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Martin boundary for $\Delta - P$

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§1. One of interesting problems concerning the Martin boundary is to determine when it is homeomorphic to the Euclidean boundary. In the present note, we give sufficient conditions for the Martin boundary of a bounded domain Ω in the n-dimensional Euclidean space $\boldsymbol{R}^n$ ($n \geqq 2$) with respect to the operator of the form $L_P = \Delta - P$ to be homeomorphic to the Euclidean boundary $\partial\Omega$, where Δ denotes the Laplace operator and P is a non-negative locally Hölder continuous function on Ω.

We say that a domain Ω has *bounded curvature* if there exists a positive number d such that for each point $Y \in \partial\Omega$ there exist points x_y and x'_y such that

$$B(x_y, d) \subset \Omega,\ B(x'_y, d) \subset C\bar{\Omega} \quad \text{and} \quad Y \in \partial B(x_y, d) \cap \partial B(x'_y, d),$$

where $B(x, d)$ denotes the open ball in $\boldsymbol{R}^n$ of radius $d > 0$ centered at x. We call d an *admissible radius* of Ω. We define a function δ_Ω on Ω by

$$\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega).$$

Our main result is the following

THEOREM 1. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ of bounded curvature. Suppose that a non-negative locally Hölder continuous function P on Ω satisfies the following condition* (*a*) *or* (*b*):

(*a*) *Ω is of $C^{1,\alpha}$-class* ($\alpha > 0$) *and*

$$\int_0 r\{\max_{\delta_\Omega(x) \geqq r} P(x)\} dr < \infty.$$

(*b*) *$P \in L^q(\Omega)$ for some $q > n/2$.*

Then the Martin boundary of Ω with respect to $L_P = \Delta - P$ is homeomorphic to the Euclidean boundary.

Related results have been given by A. Ancona ([1], Théorème 6) and H. Imai ([5], Theorem 2). Ancona showed the equivalence of the Martin boundary and the Euclidean boundary of a bounded Lipschitz domain in the half-space $\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n;\ x_n > 0\}$ with respect to an elliptic operator L of the form

$$Lu(x) = \sum a_{ij}(x) u_{ij}(x) + x_n^{-1} \sum b_i(x) u_i(x) + x_n^{\beta-2} c(x) u(x),$$

where $0<\beta<1$ and a_{ij}, b_i and c are Hölder continuous functions on $\overline{\boldsymbol{R}^n_+}$. On the other hand, Imai proved the above equivalence for an annulus $\{x\in\boldsymbol{R}^n; \lambda<|x|<1\}$ with respect to the elliptic operator L_P with any rotation free P.

In the last section, we give an example of P on a ball for which the conclusion of Theorem 1 does not hold. This example shows that conditions (a) and (b) in Theorem 1 are fairly sharp.

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§**2.** Consider the Green function $G_\Omega(x, y)$ on the domain Ω with respect to Δ. The Green function $G_\Omega^P(x, y)$ on Ω with respect to L_P exists and the equality

$$G_\Omega(x, y) = G_\Omega^P(x, y) + \int G_\Omega^P(x, z)P(z)G_\Omega(z, y)dz \tag{1}$$

holds. Note that $G_\Omega^P(x, y)=G_\Omega^P(y, x)$ and $G_\Omega(x, y)\geqq G_\Omega^P(x, y)$.

By the general theory of Martin compactification, in order to obtain the equivalence of the Martin boundary and the Euclidean boundary it is sufficient to show that quotients of the Green functions have continuous extension to the Euclidean closure and they separate points of the Euclidean boundary (cf. [3], ch. 12, [6] and [9]). Hence if the Green functions G_Ω and G_Ω^P are comparable, we can reduce the problem for L_P to that for Δ, that is to be exact, we have the following

THEOREM 2. *Let Ω be a bounded domain and suppose that the following two conditions are satisfied*:

($*$) *The Martin boundary of Ω with respect to Δ is homeomorphic to the Euclidean boundary.*

($**$) *For a fixed point x_0 in Ω, there exists a positive constant A satisfying*

$$\frac{G_\Omega(x_0, y)}{G_\Omega^P(x_0, y)} \leqq A \quad \textit{for all } y\in\Omega.$$

Then the Martin boundary of Ω with respect to L_P is homeomorphic to the Euclidean boundary.

