^{i+1} \Lambda x$ *for* $x \in H^*(M; Z_2)$, *where* $v^i x = j^* v^i \cdot \pi^* x$ (π: $PM \to M$ *is the projection*).

PROOF. The relations (1) and (2) are easily obtained by chasing the constructions of Λx and $\Delta(x, y)$ given in [7]. The relation (3) follows from the equality $\delta x = v \Lambda x$ (δ: $H^{i-1}(M) = H^{i-1}(\Delta M) \to H^i(\Lambda^2 M, \Delta M)$) in [7, Lemma 6], by noticing that the restriction of the projection $N \to M$ on PM is equal to π and the one on ΔM is the identity $\Delta M \to M$. q.e.d.

§2. J_{2n}, I, E and $[M \subseteq R^{2n-1}]$

The following results are well-known:

(2.1) *Let* $v \in H^1(PM; Z_2)$ *be the first Stiefel-Whitney class of the double covering* $SM \to PM$. *Then* $1, v, \ldots, v^{n-1}$ *form a base of the* $H^*(M; Z_2)$*-module* $H^*(PM; Z_2)$ *with the relation*

$$v^n = \sum_{i=1}^n v^{n-i} w_i \quad (w_i = w_i(M)).$$

(2.2) $[M \subseteq R^{2n}] = H^{2n-1}(PM; Z)$ *in Theorem* 1.1 *and Proposition* 1.2 *is isomorphic to* Z *if* n *is even and* Z_2 *if* n *is odd.*

(2.3) ([3], [5] and [11]) $[M \subset R^{2n}] = H^{2n-1}(M^*; Z)$ *in Theorem* 1.1 *and Proposition* 1.2 *is isomorphic to*

$H^{n-1}(M; Z)$ *if n is odd and* $w_1=0$,

$Z+K$ $(K=\mathrm{Ker}\,(Sq^1: H^{n-1}(M; Z_2)\to H^n(M; Z_2)))$ *if n is even and* $w_1\neq 0$,

$H^{n-1}(M; Z_2)$ *otherwise.*

PROOF OF MAIN THEOREM ON J_{2n}. By the results stated in §1, we have a commutative diagram

$$\begin{array}{ccccccc} [M\subset R^{2n}] & \xrightarrow{J_{2n}} & [M\subseteq R^{2n}] & & & & \\ \| & & \| & & & & \\ H^{2n-1}(M^*; Z) & \xrightarrow{j^*} & H^{2n-1}(PM; Z) & \xrightarrow{\partial} & H^{2n}(A^2M, \Delta M; Z) & \longrightarrow & 0, \end{array}$$

where the lower sequence is exact by Lemma 1.4, while by Proposition 5.2(2),

$$H^{2n}(A^2M, \Delta M; Z) = \begin{cases} Z & \text{if } n \text{ is even and } w_1 = 0, \\ Z_2 & \text{otherwise.} \end{cases}$$

Thus if n is even and $w_1\neq 0$ then $\mathrm{Im}\,(j^*: Z+K\to Z)=2Z$, and if it is not then $j^*=0$. q.e.d.

We now recall that the filtration

$$[M\subseteq R^{2n-1}] = [PM, P^{2n-2}; \eta] = F_0 \supset F_1 \supset 0 \quad (F_i = F_i(\eta))$$

satisfies

$$F_0/F_1 = H^{2n-2}(PM; Z[v]),$$

$$F_1 = \mathrm{Coker}\,(\Theta: H^{2n-3}(PM; Z[v]) \longrightarrow H^{2n-1}(PM; Z_2))$$

where $\Theta=Sq^2\tilde{\rho}_2+(n-1)v^2\tilde{\rho}_2$.

The twisted integral cohomology of PM is investigated by R. D. Rigdon and is given as follows:

PROPOSITION 2.4 (Rigdon [11, Prop. 9.2 and 9.13]). *Let* $M\in H^n(M; Z_2)$ *be the generator. Then*

(1) *if n is even, there exist isomorphisms*

$$H^{2n-1}(PM; Z[v]) = Z_2,$$

$$\theta: H^{n-1}(M; Z_2) \cong H^{2n-2}(PM; Z[v]), \quad \theta(x) = \tilde{\beta}_2(v^{n-2}x)\ (x\in H^{n-1}(M; Z_2))\,;$$

(2) *if n is odd, there exist isomorphisms*

$$H^{2n-1}(PM; Z[v]) = Z,$$

$$\theta: H^{n-1}(M; Z[w_1]) + H^n(M; Z_2) \cong H^{2n-2}(PM; Z[v]),$$

$\theta(M)=\tilde{\beta}_2(v^{n-3}M),\ \tilde{\rho}_2\theta(y)=(v^{n-1}+v^{n-2}w_1)\tilde{\rho}_2 y\ (y\in H^{n-1}(M;Z[v]))$.*)

Let $M'\in H^{n-1}(M;Z_2)$ be the element with $Sq^1M'=M$ when $w_1\neq 0$ and let $K=\mathrm{Ker}(Sq^1\colon H^{n-1}(M;Z_2)\to H^n(M;Z_2))$. Then $H^{2n-2}(PM;Z[v])$ is the following form by Proposition 2.4, ($Z_r\langle a\rangle$ denotes the cyclic group of order r generated by a):

(2.5)
$$\begin{aligned} F_0/F_1 &= \theta H^{n-1}(M;Z_2) && \textit{if } n \textit{ is even and } w_1=0,\\ &= \theta K + Z_2\langle\theta M'\rangle && \textit{if } n \textit{ is even and } w_1\neq 0,\\ &= \theta H^{n-1}(M;Z[w_1]) + Z_2\langle\theta M\rangle && \textit{if } n \textit{ is odd.}\end{aligned}$$

Further, by studying Θ, we have

(2.6)
$$F_1=\mathrm{Coker}\,\Theta=\begin{cases} H^{2n-1}(PM;Z_2)=Z_2 & \textit{if } n \textit{ is odd and } w_1=0, \textit{ or } n\equiv 2(4),\\ 0 & \textit{otherwise.}\end{cases}$$

In case of $F_1=Z_2$, the group extension ϕ_2 of $0\to F_1\to F_0\to F_0/F_1\to 0$ is given by

$$\phi_2=Sq^2\tilde{\beta}_2^{-1}+(n-1)v^2\tilde{\beta}_2^{-1}+Sq^1\tilde{\rho}_2\colon \{z\in F_0/F_1 \mid 2z=0\}=\tilde{\beta}_2H^{2n-3}(PM;Z_2)\longrightarrow F_1=H^{2n-1}(PM;Z_2),$$

which is proved by using [10, Th. 4.1] (cf. [9, Cor. 3.7]), and so we have the following:

