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Semi-linear boundary value problems with respect to an ideal boundary on a self-adjoint harmonic space

Dedicated to Professor Makoto Ohtsuka on the occasion of his 60th birthday

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Introduction.

In the paper [6], the author discussed boundary value problems for the linear equation $\Delta u - qu = 0$ ($q \geqq 0$) with respect to an ideal boundary of a locally euclidean space X. There we considered a resolutive compactification X^* of X and boundary conditions of the form

$$(1) \qquad \begin{cases} u = \tau & \text{on} \quad \partial^* X \setminus \Lambda \\ u \text{ has normal derivative } \beta u + \gamma \text{ on } \Lambda \end{cases}$$

for a subset Λ of the ideal boundary $\partial^* X = X^* \setminus X$ and for given functions τ on $\partial^* X$ and β, γ on Λ ($\beta \geqq 0$). The notion of normal derivatives on an ideal boundary is defined by means of Green's formula (cf. [5], [6]; also [2], [4]) and its definition relies on the notion of Dirichlet integrals. Thus, once the notion of Dirichlet integrals is introduced on abstract harmonic spaces (see [7], [8]), we can define normal derivatives of functions on harmonic spaces with respect to an ideal boundary. In fact, Kori [4] and the author [8; §8] discussed Neumann problems on a self-adjoint (or, symmetric) harmonic space with respect to the Martin boundary and the Royden boundary, respectively.

The purpose of the present paper is to discuss semi-linear boundary value problems on a self-adjoint harmonic space $(X, \mathscr{H})$ with respect to the ideal boundary $\partial^* X$ of a resolutive compactification X^* of X, with boundary conditions of type (1), but with non-linear form. As in [6], we seek solutions of the form $u = H_\varphi + g$ with a function φ on $\partial^* X$ and a function of potential type g on X, where H_φ denotes the Dirichlet solution. We regard φ as the boundary values (or the trace) of u.

In the special case where X is a Riemannian manifold and $\mathscr{H}$ is given by $\mathscr{H}(U) = \{u \in \mathscr{C}^2(U) \mid \Delta u = 0\}$, where Δ is the Laplace-Beltrami operator, our boundary value problem includes the problem of the type

$$\begin{cases} \Delta u(x) = F(x, u(x)) & \text{on} \quad X \\ \varphi = \tau & \text{on} \quad \partial^* X \setminus \Lambda \\ \partial_n u = \beta(\xi, \varphi(\xi)) & \text{on} \quad \Lambda. \end{cases}$$

Here, F (resp. β) is a function on $X \times \boldsymbol{R}$ (resp. $\Lambda \times \boldsymbol{R}$) which is locally Lipschitz continuous in the second variable, τ is a given boundary function and $\partial_n u$ denotes the normal derivative of u with respect to $\partial^* X$.

To the proof of our main existence theorem (Theorem 3), we apply the so called monotone iteration method, which is used to prove similar results for problems on euclidean domains with smooth boundary (see, e.g., [3] and [10]). However, the final convergence arguments in our case are more potential theoretic.

§1. Preliminaries

Throughout this paper, let $(X, \mathscr{H})$ be a self-adjoint P-harmonic space (see [7], [8]). We assume that X is connected and has a countable base, and that $1 \in \mathscr{H}(X)$. By definition there exists a symmetric Green function $G(x, y)$ on X. For a non-negative measure μ on X, we denote

$$G\mu(x) = \int_X G(x, y) d\mu(y), \quad x \in X.$$

We know that $G\mu$ is a potential on X if $G\mu \not\equiv +\infty$. In this case $G\mu(x)$ is finite q.e., i.e., except on a polar set. Thus, if ν is a signed measure on X such that $G|\nu| \not\equiv +\infty$, then $G\nu = G\nu^+ - G\nu^-$ is defined q.e. on X.

For an open set U in X, let $\mathscr{M}_C^+(U)$ be the set of non-negative measures μ on U such that for each compact set K in U, $x \mapsto \int_K G(x, y) d\mu(y)$ is continuous on X. Let $\mathscr{M}_C(U) = \{\nu \mid |\nu| \in \mathscr{M}_C^+(U)\}$. Then the mappings

$$\mathscr{M}_C^+ \colon U \longmapsto \mathscr{M}_C^+(U) \quad \text{and} \quad \mathscr{M}_C \colon U \longmapsto \mathscr{M}_C(U)$$

are sheaves of measures. We further consider the following classes of measures on X:

$$\mathscr{M}_B^+ = \{\mu \geqq 0 \mid G\mu \text{ is bounded on } X\}, \quad \mathscr{M}_{BC}^+ = \mathscr{M}_B^+ \cap \mathscr{M}_C^+(X),$$

$$\mathscr{M}_E^+ = \left\{\mu \geqq 0 \,\middle|\, \int G\mu \, d\mu < +\infty\right\}, \quad \mathscr{M}_{EC}^+ = \mathscr{M}_E^+ \cap \mathscr{M}_C^+(X),$$

$$\mathscr{M}_{BFC}^+ = \{\mu \in \mathscr{M}_{BC}^+ \mid \mu(X) < +\infty\}, \quad \mathscr{M}_{EFC}^+ = \{\mu \in \mathscr{M}_{EC}^+ \mid \mu(X) < +\infty\}.$$

Note that $\mathscr{M}_{BFC}^+ \subset \mathscr{M}_{EFC}^+$. For $Z = B$, BC, E, EC, BFC or EFC, let

$$\mathscr{M}_Z = \{\nu \mid |\nu| \in \mathscr{M}_Z^+\},$$

$$\mathscr{P}_Z = \{G\mu \mid \mu \in \mathscr{M}_Z^+\}, \quad \mathscr{Q}_Z = \{G\nu \mid \nu \in \mathscr{M}_Z\} = \mathscr{P}_Z - \mathscr{P}_Z.$$

It is easy to see that if $\mu \in \mathscr{M}_Z^+$ and $|\nu| \leqq \mu$, then $\nu \in \mathscr{M}_Z$.

As in [7] and [8], let $\mathscr{R}$ be the sheaf of functions which are locally expressible as the difference of two continuous superharmonic functions. There is a canonical measure representation σ (see [8, p. 69]), i.e., a sheaf homomorphism $\mathscr{R}\to\mathscr{M}_C$ with linear structures in $\mathscr{R}(U)$ and $\mathscr{M}_C(U)$ such that for $f\in\mathscr{R}(U)$, $f-\int_V G(\cdot, y)d\sigma(f)(y)$ is harmonic on V for any relatively compact open set V with $\bar{V}\subset U$. In particular $\sigma(h+Gv)=v$ for $h\in\mathscr{H}(X)$ and $v\in\mathscr{M}_{BC}\cup\mathscr{M}_{EC}$. We shall also write $\sigma(f)=v$ if $f=h+Gv$ with $h\in\mathscr{H}(X)$ and $v\in\mathscr{M}_B\cup\mathscr{M}_E$.

For f, $g\in\mathscr{R}(U)$, their mutual gradient measure is defined by

$$\delta_{[f,g]}=\frac{1}{2}\{f\sigma(g)+g\sigma(f)-\sigma(fg)\}$$

and the gradient measure of $f\in\mathscr{R}(U)$ by

$$\delta_f=\delta_{[f,f]}=f\sigma(f)-\frac{1}{2}\sigma(f^2)$$

(see [7], [8]). We know that $\delta_f\geqq 0$ on U for any $f\in\mathscr{R}(U)$. The Dirichlet integral of $f\in\mathscr{R}(X)$ is given by $D[f]=\delta_f(X)$. Let

$$\mathscr{D}_C=\{f\in\mathscr{R}(X)\mid D[f]<+\infty\},$$
$$\mathscr{H}_D=\{f\in\mathscr{H}(X)\mid D[f]<+\infty\}.$$

$\mathscr{D}_C$ is a linear space and $\mathscr{H}_D$ is a linear subspace of $\mathscr{D}_C$. For f, $g\in\mathscr{D}_C$, $D[f, g]=\delta_{[f,g]}(X)$ is well-defined and it is a symmetric bilinear form on $\mathscr{D}_C$. We know that $D[f]=0$ implies $f=$const. ([8; Theorem 5.4]). For a continuous potential p on X, $D[p]<+\infty$ if and only if $p\in\mathscr{P}_{EC}$ ([8; Theorem 4.3 and Proposition 6.5]). Thus, $\mathscr{H}_D+\mathscr{Q}_{EC}\subset\mathscr{D}_C$.

