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Diffusion approximations of some stochastic difference equations II

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1. Introduction

This paper is concerned with the diffusion approximations of certain stochastic difference equations expressed in the following manner

$$(1)\qquad \begin{cases} X^{\varepsilon}_{n+1} - X^{\varepsilon}_{n} = \varepsilon F(n, X^{\varepsilon}_{n}, \omega) + \varepsilon^2 G(n, X^{\varepsilon}_{n}, \omega), \qquad (n = 0, 1, 2,\dots) \\ X^{\varepsilon}_{0} = X_0 \in R^d, \end{cases}$$

where $\{F(n, x, \omega)\}$ $(E(F(n, x, \omega))=0)$ and $\{G(n, x, \omega)\}$ are certain random fields on a probability space $(\Omega, \mathscr{B}, P)$. This paper is a continuation of the author's previous paper H. Watanabe [12] which deals with the case in which random fields are derived from some Markov chains. In this paper, we deal with the case in which given random fields are not necessarily Markovian, but mixing and stationary. The methods of the proof base on a paper of Kesten and Papanicolaou [6]. Here, we need more stringent conditions than in the continuous parameter case, about boundedness and differentiability of random fields. Iizuka and Matsuda [5] have considered the special case of (1). Here, we will show that their results can be derived as the discrete versions of [2], [6], [7], [9]. In the course of the preparations of this paper, a paper of Kushner and Huang [8] has appeared. They have considered the same problem as ours under ϕ-mixing conditions, while in our paper we assume strong mixing conditions.

2. Assumptions and main results

Let $(\Omega, \mathscr{B}, P)$ be a probability space. We introduce the following conditions.

I) $F(n, x, \omega)$ and $G(n, x, \omega)$: $R^d \times \Omega \to R^d (x \in R^d; \omega \in \Omega)$ are jointly measurable with respect to $\mathscr{B} \times \mathscr{B}(R^d)$ for each $n \in N = \{0, 1, 2,\dots\}$, where $\mathscr{B}(R^d) =$ σ-algebra of Borel sets in R^d.

II) For P almost all ω, the random field $F(n, x, \omega) = (F_1(n, x, \omega), F_2(n, x, \omega),\dots, F_d(n, x, \omega))$ (respectively, $G(n, x, \omega)$) are six (one) times continuously differentiable with respect to $x = (x_1,\dots, x_d)$.

III) $\{F(n, x, \omega), n \in N\}$ and $\{G(n, x, \omega), n \in N\}$ are strictly stationary for each fixed x.

IV) For each fixed $M<\infty$, there exist constants $C=C(M)$ and C_M independent of n such that

a) $E(\sup_{|x|\leqq M}|D^{\beta}F_l(n, x, \omega)|^8)\leqq C, \qquad 0\leqq|\beta|\leqq 6,$

b) $E(\sup_{|x|\leqq M}|D^{\gamma}G_l(n, x, \omega)|^{8-4|\gamma|})\leqq C, \qquad 0\leqq|\gamma|\leqq 1,$

c) $P(\max_{|x|\leqq M}|F_l(n, x, \omega)|\leqq C_M)=1,$

and

d) $P(\max_{|x|\leqq M}|G_l(n, x, \omega)|\leqq C_M)\leqq 1.$

V) $\int_{\Omega}F(n, x, \omega)P(d\omega)=E(F(n, x, \omega))=0.$

REMARK. Under the conditions IV) and V)

$$E(D^{\beta}F(n, x, \omega))=0, \qquad 0\leqq|\beta|\leqq 6.$$

VI) Let

$$\mathscr{M}_l^m(M)=\sigma\{F(n, x, \omega), G(n, x, \omega), l\leqq n\leqq m, |x|\leqq M\},$$

and

$$\beta(n, M)=\sup_{l\leqq 0}\sup_{A\in\mathscr{M}_0^l(M)B\in\mathscr{M}_{l+n}^{\infty}(M)}|P(A\cap B)-P(A)P(B)|.$$

Then,

$$\sum_{n=1}^{\infty}(\beta(n, M))^{1/p}<\infty \qquad \text{for} \quad p=6+2d.$$

VII) The following limits exist uniformly on compact sets and are continuous functions of x.[1]

$$a_{kl}(x)=\sum_{k,l}'^{\,2)}\sum_{n=1}^{\infty}E(F_k(0, x, \omega)F_l(n, x, \omega))+E(F_k(0, x, \omega)F_l(0, x, \omega)),$$

$$b_{kl}(x)=\sum_{n=1}^{\infty}E(F_k(0, x, \omega)\partial F_l/\partial x_k(n, x, \omega)),$$

and

$$c_k(x)=E(G_k(n, x, \omega)).$$

We put for $f\in C^2(R^d)$

$$\mathscr{L}f(x)=1/2\sum_{k,l=1}^{d}a_{kl}(x)\partial^2 f/\partial x_k\partial x_l+\sum_{l=1}^{d}\left(\sum_{k=1}^{d}b_{kl}(x)+c_l(x)\right)\partial f/\partial x_l.$$

The martingale problem associated with the generator $\mathscr{L}$ and starting at $X\in R^d$ has a unique solution R on $(C([0, \infty), R^d), \mathscr{B}(C))$ such that $R(X(0)=X)=1$, where $\mathscr{B}(C)$ is the topological Borel field of $C([0, \infty), R^d)$.

Define $X^{\varepsilon}(t)$ for $\{X_j^{\varepsilon}\}$ given by (1) as follows

1) Under the conditions I)–VI), we can show these facts.

2) $\sum_{kl}' 0_{k,l}\equiv 0_{k,l}+0_{l,k}$

(2) $X^\varepsilon(t) = X_j^\varepsilon + (t-j\varepsilon^2)/\varepsilon^2(X_{j+1}^\varepsilon - X_j^\varepsilon)$ if $j\varepsilon^2 \leqq t < (j+1)\varepsilon^2$, $(j=0, 1, 2,\dots)$,

$X^\varepsilon(0) = X_0$.

Let R^ε be the measure on $C([0, \infty), R^d)$ generated by the stochasitc process $\{X^\varepsilon(t)\}$.

THEOREM1. *We assume that the above stated conditions on* $\{F\}$ *and* $\{G\}$ *are satisfied. Then* R^ε *converges weakly to the probability measure* R *on* $C([0, \infty), R^d)$ *when* $\varepsilon \to 0$.

3. Proof of Theorem 1

Let $\phi_M(x)$ be a $C^\infty(R^d \to R)$ function such that $0 \leqq \phi_M \leqq 1$,

$$\phi_M(x) = \begin{cases} 1 & \text{if } |x| \leqq M/2, \\ 0 & \text{if } |x| \geqq M, \end{cases} \qquad (M \geqq 1)$$

and such that the grandient $\phi_M(x)$ is bounded uniformly in x. Define the truncated fields $F^M(n, x, \omega) = \phi_M(x)F(n, x, \omega)$ and $G^M(n, x, \omega) = \phi_M(x)G(n, x, \omega)$. Also define stochastic processes $X_n^{\varepsilon,M}(\omega)$ successively,

$$X_{n+1}^{\varepsilon,M}(\omega) = X_n^{\varepsilon,M}(\omega) + \varepsilon F^M(n, X_n^{\varepsilon,M}(\omega), \omega) + \varepsilon^2 G^M(n, X_n^{\varepsilon,M}(\omega), \omega),$$
$$X_0^{\varepsilon,M}(\omega) = X_0, \qquad (|X_0| \leqq M).$$

We define $X^{\varepsilon,M}(t)$ as in (2). From the definitions, we know that there hold

$|X_n^{\varepsilon,M}(\omega)| \leqq 2M$ for all $n = 1, 2,\dots$, and for sufficiently small $\varepsilon > 0$,

and

$|X^{\varepsilon,M}(t)| \leqq 2M$ for all $t > 0$ and for sufficiently small $\varepsilon > 0$.

Let $Q^{\varepsilon,M}$ be the measure induced by $X^{\varepsilon,M}(t)$ on $D([0, \infty), R^d)$. At first, we show that for each fixed $M > 0$, the family of measures $\{Q^{\varepsilon,M}\}_{\varepsilon>0}$ is tight on $D([0, \infty), R^d)$.

We list the following lemmas from Kesten and Papanicolaou [6] which we will use frequently. Their proofs will be omitted, because they are similar to [6].

LEMMA 1. *Let* $U(n, x, \omega)$ *be* $\mathcal{M}_n^\infty$[3]*-measurable for each fixed* $x(|x| \leqq 2M)$ *and such that* $E\{U(n, x, \omega)\} = 0$. *Let* $V(m, \omega)$, $m \leqq n$, *be an* $\mathcal{M}_m^m$*-measurable random variable. Then, for* $0 < \gamma \leqq 1$, *there eixsts a constant* $C_1 = C_1(\gamma, d)$ *such that for all* $0 \leqq l \leqq m \leqq n$

$$|E\{U(n, X_n^{\varepsilon,M})V(m)\}| \leqq C_1 E\{\textstyle\sum_{|\beta| \leqq 1} \sup_{|x| \leqq 2M} |D^\beta U(n, x)|^4\}^{1/4}$$
$$\times\ E(|V(m)|^4)^{1/4}\{\beta(n-m)\}^{\gamma/(2+\gamma+\gamma d)}.$$

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3) In the following, in $\mathcal{M}:(M)$, we will omit M and simply denote by $\mathcal{M}:$, also for $\beta(\cdot, M)$.