(2.7) *The group extension* ϕ_2 *is trivial except for*

$$\begin{aligned} \phi_2(\theta M') &= v^{n-1}M && \textit{if } n\equiv 2(4) \textit{ and } w_1\neq 0,\\ \phi_2(\theta M) &= v^{n-1}M && \textit{if } n\equiv 3(4) \textit{ and } w_1=0.\end{aligned}$$

THEOREM 2.8 (Bausum [1, Th. 37 and Prop. 41], Larmore-Thomas [10, Th. 5.1], Rigdon [11, Th. 10.4]). *Let* $n\geqq 4$. *Then the group* $[M\subseteq R^{2n-1}]=[PM, P^{2n-2};\eta]$ *is as follows*:

$$\begin{aligned} [M\subseteq R^{2n-1}] &= \theta H^{n-1}(M;Z_2) && \textit{if } n\equiv 0(4),\\ &= \theta H^{n-1}(M;Z_2)+Z_2 && \textit{if } n\equiv 2(4) \textit{ and } w_1=0,\\ &= \theta K+Z_4 && \textit{if } n\equiv 2(4) \textit{ and } w_1\neq 0,\\ &= \theta H^{n-1}(M;Z)+Z_2+Z_2 && \textit{if } n\equiv 1(4) \textit{ and } w_1=0,\\ &= \theta H^{n-1}(M;Z)+Z_4 && \textit{if } n\equiv 3(4) \textit{ and } w_1=0,\\ &= \theta H^{n-1}(M;Z[w_1])+Z_2 && \textit{if } n\equiv 1(2) \textit{ and } w_1\neq 0.\end{aligned}$$

*) This relation is different from that of Rigdon [11], but his relation can be modified as stated in the proposition by chasing his construction of θ.

PROOF OF MAIN THEOREM ON I AND E. By (2.2), (2.4) and Proposition 1.3(1), we see that

(2.9) ([11, Th. 10.4]) *I is trivial if n is even.*

Assume that n is odd and consider the homomorphism

$$\rho_2 i_\# \theta\colon H^{n-1}(M; Z[w_1]) + H^n(M; Z_2) \cong H^{2n-2}(PM; Z[v])(=F_0/F_1)$$
$$\xrightarrow{i_\#} H^{2n-1}(PM; Z)(=Z_2) \xrightarrow{\rho_2} H^{2n-1}(PM; Z_2).$$

Then the relation $\rho_2 i_\# \theta(x, y) = v^{n-1}y$ follows from Propositions 2.4, 1.3(1) and (2.1). Therefore, by (2.6–8), we have the equalities

$$I(a, b, c) = b \quad \text{if} \quad n \equiv 1(4) \quad \text{and} \quad w_1 = 0,$$
$$I(a, b) \equiv b(2) \quad \text{if} \quad n \equiv 3(4) \quad \text{and} \quad w_1 = 0,$$
$$I(a, b) = b \quad \text{if} \quad n \equiv 1(2) \quad \text{and} \quad w_1 \neq 0.$$

These and (2.9) show the desired results on I. The results on E is proved by R. D. Ridgon [11, Th. 11.11 and Th. 11.26]. q.e.d.

§3. $j^\#\colon F_i(\xi)/F_{i+1}(\xi) \to F_i(\eta)/F_{i+1}(\eta)$ in Proposition 1.3

In this and next sections, we investigate the homomorphism

$$J_{2n-1} = j^\#\colon [M \subset R^{2n-1}] = [M^*, P^{2n-2}; \xi] \longrightarrow [M \subseteq R^{2n-1}] = [PM, P^{2n-2}; \eta]$$

in Proposition 1.3(2), which preserves the filtrations

$$[M^*, P^{2n-2}; \xi] = F_0(\xi) \supset F_1(\xi) \supset 0, \quad [PM, P^{2n-2}; \eta] = F_0(\eta) \supset F_1(\eta) \supset 0$$

given in Proposition 1.2.

LEMMA 3.1. *$j^\# = j^*\colon F_1(\xi) = H^{2n-1}(M^*; Z_2) \to F_1(\eta) = H^{2n-1}(PM; Z_2)$ is trivial.*

PROOF. This is an immediate consequence of E. Thomas [12, Prop. 2.9(c)]. q.e.d.

Next, we study the homomorphism

$$j^\# = j^*\colon F_0(\xi)/F_1(\xi) = H^{2n-2}(M^*; Z[v])$$
$$\longrightarrow F_0(\eta)/F_1(\eta) = H^{2n-2}(PM; Z[v]),$$

where the range $H^{2n-2}(PM; Z[v])$ is given in Proposition 2.4. Hereafter, we use essentially Propositions 5.2–3 given in §5 below.

LEMMA 3.2. (1) *If n is even and $w_1=0$, then j^* is surjective.*
(2) *If n is even and $w_1 \neq 0$, then* $\mathrm{Im}\, j^* = \theta\rho_2 H^{n-1}(M; Z) = \theta K$.
(3) *If n is odd and $w_1=0$, then* $\mathrm{Im}\, j^* = \theta\beta_2 H^{n-2}(M; Z_2)$.
(4) *If n is odd and $w_1 \neq 0$, then* $\mathrm{Im}\, j^* = \theta H^{n-1}(M; Z[w_1])$.

PROOF. We prove the lemma by using the exact sequence

$$(3.3) \quad \cdots \longrightarrow H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} H^{2n-2}(PM; Z[v]) \xrightarrow{\delta} H^{2n-1}(\Lambda^2 M, \Delta M; Z[v]) \xrightarrow{i^*} H^{2n-1}(M^*; Z[v]) \xrightarrow{j^*} H^{2n-1}(PM; Z[v]) \xrightarrow{\delta} H^{2n}(\Lambda^2 M, \Delta M; Z[v]) \longrightarrow 0$$

in Lemma 1.4. In this sequence, the following is given by R. D. Rigdon [11, Prop. 11.9 and Prop. 11.19]:

$$(3.4) \quad H^{2n-1}(M^*; Z[v]) \cong H^{n-1}(M; Z) \quad \text{if } n \text{ is even and } w_1=0,$$
$$\cong Z + K \quad \text{if } n \text{ is odd and } w_1 \neq 0,$$
$$\cong H^{n-1}(M; Z_2) \quad \text{otherwise.}$$

(1) Assume that n is even and $w_1=0$. Then for any $z \in H^{n-1}(M; Z)$, we have $\delta\tilde{\beta}_2(v^{n-2}z') = \tilde{\beta}_2\delta(v^{n-2}z') = \tilde{\beta}_2(v^{n-1}\Lambda z') = \tilde{\beta}_2(\Lambda z'\Lambda z' + v^{n-2}\Lambda(Sq^1 z')) = \tilde{\beta}_2\tilde{\rho}_2\Delta(z, z) = 0$ $(\rho_2 z = z')$ by Lemma 1.5 and [7, Lemma 10]. Therefore the first δ in (3.3) is trivial by Proposition 2.4 (1) and so (1) is shown.