PROOF. For each Y in $\partial\Omega$, we put

$$M_Y = \left\{ u;\ \begin{array}{l} \text{there exists a sequence } (y_n)_{n=1}^\infty \text{ in } \Omega \text{ with} \\ \lim_{n\to\infty} y_n = Y \text{ such that } \lim_{n\to\infty} G_\Omega^P(x, y_n)/G_\Omega^P(x_0, y_n) \\ = u(x) \text{ for every } x\in\Omega. \end{array} \right\}$$

Note that $M_Y\neq\emptyset$ and $L_P u=0$ for any u in M_Y. As remarked in the beginning of the preceding paragraph, to obtain Theorem 2 it is sufficient to show that

M_Y consists of a single element and that $M_{Y_1}\neq M_{Y_2}$ if $Y_1\neq Y_2$. Let $u_j\in M_Y$ $(j=1, 2)$. By (**), we get

$$\frac{G_\Omega^P(x, y)}{G_\Omega^P(x_o, y)} \leqq A\,\frac{G_\Omega(x, y)}{G_\Omega(x_o, y)} \quad \text{on } \Omega$$

for every $y\in\Omega$. It follows from (*) that $G_\Omega(x, y)/G_\Omega(x_o, y)$ converges to a minimal Δ-harmonic function v_Y as y tends to Y (a positive Δ-harmonic function v on Ω is said to be *minimal* if for any non-negative Δ-harmonic function h, $v\geqq h$ on Ω implies $h=cv$ for some $c\geqq 0$). Hence we have

$$u_j \leqq Av_Y \quad (j=1, 2).$$

Now put $s_j=Av_Y-u_j$ $(j=1, 2)$. Then it is clear that s_j is a non-negative Δ-superharmonic function so that, by the Riesz decomposition theorem, there exist a non-negative Borel measure μ_j on Ω and a non-negative Δ-harmonic function h_j on Ω such that $s_j(x)=\int G_\Omega(x, z)d\mu_j(z)+h_j(x)$. Since $Av_Y\geqq h_j$ and v_Y is minimal, we get $h_j=c_jv_Y$ for some $c_j\geqq 0$. Furthermore $d\mu_j(x)=P(x)u_j(x)dx$, because $\Delta u_j(x)=P(x)u_j(x)$. Thus there exist $A_j>0$ such that

$$u_j(x) = A_jv_Y(x) - \int G_\Omega(x, z)P(z)u_j(z)dz \quad (j=1, 2). \tag{2}$$

Eliminating v_Y from (2), we obtain

$$A_1u_2(x) - A_2u_1(x) = \int G_\Omega(x, z)P(z)(A_2u_1(z) - A_1u_2(z))dz.$$

Since G_Ω satisfies the domination principle, the above equality gives $A_1u_2=A_2u_1$. Therefore $u_1(x_o)=u_2(x_o)=1$ implies $u_1=u_2$ on Ω. Furthermore by (*) and (2) we see that $M_{Y_1}\neq M_{Y_2}$ if $Y_1\neq Y_2$. This completes the proof.

REMARK 3. The above result holds also for more general second order elliptic differential operators.