LEMMA 2. *Let $Y(m, \omega)$ be $\mathcal{M}_0^m$-measurable, and for $x(|x| \leqq 2M)$ let $U(n, x)$ (respectively $V(l, x)$) be $\mathcal{M}_n^n(\mathcal{M}_l^l)$ measurable. Assume that $E(V(l, x))=0$.*

Put

$$W(n, l, x) = E(U(n, x)V(l, x)).$$

Then, for each $0<\gamma \leqq 1$, there exists a constant $C_2=C_2(\gamma, d)$ such that for all $0 \leqq m \leqq n \leqq l$.

$$\begin{aligned}
&|E(Y(m)[U(n, X_m^{\varepsilon,M})V(l, X_m^{\varepsilon,M}) - W(n, l, X_m^{\varepsilon,M})])| \\
&\quad \leqq C_2[E\{|Y(m)|^8\}]^{1/8}[E\{\textstyle\sum_{|\beta| \leqq 1} \sup_{|x| \leqq 2M} |D^\beta U(n, x)|^8\}^{1/8} \\
&\quad \times (E\{\textstyle\sum_{|\beta| \leqq 1} \sup_{|x| \leqq 2M} |D^\beta V(l, x)|^8\})^{1/8}\{\beta(n-m, M)\beta(l-n, M)\}^{\gamma/(4+2\gamma+2\gamma d)}.
\end{aligned}$$

Let $\Phi(s, t)=E(|X^{\varepsilon,M}(t)-X^{\varepsilon,M}(s)|^r)$, $0 \leqq s \leqq t$, $0 \leqq r \leqq 2$. If we can show that

$$E(|X^{\varepsilon,M}(u) - X^{\varepsilon,M}(t)|^2\Phi) \leqq \text{const.}\, |u - t| E(|\Phi|^8)^{1/8},$$

then, by the argument in [6, p. 108], we can obtain the tightness of the family of measures $\{Q^{\varepsilon,M}\}_{\varepsilon>0}$ for each fixed $M>0$ on $D([0, \infty); R^d)$. Since $|X^{\varepsilon,M}(u)-X^{\varepsilon,M}_{[u/\varepsilon^2]}|^2 \leqq C_M|u-[u/\varepsilon^2]| \leqq C_M|u-t|$ and $|X^{\varepsilon,M}_{[t/\varepsilon^2]+1}-X^{\varepsilon,M}(t)|^2 \leqq C_M|u-t|$, it is enough to show that $E(|X^{\varepsilon,M}_{[u/\varepsilon^2]}-X^{\varepsilon,M}_{[t/\varepsilon^2]+1}|^2\Phi) \leqq \text{const.}\, |u-t| E(|\Phi|^8)^{1/8}$.

Put $[u/\varepsilon^2]=u(\varepsilon)$, $[u/\varepsilon^2]-1=u'(\varepsilon)$ and $[u/\varepsilon^2]+1=u''(\varepsilon)$ for an arbitrary real number u. We have, for $0 \leqq t \leqq u$, by Taylor's expansion theorem,

$$\begin{aligned}
&|X^{\varepsilon,M}_{(\varepsilon)} - X^{\varepsilon,M}_{t''(\varepsilon)}|^2 \\
&= \textstyle\sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{l=1}^d [(X^{\varepsilon,M}_{j+1} - X^{\varepsilon,M}_{t''(\varepsilon)})_l^2 - (X^{\varepsilon,M}_j - X^{\varepsilon,M}_{t''(\varepsilon)})_l^2] \\
&= 2\varepsilon \textstyle\sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{l=1}^d \int_0^1 F_l^M(j, X_j, \omega)\, (X^{\varepsilon,M}_j + u(X^{\varepsilon,M}_{j+1} - X^{\varepsilon,M}_j) - X^{\varepsilon,M}_{t''(\varepsilon)})_l du \\
&\quad + 2\varepsilon^2 \textstyle\sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{l=1}^d \int_0^1 G_l^M(j, X_j, \omega)\, (X^{\varepsilon,M}_j + u\, (X^{\varepsilon,M}_{j+1} - X^{\varepsilon}_{jM}) - X^{\varepsilon,M}_{t''(\varepsilon)})_l du \\
&= A_1 + A_2.
\end{aligned}$$

By the boundedness of $X_j^{\varepsilon,M}$ and condition *b*) in IV), we have

$$|E\{A_2\Phi\}| \leqq |u-t| \text{ const. } E(|\Phi|^8)^{1/8}.$$

Also, we have, by Taylor's theorem again,

$$\begin{aligned}
&E\{A_1\Phi\} \\
&= \varepsilon^2 \textstyle\sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{l=1}^d E[F_l^M(j, X_j^{\varepsilon,M}, \omega)(F_l^M(j, X_j^{\varepsilon,M}, \omega) + \varepsilon G_l^M(j, X_j^{\varepsilon,M}, \omega))\Phi] \\
&\quad + 2\varepsilon^2 \textstyle\sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{k=t''(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^d E([F_{l_2}^M(k, x, \omega) + \varepsilon G_{l_2}^M(k, x, \omega)) \\
&\quad \times (\partial/\partial x_{l_2}(F_{l_1}^M(j, x, \omega)\, (x - X^{\varepsilon,M}_{t''(\varepsilon)})_{l_1}]_{x=X_k^{\varepsilon,M}}\Phi)
\end{aligned}$$

$$+ \varepsilon^3 \sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{k=t''(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3,=1}^{d} E([\prod_{i=2}^{3} (F_{l_i}^M (k, x, \omega) + \varepsilon G_{l_i}^M (k, x, \omega))$$
$$\times (\partial^2/\partial x_{l_3}\partial x_{l_2}) (F_{l_1}^M(j, x, \omega) (x - X_{t''(\varepsilon)}^{\varepsilon,M})_{l_1})]_{x=X_k^{\varepsilon,M}}\Phi)$$
$$+ \varepsilon^4 \sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{k=t''(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3,l_4=1}^{d} E(\prod_{i=2}^{4} (F_{l_i}^M(k, X_k^{\varepsilon,M}, \omega) + \varepsilon G_{l_i}^M(k, X_k^{\varepsilon,M}, \omega))$$
$$\times \int_0^1 (1-u)^2 (\partial^3/\partial x_{l_4}\partial x_{l_3}\partial x_{l_2})(F_{l_1}^M(j, x, \omega)(x - X_{t''(\varepsilon)l_1}^{\varepsilon,M})_{x=X_k^{\varepsilon,M}+u(X_{k+1}^{\varepsilon,M}-X_k^{\varepsilon,M})} du\Phi)$$
$$= I_1 + I_2 + I_3 + I_4.$$

By Assumption IV), we can see easily that

$$|I_i| \leqq \text{const.}\, |u-t| E(|\Phi|^8)^{1/8} \qquad (i = 1, 4).$$

By means of Lemma 1, we have

$$|I_2| \leqq 2\varepsilon^2 \sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{k=t''(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} |E([(F_{l_2}^M(k, x, \omega) + \varepsilon G_{l_2}^M(k, x, \omega))$$
$$\times (\partial/\partial x_{l_2})(F_{l_1}^M(j, x, \omega)(x - X_{t''(\varepsilon)}^{\varepsilon,M})_{l_1} + F_{l_1}^M(j, x, \omega)\delta_{l_1,l_2}]_{x=X_k^{\varepsilon,M}}\Phi)|$$
$$\leqq 4\varepsilon^2 \sum_{j=t''(\varepsilon)}^{u'(\varepsilon)} \sum_{k=t''(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} c_1 \beta(j-k)^{1/(3+d)} 2ME\, [\sum_{|\beta|\leq 2} \sup_{|x|\leq 2M} |D^\beta F_{l_1}(j, x, \omega)|^4]^{1/4}$$
$$\times E[\sup_{|x|\leqq 2M} (|F_{l_2}^M(k, x, \omega)| + \varepsilon |G_{l_2}^M(k, x, \omega)|)^8]^{1/8} E(|\Phi|^8)^{1/8}$$
$$\leqq \text{const}\, |u - t| E(|\Phi|^8)^{1/8}.$$

By the same way, we have

$$|I_3| \leqq \text{const.}\, |u-t| E(|\Phi|^8)^{1/8}.$$

Since the family $\{Q^{\varepsilon,M}\}_{\varepsilon>0}$ is tight, we can extract a sequence $\{\varepsilon_n\}$ of positive numbers tending to zero such that $Q^{\varepsilon_n,M}$ converges weakly to Q^M. Generally, there may exist many such sequences and their corresponding limit measures, say, Q_λ^M, $\lambda \in \Lambda$. Now, we show that for any Q_λ^M, $\lambda \in \Lambda$,

$$f(X(t)) - \int_0^t \mathscr{L}^M f(X(t))dt$$

is a martingale with respect to $\mathscr{F}_0^t$ $(=\sigma(X(s), s \leqq t)$ on $C([0, \infty), R^d))$, where f is any C^∞ function $f\colon R^d \to R$ with compact support and $\mathscr{L}^M$ is a generator which will be determined later. For this purpose, we prove that for any integer $n>0$ and a bounded continuous function $\Phi\colon (R^d)^n \to R$ and $0 \leqq s_1 < s_2 < \cdots < s_n \leqq s \leqq t$

$$E^{Q_\lambda^M}\{(f(X(t)) - f(X(s)))\Phi(X(s_1),\ldots, X(s_n))\}$$
$$= E^{Q_\lambda^M}\left\{\int_s^t \mathscr{L}^M f(X(u))\Phi(X(s_1),\ldots, X(s_n))du\right\}.$$