(2) Assume that n is even and $w_1 \neq 0$. Then the exact sequence (3.3) is equal to

$$H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} \theta K + Z_2 \xrightarrow{\delta} K + Z_4 \longrightarrow K + Z_2 \longrightarrow Z_2 \longrightarrow Z_2 \longrightarrow 0$$

by Proposition 2.4(1), (3.4) and Propositions 5.2–3, and so $\mathrm{Im}\,\delta = Z_2$. Now $\delta\theta K = 0$ is proved in the above case. Thus $\mathrm{Im}\, j^* = \mathrm{Ker}\,\delta = \theta K$.

(3) Assume that n is odd and $w_1=0$. Then (3.3) induces an exact sequence

$$(3.5) \quad H^{2n-2}(M^*; Z[v]) \xrightarrow{j^*} \theta G + Z_2\langle\tilde{\beta}_2(v^{n-3}M)\rangle \xrightarrow{\delta} G + Z_2\langle\tilde{\beta}_2(v^{n-2}\Lambda M)\rangle \xrightarrow{i^*} K = \rho_2 G, \ (G \cong H^{n-1}(M; Z)),$$

by Proposition 2.4(2), (3.4) and Propositions 5.2–3. Here the relation

$$\delta\tilde{\beta}_2(v^{n-3}M) = \tilde{\beta}_2(v^{n-2}\Lambda M)$$

holds by Lemma 1.5(3), and the relation

$$(3.6) \qquad \delta(\theta G) \subset G$$

holds, because $\tilde{\rho}_2\tilde{\beta}_2(v^{n-2}\Lambda M) = v^{n-1}\Lambda M$ in $H^{2n-1}(\Lambda^2 M, \Delta M; Z_2)$ by [7, Lemma

10] and $\tilde{\rho}_2\delta\theta G=\delta\tilde{\rho}_2\theta G=0$ by Lemma 1.5 and Proposition 2.4(2). Therefore the sequence (3.5) induces an exact sequence

$$(3.5)' \qquad H^{2n-2}(M^*; Z[v]) \xrightarrow{\theta^{-1}j^*} G \xrightarrow{f} G \xrightarrow{g} K \longrightarrow 0, \quad (f=\delta\theta,\ g=i^*).$$

Here $K=\rho_2 G=G/2G$. Hence $g(2G)=0$ and g induces an epimorphism g': $G/2G \to K$, which is isomorphism because $G/2G$ is finite. Therefore $2G=\mathrm{Ker}\, g=\mathrm{Im} f$. Since rank $G=$ rank $2G$ and $2G=\mathrm{Im} f$, we see that

$$\mathrm{Ker} f \subset T \text{ and } f(T)=2T \ (T \text{ is the torsion subgroup of } G)$$

by noticing that the torsion subgroup of $2G$ is equal to $2T$. Thus f determines an epimorphism

$$f|T: T \longrightarrow 2T.$$

If we can prove

$$(3.7) \qquad {}_2G\ (=\{x\in G\,|\,2x=0\}) = \beta_2 H^{n-2}(M; Z_2) \subset \mathrm{Ker} f \text{ in } (3.5)',$$

then ${}_2T(=\{x \in T|\,2x=0\})={}_2G\subset \mathrm{Ker}\,(f|\,T)$ an $f|\,T$ induces an epimorphism $T/{}_2T\to 2T$, which is isomorphic because the orders of the two groups are finite and coincident with each other. Hence $\mathrm{Ker} f={}_2G$ and Lemma 3.2(3) is proved.

To show (3.7), we notice that $\theta(\beta_2 H^{n-2}(M; Z_2))\subset\tilde{\beta}_2 H^{2n-3}(PM; Z_2)$. For any element $X\in\theta(\mathrm{Im}\,\beta_2)$, there is an element $Y\in H^{2n-3}(PM; Z_2)$ such that $\tilde{\beta}_2 Y=X$ and

$$Y = \lambda v^{n-3}M + v^{n-2}x + (v^{n-1}y + v^{n-3}Sq^2 y)$$

for some $\lambda\in Z_2$, $x\in H^{n-1}(M; Z_2)$ and $y\in H^{n-2}(M; Z_2)$ by (2.1). For $x\in H^{n-1}(M; Z_2)$, there is a relation $\tilde{\rho}_2\tilde{\beta}_2(v^{n-3}x)=v^{n-2}x$ and so $\tilde{\beta}_2(v^{n-2}x)=0$. Further the relation $\delta\tilde{\beta}_2(v^{n-1}y+v^{n-3}Sq^2y)=0$ for $y\in H^{n-2}(M; Z_2)$ follows from Lemma 1.5 and [7, Th. 11]. Thus $\delta X=\delta\tilde{\beta}_2 Y=\lambda\tilde{\beta}_2(v^{n-2}\Lambda M)$ and so $\tilde{\rho}_2\delta X=\lambda v^{n-1}\Lambda M$. This and (3.6) imply $\lambda=0$ and so $\delta X=0$. This completes the proof of (3.7).

(4) Assume that n is odd and $w_1\neq 0$. Then $\tilde{\rho}_2$: $H^{2n-1}(A^2M, \Delta M; Z[v])\to H^{2n-1}(A^2M, \Delta M; Z_2)$ is monomorphic by Proposition 5.3(iv). Further, by Lemma 1.5 and Proposition 2.4, we see that

$$\tilde{\rho}_2\delta\tilde{\beta}_2(v^{n-3}M) = v^{n-1}\Lambda M, \quad \tilde{\rho}_2\delta\theta(x) = 0 \qquad \text{for} \quad x\in H^{n-1}(M; Z[w_1]).$$

Therefore $\mathrm{Im}\, j^*=\mathrm{Ker}\,\delta=\theta H^{n-1}(M; Z[w_1])$. q.e.d.

§4. J_{2n-1}: $[M\subset R^{2n-1}]\to[M\subseteq R^{2n-1}]$

This section is a continuation of §3 and we will determine $\mathrm{Im}\, J_{2n-1}$ by using Proposition 1.3(2).