LEMMA 1.1. ([8; Proposition 2.16, Proposition 3.5 and Corollary 3.2])
(i) *If f, $g\in\mathscr{D}_C$, then* $\max(f, g)$, $\min(f, g)\in\mathscr{D}_C$ *and*

$$D[\max(f, 0), \min(f, 0)]=0,$$
$$D[\max(f, g)]+D[\min(f, g)]=D[f]+D[g].$$

(ii) *If $f\in\mathscr{D}_C$, then* $D[f-\min(f, n)]\to 0$ $(n\to\infty)$.

LEMMA 1.2. *Let u_1, $u_2\in\mathscr{H}_D$ and*
$h_1=$ the least harmonic majorant of $\max(u_1, u_2)$,
$h_2=$ the greatest harmonic minorant of $\min(u_1, u_2)$.

Then, h_1, $h_2\in\mathscr{H}_D$, $h_1-\max(u_1, u_2)\in\mathscr{P}_{EC}$ and $\min(u_1, u_2)-h_2\in\mathscr{P}_{EC}$.

This lemma follows from [8; Lemma 6.1 and Theorem 6.2].

The space $\mathscr{H}_D$ is complete with respect to the semi-norm $D[\cdot]^{1/2}$ ([8; Theorem 6.5]). $\mathscr{Q}_{EC}$ is a pre-Hilbert space with respect to the inner product $D[\cdot,\cdot]$. Let $\mathscr{D}_0$ be the completion of $\mathscr{Q}_{EC}$. Then

$$\mathscr{D}_0 = \left\{ g \;\middle|\; \begin{array}{l} \text{there is a sequence } \{g_n\} \text{ in } \mathscr{Q}_{EC} \text{ such that} \\ g_n \to g \text{ q.e. on } X \text{ and } D[g_n - g_m] \to 0 \ (n, m \to \infty) \end{array} \right\},$$

two functions which are equal q.e. being identified ([7; Theorem 6.1]). $D[g]$ and $D[g_1, g_2]$ are defined naturally for $g, g_1, g_2 \in \mathscr{D}_0$.

LEMMA 1.3. (i) ([7; Proposition 6.1]) $\mathscr{Q}_E \subset \mathscr{D}_0$.
(ii) *If* $f \in \mathscr{H}_D + \mathscr{Q}_E$ *and* $g \in \mathscr{D}_0$, *then* g *is* $|\sigma(f)|$*-summable and*

$$D[f, g] = \int_X g \, d\sigma(f). \tag{1.1}$$

In particular $D[h, g] = 0$ *if* $h \in \mathscr{H}_D$ *and* $g \in \mathscr{D}_0$.

PROOF of (ii). Equality (1.1) is shown for $f \in \mathscr{H}_D + \mathscr{Q}_{EC}$ and $g \in \mathscr{Q}_{EC}$ in [8; Theorem 5.2]. If $q \in \mathscr{Q}_E$, then we can choose $q_n \in \mathscr{Q}_{EC}$ such that $q_n \to q$ q.e. and $\int_X (q - q_n) d\sigma(q - q_n) \to 0$ $(n \to \infty)$ ([7; Lemma 1.5]). Then $D[q - q_n] \to 0$ and $\int_X g \, d\sigma(q - q_n) \to 0$ for any $g \in \mathscr{Q}_{EC}$. Hence (1.1) holds for $f \in \mathscr{H}_D + \mathscr{Q}_E$ and $g \in \mathscr{Q}_{EC}$. If $g \in \mathscr{D}_0$, then choose $g_n \in \mathscr{Q}_{EC}$ such that $g_n \to g$ q.e. and $D[g_n - g_m] \to 0$ $(n, m \to \infty)$. Then by Fatou's Lemma

$$\begin{aligned} \int_X |g_n - g| d|\sigma(f)| &\leqq \liminf_{m\to\infty} \int_X |g_n - g_m| d|\sigma(f)| \\ &= \liminf_{m\to\infty} D[|g_n - g_m|, G|\sigma(f)|] \\ &\leqq \liminf_{m\to\infty} D[g_n - g_m]^{1/2} D[G|\sigma(f)|]^{1/2} \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

Hence g is $|\sigma(f)|$-summable and (1.1) holds for $f \in \mathscr{H}_D + \mathscr{Q}_E$ and $g \in \mathscr{D}_0$.

Let $\mathscr{D} = \mathscr{H}_D + \mathscr{D}_0$, and for $f_i = h_i + g_i$ with $h_i \in \mathscr{H}_D$ and $g_i \in \mathscr{D}_0$, $i = 1, 2$, define

$$D[f_1, f_2] = D[h_1, h_2] + D[g_1, g_2].$$

Then, $\mathscr{D}$ is complete with respect to the semi-norm $D[\cdot]^{1/2}$; $D[f] = 0$ implies $f = \text{const.}$ (q.e.) ([7; Theorem 7.3]).

§ 2. Normal derivatives and a comparison principle

Let X^* be a resolutive compactification of X and let $\omega = \omega_x$ be the harmonic measure on $\partial^* X = X^* \setminus X$ (at the point $x \in X$) (cf. [1; §4]). For each $\varphi \in \mathscr{L}^1(\omega)$, let

$$H_\varphi(x) = \int_{\partial^* X} \varphi \, d\omega_x, \qquad x \in X.$$

Then $H_\varphi \in \mathscr{H}(X)$. We identify functions on $\partial^* X$ which are equal ω-a.e. on $\partial^* X$.

LEMMA 2.1. ([1; Theorem 4.5]) *For* $\varphi_1, \varphi_2 \in \mathscr{L}^1(\omega)$,

$H_{\max(\varphi_1,\varphi_2)} =$ *the least harmonic majorant of* H_{φ_1} *and* H_{φ_2},

$H_{\min(\varphi_1,\varphi_2)} =$ *the greatest harmonic minorant of* H_{φ_1} *and* H_{φ_2}.

LEMMA 2.2. *For* $\varphi \in \mathscr{L}^1(\omega)$, $H_\varphi \geqq 0$ *if and only if* $\varphi \geqq 0$ *ω-a.e. on* $\partial^* X$.

PROOF. Obviously, $\varphi \geqq 0$ ω-a.e. on $\partial^* X$ implies $H_\varphi \geqq 0$. Suppose $H_\varphi \geqq 0$. Then, by the above lemma $H_{\varphi^-} = 0$, and hence $\varphi^- = 0$ ω-a.e.

COROLLARY. *For* $\varphi \in \mathscr{L}^1(\omega)$ *and* $g \in \mathscr{Q}_B \cup \mathscr{Q}_E$, $H_\varphi + g \geqq 0$ *implies* $\varphi \geqq 0$ *ω-a.e. on* $\partial^* X$.

We consider the classes

$$\Phi_D = \{\varphi \in \mathscr{L}^1(\omega) \,|\, H_\varphi \in \mathscr{H}_D\}, \quad \Phi_{BD} = \Phi_D \cap \mathscr{L}^\infty(\omega),$$
$$\mathscr{H}_D(X^*) = \{H_\varphi \,|\, \varphi \in \Phi_D\}.$$

These are linear spaces and contain constant functions. By Lemmas 2.1 and 1.2, we see that Φ_D and Φ_{BD} are closed under max. and min. operations. Furthermore, as in the proof of [7; Theorem 7.2], we can prove

LEMMA 2.3. $\mathscr{H}_D(X^*) + \mathscr{Q}_{EC}$ *is closed under max. and min. operations. In fact, if* $f = H_\varphi + g$ *with* $\varphi \in \Phi_D$ *and* $g \in \mathscr{Q}_{EC}$, *then* $\max(f, 0) = H_{\max(\varphi,0)} + g_1$ *with* $g_1 \in \mathscr{Q}_{EC}$.

LEMMA 2.4. *For each* $x \in X$, *there is* $M_x > 0$ *such that*

$$\int \varphi^2 d\omega_x \leqq M_x(D[H_\varphi] + |H_\varphi(x)|^2)$$

for all $\varphi \in \Phi_D$.

This lemma can be proved in the same way as [5; Lemma 3] (also cf. [2], [8; Lemma 8.2]). As a consequence of this lemma, we have (cf. [5; Theorem 1])

PROPOSITION 2.1. $\mathscr{H}_D(X^*)$ *is closed in* $\mathscr{H}_D$, *so that* Φ_D *is complete with respect to the semi-norm* $\|\varphi\|_D = D[H_\varphi]^{1/2}$.