For the mean time, we omit the super-suffix ε, M in $X_k^{\varepsilon,M}$ and $X^{\varepsilon,M}(t)$. By using Taylor's theorem repeatedly, we have for $0\leqq s\leqq t$,

$$\begin{aligned}
&f(X_{t(\varepsilon)})-f(X_{s(\varepsilon)})\\
&=\varepsilon\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l=1}^{d}\int_0^1(\partial/\partial x_l)f(X_j+u(X_{j+1}-X_j))\,du(F_l^M(j,X_j,\omega)\\
&\qquad+\varepsilon G_l^M(j,X_j,\omega))\\
&=\varepsilon\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l=1}^{d}\int_0^1(F_l^M(j,X_j,\omega)(\partial/\partial x_l)f(X_j+u(X_{j+1}-X_j))\\
&\qquad-F_l^M(j\,X_j,\omega)(\partial/\partial x_l)f(X_j))du\\
&\quad+\varepsilon\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{k=s(\varepsilon)}^{j-1}\sum_{l=1}^{d}(F_l^M(j,X_{k+1},\omega)(\partial/\partial x_l)f(X_{k+1})\\
&\qquad-F_l^M(j,X_k,\omega)(\partial/\partial x_l)f(X_k))\\
&\quad+\varepsilon\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l=1}^{d}F_l^M(j,X_{s(\varepsilon)},\omega)(\partial/\partial x_l)f(X_{s(\varepsilon)})\\
&\quad+\varepsilon^2\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l=1}^{d}\int_0^1(\partial/\partial x_l)f(X_j+u(X_{j+1}-X_j))\,duG_l^M(j,X_j,\omega)\\
&=J_1+J_2+J_3+J_4.
\end{aligned}$$

Let $\Phi_\varepsilon(\omega)=\Phi(X_{s_1(\varepsilon)},X_{s_2(\varepsilon)},\dots,X_{s_n(\varepsilon)})$ and $|\Phi_\varepsilon|\leqq C_\Phi<\infty$. Then we have

$$\begin{aligned}
|E(J_3\Phi_\varepsilon)|&=|E(\varepsilon\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l=1}^{d}F_l^M(j,X_{s(\varepsilon)},\omega)(\partial/\partial x_l)f(X_{s(\varepsilon)})\Phi_\varepsilon)|\\
&\leqq\varepsilon\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l=1}^{d}C_1E(\sum_{|\beta|\leqq1}\sup_{|x|\leqq 2M}|D^\beta F_l^M(j,x,\omega)|^4)^{1/4}\\
&\quad\times(E(|\Phi_\varepsilon(\partial/\partial x_l)f(X_{s(\varepsilon)})|^4)^{1/4}\beta(j-s(\varepsilon))^{1/(3+d)}\\
&\leqq\varepsilon\,\text{const}\sum_{l=1}^{\infty}\beta(l)^{1/(3+d)}\longrightarrow 0\quad(\varepsilon\to 0).
\end{aligned}$$

(3) J_1+J_2

$$\begin{aligned}
&=(\varepsilon^2/2)\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l_1,l_2=1}^{d}F_{l_1}^M(j,X_j,\omega)(F_{l_2}^M(j,X_j,\omega)\\
&\qquad+\varepsilon G_{l_2}^M(j,X_j,\omega))(\partial^2/\partial x_{l_2}\partial x_{l_1})f(X_j)\\
&\quad+(\varepsilon^2/2)\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{l_1,l_2,l_3=1}^{d}F_{l_1}^M(j,X_j,\omega)\prod_{i=2}^{3}((F_{l_i}^M(j,X_j,\omega)\\
&\qquad+\varepsilon G_{l_i}^M(j,X_j,\omega))\\
&\quad\times\int_0^1du\int_0^u(u-v)(\partial^3/\partial x_{l_3}\partial x_{l_2}\partial x_{l_1})f(X_k+v(X_{k+1}-X_k))dv\\
&\quad+\varepsilon^2\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{k=s(\varepsilon)}^{j-1}\sum_{l_1,l_2=1}^{d}(\partial/\partial x_{l_2})(F_{l_1}^M(j,x,\omega)(\partial/\partial x_{l_1})f(x))_{x=X_k}\\
&\quad\times(F_{l_2}^M(k,X_k,\omega)+\varepsilon G_{l_2}^M(k,X_k,\omega))\\
&\quad+(\varepsilon^3/2)\sum_{j=s(\varepsilon)}^{t'(\varepsilon)}\sum_{k=s(\varepsilon)}^{j-1}\sum_{l_1,l_2,l_3=1}^{d}(\partial^2/\partial x_{l_2}\partial x_{l_3})\\
&\qquad(F_{l_1}^M(j,x,\omega)(\partial/\partial x_{l_1})f(x))_{x=X_k}
\end{aligned}$$

$$\times \prod_{i=2}^{3} (F_{l_i}^M(k, X_k, \omega) + \varepsilon G_{l_i}^M(k, X_k, \omega))$$

$$+ (\varepsilon^4/3! \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3,l_4=1}^{d} \prod_{i=2}^{4}(F_{l_i}^M(j, X_k, \omega) + \varepsilon G_{l_i}^M(j, X_k, \omega))$$

$$\times (\partial^3/\partial x_{l_4}\partial x_{l_3}\partial x_{l_2})(F_{l_1}^M(j, x, \omega)\,(\partial/\partial x_{l_1})f(x))_{x=X_k}$$

$$+ (\varepsilon^5/3!) \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3,l_4,l_5=1}^{d} \prod_{i=2}^{5} (F_{l_i}^M(k, X_k, \omega) + \varepsilon G_{l_i}^M(k, X_k, \omega))$$

$$\times \int_0^1 (1-u)^3(\partial^4/\partial x_{l_5}\partial x_{l_4}\partial x_{l_3}\partial x_{l_2})(F_{l_1}^M(j, x, \omega)\,(\partial/\partial x_{l_1})f(x))_{x=X_k+u(X_{k+1}-X_k)}du$$

$$= J_{1,1} + J_{1,2} + J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}.$$

Gathering the terms involving ε^2 in (3), we put

$$K_1 = \varepsilon^2(2^{-1} \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} F_l^M(j, X_j, \omega)F_m^M(j, X_j, \omega)\,(\partial^2/\partial x_m\partial x_l)f(X_j)$$

$$+ \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l,m=1}^{d} (\partial/\partial x_m)(F_l^M(j, x, \omega)\,(\partial/\partial x_l)f(x))_{x=X_k} \times F_m^M(k, X_k, \omega))$$

$$= \varepsilon^2 \sum_{k=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} \sum_{j=k+1}^{t'(\varepsilon)} F_l^M(j, X_k, \omega)F_m^M(k, X_k, \omega)(\partial^2/\partial x_m\partial x_l)f(X_k)$$

$$+ \varepsilon^2 \sum_{k=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} \sum_{j=k+1}^{t'(\varepsilon)} F_m^M(k, X_k, \omega)\,((\partial/\partial x_k)\, F_l^M(j, x, \omega)\,(\partial/\partial x_l)f(x))_{x=X_k}$$

$$+ (\varepsilon^2/2) \sum_{k=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} F_l^M(j, X_j, \omega)F_m^M(j, X_j, \omega)(\partial^2/\partial x_m\partial x_l)f(X_j).$$

Then we have

$$E(K_1\Phi_\varepsilon(\omega))$$

$$= \varepsilon^2 \sum_{k=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} E\{(\partial^2/\partial x_m\partial x_l)f(X_k)\Phi_\varepsilon(\omega)\,(\sum_{j=k+1}^{t'(\varepsilon)}E(F_l^M(j, x, \omega) \times F_m^M(k, x, \omega))_{x=X_k})\}$$

$$+ \varepsilon^2 \sum_{k=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l,m=1}^{d} E[\partial^2/\partial x_m\partial x_l)f(X_k)\Phi_\varepsilon(\omega) \times \{F_l^M(j, x, \omega)F_m^M(k, x, \omega)$$

$$- E(F_l^M(j, x, \omega)\, F_m^M(k, x, \omega))\}_{x=X_k}]$$

$$+ \varepsilon^2 \sum_{k=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} E\{(\partial/\partial x_l)f(X_k)\Phi_\varepsilon(\omega) \sum_{j=k+1}^{t'(\varepsilon)} [E(F_m^M(k, x, \omega) \times (\partial/\partial x_m)\, F_l^M\,(j, x, \omega))]_{x=X_k}\}$$

$$+ \varepsilon^2 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l,m=1}^{d} E[(\partial/\partial x_l)f(X_k))\Phi_\varepsilon\,(\omega) \times \{F_m^M(k, x, \omega)(\partial/\partial x_m)F_l^M(j, x, \omega)$$

$$- E(F_m^M(k, x, \omega)\,(\partial/\partial x_m)F_l^M\,(j, x, \omega))\}_{x=X_k}]$$

$$+ (\varepsilon^2/2) \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} E[(\partial^2/\partial x_m\partial x_l)f(X_j)\Phi_\varepsilon(\omega))E(F_l^M(j, x, \omega) \times F_m^M(j, x, \omega))_{x=X_j}]$$

$$+ (\varepsilon^2/2) \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} E[(\partial^2/\partial x_m \partial x_l) f(X_j) \Phi_\varepsilon(\omega) \times \{F_l^M(j, x, \omega) F_m^M(j, x, \omega)$$
$$- E(F_l^M(j, x, \omega) F_m^M(j, x, \omega))\}_{x=X_j}]$$
$$= K_{1,1} + K_{1,2} + K_{1,3} + K_{1,4} + K_{1,5} + K_{1,6}.$$

From the mixing property VI (see Theorem 17. 2.2 of [4]), it follows that

$$\sum_{l,m}' \sum_{j=k+1}^{t'(\varepsilon)} E(F_l^M(j, x, \omega) F_m^M(k, x, \omega))$$
$$+ E(F_l^M(j, x, \omega) F_m^M(j, x, \omega)) = a_{lm}^M(x) + O_\varepsilon(1)$$

and

$$\sum_{j=k+1}^{t'(\varepsilon)} E\{F_m^M(k, x, \omega)(\partial/\partial x_m) F_l^M(j, x, \omega)\} = b_{lm}^M(x) + O_\varepsilon(1),$$

where $a_{lm}^M(x) \equiv \phi_M^2(x) a_{lm}(x)$, $b_{lm}^M(x) \equiv \phi_M^2(x) b_{lm}(x) + (\partial/\partial x_m)\phi_M(x) \sum_{j=1}^{\infty} E(F_m(0, x, \omega) \times F_l(j, x, \omega))$ and $O_\varepsilon(1) \to 0$ uniformly on R^d as $\varepsilon \to 0$.