If $F_1(\eta)=0$, then $\operatorname{Im} J_{2n-1}=\operatorname{Im}(j^{\#}\colon F_0(\xi)/F_1(\xi)\to F_0(\eta)/F_1(\eta))$ and so by Proposition 1.3(2), (2.6) and Lemma 3.2, we have the following

PROPOSITION 4.1. (1) *If* $n\equiv 0(4)$ *and* $w_1=0$, *then* $\operatorname{Im} J_{2n-1}=[M\subseteq R^{2n-1}]$.
(2) *If* $n\equiv 0(4)$ *and* $w_1\neq 0$, *then* $\operatorname{Im} J_{2n-1}=\theta\rho_2 H^{n-1}(M; Z)=\theta K$.
(3) *If* $n\equiv 1(2)$ *and* $w_1\neq 0$, *then* $\operatorname{Im} J_{2n-1}=\theta H^{n-1}(M; Z[w_1])$.

In the rest of this section, we study J_{2n-1} in case when $n\equiv 1(2)$ and $w_1=0$, or $n\equiv 2(4)$. In these cases, $F_1(\eta)=H^{2n-1}(PM; Z_2)$ and we have to study the homomorphism

$$j_0^{\#}\colon \operatorname{Ker}(j^{\#}\colon F_0(\xi)/F_1(\xi)\longrightarrow F_0(\eta)/F_1(\eta))\longrightarrow \operatorname{Coker}(j^{\#}\colon F_1(\xi)\longrightarrow F_1(\eta))$$

induced from $j^{\#}\colon (F_0(\xi), F_1(\xi))\to(F_0(\eta), F_1(\eta))$. By Lemma 3.1,

$$\operatorname{Coker}(j^{\#}\colon F_1(\xi)\longrightarrow F_1(\eta)) = F_1(\eta) = H^{2n-1}(PM; Z_2) = Z_2.$$

Further by the second half of Proposition 1.3(2),

$$\operatorname{Im} j_0^{\#} = \operatorname{Im}\delta^{-1}\Theta \tag{4.2}$$

where

$$\Theta = Sq^2\tilde{\rho}_2 + (n-1)v^2\tilde{\rho}_2\colon H^{2n-2}(A^2M, \Delta M; Z[v])\longrightarrow H^{2n}(A^2M, \Delta M; Z_2).$$

Because $H^{2n}(A^2M, \Delta M; Z_2)=Z_2$,

(4.3) *the homomorphism* $\delta\colon H^{2n-1}(PM; Z_2)\to H^{2n}(A^2M, \Delta M; Z_2)$ *in* (4.2) *is an isomorphism.*

We now assume that the integral cohomology groups $H^i(M; Z)$ for $i=n$, $n-1$ are given as in (5.1). Let K_i $(i=1,\dots, 4)$ be the subgroups of $H^{2n-2}(A^2M, \Delta M; Z_2)$ defined as follows:

$$K_1 = \{A\rho_2 x A\rho_2 y \mid x, y\in H^{n-1}(M; Z)\},$$
$$K_2 = \{A\rho_2 x AM \mid x\in H^{n-2}(M; Z)\}, \ (M = \rho_2 M \text{ if } w_1 = 0),$$
$$K_3 = Z_2\langle v^{n-2}AM\rangle,$$
$$K_4 = \textstyle\sum_{i=\alpha+1}^{\beta} Z_2\langle AMA\rho_2 y_i+(r(i)/2)AM'A\rho_2 x_i\rangle \text{ if } w_1\neq 0.$$

LEMMA 4.4. *With the above notation,* $\tilde{\rho}_2 H^{2n-2}(A^2M, \Delta M; Z[v])$ *is*
(1) $\sum_{i=1}^{3} K_i$ *if n is even and* $w_1=0$,
(2) $\sum_{i=1}^{4} K_i$ *if n is even and* $w_1\neq 0$,
(3) K_1+K_2 *if n is odd and* $w_1=0$.

PROOF. (1) Assume that n is even and $w_1=0$. Then $H^{2n-2}(A^2M, \Delta M; Z_2)$ is given by [7, Th. 11] as follows:

$$H^{2n-2}(\Lambda^2M, \Delta M; Z_2) = K_1 + K_2 + K_3 + K_5,$$

where

$$K_5 = \sum_{i=\alpha+1}^{\beta} Z_2\langle \Lambda\rho_2 y_i \Lambda M\rangle .$$

By Lemma 1.5 and the relation $\tilde{\rho}_2\tilde{\beta}_2 = Sq^1 + v$, we have the relations

$$\tilde{\rho}_2\Delta(x, y) = \Lambda\rho_2 x\Lambda\rho_2 y \qquad \text{for} \quad x, y \in H^{n-1}(M; Z),$$

$$\tilde{\rho}_2\Delta(x, M) = \Lambda\rho_2 x\Lambda\rho_2 M = \Lambda\rho_2 x\Lambda M \qquad \text{for} \quad x \in H^{n-2}(M; Z),$$

$$\tilde{\rho}_2\tilde{\beta}_2(v^{n-3}\Lambda M) = v^{n-2}\Lambda M,$$

$$\tilde{\beta}_2(\Lambda\rho_2 y_i\Lambda M) = \tilde{\beta}_2\tilde{\rho}_2\Delta(y_i, \rho_{r(i)}M) = (r(i)/2)\tilde{\beta}_{r(i)}\Delta(y_i, \rho_{r(i)}M),$$

and so

$$K_1 + K_2 + K_3 \subset \tilde{\rho}_2 H^{2n-2}(\Lambda^2M, \Delta M; Z[v]).$$

On the other hand, $(r(i)/2)\tilde{\beta}_{r(i)}\Delta(y_i, \rho_{r(i)}M)$ for $\alpha < i \leqq \beta$ form a base of $\tilde{\beta}_2 H^{2n-2}(\Lambda^2M, \Delta M; Z_2)$ by Proposition 5.3(i). This completes the proof of (1).