Let Λ be an ω-measurable subset of $\partial^* X$. We write

$$\Phi_{BD}(\Lambda) = \{\varphi \in \Phi_{BD} \,|\, \varphi = 0 \ \omega\text{-a.e. on } \partial^* X \setminus \Lambda\}.$$

We denote by $\mathcal{N}(\Lambda)$ the set of all measures $\gamma=\psi\omega$ on Λ such that ψ is ω-measurable and $\int_\Lambda |\varphi|\, d|\gamma| = \int_\Lambda |\varphi\psi| d\omega < +\infty$ for all $\varphi\in\Phi_{BD}(\Lambda)$. Given $f\in\mathscr{H}_D+\mathscr{Q}_{EFC}$ and $\gamma\in\mathcal{N}(\Lambda)$, we write

$$N(f) \geqq \gamma \qquad (\text{resp. } \leqq\gamma,\ =\gamma) \quad \text{on} \quad \Lambda$$

if

$$D[H_\varphi, f] - \int_X H_\varphi\, d\sigma(f) + \int_\Lambda \varphi d\gamma \leqq 0 \qquad (\text{resp. } \geqq 0,\ =0)$$

for all $\varphi\in\Phi_{BD}^+(\Lambda)=\{\varphi\in\Phi_{BD}(\Lambda)\,|\,\varphi\geqq 0\ \omega\text{-a.e.}\}$. In case $N(f)=\gamma$, we say that f has a normal derivative γ on Λ.

THEOREM 2.1 (Comparison principle). *Let $F\colon \mathscr{R}\to\mathscr{M}_C$ be a sheaf morphism satisfying*

(F.1) *$f\leqq g$ on U implies $F(f)\leqq F(g)$ on U for any open set U in X.*

Let Λ be an ω-measurable subset of ∂^*X and let $\beta\colon \Phi_D\to\mathcal{N}(\Lambda)$ satisfy

(β.1) *for any ω-measurable subset Σ of Λ, $\varphi_1\leqq\varphi_2$ ω-a.e. on Σ implies $\beta(\varphi_1)\leqq \beta(\varphi_2)$ on Σ.*

Suppose $u=H_\varphi+g$, $v=H_\psi+q$ with $\varphi, \psi\in\Phi_D$ and $g, q\in\mathscr{Q}_{EFC}$ satisfy the following three conditions:

(a) $\sigma(u) + F(u) \geqq \sigma(v) + F(v)$ *on* X,

(b) $\varphi \geqq \psi$ *ω-a.e. on* $\partial^*X\setminus\Lambda$,

(c) $N(u-v) \leqq \beta(\varphi) - \beta(\psi)$ *on* Λ.

Then

(i) *$u\geqq v$ on X, in case $\omega(\partial^*X\setminus\Lambda)>0$;*

(ii) *$u\geqq v$ on X or $v=u+c$ with a constant $c>0$, in case $\omega(\partial^*X\setminus\Lambda)=0$; the latter occurs only when $F(u)=F(u+c)$ on X and $\beta(\varphi)=\beta(\varphi+c)$ on Λ.*

PROOF. Put $f=(u-v)^-$, $f_n=\min(f, n)$, $\varphi_0=(\varphi-\psi)^-$ and $\varphi_n=\min(\varphi_0, n)$, $n=1, 2,\ldots$. By Lemma 2.3 and condition (b), $\varphi_0\in\Phi_D^+$, $\varphi_n\in\Phi_{BD}^+(\Lambda)$ for all n and $f=H_{\varphi_0}+g_0$, $f_n=H_{\varphi_n}+g_n$ with $g_0, g_n\in\mathscr{Q}_{EC}$ $(n=1, 2,\ldots)$. By condition (c), we have

$$D[H_{\varphi_n}, u-v] - \int_X H_{\varphi_n} d[\sigma(u)-\sigma(v)] + \int_\Lambda \varphi_n\, d[\beta(\varphi)-\beta(\psi)] \geqq 0 \tag{2.1}$$

for each n. Let $\Sigma=\{\xi\in\Lambda\,|\,\varphi_0(\xi)>0\}$. Since $\varphi\leqq\psi$ on Σ, $\beta(\varphi)\leqq\beta(\psi)$ on Σ. Hence

$$\int_\Lambda \varphi_n\, d[\beta(\varphi)-\beta(\psi)] = \int_\Sigma \varphi_n\, d[\beta(\varphi)-\beta(\psi)] \leqq 0. \tag{2.2}$$

On the other hand, using Lemma 1.3, we have

$$\begin{aligned} D[H_{\varphi_n}, u-v] &= D[f_n, u-v] - D[g_n, u-v] \\ &= D[f_n, u-v] - \int_X g_n\, d[\sigma(u)-\sigma(v)] \\ &= D[f_n, u-v] + \int_X (H_{\varphi_n}-f_n) d[\sigma(u)-\sigma(v)]. \end{aligned}$$

Hence, in view of (2.1) and (2.2), we obtain

$$D[f_n, u-v] \geqq \int_X f_n\, d[\sigma(u)-\sigma(v)]. \tag{2.3}$$

Let $A=\{x\in X \mid f(x)>0\}$. Then A is an open set in X and $u<v$ on A. Hence $F(u)\leqq F(v)$ on A, and condition (a) implies $\sigma(u)\geqq\sigma(v)$ on A. Thus

$$\int_X f_n\, d[\sigma(u)-\sigma(v)] = \int_A f_n\, d[\sigma(u)-\sigma(v)] \geqq 0.$$

Hence $D[f_n, u-v]\geqq 0$ by (2.3). Since $D[f_n-f]\to 0$ $(n\to\infty)$ by Lemma 1.1, (ii), it follows that $D[f, u-v]\geqq 0$. Since $u-v=(u-v)^+-f$, by Lemma 1.1, (i) we conclude that $D[(u-v)^-]=0$. Hence $(u-v)^-=\text{const.}=c\geqq 0$, and $\varphi_0=c$ ω-a.e. by Lemma 2.2. Thus, if $\omega(\partial^*X\setminus A)>0$, then condition (b) implies that $c=0$, so that $u\geqq v$. In case $\omega(\partial^*X\setminus A)=0$, if $c=0$ then $u\geqq v$; if $c>0$, then the connectedness of X and the continuity of $u-v$ imply that $(u-v)^+=0$, so that $v=u+c$. In this last case, $\sigma(v)=\sigma(u)$, and hence condition (a) implies $F(v)=F(u+c)\geqq F(u)\geqq F(v)$, namely $F(u)=F(u+c)$. Furthermore, by (2.1) and (2.2), we see that $\beta(\varphi)=\beta(\psi)$ on A, i.e., $\beta(\varphi)=\beta(\varphi+c)$ on A.

§3. Linear boundary value problems

In this section, we consider linear boundary value problems which are generalizations of those discussed in [6].

For $\lambda\in\mathscr{M}_C^+(X)$, $\lambda\neq 0$, let

$$\mathscr{D}^\lambda = \{f\in\mathscr{D} \mid \int_X f^2 d\lambda<\infty\} = \mathscr{D}\cap\mathscr{L}^2(\lambda)$$

and

$$\mathscr{H}_D^\lambda = \{u\in\mathscr{D}_C\cap\mathscr{L}^2(\lambda) \mid \sigma(u)+u\lambda=0\}.$$

Note that any polar set is of λ-measure zero, and any $f\in\mathscr{D}$ is locally λ-summable. $\mathscr{D}^\lambda$ is a Hilbert space with respect to the inner product

$$D^\lambda[f, g] = D[f, g] + \int_X fg\, d\lambda.$$

LEMMA 3.1. *If $\lambda \in \mathscr{M}_{BC}^+$, then $\mathscr{D}_0 \subset \mathscr{D}^\lambda$; in fact*

$$\int_X g^2 d\lambda \leqq \|G\lambda\|_\infty D[g] \qquad \text{for all} \quad g \in \mathscr{D}_0. \tag{3.1}$$

PROOF. We know that (3.1) holds for $g \in \mathscr{Q}_{EC}$ ([7; Theorem 1.2] or [8; Lemma 4.2]). If $g \in \mathscr{D}_0$, then choosing $g_n \in \mathscr{Q}_{EC}$ such that $g_n \to g$ q.e. on X and $D[g_n - g] \to 0$ $(n \to \infty)$, we obtain (3.1).