Therefore, we have, applying Lemma 3.4 in [10],

$$K_{1,1} + K_{1,3} + K_{1,5}$$
$$= \varepsilon^2 \sum_{k=s(\varepsilon_n)}^{t'(\varepsilon_n)} \sum_{l,m=1}^{d} E[2^{-1}(\partial^2/\partial x_m \partial x_l) f(X_k) \Phi_{\varepsilon_n}(\omega) (a_{lm}^M(X_k) + O_{\varepsilon_n}(1))$$
$$+ (\partial/\partial x_l) f(X_k) \Phi_{\varepsilon_n}(\omega)(b_{lm}^M(X_k) + O_{\varepsilon_n}(1))]$$
$$\longrightarrow \sum_{l=1}^{d} \sum_{m=1}^{d} \int_s^t E^{Q_\lambda^M}[2^{-1} (\partial^2/\partial x_m \partial x_l) f(X(u)) \Phi(\omega) a_{lm}^M(X(u))$$
$$+ (\partial/\partial x_l) f(X(u)) \Phi(\omega) b_{lm}^M(X(u))] du, \quad (\varepsilon_n \longrightarrow 0),$$

where $Q^{\varepsilon_n, M} \Rightarrow Q_\lambda^M$ $(\varepsilon_n \to 0)$. Put $H_{l,m}(j, k, x, \omega) = F_l^M(j, x, \omega) F_m^M(k, x, \omega) - E(F_l^M(j, x, \omega) F_m^M(k, x, \omega))$. Then, $K_{1,2}$ can be written in the following way,

$$K_{1,2} = \varepsilon^2 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} E\{(\partial^2/\partial x_{l_2} \partial x_{l_1}) f(X_{s(\varepsilon)}) \times \Phi_\varepsilon(\omega) H_{l_1,l_2}(j, k, X_{s(\varepsilon)}, \omega)\}$$
$$+ \varepsilon^3 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,l_3=1}^{d} E\{\Phi_\varepsilon(\omega)(F_{l_3}^M(h, X_h, \omega) + \varepsilon G_{l_3}^M(h, X_h, \omega))$$
$$\times (\partial/\partial x_{l_3})(\partial^2/\partial x_{l_2} \partial x_{l_1}) f(x) H_{l_1,l_2}(j, k, x, \omega))_{x=X_h}\}$$
$$+ (\varepsilon^4/2!) \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,l_3,l_4=1}^{d} E\{\Phi_\varepsilon(\omega) \prod_{i=3}^{4} (F_{l_i}^M(h, X_h, \omega)$$
$$+ \varepsilon G_{l_i}^M(h, X_h, \omega))(\partial^2/\partial x_{l_4} \partial x_{l_3}((\partial^2/\partial x_{l_2} \partial x_{l_1}) f(x) H_{l_1,l_2}(j, k, x, \omega)))_{x=X_h}\}$$
$$+ (\varepsilon^5/3!) \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,\ldots,l_5=1}^{d} E\{\Phi_\varepsilon(\omega) \prod_{i=3}^{5} (F_{l_i}^M(h, X_h, \omega)$$
$$+ \varepsilon G_{l_i}^M(h, X_h, \omega))(\partial^3/\partial x_{l_5} \partial x_{l_4} \partial x_{l_3})((\partial^2/\partial x_{l_2} \partial x_{l_1}) f(x) H_{l_1,l_2}(j, k, x, \omega)))_{x=X_h}\}$$
$$+ (\varepsilon^6/4!) \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,\ldots,l_6=1}^{d} E\{\Phi_\varepsilon(\omega) \prod_{i=3}^{6} (F_{l_i}^M(h, X_h, \omega)$$

$$+ \varepsilon G^M_{l_i}(h, X_h, \omega))(\partial^4/\partial x_{l_6}\partial x_{l_5}\partial x_{l_4}\partial x_{l_3})((\partial^2/\partial x_{l_2}\partial x_{l_1})f(x)H_{l_1,l_2}(j, k, x, \omega))_{x=X_h}\}$$

$$+ (\varepsilon^7/4!) \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,\ldots,l_7=1}^{d} E\{\Phi_\varepsilon(\omega) \prod_{i=3}^{7} (F^M_{l_i}(h, X_h, \omega)$$

$$+ \varepsilon G^M_l(h, X_h, \omega))$$

$$\times \int_0^1 (1-u)^4(\partial^5/\partial x_{l_7}\partial x_{l_6}\partial x_{l_5}\partial x_{l_4}\partial x_{l_3})$$

$$((\partial^2/\partial x_{l_2}\partial x_{l_1})f(x)H_{l_1,l_2}(j, k, x, \omega)))_{x=X_h+u(X_{h+1}-X_h)}du\}$$

$$= K_{1,2,1} + K_{1,2,2} + \cdots + K_{1,2,6}.$$

By applying Lemma 2, we have

$$|K_{1,2,1}| \leqq \varepsilon^2 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} |E\{\Phi_\varepsilon(\omega)$$

$$\times (\partial^2/\partial x_{l_2}\partial x_{l_1})f(X_{s(\varepsilon)})H_{l_1,l_2}(j, k, X_{s(\varepsilon)}, \omega)\}|$$

$$\leqq \text{const. } \varepsilon^2 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} E\{|\Phi_\varepsilon(\omega)(\partial^2/\partial x_{l_2}\partial x_{l_1})f(X_{s(\varepsilon)})|^8\}^{1/8}$$

$$\times E \textstyle\sum_{|\beta|\leqq 1} \sup_{|x|\leqq 2M}|D^\beta F^M_{l_1}(j, x, \omega)|^8\}^{1/8}$$

$$\times E\{\textstyle\sum_{|\beta|\leqq 1} \sup_{|x|\leqq 2M}|D^\beta F^M_{l_2}(k, x, \omega)|^8\}^{1/8}$$

$$\times (\beta(j-k)\beta(k-s(\omega))^{1/(6+2d)}$$

$$\leqq \text{const. } \varepsilon^2 \textstyle\sum_{j_1=0}^{\infty} \beta(j_1)^{1/(6+2d)} \sum_{j_2=0}^{\infty} \beta(j_2)^{1/(6+2d)} \longrightarrow 0 \quad (\varepsilon \to 0).$$

Again applying Lemma 2, we have

$$|K_{1,2,2}| \leqq \text{const. } \varepsilon^3 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,l_3=1}^{d}$$

$$\times [E(|\Phi_\varepsilon(\omega)| \sup_{|x|\leqq 2M}|(\partial^3/\partial x_{l_3}\partial x_{l_2}\partial x_{l_1})f(x)F^M_{l_3}(h, x, \omega)|)^8)^{1/8}$$

$$+ \varepsilon E((|\Phi_\varepsilon(\omega)| \sup_{|x|\leqq 2M}|(\partial^3/\partial x_{l_3}\partial x_{l_2}\partial x_{l_1})f(x)G^M_{l_3}(h, x, \omega)|)^8)^{1/8}]$$

$$\times E\{\textstyle\sum_{|\beta|\leqq 1} \sup_{|x|\leqq 2M}|D^\beta F^M_{l_1}(j, x, \omega)|^8\}^{1/8} E\{\sum_{|\beta|\leqq 1} \sup_{|x|\leqq 2M}$$

$$\times |D^\beta F^M_{l_2}(k, x, \omega)|^8\}^{1/8}$$

$$\times (\beta(j-k)\beta(k-h))^{1/(6+2d)}$$

$$+ \text{const. } \varepsilon^3 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} \sum_{l_1,l_2,l_3=1}^{d}$$

$$\times \{E((|\Phi_\varepsilon(\omega)| \sup_{|x|\leqq 2M}|(\partial^2/\partial x_{l_2}\partial x_{l_1})f(x)F^M_{l_3}(h, x, \omega)|)^8)^{1/8}$$

$$+ \varepsilon E((|\Phi_\varepsilon(\omega)| \sup_{|x|\leqq 2M}|(\partial^2/\partial x_{l_2}\partial x_{l_1})f(x)G^M_{l_3}(h, x, \omega)|)^8)^{1/8}\}$$

$$\times E(\textstyle\sum_{|\beta|\leqq 2} \sup_{|x|\leqq 2M}|D^\beta F^M_{l_1}(j, x, \omega)|^8)^{1/8} E(\sum_{|\beta|\leqq 2} \sup_{|x|\leqq 2M}$$

$$\times |D^\beta F^M_{l_2}(k, x, \omega)|^8)^{1/8}$$

$$\times (\beta(j-k)\beta(k-h))^{1/(6+2d)}$$

$$\leqq \text{const. } \varepsilon^3 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{h=s(\varepsilon)}^{k-1} (\beta(j-k)\beta(k-h))^{1/(6+2d)}$$

$\leqq \text{const.}\, \varepsilon^3(t(\varepsilon) - s(\varepsilon))(\sum_{v=0}^{\infty} \beta(v)^{1/(6+2d)})^2$

$\leqq \text{const.}\, \varepsilon|t-s| \longrightarrow 0, \quad (\varepsilon\to 0).$

If we recall that $E(D^\alpha F(n, x, \omega))=0$, $|\alpha|\leqq 6$, we can prove that $|K_{1,2,i}|\to 0$ $(\varepsilon\to 0)$ for $i=3, 4, 5$.