(2) Assume that n is even and $w_1 \neq 0$. Then we have, in the same way as the above proof,

$$H^{2n-2}(\Lambda^2M, \Delta M; Z_2) = \sum_{i=1}^{4} K_i + K_6, \quad K_6 = \{\Lambda M'\Lambda x \mid x \in H^{n-1}(M; Z_2)\},$$

and

$$K_1 + K_3 \subset \tilde{\rho}_2 H^{2n-2}(\Lambda^2M, \Delta M; Z[v]).$$

Moreover, we have the relations

$$\tilde{\rho}_2\tilde{\beta}_2(\Lambda\rho_2 x\Lambda M') = \Lambda\rho_2 x\Lambda M,$$

$$\tilde{\rho}_2\tilde{\beta}_2(\Lambda M'\Lambda\rho_2 y_i) = \Lambda M\Lambda\rho_2 y_i + (r(i)/2)\Lambda M'\Lambda\rho_2 x_i \qquad \text{for} \quad \alpha < i \leqq \beta,$$

and so $K_2 + K_4 \subset \tilde{\rho}_2 H^{2n-2}(\Lambda^2M, \Delta M; Z[v])$. On the other hand, we see that $\dim_{Z_2} \tilde{\beta}_2 H^{2n-2}(\Lambda^2M, \Delta M; Z_2) = \beta + 1$ by Proposition 5.3(ii) and $\dim_{Z_2} K_6 = \beta + 1$. This implies (2).

(3) is obtained by the method similar to those of the above cases. q.e.d.

LEMMA 4.5. *$\mathrm{Im}\, j_0^\#(\subset H^{2n-1}(PM; Z_2) = Z_2)$ is given as follows:*

(1) *When $n \equiv 2(4)$ and $w_1 = 0$, $\mathrm{Im}\, j_0^\# = 0$ if and only if $w_2\rho_2 H^{n-2}(M; Z) = 0$.*

(2) *When $n \equiv 2(4)$ and $w_1 \neq 0$, $\mathrm{Im}\, j_0^\# = 0$ if and only if $w_2 + w_1^2 = 0$.*

(3) *When $n \equiv 1(2)$ and $w_1 = 0$, $\mathrm{Im}\, j_0^\# = 0$ if and only if $w_2\rho_2 H^{n-2}(M; Z) = 0$.*

PROOF. The $(Sq^2 + (n-1)v^2)$-image of K_i $(i = 1, \dots, 4)$ are easily obtained by using [7, Lemmas 7 and 10] as follows:

$$(Sq^2+(n-1)v^2)(K_1+K_3)=0,$$
$$(Sq^2+(n-1)v^2)K_2=\{\Lambda Sq^2\rho_2 x\Lambda M \mid x\in H^{n-2}(M; Z)\},$$
$$(Sq^2+v^2)(\Lambda M\Lambda\rho_2 y_i+(r(i)/2)\Lambda M'\Lambda\rho_2 x_i)=\Lambda M\Lambda Sq^2\rho_2 y_i \quad (\alpha<i\leqq\beta).$$

Using these relations and the well-known fact that $Sq^2x=(w_2+w_1^2)x$ for $x\in H^{n-2}(M; Z_2)$, we have

$$\begin{aligned}\Theta H^{2n-2}(\Lambda^2M, \Delta M; Z[v]) &= \{\Lambda w_2\rho_2 x\Lambda M \mid x\in H^{n-2}(M; Z)\} \quad \text{in cases (1) and (3),}\\ &= \{\Lambda(w_2+w_1^2)x\Lambda M \mid x\in H^{n-2}(M; Z_2)\} \quad \text{in case (2).}\end{aligned}$$

This and (4.3) show the lemma. q.e.d.

We are now ready to determine $\operatorname{Im} J_{2n-1}$ for $n\equiv 1(2)$ and $w_1=0$, or $n\equiv 2(4)$.

PROPOSITION 4.6. (1) *Assume that* $n\equiv 2(4)$ *and* $w_1=0$. *Then*

$$\operatorname{Im} J_{2n-1}=\begin{cases}[M\subseteq R^{2n-1}] & \text{if } w_2\rho_2H^{n-2}(M; Z)\neq 0,\\ \theta H^{n-1}(M; Z_2) & \text{otherwise.}\end{cases}$$

(2) *Assume that* $n\equiv 2(4)$ *and* $w_1\neq 0$. *Then*

$$\operatorname{Im} J_{2n-1}=\begin{cases}\theta K+Z_2 & \text{if } w_2+w_1^2\neq 0,\\ \theta K & \text{otherwise.}\end{cases}$$

(3) *Assume that* $n\equiv 1(2)$ *and* $w_1=0$. *Then*

$$\begin{aligned}\operatorname{Im} J_{2n-1} &= \theta\beta_2H^{n-2}(M; Z_2) && \text{if } w_2\rho_2H^{n-2}(M; Z)=0,\\ &= \theta\beta_2H^{n-2}(M; Z_2)+0+Z_2 && \text{if } n\equiv 1(4) \text{ and } w_2\rho_2H^{n-2}(M; Z)\neq 0,\\ &= \theta\beta_2H^{n-2}(M; Z_2)+Z_2 && \text{otherwise.}\end{aligned}$$

PROOF. This is an immediate consequence of Lemmas 3.1, 3.2, 4.5 and (2.7). q.e.d.

Propositions 4.1 and 4.6 give the results on J_{2n-1} in Main Theorem. Thus Main Theorem in the introduction is proved.

§5. Appendix on $H^i(\Lambda^2M, \Delta M; Z[v])$ for $i\geqq 2n-3$

In the previous sections, the cohomology of $(\Lambda^2M, \Delta M)$ plays an important part. L. L. Larmore [7] investigated it but the author can not understand the proof of [7, Th. 20]. Therefore we should like to try to describe the cohomology groups $H^{2n}(\Lambda^2 M, \Delta M; Z)$ and $H^i(\Lambda^2 M, \Delta M; Z[v])$ for $i \geqq 2n-3$ by using the notations and the results stated in [7, pp. 908–915]. We note that Propositions 5.4–5 for $i=2n-2$, $2n-3$ are not used in this paper and are prepared for the forthcoming paper [14].

Let M be a closed connected n-manifold and assume that

$$(5.1) \qquad H^n(M; Z) = Z\langle M\rangle \text{ if } w_1=0, \quad = Z_2\langle \beta_2 M'\rangle \ (Sq^1 M' = M) \text{ if } w_1 \neq 0,$$

$$H^m(M; Z) = \textstyle\sum_{i=1}^{\gamma(m)} Z_{r(m,i)}\langle x_{m,i}\rangle \text{ (direct sum) for } m \leqq n-1,$$

$$x_{m,i} = \beta_{r(m,i)} y_{m,i} \ (y_{m,i} \in H^{m-1}(M; Z_{r(m,i)}) \text{ for } \alpha(m) < i \leqq \gamma(m),$$

where the order $r(m, i)$ is infinite for $1 \leqq i \leqq \alpha(m)$, a power of 2 for $\alpha(m) < i \leqq \beta(m)$ and a power of an odd prime for $\beta(m) < i \leqq \gamma(m)$, and if $\alpha(m) < i < j$ then either $(r(m, i), r(m, j)) = 1$ or $r(m, i) \mid r(m, j)$ holds.