LEMMA 3.2. ([7; Lemma 1.10]) *If $\lambda \in \mathscr{M}_{BC}^+$ and $f \in \mathscr{L}^2(\lambda)$, then $f\lambda \in \mathscr{M}_E$ and*

$$D[G(f\lambda)] \leqq \|G\lambda\|_\infty \int_X f^2 d\lambda.$$

COROLLARY 1. *If $\lambda \in \mathscr{M}_{BC}^+$, $f \in \mathscr{L}^2(\lambda)$ and f is locally bounded (in particular, f is continuous) on X, then $f\lambda \in \mathscr{M}_{EC}$.*

COROLLARY 2. *If $\lambda \in \mathscr{M}_{BC}^+$ (resp. $\mathscr{M}_{BFC}^+$), then $\mathscr{H}_D^\lambda \subset \mathscr{H}_D + \mathscr{Q}_{EC}$ (resp. $\mathscr{H}_D + \mathscr{Q}_{EFC}$). In fact, $u \in \mathscr{H}_D^\lambda$ implies $u + G(u\lambda) \in \mathscr{H}_D$ and $G(u\lambda) \in \mathscr{Q}_{EC}$ (resp. $\mathscr{Q}_{EFC}$).*

LEMMA 3.3. *Let $\lambda \in \mathscr{M}_{BC}^+$. If $v \in \mathscr{D}^\lambda$ and $\mathscr{D}^\lambda[v, g] = 0$ for all $g \in \mathscr{D}_0$, then $v + G(v\lambda) \in \mathscr{H}_D$* (modifying the values of v on a polar set, if necessary). *If furthermore $v + p \geqq 0$ (q.p.) for some $p \in \mathscr{P}_E$, then $v \geqq 0$ (q.p.).*

PROOF. By Lemma 3.2, $v\lambda \in \mathscr{M}_E$. Put $h = v + G(v\lambda)$. Then $h \in \mathscr{D}$ and for any $g \in \mathscr{D}_0$,

$$D[h, g] = D[v, g] + D[G(v\lambda), g] = -\int_X vg\, d\lambda + \int_X gv\, d\lambda = 0$$

by Lemma 1.3. It follows that $h \in \mathscr{H}_D$ (by modifying the values of v on a polar set).

Next, suppose $v + p \geqq 0$ with $p \in \mathscr{P}_E$. We can write

$$v^- = -\min(v, 0) = -\min(h + G(v^-\lambda), G(v^+\lambda)) + G(v^+\lambda).$$

$g = \min(h + G(v^-\lambda), G(v^+\lambda))$ is superharmonic on X. Since $h \geqq -p - G(v^-\lambda)$, $h \geqq 0$. Hence g is a potential dominated by $G(v^+\lambda)$. Since $v^+\lambda \in \mathscr{M}_E$, it follows that $g \in \mathscr{P}_E$. Hence $v^- \in \mathscr{Q}_E$. On the other hand, $\sigma(v^-) = -\sigma(g) + v^+\lambda \leqq v^+\lambda$. Hence, by Lemma 1.3,

$$D[v^-] = \int v^- d\sigma(v^-) \leqq \int v^- v^+ d\lambda = 0.$$

Thus, $v^- = 0$, and hence $v \geqq 0$.

PROPOSITION 3.1. *Let $\lambda \in \mathscr{M}_{BC}^+$, $\lambda \neq 0$. Then*

$$\mathscr{H}_D^\lambda = \{u \in \mathscr{D}^\lambda \mid D^\lambda[u, g]=0 \textit{ for all } g \in \mathscr{D}_0\},$$

so that $\mathscr{H}_D^\lambda$ *is a closed linear subspace of* $\mathscr{D}^\lambda$ *and* $\mathscr{D}^\lambda = \mathscr{H}_D^\lambda \oplus \mathscr{D}_0$.

PROOF. First, let $u \in \mathscr{H}_D^\lambda$. By Corollary 2 to Lemma 3.2, $u \in \mathscr{H}_D + \mathscr{Q}_{EC}$. Hence by Lemma 1.3

$$D^\lambda[u, g] = D[u, g] + \int ug\, d\lambda = \int g\, d\sigma(u) + \int gu\, d\lambda = 0$$

for any $g \in \mathscr{D}_0$.

Conversely, assume that $u \in \mathscr{D}^\lambda$ and $D^\lambda[u, g]=0$ for all $g \in \mathscr{D}_0$. By Lemma 3.3, $h = u + G(u\lambda) \in \mathscr{H}_D$. If $u \geqq 0$ on X, then $G(u\lambda) \geqq 0$, so that $0 \leqq u \leqq h$. Since h is continuous, u is locally bounded on X. Hence, by Corollary 1 to Lemma 3.2, $G(u\lambda) \in \mathscr{Q}_{EC}$, and hence $u \in \mathscr{D}_C$. Since $\sigma(u) = -u\lambda$, it follows that $u \in \mathscr{H}_D^\lambda$. In the general case, we consider u^+. We know that $u^+ \in \mathscr{H}_D + \mathscr{Q}_E$ (cf. [7; Theorem 7.2]). Let $\tilde{u}$ be the orthogonal projection (with respect to $D^\lambda[\cdot,\cdot]$) of u^+ to the space $\{u \in \mathscr{D}^\lambda \mid D^\lambda[u, g]=0$ for all $g \in \mathscr{D}_0\}$. By Lemmas 3.2 and 3.3, we see that $\tilde{u} \in \mathscr{H}_D + \mathscr{Q}_E$. Hence $\tilde{u} - u^+ \in \mathscr{D}_0 \cap (\mathscr{H}_D + \mathscr{Q}_E) = \mathscr{Q}_E$. Thus, $\tilde{u} \geqq -p$ for some $p \in \mathscr{P}_E$, and hence $\tilde{u} \geqq 0$ by Lemma 3.3. Hence, by the above result, we have $\tilde{u} \in \mathscr{H}_D^\lambda$. Since $D^\lambda[\tilde{u}-u, g]=0$ for all $g \in \mathscr{D}_0$ and $\tilde{u}-u \geqq \tilde{u} - u^+ \in \mathscr{Q}_E$, by the same argument as above we see that $\tilde{u}-u \in \mathscr{H}_D^\lambda$. Hence $u \in \mathscr{H}_D^\lambda$.

LEMMA 3.4. *Let* $\lambda \in \mathscr{M}_{BFC}^+$. *Then, given* $\mu \in \mathscr{M}_{BC}$, *there exists a unique* $g \in \mathscr{Q}_{BC}$ *such that* $\sigma(g) + g\lambda = \mu$ *on* X. *If* $\mu \in \mathscr{M}_{BFC}$, *then* $g \in \mathscr{Q}_{BFC}$ *and* $D^\lambda[g] \leqq D[G|\mu|]$.

PROOF. The unique existence of $g \in \mathscr{Q}_{BC}$ is given in [9; Proposition 1.4]. Since

$$\pm\, [\sigma(g) + g\lambda] = \pm\, \mu \leqq |\mu| = \sigma(G|\mu|) \leqq \sigma(G|\mu|) + (G|\mu|)\lambda,$$

a comparison theorem (see, e.g., [9; Proposition 2.1], or our Theorem 2.1 with $A = \phi$) implies that $|g| \leqq G|\mu|$. Since g is bounded and $\int d\lambda < +\infty$, $g\lambda \in \mathscr{M}_{BFC}$. Hence, if $\mu \in \mathscr{M}_{BFC}$, then $\sigma(g) \in \mathscr{M}_{BFC}$, and hence $g \in \mathscr{Q}_{BFC}$. Furthermore,

$$D^\lambda[g] = D[g] + \int g^2 d\lambda = \int g d\sigma(g) + \int g^2 d\lambda = \int g d\mu \leqq \int (G|\mu|) d|\mu| = D[G|\mu|].$$

In what follows in this section, let $\lambda \in \mathscr{M}_{BFC}^+$, $\lambda \neq 0$ and A be an ω-measurable subset of $\partial^* X$. Set

$$\Phi_D^\lambda = \{\varphi \in \Phi_D \mid H_\varphi \in \mathscr{L}^2(\lambda)\},$$

$$\Phi_D^\lambda(A) = \{\varphi \in \Phi_D^\lambda \mid \varphi = 0\ \omega\text{-a.e. on } \partial^* X \setminus A\}.$$

For each $\varphi \in \Phi_D^\lambda$, let H_φ^λ be the orthogonal projection of H_φ to $\mathscr{H}_D^\lambda$ with respect to

$D^{\lambda}[\cdot, \cdot]$. By Corollary 2 to Lemma 3.2,

$$H_{\varphi} - H_{\Phi}^{\lambda} = G(H_{\varphi}^{\lambda}\lambda) \in \mathscr{Q}_{EFC}.$$

Since $\mathscr{H}_D^{\lambda}$ is complete with respect to the norm $D^{\lambda}[\cdot]^{1/2}$ and $\mathscr{H}_D(X^*)$ is closed in $\mathscr{H}_D$, we easily obtain (cf. [6; Lemma 5.2])

LEMMA 3.5. *Φ_D^{λ} is a Hilbert space with respect to the inner product $\langle\varphi, \psi\rangle_{D,\lambda} = D^{\lambda}[H_{\varphi}^{\lambda}, H_{\psi}^{\lambda}]$. $\Phi_D^{\lambda}(\Lambda)$ is a closed subspace of Φ_D^{λ}.*