As for a parabolic operator, we make the following

REMARK 4. Let D be a bounded mixed-Lipschitz domain in $\boldsymbol{R}^n\times\boldsymbol{R}$ as defined in [8]. Let $(X, T)\in D$ and put $D_T=D\cap\{(x, t); t<T\}$. Then by a theorem of J. T. Kemper ([8], Theorem 1.10), we can interpret the parabolic boundary ∂_pD_T as the Martin boundary of D_T with respect to the heat equation. Here the *parabolic boundary* ∂_pD_T is the set consisting of all points on the Euclidean boundary of D_T which can be connected to some point in the interior by a continuous curve along which time t decrease as the boundary point is approached. Analogously to Theorem 2, we see the following: Let $P(x, t)$ be a non-negative locally Hölder continuous function on D with $P(x, t)\leqq M$ for some constant

$M>0$. Then the Martin boundary of D_T with respect to $H_P=\Delta-\partial/\partial t-P(x, t)$ is identified with the parabolic boundary $\partial_p D_T$. In fact, denoting by $G_{D_T}((x, t), (y, s))$, $G_{D_T}^P((x, t), (y, s))$ and $G_{D_T}^M((x, t), (y, s))$ the Green functions on D_T with respect to Δ-$\partial/\partial t$, H_P and H_M respectively, we have $G_{D_T}\geqq G_{D_T}^P\geqq G_{D_T}^M$ and $G_{D_T}^M((x, t), (y, s))=e^{M(s-t)}G_{D_T}((x, t), (y, s))$. Therefore $G_{D_T}/G_{D_T}^P$ is bounded and hence an argument similar to the proof of Theorem 2 leads to the above conclusion. As for the potential theoretic properties of the heat equation, we refer to [12].

§3. In this section, we prepare some lemmas which we shall require in the the proof of Theorem 1. We prove first the following

LEMMA 5. *Let Ω be a bounded domain of bounded curvature and let $x_0 \in \Omega$. Then there exist positive numbers c_0 and d_0 such that*

$$c_0^{-1}\delta_\Omega(y) \leqq G_\Omega(x_0, y) \leqq c_0\delta_\Omega(y) \tag{3}$$

for any $y \in \Omega$ with $\delta_\Omega(y)\leqq d_0$.

PROOF. Put $d_1=(1/2)\min(\delta_\Omega(x_0), d_\Omega)$, where d_Ω is an admissible radius of Ω, and set $F=\{y \in \Omega;\ \delta_\Omega(y) \leqq d_1\}$. For any $y \in F$, there exist $Y_y \in \partial\Omega$, $x_y \in \Omega$ and $x'_y \in C\bar{\Omega}$ such that $\delta_\Omega(y)=\operatorname{dist}(y, Y_y)$, $B(x_y, d_\Omega)\subset\Omega$, $B(x'_y, d_\Omega)\subset C\bar{\Omega}$ and $Y_y \in \partial B(x_y, d_\Omega) \cap \partial B(x'_y, d_\Omega)$. It is easy to see that y lies on the segment connecting x_y and Y_y. We now choose a domain U and a positive number d_0 satisfying

$$\{x \in \Omega;\ \delta_\Omega(x) \geqq d_1\} \subset U \subset \{x \in \Omega;\ \delta_\Omega(x) > d_0\}.$$

Since a set $\{x_y;\ y \in F\}$ is contained in the compact set $\{x \in \Omega;\ \delta_\Omega(x)\geqq d_1\}$, it follows from the Harnack inequality that there exists $c>0$ satisfying

$$c^{-1}G_\Omega(x_y, y) \leqq G_\Omega(x_0, y) \leqq cG_\Omega(x_y, y)$$

for any y with $\delta_\Omega(y)\leqq d_0$. Since $B=B(x_y, d_\Omega)\subset\Omega\subset D=C\overline{B(x'_y, d_\Omega)}$,

$$c^{-1}G_B(x_y, y) \leqq G_\Omega(x_0, y) \leqq cG_D(x_y, y).$$

Describing the function G_B and G_D explicitly and evaluating the values $G_B(x_y, y)$ and $G_D(x_y, y)$, we obtain our assertion without difficulty.