Furthermore, for the simplicity,

(5.1)′ denote $\alpha(m)$, $\beta(m)$, $\gamma(m)$, $r(m, i)$, $x_{m,i}$ and $y_{m,i}$ in (5.1) respectively by

$$\alpha,\ \beta,\ \gamma,\ r(i),\ x_i \text{ and } y_i \qquad \text{when } m=n-1,$$

$$\alpha',\ \beta',\ \gamma',\ r'(i),\ x'_i \text{ and } y'_i \qquad \text{when } m=n-2.$$

Then we have the following propositions, where (i)'s,..., (iv)'s hold respectively when

(i) n is even and $w_1=0$, (ii) n is even and $w_1 \neq 0$,

(iii) n is odd and $w_1=0$, (iv) n is odd and $w_1 \neq 0$.

PROPOSITION 5.2. (1) *$H^{2n}(\Lambda^2 M, \Delta M; Z[v])$ is*

(i) $Z_2\langle \tilde{\beta}_2(v^{n-1}\Lambda\rho_2 M)\rangle$, (ii) $Z_2\langle \tilde{\beta}_2(v^{n-1}\Lambda M)\rangle$,

(iii) $Z\langle \Delta(M, M)\rangle$, (iv) $Z_2\langle \tilde{\beta}_2(\Lambda M'\Lambda M)\rangle$.

(2) *$H^{2n}(\Lambda^2 M, \Delta M; Z)$ is*

(i) $Z\langle \Lambda M\Lambda M\rangle$, (ii) $Z_2\langle \beta_2(\Lambda M'\Lambda M)\rangle$,

(iii) $Z_2\langle \beta_2(v^{n-1}\Lambda\rho_2 M)\rangle$, (iv) $Z_2\langle \beta_2(v^{n-1}\Lambda M)\rangle$.

PROPOSITION 5.3. *$H^{2n-1}(\Lambda^2 M, \Delta M; Z[v])$ is*

(i) G, (ii) $Z_4\langle (1/2)\tilde{\beta}_2(v^{n-1}\Lambda M')\rangle + K$,

(iii) $Z_2\langle \tilde{\beta}_2(v^{n-2}\Lambda\rho_2 M)\rangle + G$,

(iv) $Z_2\langle \tilde{\beta}_2(v^{n-2}\Lambda M)\rangle + K$, *and* $\tilde{\rho}_2$: $H^{2n-1}(\Lambda^2 M, \Delta M; Z[v]) \to H^{2n-1}(\Lambda^2 M, \Delta M; Z_2)$ *is monomorphic,* *where*

$$G=\sum_{i=1}^{\alpha} Z\langle\Delta(x_i, M)\rangle+\sum_{i=\alpha+1}^{\gamma} Z_{r(i)}\langle\tilde{\beta}_{r(i)}\Delta(y_i, \rho_{r(i)}M)\rangle\ (\cong H^{n-1}(M; Z)),$$

$$K=\sum_{i=1}^{\beta} Z_2\langle\tilde{\beta}_2\Delta(M', \rho_2 x_i)\rangle\ (\cong \mathrm{Im}\,(\rho_2: H^{n-1}(M; Z)\longrightarrow H^{n-1}(M; Z_2))).$$

PROPOSITION 5.4. *$H^{2n-2}(\Lambda^2M, \Delta M; Z[v])$ is*

(i) $Z_2\langle\tilde{\beta}_2(v^{n-3}\Lambda\rho_2M)\rangle + G_1 + G_2 + G_3 + G_4 + G_6$,

(ii) $Z_2\langle\tilde{\beta}_2(v^{n-3}\Lambda M)\rangle + G_1 + G_2 + G_3 + G_4 + G_7$,

(iii) $G_1 + G_3 + G_5 + G_6$,

(iv) $Z_2\langle\tilde{\beta}_2(v^{n-2}\Lambda M')\rangle + G_1 + G_3 + G_5 + G_7$,

where

$$G_1 = \sum_{1\leqq i<j\leqq\alpha} Z\langle\Delta(x_i, x_j)\rangle, \qquad G_2 = \sum_{i=1}^{\alpha} Z\langle\Delta(x_i, x_i)\rangle,$$

$$G_3 = (\sum_{1\leqq i\leqq\alpha<j\leqq\gamma} + \sum_{\alpha<j<i\leqq\gamma})Z_{r(j)}\langle\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_i)\rangle,$$

$$G_4 = \sum_{i=\alpha+1}^{\gamma} Z_{r(i)}\langle\tilde{\beta}_{r(i)}\Delta(y_i, \rho_{r(i)}x_i)\rangle, \qquad G_5 = \sum_{i=1}^{\beta} Z_2\langle\tilde{\beta}_2(v^{n-2}\Lambda\rho_2 x_i)\rangle,$$

$$G_6 = \sum_{k=1}^{\alpha'} Z\langle\Delta(x'_k, M)\rangle + \sum_{k=\alpha'+1}^{\gamma'} Z_{r'(k)}\langle\tilde{\beta}_{r'(k)}\Delta(y'_k, \rho_{r'(k)}M)\rangle\ (\cong H^{n-2}(M; Z)),$$

$$G_7 = \sum_{k=1}^{\beta'} Z_2\langle\tilde{\beta}_2\Delta(M', \rho_2 x'_k)\rangle + \sum_{i=\alpha+1}^{\beta} Z_2\langle\tilde{\beta}_2\Delta(M', \rho_2 y_i)\rangle\ (\cong H^{n-2}(M; Z_2)).$$

PROPOSITION 5.5. *$\tilde{\rho}_2H^{2n-3}(\Lambda^2M, \Delta M; Z[v])/\delta H^{2n-4}(PM; Z_2)$ is isomorphic to*