The corresponding norm in Φ_D^{λ} is denoted by $\|\cdot\|_{D,\lambda}$, i.e.,

$$\|\varphi\|_{D,\lambda} = \langle\varphi, \varphi\rangle_{D,\lambda}^{1/2} = D^{\lambda}[H_{\varphi}^{\lambda}]^{1/2}.$$

LEMMA 3.6. *For $\varphi \in \Phi_D^{\lambda}$,*

$$\int_X H_{\varphi}^2\, d\lambda \leqq (1 + \|G\lambda\|_{\infty})^2 \int_X (H_{\varphi}^{\lambda})^2 d\lambda.$$

PROOF. Since $H_{\varphi} - H_{\varphi}^{\lambda} = G(H_{\varphi}^{\lambda}\lambda) \in \mathscr{Q}_{EFC} \subset \mathscr{D}_0$, using Lemmas 3.1 and 3.2 we have

$$\int_X (H_{\varphi} - H_{\varphi}^{\lambda})^2\, d\lambda = \int_X G(H_{\varphi}^{\lambda}\lambda)^2 d\lambda$$

$$\leqq \|G\lambda\|_{\infty} D[G(H_{\varphi}^{\lambda}\lambda)] \leqq \|G\lambda\|_{\infty}^2 \int_X (H_{\varphi}^{\lambda})^2 d\lambda.$$

Hence

$$\int_X H_{\varphi}^2\, d\lambda \leqq (1 + \|G\lambda\|_{\infty})^2 \int_X (H_{\varphi}^{\lambda})^2 d\lambda.$$

THEOREM 3.1. *Let $\mu \in \mathscr{M}_{BFC}$, $\tau \in \Phi_D^{\lambda}$, β, $\gamma \in \mathscr{N}(\Lambda)$, $\beta \geqq 0$, $\int_{\Lambda} d\beta < +\infty$ and assume that $\int_{\Lambda} \tau^2 d\beta < +\infty$,*

$[\mu]$: there is $a(\mu) > 0$ such that

$$\left|\int_X H_{\psi}\, d\mu\right| \leqq a(\mu)\|\psi\|_{D,\lambda} \qquad \text{for all} \quad \psi \in \Phi_D^{\lambda}(\Lambda),$$

$[\beta\text{-}\gamma]$: there is $b(\beta, \gamma) > 0$ such that

$$\left|\int_{\Lambda} \psi d\gamma\right| \leqq b(\beta, \gamma)\left\{\int_{\Lambda} \psi^2 d\beta + \|\psi\|_{D,\lambda}^2\right\}^{1/2} \qquad \text{for all} \quad \psi \in \Phi_D^{\lambda}(\Lambda) \cap \mathscr{L}^2(\beta).$$

Then there exists a unique $u = H_{\varphi} + g$ with $\varphi \in \Phi_D^{\lambda}$ and $g \in \mathscr{Q}_{EFC}$ which satisfies

$$\begin{cases} \sigma(u) + u\lambda = \mu & \text{on} \quad X \\ \varphi = \tau & \omega\text{-a.e.} \quad \text{on} \quad \partial^* X \setminus \Lambda \\ N(u) = \varphi\beta + \gamma & \text{on} \quad \Lambda. \end{cases}$$

Furthermore,

$$D^\lambda[u]^{1/2} \leqq 2\|\tau\|_{D,\lambda} + (2+\|G\lambda\|_\infty)D[G|\mu|]^{1/2} + a(\mu) + \left(\int_A \tau^2 d\beta\right)^{1/2} + b(\beta, \gamma).$$

PROOF. The uniqueness follows from Theorem 2.1, since $\lambda \neq 0$. By Lemma 3.4, there is $q \in \mathscr{Q}_{BFC}$ such that $\sigma(q)+q\lambda=\mu$ on X. We define a linear form l on $\Phi_D^\lambda(A) \cap \mathscr{L}^2(\beta)$ by

$$l(\psi) = -\langle\psi, \tau\rangle_{D,\lambda} - \int_X H_\psi q\, d\lambda + \int_X H_\psi d\mu - \int_A \psi\tau d\beta - \int_A \psi d\gamma.$$

By Lemma 3.5, we see that $\Phi_D^\lambda(A) \cap \mathscr{L}^2(\beta)$ is a Hilbert space with respect to the inner product

$$\langle\varphi, \psi\rangle_{D,\lambda,\beta} = \langle\varphi, \psi\rangle_{D,\lambda} + \int_A \varphi\psi d\beta.$$

Let $\|\varphi\|_{D,\lambda,\beta} = \langle\varphi, \varphi\rangle_{D,\lambda,\beta}^{1/2}$. We shall show that l is continuous on $\Phi_D^\lambda(A) \cap \mathscr{L}^2(\beta)$ with respect to this norm.

First, we have

$$|\langle\psi, \tau\rangle_{D,\lambda}| \leqq \|\tau\|_{D,\lambda}\|\psi\|_{D,\lambda} \leqq \|\tau\|_{D,\lambda}\|\psi\|_{D,\lambda,\beta}.$$

By Lemmas 3.4, 3.5 and 3.6,

$$\begin{aligned}\left|\int_X H_\psi q d\lambda\right| &\leqq \left(\int_X H_\psi^2 d\lambda\right)^{1/2}\left(\int q^2 d\lambda\right)^{1/2} \\ &\leqq (1+\|G\lambda\|_\infty)\left\{\int_X (H_\psi^\lambda)^2 d\lambda\right\}^{1/2} D[G|\mu|]^{1/2} \\ &\leqq (1+\|G\lambda\|_\infty)D[G|\mu|]^{1/2}\|\psi\|_{D,\lambda,\beta}.\end{aligned}$$

By Schwarz inequality,

$$\left|\int_A \psi\tau d\beta\right| \leqq \left(\int_A \tau^2 d\beta\right)^{1/2}\left(\int_A \psi^2 d\beta\right)^{1/2} \leqq \left(\int_A \tau^2 d\beta\right)^{1/2} \|\psi\|_{D,\lambda,\beta}.$$

Thus, in view of conditions [μ] and [β-γ], we see that l is continuous with the operator norm

$$(3.2)\quad \|l\| \leqq \|\tau\|_{D,\lambda} + (1+\|G\lambda\|_\infty)D[G|\mu|]^{1/2} + a(\mu) + \left(\int_A \tau^2 d\beta\right)^{1/2} + b(\beta, \gamma).$$

Hence, there is $\varphi_0 \in \Phi_D^\lambda(A) \cap \mathscr{L}^2(\beta)$ such that $\|\varphi_0\|_{D,\lambda,\beta} = \|l\|$ and $l(\psi) = \langle\varphi_0, \psi\rangle_{D,\lambda,\beta}$ for all $\psi \in \Phi_D^\lambda(A) \cap \mathscr{L}^2(\beta)$, i.e.,

$$(3.3)\quad \begin{cases} D^{\lambda}[H^{\lambda}_{\varphi_0}, H^{\lambda}_{\psi}] + \int_{\Lambda} \varphi_0 \psi d\beta \\ = - D^{\lambda}[H^{\lambda}_{\psi}, H^{\lambda}_{\tau}] - \int_X H_{\psi} q\, d\lambda + \int_X H_{\psi} d\mu - \int_{\Lambda} \psi\tau d\beta - \int_{\Lambda} \psi d\gamma \end{cases}$$

for all $\psi \in \Phi^{\lambda}_D(\Lambda) \cap \mathscr{L}^2(\beta)$. Since $\int_{\Lambda} d\beta < +\infty$, $\Phi_{BD}(\Lambda) \subset \Phi^{\lambda}_D(\Lambda) \cap \mathscr{L}^2(\beta)$, so that (3.3) holds for all $\psi \in \Phi_{BD}(\Lambda)$. Now, let $\varphi = \varphi_0 + \tau$ and $u = H^{\lambda}_{\varphi} + q$. Then $\varphi \in \Phi^{\lambda}_D$ and $\varphi = \tau$ ω-a.e. on $\partial^* X \setminus \Lambda$. Let

$$g = H^{\lambda}_{\varphi} - H_{\varphi} + q = u - H_{\varphi}.$$

By Corollary 2 to Lemma 3.2, $H^{\lambda}_{\varphi} - H_{\varphi} \in \mathscr{Q}_{EFC}$, so that $g \in \mathscr{Q}_{EFC}$. Furthermore

$$\sigma(u) + u\lambda = \sigma(q) + q\lambda = \mu \qquad \text{on} \quad X.$$

For any $\psi \in \Phi_{BD}(\Lambda)$, by (3.3) we have

$$D^{\lambda}[H^{\lambda}_{\varphi}, H^{\lambda}_{\psi}] + \int_X H_{\psi} q\, d\lambda - \int_X H_{\psi} d\mu + \int_{\Lambda} \psi\varphi d\beta + \int_{\Lambda} \psi d\gamma = 0.$$

Since

$$\begin{aligned} D^{\lambda}[H^{\lambda}_{\varphi}, H^{\lambda}_{\psi}] &= D^{\lambda}[H^{\lambda}_{\varphi}, H_{\psi}] \\ &= D[H^{\lambda}_{\varphi}, H_{\psi}] + \int_X H^{\lambda}_{\varphi} H_{\psi} d\lambda \\ &= D[H_{\psi}, u] + \int_X H_{\psi} u\, d\lambda - \int_X H_{\psi} q\, d\lambda, \end{aligned}$$

it follows that

$$D[H_{\psi}, u] - \int_X H_{\psi} d\sigma(u) + \int_{\Lambda} \psi\varphi d\beta + \int_{\Lambda} \psi d\gamma = 0$$

for any $\psi \in \Phi_{BD}(\Lambda)$, i.e.,

$$N(u) = \varphi\beta + \gamma \qquad \text{on} \quad \Lambda.$$

Hence this u is the required solution.