Since

$$E\{\Phi_\varepsilon(\omega) \prod_{i=3}^{7} (F_{l_i}^M(h, X_h, \omega) + \varepsilon G_{l_i}^M(h, X_h, \omega)) \times (\partial^5/\partial x_{l_7}\partial x_{l_6}\partial x_{l_5}\partial x_{l_4}\partial x_{l_3}) ((\partial^2/\partial x_{l_2}\partial x_{l_1}) f(x) H_{l_1,l_2}(j, k, x, \omega)))_{x=X_h+u(X_{h+1}-X_h)}\}$$

is bounded, we have

$$|K_{1,2,6}| \leqq \text{const.}\, \varepsilon(\varepsilon^2(t(\varepsilon) - s(\varepsilon)))^3 \leqq \text{const.}\, \varepsilon|t-s|^3 \longrightarrow 0, \qquad (\varepsilon\to 0).$$

By the same method as for $K_{1,2}$ we can see that $K_{1,4}\to 0$, $(\varepsilon\to 0)$. For $K_{1,6}$, we make the following transformation.

$$\begin{aligned} K_{1,6} &= 2^{-1}\varepsilon^2 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l,m=1}^{d} E(\partial^2/\partial x_m\partial x_l) f(X_{s(\varepsilon)})\Phi_\varepsilon(\omega)(H_{l,m}(j, j, X_j, \omega)) \\ &+ 2^{-1}\varepsilon^3 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3=1}^{d} E[(\partial^3/\partial x_{l_3}\partial x_{l_2}\partial x_{l_1}) f(X_k)\Phi_\varepsilon(\omega) \\ &\times (F_{l_3}^M(k, X_k, \omega) + \varepsilon G_{l_3}^M(k, X_k, \omega)) H_{l_1,l_2}(j, j, X_j, \omega)] \\ &+ 4^{-1}\varepsilon^4 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3,l_4=1}^{d} E[(\partial^4/\partial x_{l_4}\partial x_{l_3}\partial x_{l_2}\partial x_{l_1}) f(X_k)\Phi_\varepsilon(\omega) \\ &\times \prod_{i=3}^{4} (F_{l_i}^M(k, X_k, \omega) + \varepsilon G_{l_i}^M(k, X_k, \omega) H_{l_1,l_2}(j, j, X_j, \omega)] \\ &+ 4^{-1}\varepsilon^5 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{j=s(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3,l_4,l_5=1}^{d} E\Big[\int_0^1 (1-u)^2 (\partial^5/\partial x_{l_5}\partial x_{l_4}\partial x_{l_3}\partial x_{l_2}\partial x_{l_1}) f \\ &\times (X_k + u(X_{k+1} - X_k))\, du\Phi_3(\omega) \prod_{i=3}^{5} (F_{l_i}^M(k, X_k, \omega) \\ &\qquad + \varepsilon G_{l_i}^M(k, X_k, \omega)) H_{l_1,l_2}(j, j, X_j, \omega)\Big] \\ &= K_{1,6,1} + K_{1,6,2} + K_{1,6,3} + K_{1,6,4}. \end{aligned}$$

By applying Lemma 1 to $K_{1,6,1}$, we have

$$K_{1,6,1} \leqq \text{const.}\, \varepsilon^2 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \beta(j - s(\varepsilon))^{1/(3+d)} \longrightarrow 0, \qquad (\varepsilon^2\to 0).$$

Also, we can show that $K_{1,6,i}\to 0$, $(\varepsilon\to 0)$, $i=2, 3, 4$.

We put

$$\begin{aligned} &J_{1,1} + J_{2,1} \\ &= K_1 + \varepsilon^3 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l_1,l_2,=1}^{d} F_{l_1}^M(j, X_j, \omega) G_{l_2}^M(j, X_j, \omega) (\partial^2/\partial x_{l_2}\partial x_{l_1}) f(X_j) \\ &+ \varepsilon^3 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} (\partial/\partial x_{l_2}) (F_{l_1}^M(j, x, \omega) (\partial/\partial x_{l_2}) f(x))_{x=Xk} \\ &\qquad \times G_{l_2}^M(k, X_k, \omega) \\ &= K_1 + K_2 + K_3. \end{aligned}$$

Since $E\{F_{l_1}^M(j, X_j, \omega)G_{l_2}^M(j, X_j, \omega)(\partial^2/\partial x_{l_2}\partial x_{l_1})f(X_j)\Phi_\varepsilon(\omega)\}$ is bounded, we have $E\{K_2\Phi_\varepsilon\}\to 0$, $(\varepsilon\to 0)$.

Furthermore, we have

$$
\begin{aligned}
&|E\{K_3\Phi_\varepsilon\}| \\
&\leqq \varepsilon^3 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} E\{(|\Phi_\varepsilon(\omega)|\sup_{|x|\leqq 2M}|(\partial^2/\partial x_{l_2}\partial x_{l_1})f(x) \\
&\qquad\qquad\qquad\qquad \times G_{l_2}^M(k,x,\omega)|)^4\}^{1/4} \\
&\quad \times E\{\textstyle\sum_{|\beta|\leqq 1}\sup_{|x|\leqq 2M}|D^\beta F_{l_1}^M(j,x,\omega)|^4\}^{1/4}\beta(j-k)^{1/(3+d)} \\
&\quad + \varepsilon^3 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2=1}^{d} E\{(|\Phi_\varepsilon(\omega)|\sup_{|x|\leqq 2M}|(\partial/\partial x_{l_1})f(x) \\
&\qquad\qquad\qquad\qquad \times G_{l_2}^M(k,x,\omega|)^4\}^{1/4} \\
&\quad \times E\{\textstyle\sum_{|\beta|\leqq 2}\sup_{|x|\leqq 2M}|D^\beta F_{l_1}^M(j,x,\omega)|^4\}^{1/4}\beta(j-k)^{1/(3+d)} \\
&\leqq \text{const. } \varepsilon \textstyle\sum_{l=0}^{\infty}\beta(l)^{1/(3+d)} \longrightarrow 0, \quad (\varepsilon\to 0).
\end{aligned}
$$

For $J_{1,2}$, we have

$$
\begin{aligned}
&E(J_{1,2}\Phi_\varepsilon(\omega)) \\
&= \varepsilon^3 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l_1,l_2,l_3=1}^{d} E\{F_{l_1}^M(j,X_j,\omega)\prod_{i=2}^{3}(F_{l_i}^M(j,X_j,\omega) \\
&\qquad\qquad\qquad\qquad + \varepsilon G_{l_i}^M(j,X_j,\omega))\Phi_\varepsilon(\omega) \\
&\quad \times \int_0^1 du\int_0^u (u-v)^2(\partial^3/\partial x_{l_3}\partial x_{l_2}\partial x_{l_1})f(X_k+v(X_{k+1}-X_k))dv\}.
\end{aligned}
$$

Since $E\{\ \}$ is bounded by assumptions, we have

$$|E\{J_{1,2}\Phi_\varepsilon(\omega)\}| \leqq \text{const. } \varepsilon \longrightarrow 0, \qquad (\varepsilon\to 0).$$

In the same way, we have $E\{J_{2,4}\Phi_\varepsilon(\omega)\}\to 0$ $(\varepsilon\to 0)$.

Concerning $J_{2,2}$, since $E(D^\alpha F(j, x, \omega))=0$, we can apply Lemma 1 to this situation and we have

$$
\begin{aligned}
&|E\{J_{2,2}\Phi_\varepsilon(\omega)\}| \\
&\leqq (\varepsilon^3/2) \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \sum_{l_1,l_2,l_3=1}^{d} |E\{\Phi_\varepsilon(\omega) \\
&\qquad\qquad \times((\partial^2/\partial x_{l_3}\partial x_{l_2})(F_{l_1}^M(j,x,\omega)(\partial/\partial x_{l_1})f(x)))_{s=Xk} \\
&\quad \times \textstyle\prod_{i=2}^{3}(F_{l_i}^M(k,X_k,\omega)+\varepsilon G_{l_i}^M(k,X_k,\omega)\}| \\
&\leqq \varepsilon^2 \textstyle\sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{k=s(\varepsilon)}^{j-1} \text{const. } \beta(j-k)^{1/(3+d)} \\
&= 0(\varepsilon) \longrightarrow 0, \quad (\varepsilon\to 0).
\end{aligned}
$$

In the same way we have $E\{J_{2,3}\Phi_\varepsilon(\omega)\}\to 0$ $(\varepsilon\to 0)$.