(i) H, (ii) $H + H_4$, (iii) $H + H_5$, (iv) $H + H_4 + H_5$,

where

$$H = H_1 + H_2 + H_3,$$

$$H_1 = \begin{cases} \{\Lambda\rho_2 x\Lambda\rho_2 M \mid x\in H^{n-3}(M; Z)\} & \text{if } w_1 = 0, \\ \{\Lambda\rho_2 x\Lambda M \mid x\in H^{n-3}(M; Z)\} & \text{if } w_1 \neq 0, \end{cases}$$

$$H_2 = \{\Lambda\rho_2 x\Lambda\rho_2 y \mid x\in H^{n-2}(M; Z),\ y\in H^{n-1}(M; Z)\},$$

$$H_3 = \sum_{\alpha<i<j\leqq\beta} Z_2\langle\Lambda\rho_2 x_i\Lambda\rho_2 y_j+(r(j)/r(i))\Lambda\rho_2 y_i\Lambda\rho_2 x_j\rangle,$$

$$H_4 = \sum_{k=\alpha'+1}^{\beta'} Z_2\langle\Lambda\rho_2 y'_k\Lambda M+(r'(k)/2)\Lambda\rho_2 x'_k\Lambda M'\rangle,$$

$$H_5 = \sum_{i=\alpha+1}^{\beta} Z_2\langle\Lambda\rho_2 y_i\Lambda\rho_2 x_i\rangle.$$

To prove these propositions, we use the following results frequently:

(5.6) ([7, p. 914]) *For any cyclic group G, there is an exact sequence*

$$\cdots\longrightarrow H^{i-1}(\Lambda^2M, \Delta M; G)\xrightarrow{V\cdot} H^i(\Lambda^2M, \Delta M; G[v])$$

$$\xrightarrow{\pi^*} H^i(M^2, \Delta M; G)\longrightarrow H^i(\Lambda^2M, \Delta M; G)\longrightarrow\cdots,$$

where $\pi: (M^2, \Delta M) \to (A^2M, \Delta M)$ *is the natural projection,* v *is the first Stiefel-Whitney class of the double covering* $M^2 - \Delta M \to A^2M - \Delta M = M^*$ *and* $V = \tilde{\beta}_2(1) \in H^1(M^*; Z[v])$.

(5.7) *For any positive integer p, there is the Bockstein exact sequence*

$$\cdots \longrightarrow H^{i-1}(A^2M, \Delta M; Z_p[v]) \xrightarrow{\tilde{\beta}_p} H^i(A^2M, \Delta M; Z[v]) \xrightarrow{\times p} H^i(A^2M, \Delta M; Z[v]) \xrightarrow{\tilde{\rho}_p} H^i(A^2M, \Delta M; Z_p[v]) \xrightarrow{\tilde{\beta}_p} \cdots.$$

(5.8) ([7, Remark 13]) *For any odd prime* p, $\pi^*: H^*(A^2M, \Delta M; Z_p[v]) \to H^*(M^2, \Delta M; Z_p)$ *is monomorphic and*

$$\operatorname{Im} \pi^* = \{x \mid x \in H^*(M^2, \Delta M; Z_p),\ t^*x = -x\},$$

where $t: (M^2, \Delta M) \to (M^2, \Delta M)$ *is a map defined by* $t(x, y) = (y, x)$.

(5.9) (cf. [7, p. 914]) *For* $x \in H^r(M; Z_t)$ *and* $y \in H^s(M; Z_t)$ $(t \leqq \infty)$,

$$\pi^*\Delta(x, y) = x \otimes y - (-1)^{rs} y \otimes x \quad \textit{and} \quad \pi^* \Lambda x = x \otimes 1 - 1 \otimes x,$$

and moreover the order of $\Delta(x, y)$ *for* $x \neq y$ *is the greatest common factor of those of* x *and* y, *and the order of* Λx *is equal to that of* x.

We now sketch the proofs of Propositions 5.2–5.

By (5.6), the following relation holds:

$$\operatorname{rank} H^i(A^2M, \Delta M; Z[v]) + \operatorname{rank} H^i(A^2M, \Delta M; Z) = \operatorname{rank} H^i(M^2, \Delta M; Z).$$

By using (5.9) and Lemma 1.5(1), we can choose generators (mod torsions) of $H^i(A^2M, \Delta M; Z[v])$ and $H^i(A^2M, \Delta M; Z)$. In particular we have

LEMMA 5.10. *There hold the following congruences mod torsions*:

(1) $H^{2n}(A^2M, \Delta M; Z) \equiv \begin{cases} Z\langle \Lambda M \Lambda M \rangle & \textit{if } n \textit{ is even and } w_1 = 0, \\ 0 & \textit{otherwise.} \end{cases}$

(2) $H^{2n}(A^2M, \Delta M; Z[v]) \equiv \begin{cases} Z\langle \Delta(M, M) \rangle & \textit{if } n \textit{ is odd and } w_1 = 0, \\ 0 & \textit{otherwise.} \end{cases}$

(3) $H^{2n-1}(A^2M, \Delta M; Z[v]) \equiv \begin{cases} \sum_{i=1}^{\alpha} Z\langle \Delta(x_i, M) \rangle & \textit{if } w_1 = 0, \\ 0 & \textit{if } w_1 \neq 0. \end{cases}$

(4) $H^{2n-2}(A^2M, \Delta M; Z[v])$ *is congruent mod torsion to the direct sum of* G_1 *and*

$$\begin{array}{ll} G_2 & \textit{if } n \textit{ is even}, \\ \sum_{k=1}^{\alpha'} Z\langle \Delta(x'_k, M) \rangle & \textit{if } w_1 = 0. \end{array}$$

To determine the odd torsion subgroup of $H^i(\Lambda^2M, \Delta M; Z[v])$, let p be an odd prime. Then the Z_p-base of $H^i(\Lambda^2M, \Delta M; Z_p[v])$ can be determined by (5.8–9). Thus the p-primary component and its generators of $H^i(\Lambda^2M, \Delta M; Z[v])$ are determined by the exact sequence (5.7) for odd prime p and [7, Remark 16]. In particular we have

LEMMA 5.11. *Denote by T_o^i the odd torsion subgroup of $H^i(\Lambda^2M, \Delta M; Z[v])$. Then*

(1) $T_o^{2n} = 0$;

(2) $T_o^{2n-1} = \begin{cases} (G)_o = \sum_{i=\beta+1}^{\gamma} Z_{r(i)}\langle\tilde{\beta}_{r(i)}\Delta(y_i, \rho_{r(i)}M)\rangle & \text{if } w_1 = 0, \\ 0 & \text{if } w_1 \neq 0; \end{cases}$

(3) T_o^{2n-2} *is the direct sum of*

$$(G_3)_o = (\textstyle\sum_{1\leqq i\leqq\alpha,\beta<j\leqq\gamma} + \sum_{\beta<j<i\leqq\gamma})Z_{r(j)}\langle\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_i)\rangle$$

and

$$(G_4)_o = \textstyle\sum_{i=\beta+1}^{\gamma} Z_{r(i)}\langle\tilde{\beta}_{r(i)}\Delta(y_i, \rho_{r(i)}x_i)\rangle \qquad \text{if } n \text{ is even},$$

$$(G_6)_o = \textstyle\sum_{k=\beta'+1}^{\gamma'} Z_{r'(k)}\langle\tilde{\beta}_{r'(k)}\Delta(y'_k, \rho_{r'(k)}M)\rangle \qquad \text{if } w_1 = 0.$$

The proof of (1) of Proposition 5.2 is given by using Lemmas 5.10–11, (5.7) for $p=2$ and [7, Th. 11], and that of (2) is given by using the ordinary Bockstein exact sequence instead of (5.7).