Since $u = H^{\lambda}_{\varphi_0} + H^{\lambda}_{\tau} + q$, (3.2) and Lemma 3.4 imply

$$\begin{aligned} D^{\lambda}[u]^{1/2} &\leqq \|\varphi_0\|_{D,\lambda} + \|\tau\|_{D,\lambda} + D^{\lambda}[q]^{1/2} \\ &\leqq 2\|\tau\|_{D,\lambda} + (2 + \|G\lambda\|_{\infty}) D[G|\mu|]^{1/2} + a(\mu) + \left(\int_{\Lambda} \tau^2 d\beta\right)^{1/2} + b(\beta, \gamma). \end{aligned}$$

§4. Semi-linear boundary value problems

We now prove our main existence theorem for semi-linear problems.

THEOREM 4.1. *Let* $F\colon \mathscr{R}\to\mathscr{M}_C$ *be a sheaf morphism satisfying*

(F.2) $F(0)\in\mathscr{M}_{BFC}$ *and for each* $M>0$ *there is* $\lambda_M\in\mathscr{M}^+_{BFC}$ *such that*

$$|F(f_1)-F(f_2)| \leqq (f_2-f_1)\lambda_M \qquad \text{on} \quad X$$

whenever $f_1, f_2\in\mathscr{R}(X)$ *and* $-M\leqq f_1\leqq f_2\leqq M$ *on* X.

Let Λ *be an* ω*-measurable subset of* ∂^*X *and let* $\beta\colon \Phi_D\to\mathscr{N}(\Lambda)$ *satisfy*

(β.2) $\int_\Lambda d|\beta(0)|<\infty$ *and for each* $M>0$ *there is* $\alpha_M\in\mathscr{N}^+(\Lambda)$ *such that* $\int_\Lambda d\alpha_M<\infty$ *and*

$$|\beta(\varphi_1)-\beta(\varphi_2)| \leqq (\varphi_2-\varphi_1)\alpha_M \qquad \text{on} \quad \Lambda$$

whenever $\varphi_1, \varphi_2\in\Phi_D$, $-M\leqq\varphi_1\leqq\varphi_2\leqq M$ *on* Λ.

Let $\tau\in\Phi_{BD}$ *and suppose there are* $u_0=H_{\varphi_0}+g_0$ *and* $\tilde{u}_0=H_{\tilde{\varphi}_0}+\tilde{g}_0$ *with* φ_0, $\tilde{\varphi}_0\in\Phi_{BD}$ *and* g_0 $\tilde{g}_0\in\mathscr{Q}_{BFC}$ *such that* $\tilde{u}_0\leqq u_0$ *on* X,

$$(4.1)\qquad \begin{cases} \sigma(u_0) + F(u_0) \geqq 0 \geqq \sigma(\tilde{u}_0) + F(\tilde{u}_0) & \text{on} \quad X, \\ \varphi_0 \geqq \tau \geqq \tilde{\varphi}_0 & \omega\text{-a.e. on} \quad \partial^*X\setminus\Lambda, \\ N(u_0) \leqq \beta(\varphi_0) \quad \text{and} \quad N(\tilde{u}_0) \geqq \beta(\tilde{\varphi}_0) & \text{on} \quad \Lambda. \end{cases}$$

Then there exist $u^*=H_{\varphi^*}+g^*$ *and* $\hat{u}=H_{\hat{\varphi}}+\hat{g}$ *with* φ^*, $\hat{\varphi}\in\Phi_{BD}$ *and* g^*, $\hat{g}\in\mathscr{Q}_{BFC}$ *such that*

(i) $\tilde{u}_0\leqq\hat{u}\leqq u^*\leqq u_0$ *on* X;

(ii) $u=u^*$ *and* $\varphi=\varphi^*$ (*resp.* $u=\hat{u}$ *and* $\varphi=\hat{\varphi}$) *satisfy*

$$(4.2)\qquad \begin{cases} \sigma(u) + F(u) = 0 & \text{on} \quad X, \\ \varphi = \tau & \omega\text{-a.e. on} \quad \partial^*X\setminus\Lambda, \\ N(u) = \beta(\varphi) & \text{on} \quad \Lambda; \end{cases}$$

(iii) *if* $u=H_\varphi+g$ *with* $\varphi\in\Phi_{BD}$ *and* $g\in\mathscr{Q}_{BFC}$ *satisfies* (4.2) *and* $\tilde{u}_0\leqq u\leqq u_0$, *then* $\hat{u}\leqq u\leqq u^*$.

PROOF. Since u_0, $\tilde{u}_0$ are bounded, there is $M>0$ such that

$$-M \leqq \tilde{u}_0 \leqq u_0 \leqq M \qquad \text{on} \quad X.$$

Put $\lambda=|F(0)|+\lambda_M$ if $|F(0)|+\lambda_M\neq 0$. Then $\lambda\in\mathscr{M}^+_{BFC}$. If $|F(0)|+\lambda_M=0$, then take any $\lambda\in\mathscr{M}^+_{BFC}$ with $\lambda\neq 0$. For any $f\in\mathscr{R}(X)$ with $|f|\leqq M$, $|F(f)|\leqq|F(0)|+M\lambda_M\leqq(1+M)\lambda$, so that $f\lambda-F(f)\in\mathscr{M}_{BFC}$ and

$$(4.3)\qquad |f\lambda-F(f)| \leqq (1+2M)\lambda.$$

Next, put $\beta=|\beta(0)|+\alpha_M$. Then $\beta\in\mathscr{N}^+(\Lambda)$ and $\int_\Lambda d\beta<\infty$. If $\varphi\in\Phi_D$ and $|\varphi|\leqq M$, then $|\beta(\varphi)|\leqq|\beta(0)|+M\alpha_M\leqq(1+M)\beta$, so that

(4.4) $$|\beta(\varphi)-\varphi\beta| \leqq (1+2M)\beta.$$

Now we define a sequence $\{u_n\}$ of functions on X by induction as follows. Suppose $u_0, u_1,\dots, u_{n-1}$ $(n\geqq 1)$ have been so chosen that $u_j=H_{\varphi_j}+g_j$ with $\varphi_j\in\Phi_{BD}$ and $g_j\in\mathscr{Q}_{EFC}$, $j=0, 1,\dots, n-1$,

$$\tilde{u}_0 \leqq u_{n-1} \leqq \cdots \leqq u_1 \leqq u_0 \quad \text{on} \quad X$$

and

(4.5) $$\begin{cases} \sigma(u_j) + F(u_j) \geqq 0 & \text{on} \quad X, \\ \varphi_j \geqq \tau & \omega\text{-a.e.} \quad \text{on} \quad \partial^*X \setminus \Lambda, \\ N(u_j) \leqq \beta(\varphi_j) & \text{on} \quad \Lambda, \end{cases}$$

$j=0, 1,\dots, n-1$. Note that by Lemma 2.2,

$$-M \leqq \tilde{\varphi}_0 \leqq \varphi_{n-1} \leqq \cdots \leqq \varphi_1 \leqq \varphi_0 \leqq M \qquad \omega\text{-a.e.} \quad \text{on} \quad \partial^*X.$$