Let us consider J_4. J_4 can be transformed in the following form

$$J_4 = \varepsilon^2 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l=1}^d (\partial/\partial x_l) f(X_j) G_l^M(j, X_j, \omega)$$

$$+ \varepsilon^2 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l=1}^d \int_0^1 (\partial/\partial x_l) f(X_j + u(X_{j+1} - X_j))$$

$$- (\partial/\partial x_l) f(X_j)) du G_l^M(j, X_j, \omega)$$

$$= \varepsilon^2 \sum_{l=1}^d \left(\sum_{j=s(\varepsilon)}^{s(\varepsilon)+T-1} + \cdots + \sum_{j=s(\varepsilon)+nT}^{t'(\varepsilon)}\right) (\partial/\partial x_l) f(X_j) G_l^M(j, X_j, \omega))$$

$$+ \varepsilon^3 \sum_{j=s(\varepsilon)}^{t'(\varepsilon)} \sum_{l_1, l_2=1}^d \int_0^1 du \int_0^u (\partial^2/\partial x_{l_2} \partial x_{l_1}) f(X_j + v(X_{j+1} - X_j)) dv$$

$$\times G_{l_1}^M(j, X_j, \omega)(F_{l_2}^M(j, X_j, \omega) + \varepsilon G_{l_2}^M(j, X_j, \omega))$$

$$= J_{4,1} + J_{4,2},$$

where we put $n = [(t(\varepsilon) - s(\varepsilon))/T]$ for large integer $T > 0$. From Condition IX), we can easily deduce that

$$E(J_{4,2} \Phi_\varepsilon(\omega)) \longrightarrow 0, \quad \text{as} \quad \varepsilon \to 0.$$

Let us consider $J_{4,1}$. We have

$$E(J_{4,1} \Phi_\varepsilon(\omega)) = \varepsilon^2 \sum_{l=1}^d \left(\sum_{k=1}^{n-1} \sum_{j=kT}^{(k+1)T-1} + \sum_{j=nT}^{t(\varepsilon)-s(\varepsilon)-1}\right) E(\partial/\partial x_l) f(X_{kT+s(\varepsilon)})$$

$$\times G_l^M(j + s(\varepsilon), X_{kT+s(\varepsilon)}, \omega) \Phi_\varepsilon(\omega))$$

$$+ \varepsilon^2 \sum_{l=1}^d \left(\sum_{k=0}^{n-1} \sum_{j=kT}^{(k+1)T-1} \sum_{h=kT}^{j-1} + \sum_{j=nT}^{t(\varepsilon)-s(\varepsilon)-1} \sum_{h=nT}^{j-1}\right.$$

$$\times E([(\partial/\partial x_l) f(X_{h+1+s(\varepsilon)}) G_l^M(j + s(\varepsilon), X_{h+1+s(\varepsilon)}, \omega)$$

$$- (\partial/\partial x_l) f(X_{h+s(\varepsilon)}) G_l^M(j + s(\varepsilon), X_{h+s(\varepsilon)}, \omega)] \Phi_\varepsilon(\omega))$$

$$= \varepsilon^2 \sum_{l=1}^d \left(\sum_{k=0}^{n-1} \sum_{j=kT}^{(k+1)T-1} + \sum_{j=nT}^{t(\varepsilon)-s(\varepsilon)-1}\right) E\{(\partial/\partial x_l) f(X_{kT+s(\varepsilon)}) \Phi_\varepsilon(\omega)$$

$$\times E(G_l^M(j + s(\omega), x, \omega))_{s = X_{kT+s(\varepsilon)}}\}$$

$$+ \varepsilon^2 \sum_{l=1}^d \left(\sum_{k=0}^{n-1} \sum_{j=kT}^{(k+1)T-1} + \sum_{j=nT}^{t(\varepsilon)-s(\varepsilon)-1}\right) E\{(\partial/\partial x_l) f(X_{kT+s(\varepsilon)}) \Phi_\varepsilon(\omega)$$

$$\times (G_l^M(j + s(\varepsilon), x, \omega) - E(G_l^M(j + s(\varepsilon), x, \omega)))_{x = X_{kT+s(\varepsilon)}}\}$$

$$+ \varepsilon^3 \sum_{l_1=1}^d \sum_{l_2=1}^d \left(\sum_{k=0}^{n-1} \sum_{j=kT}^{(k+1)T-1} \sum_{h=kT}^{j-1} + \sum_{j=nT}^{t(\varepsilon)-s(\varepsilon)-1} \sum_{h=nT}^{j-1}\right)$$

$$\times E\left[\Phi_\varepsilon(\omega) \left(\left(\int_0^1 (\partial/\partial x_{l_2}) \left((\partial/\partial x_{l_1}) f(x) \left(G_l^M(j + s(\varepsilon), x, \omega)\right)\right.\right.\right.\right.$$

$$- E(G_l^M(j + s(\varepsilon), x, \omega)))_{x = X_{h+s(\varepsilon)} + u(X_{h+1+s(\varepsilon)} - X_{h+s(\varepsilon)})})) du$$

$$\left.\left.\times (F_{l_2}^M(h + s(\varepsilon), X_{h+s(\varepsilon)}, \omega) + \varepsilon G_l^M(h + s(\varepsilon), X_{h+s(\varepsilon)}, \omega)\right)\right]$$

$$= L_1 + L_2 + L_3.$$

Now choose T such that $\varepsilon T \to 0$ $(\varepsilon \to 0)$, but $T \to \infty$ (for example, take as $T = [\varepsilon^{-1/2}]$). Since the integrand in L_3 is bounded by Condition IV), we can see that $L_3 \to 0$ $(\varepsilon \to 0)$.

Since $E(G_l^M(j+s(\varepsilon), x, \omega)) = \phi_M(x)c_l(x) \equiv c_l^M(x)$ by definition,

$$L_1 = \varepsilon_n^2 T \sum_{l=1}^d [(\sum_{k=0}^{n-1} E(\partial/\partial x_l) f(X_{kT+s(\varepsilon_n)}) \Phi_{\varepsilon_n}(\omega) c_l^M(X_{kT+s(\varepsilon_n)}))$$
$$+ T^{-1}(t(\varepsilon_n) - s(\varepsilon_n) - nT) E(\partial/\partial x_l) f(X_{nT+s(\varepsilon_n)}) \Phi_{\varepsilon_n}(\omega) c_l^M(X_{nT+s(\varepsilon_n)}))]$$
$$\longrightarrow \sum_{l=1}^d \int_0^t E^{Q_\lambda^M}(\partial/\partial x_l) f(x(u)) c_l^M(x(u)) \Phi(\omega) du.$$

Let $K(n, x, \omega) = G_l^M(n, x, \omega) - c_l^M(x)$, then $E(K(n, x, \omega)) = 0$ and $K(n, x, \omega)$ is $\mathscr{M}_n^\infty$-measurable. Therefore, we can apply Lemma 1 to L_2. Hence, we have

$$|L_2| \leqq \varepsilon^2 \sum_{l=1}^d (\sum_{k=0}^{n-1} \sum_{j=kT}^{(k+1)T-1} + \sum_{j=nT,(k=n)}^{t(\varepsilon)-s(\varepsilon)-1})$$
$$\times \text{const. } E\{\sum_{|\beta| \leqq 1} \sup_{|x| \leqq 2M} |D^\beta K(j+s(\varepsilon), x, \omega)|^4\}^{1/4}$$
$$\times E(|(\partial/\partial x_l) f(x_{kT+s(\varepsilon)}) \Phi_\varepsilon(\omega)|^4)^{1/4} (\beta(j-kT))^{1/(3+d)}$$
$$\leqq \text{const. } \varepsilon^2 \cdot \varepsilon^{-2} T^{-1} \sum_{v=0}^\infty \beta(v)^{1/(3+d)} = \text{const. } T^{-1} \longrightarrow 0, \quad (\varepsilon \to 0).$$

Thus, we have, for any bounded $\mathscr{F}_0^s$-measurable $\Phi(\omega)$,

$$E^{Q_\lambda^M}((f(x(t)) - f(x(s))\Phi(\omega)) = E^{Q_\lambda^M} \left\{ \int_s^t [\{\sum_{l,m=1}^d 2^{-1}(\partial^2/\partial x_m \partial x_l) f(x(u)) a_{ml}^M(x(u)) \right.$$
$$\left. + \sum_{l,m=1}^d (\partial/\partial x_l) f(x(u)) b_{lm}^M(x(u)) + c_l^M(x(u))\} \Phi(\omega)] du \right\},$$

from which we can deduce that any Q_λ^M, $\lambda \in M$, are solutions of the martingale problem such that

$$f(x(t)) - \int_0^t \mathscr{L}^M f(x(u)) du$$

is a martingale on $(C([0, \infty), R^d), \mathscr{F}_0^t)$[4], where $\mathscr{L}^M = 2^{-1} \sum_{l,m=1}^d (a_{lm}^M(x)(\partial^2/\partial x_l \partial x_m) + \sum_{l,m=1}^d (b_{lm}^M(x) + c_l^M(x))(\partial/\partial x_l)$

We have uniformly on each compact set

$$\lim_{M\to\infty} a_{kl}^M(x) = a_{kl}(x), \lim_{M\to\infty} b_{kl}^M(x) = b_{kl}(x) \text{ and } \lim_{M\to\infty} c_k^M(x) = c_k(x),$$

which are continuous.

By Assumption VII), applying Theorem 11.4 in Stroock and Varadhan ([11], p. 264) to Q_λ^M, Q_λ^M weakly converges to R as $M \to \infty$.

If we repeat the argument in Kesten and Papanicolaou [6], we can show that R^ε weakly converges to R on $C([0, \infty), R^d)$.

4. Special cases

If the random fields $\{F(n, x, \omega)\}$ and $\{G(n, x, \omega)\}$ take the special forms,

4) As in [6], we can show that each Q_λ^M is concentrated on C.

assumptions can be much simplified.

In this section, we consider stochastic difference equations defined by

$$X_{n+1}^{\varepsilon} = X_n^{\varepsilon} + \varepsilon F^{(1)}(n, X_n^{\varepsilon}, \omega) + \varepsilon^2 F^{(2)}(n, X_n^{\varepsilon}, \omega) + \varepsilon^3 F^{(3)}(n, X_n^{\varepsilon}, \varepsilon, \omega), \; X_0^{\varepsilon} = X_0.$$

We introduce the following conditions.