In the rest of this section, we study the 2-primary components of $H^i(\Lambda^2M, \Delta M; Z[v])$ for $2n-3\leqq i\leqq 2n-1$. First we consider the case (ii) n is even and $w_1\neq 0$. By (5.7) for $p=2$, Lemmas 5.10–11 and [7, Th. 11], we have

$$H^{2n-1}(\Lambda^2M, \Delta M; Z[v]) = K + Z_s \ (K = \textstyle\sum_{i=1}^{\beta} Z_2\langle\tilde{\beta}_2\Delta(M', \rho_2x_i)\rangle),$$

$$\tilde{\rho}_2Z_s = Z_2\langle v^{n-1}\Lambda M + \Lambda M'\Lambda M\rangle, \quad \text{for some integer} \quad s \geqq 2.$$

In the exact sequence (3.3), both groups $H^{2n-2}(PM; Z[v])$ and $H^{2n-1}(M^*; Z[v])$ are isomorphic to $H^{n-1}(M; Z_2)$ by Proposition 2.4 and (3.4) and so $s\leqq 4$. On the other hand,

$$\tilde{\rho}_2\tilde{\beta}_2H^{2n-2}(\Lambda^2M, \Delta M; Z_2)\not\ni v^{n-1}\Lambda M + \Lambda M'\Lambda M$$

follows briefly. Thus $s\geqq 4$ and so $s=4$. Moreover by (5.7) for $p=2$, (5.9) and Lemmas 1.5 and 5.10, we see that

$$Z_s = Z_4\langle(1/2)\tilde{\beta}_2(v^{n-1}\Lambda M')\rangle,$$

and

$$H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]) \equiv Z_2\langle\tilde{\beta}_2(v^{n-3}\Lambda M)\rangle + G_1 + G_2 + G_7$$
$$+ (\textstyle\sum_{1\leqq i\leqq\alpha<j\leqq\beta} + \sum_{\alpha<j<i\leqq\beta})Z_{r(j)}\langle\Delta(x_j, x_i)\rangle + \sum_{j=\alpha+1}^{\beta} Z_{s(j)}\langle\Delta(x_j, x_j)\rangle$$
$$\text{mod odd torsion,}$$

where $s(j)$ is the order of $\Delta(x_j, x_j)$. As for the element $\Delta(x_j, x_i)$, if $i\neq j$ then $\pi^*\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_i)=\pi^*\Delta(x_j, x_i)$ by (5.9) and hence

$$\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_i) = \Delta(x_j, x_i) + VX_{j,i} \text{ for some } X_{j,i}\in H^{2n-3}(\Lambda^2 M, \Delta M; Z)$$

by (5.6). Further we see easily that

$$\tilde{\rho}_2\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_i) = \Lambda\rho_2 x_j\Lambda\rho_2 x_i + v\rho_2 X_{j,i} \neq 0$$

by Lemma 1.5. Therefore we can replace $\Delta(x_j, x_i)$ by $\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_i)$. If $i=j$ and $r(j)=2$, then

$$\tilde{\rho}_2\tilde{\beta}_2\Delta(y_j, \rho_2 x_j) = \Lambda\rho_2 x_j\Lambda\rho_2 x_j = \tilde{\rho}_2\Delta(x_j, x_j)$$

by Lemma 1.5, and so $s(j)=2$ and $\Delta(x_j, x_j)$ can be replaced by $\tilde{\beta}_2\Delta(y_j, \rho_2 x_j)$. If $i=j$ and $r(j)\geqq 4$, then we see easily that

$$\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_j) = \Delta(x_j, x_j) + VY_j \quad (Y_j\in H^{2n-3}(\Lambda^2 M, \Delta M; Z))$$

by (5.6) and (5.9), and that

$$\tilde{\beta}_2(\Lambda\rho_2 y_j\Lambda\rho_2 x_j) \neq 0$$

by (5.7) for $p=2$. Using Lemma 1.5 and the relation $\tilde{\beta}_2\tilde{\rho}_2=(r/2)\tilde{\beta}_r$: $H^{i-1}(\Lambda^2 M, \Delta M; Z_r[v])\to H^i(\Lambda^2 M, \Delta M; Z[v])$, we see that

$$(r(j)/2)\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_j) = \tilde{\beta}_2\tilde{\rho}_2\Delta(y_j, \rho_{r(j)}x_j)\,.$$

The above three relations imply that $s(j)=r(j)$ and $\Delta(x_j, x_j)$ can be replaced by $\tilde{\beta}_{r(j)}\Delta(y_j, \rho_{r(j)}x_j)$. This completes the proofs of (ii)'s of Propositions 5.3–4. The proof of Proposition 5.5(ii) is given by Lemma 1.5(3) and (5.7) for $p=2$ immediately.

The proofs of (i)'s, (iii)'s and (iv)'s of Propositions 5.3–5 are similar to, but simpler than, those of (ii)'s except the results concerning H_5 of Proposition 5.5 for odd n.

Let n be odd. By simple calculations, using Lemma 1.5, Proposition 5.4 and (5.9), we see that

$$\tilde{\beta}_2(\Lambda\rho_2 y_j\Lambda\rho_2 x_j)\in \operatorname{Ker}\pi^* \qquad (\subset H^{2n-2}(\Lambda^2 M, \Delta M; Z[v]))$$

and

$$\operatorname{Ker}\pi^* = \{\tilde{\beta}_2(v^{n-2}\Lambda x)\,|\, x\in H^{n-1}(M; Z_2)\}\,.$$

This implies that there is an element $X_j \in H^{n-1}(M; Z_2)$ such that

$$\Lambda\rho_2 y_j \Lambda\rho_2 x_j + v^{n-2}\Lambda X_j \in \operatorname{Im} \tilde{\rho}_2.$$

Using this result, Lemma 1.5(3) and (5.7) for $p=2$, we have Proposition 5.5(iii)–(iv) completely.

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