LEMMA 6. *Let Ω be a bounded domain of $C^{1,\alpha}$-class ($\alpha>0$) and let $x_0 \in \Omega$. Set $\Omega_r=\{x \in \Omega;\ G_\Omega(x_0, x)>r\}$. Then there exist positive numbers r_0 and c'_0 such that for any r with $0\leqq r\leqq r_0$, Ω_r is a domain and*

$$-(\partial/\partial n_y)G_{\Omega_r}(x_0, y) \geqq c_0'^{-1} \qquad \text{for any} \quad y \in \partial\Omega_r, \tag{4}$$

where $\partial/\partial n$ denotes the exterior normal derivative. Furthermore

$$dS_r \leqq c'_0 d\omega_0^r, \tag{5}$$

where dS_r is the surface element on $\partial\Omega_r$ and $d\omega_0^r$ is the harmonic measure on $\partial\Omega_r$ at $x_0 \in \Omega_r$.

In fact, since $G_{\Omega_r}(x_0, x) = G_\Omega(x_0, x) - r$, we see that the function $-(\partial/\partial n_y)G_{\Omega_r}\cdot(x_0, y)$ $(y \in \partial\Omega_r)$ is continuous and positive on $\{y \in \bar{\Omega};\ G_\Omega(x_0, y) \leqq r_0\}$ (cf. [13], Theorems 2.4 and 2.5). Hence there exists $c_0' > 0$ which satisfies (4). By the equality

$$d\omega_0^r(y) = -(\partial/\partial n_y)G_{\Omega_r}(x_0, y)dS_r(y)\ (0 < r < r_0)$$

(see [6], p. 310) and (4), we obtain (5) immediately.

The following lemmas are given in [2].

LEMMA 7 ([2], *Theorem* 1). *Let Ω be a bounded domain of bounded curvature. If $u>0$ is Δ-superharmonic in Ω, $x_0 \in \Omega$ and $0<q<n/(n-1)$, then $u \in L^q(\Omega)$ and*

$$\int u^q(x)dx \leqq A_1 u^q(x_0),$$

where A_1 is a positive constant depending only on Ω, x_0 and q.

LEMMA 8 ([2], *Theorem* 3). *Let Ω be a bounded domain of bounded curvature. If $u>0$ is Δ-superharmonic in Ω, $x_0 \in$ and $0<q<n/(n-2)$ (if $n=2$, the last term means ∞), then $\delta_\Omega u \in L^q(\Omega)$ and*

$$\int (\delta_\Omega(x)u(x))^q dx \leqq A_2 u^q(x_0),$$

where A_2 is a positive constant depending only on Ω, x_0 and q.

§4. Keeping the notation of previous sections, we devote ourselves to the proof of Theorem 1 in this section. Let us assume that Ω and P satisfy all the conditions in Theorem 1. Then it is well-known that the Martin boundary of Ω with respect to Δ is homeomorphic to the Euclidean boundary (see, for example, [4]) and hence by Theorem 2 it is sufficient to show the boundedness of $G_\Omega(x_0, x)/G_\Omega^P(x_0, x)$. For this, we first show that for any $Y \in \partial\Omega$ and any $(y_n)_{n=1}^\infty$ in Ω with $\lim_{n\to\infty} y_n = Y$

$$(6) \qquad \lim_{n\to\infty} \int G_\Omega^P(x_0, z)P(z)\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}dz = \int G_\Omega^P(x_0, z)P(z)v_Y(z)dz,$$

where $v_Y(z) = \lim_{n\to\infty} G_\Omega(z, y_n)/G_\Omega(x_0, y_n)$. Since $G_\Omega(z, y_n)/G_\Omega(x_0, y_n)$ converges uniformly on any compact set as $n\to\infty$, in order to obtain (6), it is therefore sufficient to show that for any $\varepsilon>0$, there exists a compact set K in Ω such that

(7) $$\int_{CK} G_\Omega^P(x_0, z)P(z)\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}\,dz < \varepsilon$$

for all $n \geqq 1$.