I′) The random fields $F^{(i)}(n, x, \omega) = (F_1^{(i)}(n, x, \omega), \dots, F_d^{(i)}(n, x, \omega)\}$ $(i=1,2)$ are expressed in the form

$$F_k^{(i)}(n, x, \omega) = \sum_{l=1}^{d_i} g_{kl}^{(i)}(x)\xi_l^{(i)}(n, \omega), \qquad (k = 1, 2, \dots, d),$$

where $\{\xi_l^{(i)}(n, \omega), l=1, \dots, d_i\}$, $n=0, 1, 2, \dots$ are real $\mathscr{B}$-measurable functions and $g_{kl}^{(i)}(x)$ are real $\mathscr{B}(R^d)$-measurable functions. $F^{(3)}(n, x, \varepsilon, \omega) = (F_1^{(3)}(n, x, \varepsilon, \omega), \dots, F_d^{(3)}(n, x, \varepsilon, \omega))$ is real $\mathscr{B}$-measurable for $n=0, 1, 2, \dots, x \in R^d$.

II′) $\{g_{kl}^{(1)}(x)\}_{k=1,2,\dots,d, l=1,\dots,d_1}: R^d \to R$ (respectively $g_{kl}^{(2)}(x)$) are six (one) times continuously differentiable with respect to $x = (x_1, x_2, \dots, x_d)$.

III′) $\{\xi_j^{(i)}(n, \omega), j=1, \dots, d\}$ $(i=1, 2)$ and $\{F^{(3)}(n, \cdot, \varepsilon, \omega)\}$ are bounded strictly stationary vector processes and $E(\xi_j^{(1)}(n, \omega)) = 0$.

IV′) Let $\mathscr{M}_l^m = \sigma\{\xi_j^{(i)}(n, \omega), F_j^{(3)}(n, x, \varepsilon, \omega), (j=1, 2, \dots, d_i), (i=1, 2), l \leqq n \leqq m\}$ and

$$\alpha(n) = \sup_{l \geqq 0} \sup_{A \in \mathscr{M}_l^m, B \in \mathscr{M}_{l+n}^{\infty}} |P(A \cap B) - P(A)P(B)|.$$

Then

$$\sum_{n=1}^{\infty} \alpha(n)^{1/2} < \infty.$$

V′) Let

$$\begin{aligned} a_{kl}(x) &= \sum_{k,l}' \sum_{n=1}^{\infty} E(F_k^{(1)}(0, x, \omega) F_l^{(1)}(n, x, \omega))) + E(F_k^{(1)}(0, x, \omega) F_l^{(1)}(0, x, \omega)) \\ &= \sum_{k,l}' \sum_{u,v=1}^{d_1} \sum_{n=1}^{\infty} E(\xi_u^{(1)}(0, \omega)\xi_v^{(1)}(n, \omega)) g_{ku}^{(1)}(x) g_{lv}^{(1)}(x) \\ &\quad + \sum_{u,v=1}^{d_1} E(\xi_u^{(1)}(0, \omega)\xi_v^{(1)}(0, \omega) g_{ku}^{(1)}(x) g_{lv}^{(1)}(x) \\ &= \sum_{k,l}' \sum_{u,v=1}^{d_1} R_{uv}^{(1)} g_{ku}^{(1)}(x) g_{lv}^{(1)}(x) + \sum_{u,v=1}^{d_1} E(\xi_u^{(1)}(0, \omega)\xi_v^{(1)}(0, \omega)) g_{ku}^{(1)}(x) g_{lv}^{(1)}(x), \end{aligned}$$

where we put $R_{uv}^{(1)} = \sum_{n=1}^{\infty} E(\xi_u^{(1)}(0)\, \xi_v^{(1)}(n))$. Also, we put

$$\begin{aligned} b_{kl}(x) &= \sum_{n=1}^{\infty} E(F_k^{(1)}(0, x, \omega)(\partial/\partial x_k) F_l^{(1)}(n, x, \omega)) \\ &= \sum_{u,v=1}^{d_1} R_{uv}^{(1)} g_{ku}^{(1)}(x)(\partial/\partial x_k) g_{lv}^{(1)}(x) \end{aligned}$$

and

$$\begin{aligned} c_k(x) &= E(F_k^{(2)}(n, x, \omega)) = \sum_{u=1}^{d_2} E(_u^{(2)}(n)) g_{ku}^{(2)}(x) \\ &= \sum_{u=1}^{d_2} c_u g_{ku}^{(2)}(x), \end{aligned}$$

where $c_u = E(\zeta_u^{(2)}(n))$. We put, for $f \in C^2(R^d)$,

$$\mathscr{L} f(x) = 2^{-1} \sum_{k,l=1}^d a_{kl}(x)(\partial^2/\partial x_k \partial x_l) f + \sum_{l=1}^d (\sum_{k=1}^d b_{kl}(x) + c_l(x))(\partial/\partial x_l) f$$

Then, the solution of the martingale problem associated with generator $\mathscr{L}$ has a unique solution R on $(C[0, \infty), R^d), \mathscr{B}(C))$.

We define $X^\varepsilon(t)$ as in §2. Let R^ε be the measure on $C([0, \infty), R^d)$ generated by the stochastic process $\{X^\varepsilon(t)\}$. Then, we have the following theorem.

THEOREM 2. *We assume that conditions I′)–V′) are satisfied. Then* R^ε *converges weakly to the probability measure* R *on* $C([0, \infty), R^d)$ *such that* $R(X(0) = X_0) = 1$, $X_0 \in R^d$.

The proof of Theorem 2 is analogous to the one of Theorem 1. It is necessary to replace Lemmas 1 and 2 by the following Lemmas 3 and 4.

LEMMA 3. *If* ξ *is measurable with respect to* $\mathscr{M}_0^m$ *with* $E(\xi) = 0$ *and* η *is measurable with respect to* $\mathscr{M}_{n+m}^\infty$ $(n > 0)$ *and if* $|\xi| \leqq C_1$, $|\eta| \leqq C_2$ *with probability one, then we have*

$$|E(\xi\eta)| \leqq 4C_1 C_2 \alpha(n).$$

PROOF. This is Theorem 17.2.1 in Ibragimov and Linnik [4, p. 306].

LEMMA 4. *Let* ξ_i *be* $\mathscr{M}_0^i$ *measurable and* η_j *(respectively* ζ_k*) be* $\mathscr{M}_j^j(\mathscr{M}_k^k)$*-measurable. Assume that* $E(\zeta_k) = 0$ *and*

$$P(|\xi_i| \leqq C_1, |\eta_j| \leqq C_2, |\zeta_k| \leqq C_3) = 1.$$

Then, for all $i \leqq j \leqq k$, *we have*

$$|E\{\xi_i(\eta_j\zeta_k - E(\eta_j\zeta_k))\}| \leqq 8C_1C_2C_3(\alpha(j-i))^{1/2}(\alpha(k-j))^{1/2}.$$

PROOF. $w_{jk} = \eta_j\zeta_k - E(\eta_j\zeta_k)$ is bounded and $\mathscr{M}_j^\infty$-measurable and $E(w_{jk}) = 0$. Therefore by Lemma 3, we have

$$|E\{\xi_i(\eta_j\zeta_k - E(\eta_j\xi_k))\}| \leqq 8C_1C_2C_3\alpha(j-i).$$

On the other hand, $\xi_i\eta_j$ is $\mathscr{M}_0^j$-measurable and ζ_k is $\mathscr{M}_k^\infty$-measurable. Hence

$$|E\{\xi_i\eta_j\zeta_k\}| \leqq 4C_1C_2C_3\alpha(k-j).$$

Also, we have

$$|E\{\eta_j\zeta_k\}| \leqq 4C_1C_2\alpha(k-j).$$

Therefore, we have

$$|E\{\xi_i(\eta_j\zeta_k - E(\eta_j\zeta_k)\}|^2 \leqq 64C_1^2C_2^2C_3^2(\alpha(j-i)\alpha(k-j)),$$

from which we obtain the conclusion of Lemma 3.

We will omit the proof of Theorem 2.

If $d=1$, $g^{(1)}(x)=1$ and $F^{(i)}(n, x, \omega)=0$ $(i=2, 3)$, then Theorem 2 gives Theorem 4.1 in Davydov [3]. If $\{\xi_{nk}\}$ are orthogonal random variables with unit variance and $F^{(i)}(n, x, \omega)=0$ $(i=2, 3)$, then

$$\mathscr{L}f(x) = \sum_{k,l=1}^{d} \sum_{u=1}^{d} g_{ku}^{(1)}(x)g_{lu}^{(1)}(x)(\partial^2/\partial x_k \partial x_l)f,$$

which is the generator of the solution process of the stochastic differential equation

$$dX_k(t) = \sum_{u=1}^{d} g_{ku}^{(1)}(x(t))dB_u(t), \qquad (k = 1, 2,\dots, d).$$

5. Generalizations

In this section, we will try some extension of the results of the preceeding sections. In this section, we assume that all random fields depend on ε. We introduce the following conditions.