Let $\mu_n=u_{n-1}\lambda-F(u_{n-1})$ and $\gamma_n=\beta(\varphi_{n-1})-\varphi_{n-1}\beta$. Since $|u_{n-1}|\leqq M$ and $|\varphi_{n-1}|\leqq M$ ω-a.e. on ∂^*X, $|\mu_n|\leqq(1+2M)\lambda$ and $|\gamma_n|\leqq(1+2M)\beta$ by (4.3) and (4.4). Hence in view of Lemma 3.6, μ_n and γ_n satisfy conditions [μ] and [β-γ] in Theorem 3.1, respectively. Since τ is bounded, $\int_\Lambda \tau^2 d\beta<\infty$. Therefore, by Theorem 3.1, there exist unique $\varphi_n\in\Phi_D^\lambda$ and $g_n\in\mathscr{Q}_{EFC}$ such that $u_n=H_{\varphi_n}+g_n$ satisfies

(4.6) $$\begin{cases} \sigma(u_n) + u_n\lambda = u_{n-1}\lambda - F(u_{n-1}) & \text{on} \quad X \\ \varphi_n = \tau \quad \omega\text{-a.e.} \quad \text{on} \quad \partial^*X\setminus\Lambda \\ N(u_n) = \varphi_n\beta + \beta(\varphi_{n-1}) - \varphi_{n-1}\beta & \text{on} \quad \Lambda. \end{cases}$$

We shall show that if $v=H_\psi+q$ with $\psi\in\Phi_{BD}$ and $q\in\mathscr{Q}_{BFC}$ satisfies

$$-M \leqq v \leqq u_{n-1} \qquad \text{on} \quad X$$

and

$$\begin{cases} \sigma(v) + F(v) \leqq 0 & \text{on} \quad X \\ \psi \leqq \tau & \omega\text{-a.e.} \quad \text{on} \quad \partial^*X\setminus\Lambda \\ N(v) \geqq \beta(\psi) & \text{on} \quad \Lambda, \end{cases}$$

then $v\leqq u_n\leqq u_{n-1}$; in particular, $\tilde{u}_0\leqq u_n\leqq u_{n-1}$ by (4.1).

Since $-M\leqq v\leqq u_{n-1}\leqq M$ and $\lambda\geqq\lambda_M$, condition (F.2) implies

$$\begin{aligned} &\{\sigma(u_n)+u_n\lambda\} - \{\sigma(v)+v\lambda\} \\ &\qquad \geqq u_{n-1}\lambda - F(u_{n-1}) - v\lambda + F(v) \geqq 0 \qquad \text{on} \quad X. \end{aligned}$$

Obviously,

$$\varphi_n = \tau \geqq \psi \qquad \omega\text{-a.e.} \quad \text{on} \quad \partial^* X \setminus \Lambda.$$

Since $-M \leqq \psi \leqq \varphi_{n-1} \leqq M$ and $\beta \geqq \beta_M$, condition (β.2) implies

$$N(u_n - v) \leqq \varphi_n \beta + \beta(\varphi_{n-1}) - \varphi_{n-1}\beta - \beta(\psi) \leqq (\varphi_n - \psi)\beta \qquad \text{on} \quad \Lambda.$$

Hence by Theorem 2.1 (with $F(f)=f\lambda$ and $\beta(\varphi)=\varphi\beta$), we conclude that $v \leqq u_n$ on X.

On the other hand, by (4.5) with $j=n-1$ and (4.6), we have

$$\begin{cases} \{\sigma(u_n)+u_n\lambda\} - \{\sigma(u_{n-1})+u_{n-1}\lambda\} \leqq 0 & \text{on} \quad X, \\ \varphi_n - \varphi_{n-1} \leqq 0 \quad \omega\text{-a.e.} \quad \text{on} \quad \partial^* X \setminus \Lambda, \\ N(u_n - u_{n-1}) \geqq (\varphi_n - \varphi_{n-1})\beta \quad \text{on} \quad \Lambda. \end{cases}$$

Hence, again by Theorem 2.1, $u_n \leqq u_{n-1}$ on X.

By (4.6), (F.2) and (β.2), we see that (4.5) holds for $j=n$. Therefore, by induction, we obtain $\{u_n\}$ such that $u_n = H_{\varphi_n} + g_n$ with $\varphi_n \in \Phi_{BD}$ and $g_n \in \mathscr{Q}_{EFC}$, and

$$v \leqq \cdots \leqq u_n \leqq u_{n-1} \leqq \cdots \leqq u_1 \leqq u_0 \qquad \text{on} \quad X$$

for any v as above. In particular, we may take $v=\tilde{u}_0$, and hence $\{u_n\}$ is bounded below. Let $\varphi^* = \lim_{n\to\infty} \varphi_n$ and $u^* = \lim_{n\to\infty} u_n$. Then H_{φ^*} exists and $H_{\varphi_n} \to H_{\varphi^*}$ by Lebesgue's convergence theorem. Let $g^* = u^* - H_{\varphi^*}$. Then $g_n \to g^*$ $(n\to\infty)$. By Lebesgue's convergence theorem, $G(u_n\lambda)$ decreases to $G(u^*\lambda)$. Since u^* is bounded, $G(u^*\lambda)$ is continuous. Hence, by Dini's theorem, $G(u_n\lambda)$ converges to $G(u^*\lambda)$ locally uniformly on X. Since

$$g_n = G(\sigma(u_n)) = G(u_{n-1}\lambda) - G(u_n\lambda) - G(F(u_{n-1})),$$

$g^* = -\lim_{n\to\infty} G(F(u_n))$. Furthermore, by (F.2), we see that $\{G(F(u_n))\}$ converges locally uniformly on X.

Since $F(u_n) \leqq (1+M)\lambda$, $(1+M)G\lambda - G(F(u_n)) \in \mathscr{P}_{BC}$. It then follows that $(I+M)G\lambda + g^* \in \mathscr{P}_{BC}$, which shows that $g^* \in \mathscr{Q}_{BC} \subset \mathscr{R}(X)$. Hence $u^* \in \mathscr{R}(X)$. Furthermore, by (F.2), $G(F(u_n)) \to G(F(u^*))$ $(n\to\infty)$, so that $g^* = -G(F(u^*))$. Hence $\sigma(u^*) = \sigma(g^*) = -F(u^*)$ on X. Since $|u^*| \leqq M$, $F(u^*) \in \mathscr{M}_{BFC}$, and hence $g^* \in \mathscr{Q}_{BFC}$. Obviously, $\varphi^* = \tau$ ω-a.e. on $\partial^* X \setminus \Lambda$.

Next, we show that $\varphi^* \in \Phi_{BD}$ and $N(u^*) = \beta(\varphi^*)$ on Λ. If $m > n \geqq 1$, then by (4.6)

$$\begin{cases} \sigma(u_m - u_n) + (u_m - u_n)\lambda = (u_{m-1} - u_{n-1})\lambda - F(u_{m-1}) + F(u_{n-1}) \quad \text{on} \quad X \\ \varphi_m - \varphi_n = 0 \quad \omega\text{-a.e.} \quad \text{on} \quad \partial^* X \setminus \Lambda \\ N(u_m - u_n) = (\varphi_m - \varphi_n)\beta - (\varphi_{m-1} - \varphi_{n-1})\beta + \beta(\varphi_{m-1}) - \beta(\varphi_{n-1}) \quad \text{on} \quad \Lambda. \end{cases}$$

Let $\mu_{m,n} = (u_{m-1} - u_{n-1})\lambda - F(u_{m-1}) + F(u_{n-1})$. Then

$$0 \geqq \mu_{m,n} \geqq -2(u_{n-1}-u_{m-1})\lambda.$$

Hence in view of Lemma 3.6 $\mu_{m,n}$ satisfies condition $[\mu]$ in Theorem 3.1 with

$$a(\mu_{m,n}) = 2(1+\|G\lambda\|_\infty)\left\{\int (u_{n-1}-u_{m-1})^2 d\lambda\right\}^{1/2}.$$

Similarly, if we put $\gamma_{m,n}=(\varphi_{n-1}-\varphi_{m-1})\beta-\{\beta(\varphi_{n-1})-\beta(\varphi_{m-1})\}$, then $0\leqq\gamma_{m,n}\leqq 2(\varphi_{n-1}-\varphi_{m-1})\beta$, so that $\gamma_{m,n}$ satisfies condition $[\beta\text{-}\gamma]$ in Theorem 3.1 with

$$b(\beta, \gamma_{m,n}) = 2\left\{\int (\varphi_{n-1}-\varphi_{m-1})^2 d\beta\right\}^{1/2}.$$

Hence, by Theorem 3.1,

$$D^\lambda[u_n-u_m]^{1/2} \leqq (2+\|G\lambda\|_\infty)\left\{D[G(u_{n-1}-u_{m-1})\lambda]^{1/2}+2\left\{\int (u_{n-1}-u_{m-1})^2 d\lambda\right\}^{1/2}\right\}$$
$$+2\left\{\int (\varphi_{n-1}-\varphi_{m-1})^2 d\beta\right\}^{1/2},$$

which tends to 0 as $m, n\to\infty$, by Lebesgue's dominated convergence theorem. Here note that

$$D[G(u_{n-1}-u_{m-1})\lambda] = \int \{G(u_{n-1}\lambda)-G(u_{m-1}\lambda)\}(u_{n-1}-u_{m-1})d\lambda.$$

Therefore

$$D[H_{\varphi_n}-H_{\varphi_m}] \leqq D[u_n-u_m] \leqq D^\lambda[u_n-u_m] \longrightarrow 0 \qquad (n, m\to\infty),$$

that is, $\{\varphi_n\}$ is a Cauchy sequence in Φ_D. By Proposition 2.1 and Lemma 2.4, we conclude that $\varphi^*\in\Phi_D$. Since it is bounded, $\varphi^*\in\Phi_{BD}$. Also, we see that $u^*\in\mathscr{D}$ and $D[u_n-u^*]\to 0$ $(n\to\infty)$.