In case condition (a) is satisfied: For a given $\varepsilon>0$, we choose $r_1>0$ and $t_1>0$ such that $r_1<r_0$, $t_1 \leqq c_0 r_1$, $\Omega_{t_1} \supset \{z \in \Omega;\ \delta_\Omega(z) \geqq r_1\}$ and

$$\int_0^{r_1} r\{\max_{\delta_\Omega(z)\geqq r} P(z)\}dr < \varepsilon/c_0(c_0')^2,$$

where r_0, c_0 and c_0' are the constants in Lemmas 5 and 6. Then

(8) $$\begin{aligned}&\int_{C\Omega_{t_1}} G_\Omega^P(x_0, z)P(z)\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}\,dz \\ &= \int_0^{t_1}\int_{\partial\Omega_t} G_\Omega^P(x_0, z)P(z)\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}\left|\frac{\partial}{\partial n_z}G_{\Omega_t}(x_0, z)\right|^{-1} dS_t(z)\,dt.\end{aligned}$$

By the facts $G_\Omega^P(x_0, z) \leqq G_\Omega(x_0, z)$, (3), (4) and (5), we see that (8) is dominated by

$$\begin{aligned}&(c_0')^2\int_0^{t_1} t\{\max_{z\in\partial\Omega_t} P(z)\}\int_{\partial\Omega_t}\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}\,d\omega_0^t(z)\,dt \\ &\leqq (c_0')^2 c_0\int_0^{t_1/c_0} r\{\max_{\delta_\Omega(z)\geqq r} P(z)\}\,dr < \varepsilon.\end{aligned}$$

Thus (7) is obtained.

In case condition (b) is satisfied: Suppose that $P \in L^q(\Omega)$ with some $q>n/2$. Then there exists $p>0$ satisfying $p<n/(n-2)$ and $1/p+1/q=1/r<1$. By Lemmas 5 and 8, we get for any $n \geqq 1$,

$$\begin{aligned}&\left(\int_{\delta_\Omega(z)\leqq r_0}\left(G_\Omega(x_0, z)P(z)\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}\right)^r dz\right)^{1/r} \\ &\leqq c_0\left(\int\left(\delta_\Omega(z)\frac{G_\Omega(z, y_n)}{G_\Omega(x_0, y_n)}\right)^q dz\right)^{1/p}\left(\int P(z)^p dz\right)^{1/q} \\ &\leqq c_0 A_2^{1/p}\left(\int P(z)^q dz\right)^{1/q},\end{aligned}$$

which leads to (7) immediately.

Remark here that in both cases (7) is valid even if $G_\Omega^P(x_0, z)$ is replaced with $G_\Omega(x_0, z)$.

Next we shall verify the condition (**) in Theorem 2. Suppose on the contrary that there exists a sequence $(y_n)_{n=1}^\infty$ in Ω with $\lim_{n\to\infty} y_n = Y \in \partial\Omega$ such that $G_\Omega^P(x_0, y_n)/G_\Omega(x_0, y_n) \to 0$ as $n\to\infty$. Then by the Harnack inequality for any $x \in \Omega$

$$\lim_{n\to\infty}\frac{G_\Omega^P(x, y_n)}{G_\Omega(x_0, y_n)} = 0.$$

Hence by (1) and (6) we have $v_Y(x) = \int G_\Omega^P(x, z)P(z)v_Y(z)dz$. On the other hand, the above remark about (7) implies that $\int G_\Omega(x, z)P(z)v_Y(z)dz \not\equiv \infty$, namely, this is a potential on Ω. Since

$$v_Y(x) = \int G_\Omega^P(x, z)P(z)v_Y(z)dz \leqq \int G_\Omega(x, z)P(z)v_Y(z)dz$$

and v_Y is Δ-harmonic, it follows that v_Y vanishes identically, which is absurd. Thus (**) in Theorem 2 is fulfilled. This completes the proof of Theorem 1.

§5. Using the inequality in (**) and Lemmas 7 and 8, we obtain the global integrability of non-negative L_P-superharmonic functions. Recall that u is *L_P-superharmonic* if u is lower semi-continuous and $-L_P u \geqq 0$ in the sense of distribution (see [7], Theorem 2).