The last equality in (4.6) means that

$$\text{(4.7)}\quad D[H_\varphi, u_n] - \int_X H_\varphi d\sigma(u_n) + \int_A \varphi(\varphi_n-\varphi_{n-1})d\beta + \int_A \varphi d\beta(\varphi_{n-1}) = 0$$

for all $\varphi\in\Phi_{BD}(A)$. As we have seen above, $D[H_\varphi, u_n]\to D[H_\varphi, u^*]$ $(n\to\infty)$. Since $\sigma(u_n)-\sigma(u^*)=(u_{n-1}-u_n)\lambda-F(u_{n-1})+F(u^*)$,

$$|\sigma(u_n)-\sigma(u^*)| \leqq 2(u_{n-1}-u^*)\lambda$$

by (F.2), so that

$$\left|\int_X H_\varphi d\sigma(u_n) - \int_X H_\varphi d\sigma(u^*)\right| \leqq 2\int_X |H_\varphi|(u_{n-1}-u^*)d\lambda \longrightarrow 0 \qquad (n\to\infty).$$

By $(\beta.2)$, we have

$$\left|\int_A \varphi d\beta(\varphi_{n-1}) - \int_A \varphi d\beta(\varphi^*)\right| \leqq \int_A |\varphi|(\varphi_{n-1}-\varphi^*)d\beta \longrightarrow 0 \qquad (n\to\infty).$$

Obviously, $\int_A \varphi(\varphi_n-\varphi_{n-1})d\beta\to 0$ $(n\to\infty)$. Hence, letting $n\to\infty$ in (4.7), we obtain

$$D[H_\varphi, u^*] - \int_X H_\varphi d\sigma(u^*) + \int_A \varphi d\beta(\varphi^*) = 0$$

for all $\varphi\in\Phi_{BD}(A)$, which means that $N(u^*)=\beta(\varphi^*)$ on A.

Thus, $u=u^*$ and $\varphi=\varphi^*$ satisfy (4.2), and furthermore if $u=H_\varphi+g$ with $\varphi\in\Phi_{BD}$ and $g\in\mathscr{Q}_{BFC}$ is another solution of (4.2) such that $\tilde{u}_0\leqq u\leqq u_0$, then taking $v=u$ in the above argument we see that $u\leqq u^*$.

Starting with $\tilde{u}_0$ instead of u_0, we similarly obtain $\hat{u}=H_{\hat\varphi}+\hat{g}$ satisfying (i), (ii), (iii) of the theorem.

For a Radon measure ν on X such that $\int d|\nu|<+\infty$, let ω_ν denote the element of $\mathscr{N}(\partial^*X)$ defined by the linear form $\varphi\mapsto\int_X H_\varphi d\nu$ for $\varphi\in\mathscr{C}(\partial^*X)$, i.e., $\int_{\partial^*X}\varphi d\omega_\nu=\int_X H_\varphi d\nu$ for $\varphi\in\mathscr{C}(\partial^*X)$. Then the last equality holds for all $\varphi\in\mathscr{L}^\infty(\omega)$. If $\nu\in\mathscr{M}_{EFC}$, then $N(G\nu)=\omega_\nu$ on ∂^*X.

THEOREM 4.2. *Suppose F: $\mathscr{R}\to\mathscr{M}_C$ satisfies* (F.1) *and* (F.2) *and β: $\Phi_D\to\mathscr{N}(A)$ satisfies* (β.1) *and* (β.2). *Suppose furthermore that there exist t_0, $\tilde{t}_0\in\mathbf{R}$ and φ_0, $\tilde{\varphi}_0\in\Phi_{BD}$ such that*

(4.8) $N(H_{\varphi_0})\leqq\omega_{F(t_0)}+\beta(t_0)$ and $N(H_{\tilde\varphi_0})\geqq\omega_{F(\tilde{t}_0)}+\beta(\tilde{t}_0)$ on A.

Then, given $\tau\in\Phi_{BD}$, there exists $u=H_\varphi+g$ with $\varphi\in\Phi_{BD}$ and $g\in\mathscr{Q}_{BFC}$ which satisfies (4.2). *u is uniquely determined if $\omega(\partial^*X\setminus A)>0$. In case $\omega(\partial^*X\setminus A)=0$, if $\hat{u}=H_{\hat\varphi}+\hat{g}$ is another solution of* (4.2), *then $\hat{u}=u+c$ with a constant c such that $F(u+c)=F(u)$ on X and $\beta(\varphi+c)=\beta(\varphi)$ on A.*

PROOF. The last assertion follows from Theorem 2.1. To prove the existence, we may assume that $\tilde{t}_0\leqq t_0$ by virture of (F.1) and (β.1). Furthermore, by adding constants, we may assume

$$\inf_{\partial^*X}\varphi_0 \geqq \max(t_0, \sup_{\partial^*X}\tau) + \|G(F(t_0)^+)\|_\infty,$$

$$\sup_{\partial^*X}\tilde\varphi_0 \leqq \min(\tilde{t}_0, \inf_{\partial^*X}\tau) - \|G(F(\tilde{t}_0)^-)\|_\infty.$$

Let $u_0=H_{\varphi_0}-G(F(t_0))$ and $\tilde{u}_0=H_{\tilde\varphi_0}-G(F(\tilde{t}_0))$. Then

$$u_0\geqq\max(t_0, \sup_{\partial^*X}\tau) \quad\text{and}\quad \tilde{u}_0\leqq\min(\tilde{t}_0, \inf_{\partial^*X}\tau).$$

Hence $\tilde{u}_0\leqq u_0$,

$$\sigma(u_0) + F(u_0) = -F(t_0) + F(u_0) \geqq 0$$

and

$$\sigma(\tilde{u}_0) + F(\tilde{u}_0) = - F(\tilde{t}_0) + F(\tilde{u}_0) \leqq 0$$

on X;

$$\varphi_0 \geqq \tau \geqq \tilde{\varphi}_0 \qquad \omega\text{-a.e.} \quad \text{on} \quad \partial^* X \setminus \Lambda.$$

Furthermore, by (4.8), we have

$$N(u_0) \leqq \beta(t_0) \leqq \beta(\varphi_0) \quad \text{and} \quad N(\tilde{u}_0) \geqq \beta(\tilde{t}_0) \geqq \beta(\tilde{\varphi}_0) \qquad \text{on} \quad \Lambda.$$

Therefore, by Theorem 4.1, (4.2) has a solution $u=H_\varphi+g$ with $\varphi \in \Phi_{BD}$ and $g \in \mathcal{Q}_{BFC}$.

REMARKS. (i) In Theorem 4.2, condition (4.8) is also necessary for the existence of a solution of (4.2). In fact, if $u=H_\varphi+g$ is a solution of (4.2), then $N(H_\varphi)=\omega_{F(u)}+\beta(\varphi)$ on Λ, so that (4.8) is valid with $\varphi_0=\tilde{\varphi}_0=\varphi$, $t_0=\sup_X u$ and $\tilde{t}_0=\inf_X u$.

(ii) If $F(f_0)=0$ for some bounded function $f_0 \in \mathscr{R}(X)$, then (4.8) can be replaced by

$$N(H_{\varphi_0}) \leqq \beta(t_0) \quad \text{and} \quad N(H_{\tilde{\varphi}_0}) \leqq \beta(\tilde{t}_0)$$

for some φ_0, $\tilde{\varphi}_0 \in \Phi_{BD}$ and t_0, $\tilde{t}_0 \in \boldsymbol{R}$; in particular, if $F(f_0)=0$ for some bounded $f_0 \in \mathscr{R}(X)$ and $\beta(\psi_0)=0$ on Λ for some $\psi_0 \in \Phi_{BD}$, then (4.8) is satisfied (with $\varphi_0=\tilde{\varphi}_0=0$).

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