COROLLARY 9. *Let Ω and P be as in Theorem 1. If u is non-negative L_P-superharmonic, $x_0 \in \Omega$, $0 < q_1 < n/(n-1)$ and $0 < q_2 < n/(n-2)$, then there exist positive constants B_1 and B_2 independent of u such that*

$$\int u^{q_1}(x)dx \leqq B_1 u^{q_1}(x_0)$$

and

$$\int (\delta_\Omega(x)u(x))^{q_2}dx \leqq B_2 u^{q_2}(x_0).$$

§6. Finally, we give an example of P such that the Martin boundary with respect to L_P is not homeomorphic to the Euclidean boundary. Let Ω be the open ball in $\boldsymbol{R}^2$ of radius 1/2 centered at (1/2, 0), and let $P_k(x) = |x|^{-2(1+k)}$ $(0 < k < 1)$ on Ω. For any $|t| < \pi/8$, we put

$$s_{k,t}(x) = \exp((\cos k(\theta - t)/kr^k) \quad (x = re^{i\theta}, |\theta| < \pi),$$

$c_k(x) = \exp((\cos(\pi/4)k)/kr^k)$ and

$$\Omega_{k,t} = \Omega \cap \{x = re^{i\theta}; r < ((\sin^2(\pi/4)k)/k\cos(\pi/4)k)^{1/k}, |\theta - t| < \pi/4\}.$$

Then $L_{P_k}s_{k,t} = 0$ on Ω (see [10], §5.2), and besides, it is easy to check that $-L_{P_k}c_k \geqq 0$ on $\Omega_{k,t}$, or c_k is L_{P_k}-superharmonic on $\Omega_{k,t}$. We denote by $\tilde{s}_{k,t}$ the Dirichlet solution on Ω for L_{P_k} with the boundary values $s_{k,t}(Y)$ $(Y \in \partial\Omega \backslash \{O\})$ and 0 at $O = (0, 0)$. Then $h_{k,t} = s_{k,t} - \tilde{s}_{k,t}$ is positive L_{P_k}-harmonic on Ω and $\lim_{x \to Y} h_{k,t}(x) = 0$ for all $Y \in \partial\Omega \backslash \{O\}$. Since there exists a constant $A > 0$ satisfying $s_{k,t}(x) \leqq Ac_k(x)$ for $x \in \partial\Omega_{k,t} \backslash \{O\}$, we see that $\tilde{s}_{k,t} \leqq Ac_k$ on $\Omega_{k,t}$. This observation implies

$$(9) \qquad \begin{cases} \lim_{r \to 0} h_{k,t}(re^{it}) \exp(-1/kr^k) = 1, \\ \lim_{r \to 0} h_{k,t}(re^{i\theta}) \exp(-1/kr^k) = 0 \quad \text{if} \quad \theta \neq t. \end{cases}$$

Now set

$$M=\left\{h;\begin{array}{ll}h\text{ is non-negative }L_{P_k}\text{-harmonic on }\Omega\text{ and}\\ \lim_{x\to Y}h(x)=0\qquad\text{for all}\quad Y\in\partial\Omega\backslash\{O\}.\end{array}\right\}$$

Then by the general theory of Martin boundary (see [6] and [9]), if the Martin boundary and the Euclidean boundary are homeomorphic, then M ought to be one dimensional; however, $(h_{k,t})_{|t|<\pi/8}$ are elements in M and it follows from (9) that they are not proportional to each other. This implies that the Martin boundary of Ω with respect to L_{P_k} can not be identified with the Euclidean boundary.

REMARK 10. In the above example, if $k<0$, then P_k satisfies both conditions (a) and (b) in Theorem 1. Furthermore, by a recent result of T. Tada [11], the Martin boundary for L_{P_k} is identified with the Euclidean boundary even if $k=0$.

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