VI″) Let $\mathscr{M}_l^m(\varepsilon, M)=\sigma\{F(n, x, \varepsilon, \omega),\ G(n, x, \varepsilon, \omega),\ l\leqq n\leqq m\ |x|\leqq M\}$ and

$$\beta(n, \varepsilon, M) = \sup_{l\geq 0} \sup_{A\in\mathscr{M}_0^l(\varepsilon,M), B\in\mathscr{M}_{l+n}^\infty(\varepsilon,M)} |P(A\cap B) - P(A)P(B)|.$$

There exists a positive non-decreasing function $L(\varepsilon)$ such that $L(\varepsilon)\to\infty$ $(\varepsilon\to 0)$ and $\varepsilon^2 L(\varepsilon)\to 0$, $(\varepsilon\to 0)$, and such that

$$\lim_{t\to 0} (\varepsilon/L(\varepsilon))^{1/2} \sum_{n=0}^{[t\varepsilon^{-2}L(\varepsilon)^{-1}]} \beta(n, \varepsilon, M)^{1/(6+2d)} = 0$$

and the following limits exist uniformly on any compact sets of x, independent of t

(4) $$\lim_{\varepsilon\to 0} 1/L(\varepsilon) \sum_{k,l}' \sum_{n=1}^{[t\varepsilon^{-2}L(\varepsilon)^{-1}]} E(F_k(0, x, \varepsilon, \omega)F_l(n, x, \varepsilon, \omega)) = a_{kl}^*(x),$$

and

(5) $$\lim_{\varepsilon\to 0} 1/L(\varepsilon) \sum_{n=1}^{[t\varepsilon^{-2}L(\varepsilon)^{-1}]} E(F_k(0, x, \varepsilon, \omega)(\partial/\partial x_k)F_l(n, x, \varepsilon, \omega)) = b_{kl}^*(x).$$

IV‴) Let $\mathscr{M}_l^m(\varepsilon)=\sigma\{\xi_j^{(i)}(n, \varepsilon, \omega),\ F_j^{(3)}(n, x, \varepsilon, \omega),\ (j=1, 2,\dots, d_i),\ (i=1, 2),\ l\leqq m\leqq n\}$ and

$$\alpha(n, \varepsilon) = \sup_{l\geq 0} \sup_{A\in\mathscr{M}_0^l(\varepsilon), B\in\mathscr{M}_{l+n}^\infty(\varepsilon)} |P(A\cap B) - P(A)P(B)|.$$

There exists a positive non-decreasing function $L(\varepsilon)$ such that $L(\varepsilon)\to\infty$, $(\varepsilon\to 0)$, and $\varepsilon^2 L(\varepsilon)\to 0$, $(\varepsilon\to 0)$, and such that

$$\lim_{\varepsilon\to 0} (\varepsilon/L(\varepsilon))^{1/2} \sum_{n=1}^{[t\varepsilon^{-2}L(\varepsilon)^{-1}]} \alpha(n, \varepsilon)^{1/2} = 0$$

and the following limit which is independent of t exists,

(6) $$\lim_{\varepsilon\to 0} L(\varepsilon)^{-1} \sum_{n=1}^{[t\varepsilon^{-2}L(\varepsilon)^{-1}]} E(\xi_u^{(1)}(0, \varepsilon)\xi_v^{(1)}(n, \varepsilon)) = R_{uv}^{*(1)}.$$

Define $X_*^\varepsilon(t)$, as follows,

$$X_*^\varepsilon(t) = X_j^\varepsilon + (t-j\varepsilon^2 L(\varepsilon))/\varepsilon^2 L(\varepsilon), \qquad \text{if} \quad j\varepsilon^2 L(\varepsilon) \le t < (j+1)\varepsilon^2 L(\varepsilon).$$

Let R_*^ε be the measure on $C([0, \infty), R^d)$ induced by the stochastic process $\{X_*^\varepsilon(t)\}$.

If we consider $f(X_{[t\varepsilon^{-2}L(\varepsilon)^{-1}]}^{\varepsilon,M})-f(X_{[s\varepsilon^{-2}L(\varepsilon)]}^{\varepsilon,M})$ in place of $f(X_{t(\varepsilon)}^{\varepsilon,M})-f(X_{s(\varepsilon)}^{\varepsilon,M})$ in the proof of Theorem 1 and make use of the assumptions introduced in the above, we can prove the following theorems in analogous ways as in Theorem 1.

THEOREM 3. *We assume that I)–V) in §2 and VI″), where in Assumption IV), constants $C=C(M)$ and C_M are independent of n and ε. We put $f\in C_d^2(R^d)$*

$$\mathscr{L}^*f(x) = 2^{-1} \sum_{k,l=1}^d a_{kl}^*(x)(\partial^2/\partial x_k\partial x_l)f + \sum_{k,l=1}^d b_{kl}^*(x)(\partial/\partial x_l)f,$$

where $a_{kl}^*(x)$ and $b_{kl}^*(x)$ are defined in VI″). Furthermore, we assume that the solution of the martingale problem associated with generator $\mathscr{L}^*$ and starting at $X\in R^d$ has a unique solution R_* on $(C([0, \infty), R^d), \mathscr{B}(C))$ such that $R_*(X(0)=X)=1$, $X\in R^d$. Then, R_*^ε weakly converges to R_*.

THEOREM 4. *We assume that I′)–III′) in §4 and IV‴), where in III′) $\{\xi_j^{(i)}(n, \varepsilon), j=1,\dots, d\}$ $(i=1, 2)$ are bounded also with respect to ε. Let*

$$a_{kl}^* = \sum_{k,l}' \sum_{u,v=1}^{d_1} R_{uv}^{*(1)} g_{ku}^{(1)}(x) g_{lu}^{(1)}(x)$$

and

$$b_{kl}^*(x) = \sum_{u,v=1}^{d_1} R_{uv}^{*(1)}(x)\,(\partial/\partial x_k) g_{lu}^{(1)}(x),\ g_{kv}^{(1)}(x).$$

We define $\mathscr{L}^*f(x)$ as in Theorem 3 for $f\in C^2(R^d)$. Also, we assume that the solution of the martingale problem associated with generator $\mathscr{L}^*$ starting at $X\in R^d$ has a unique solution R_* on $(C([0, \infty), R^d), \mathscr{B}(C))$ such that $R_*(X(0)=X)=1$. Then, R_*^ε weakly converges to R^*.

In Theorem 4, we take a $d\times d$ symmetric, non-negative definite matrix σ_{uk}^* $(u, k=1,\dots, d)$ such that

$$R_{uv}^* = \sum_{k=1}^d \sigma_{uk}^*\sigma_{vk}^*.$$

Now put

$$y_u(t) = \sum_{k=1}^d \sigma_{uk}^* B_k(t),$$

where $B(t)=(B_1(t),\dots, B_d(t))$ is a d-dimensional Brownian motion. Then, the corresponding stochastic differential equation to the diffusion process associated with the generator $\mathscr{L}^*$ is written as follows;

$$dx_i(t) = \sum_{u=1}^{d} g_{iu}^{(1)}(t)\circ dy_u(t),$$

where $\circ$ means the stochastic integral in the sense of Stratonovich.

Now consider the case $d=1$. Let $\delta(\varepsilon)>0$ and $\delta(\varepsilon)\to 0$ $(\varepsilon\to 0)$. Assume that $E(\xi^{(1)}(0, \varepsilon)\xi^{(1)}(n, \varepsilon))\sim\exp(-\delta(\varepsilon)n)(n\to\infty)$. Then the limit

$$\lim_{\varepsilon\to 0}\delta(\varepsilon)\sum_{n=1}^{\infty} E(\xi^{(1)}(0, \varepsilon)\xi^{(1)}(n, \varepsilon)) = R^{*(1)}$$

exists. Therefore, we can take $L(\varepsilon)=1/\delta(\varepsilon)$. Assume that $\alpha(n, \varepsilon)\sim\exp(-\delta(\varepsilon)n)$

Then, also we have

$$\sum_{n=1}^{\infty}\alpha(n, \varepsilon)^{1/2} \sim 1/\delta(\varepsilon), \qquad (\varepsilon\to 0).$$

Therefore

$$\lim_{\varepsilon\to 0}(\varepsilon\delta(\varepsilon))^{1/2}\sum_{n=1}^{[t\delta(\varepsilon)\varepsilon^{-2}]}\alpha(n, \varepsilon)^{1/2} = 0$$

if and only if $\lim_{\varepsilon\to 0}\varepsilon/\delta(\varepsilon)=0$.

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Supplement

We propose another version of Theorems 1 and 2. At first, we remark that if $\{X_n^\varepsilon\}$ defined by (1) is known to be bounded, then Assumptions (c) and (d) of IV) of Theorem 1 in Section 2 can be dropped. Many examples in population genetics are the case.

More precisely, we introduce the following conditions.

0) $\{X_n^{\varepsilon,M}(\omega)\}$ (which is defined in section 3) is bounded for each fixed $M>0$, with respect to ε, n and ω.

$\overline{\text{VI}}$) For each $M<\infty$, there exists a constant $C=C(M)$ independent of n such that

a) $E(\sup_{|x|\leqq M}|D^\beta F_l(n, x, \omega)|^8) \leqq C, \qquad 0 \leqq |\beta| \leqq 6,$

b) $E(\sup_{|x|\leqq M}|D^\gamma G_l(n, x, \omega)|^{8-4|\gamma|}) \leqq C, \quad 0\leqq|\gamma| = 1.$

It is clear that if $\{X_n^\varepsilon\}$ is bounded, condition 0) is satisfied.

THEOREM 1′. *We assume that conditions* 0), *I*), *II*), *III*), $\overline{VI}$), *V*) *and VI*) *are satisfied with respect to* $\{F\}$ *and* $\{G\}$. *Then, the conclusion of Theorem* 1 *holds.*

PROOF. In section 3 (Proof of Theorem 1), we used conditions c) and d) in two places. The first one is to say about boundedness of $\{X_n^{\varepsilon,M}\}$ which is equivalent to condition 0). The second one is to say that the inequality

$$|X^{\varepsilon,M}(u) - X^{\varepsilon,M}_{[u/\varepsilon^2]}| \leqq C_M|u-t|$$

and etc. hold.

In this situation, we proceed as follows.

$$E(|X^{\varepsilon,M}(u) - X^{\varepsilon,M}_{[u/\varepsilon^2]}|^2\Phi) \leqq |t-u|2[E\{|F([u/\varepsilon^2], X^{\varepsilon,M}_{[u/\varepsilon^2]}, \omega)|^4\}^{1/4} + \varepsilon^2 E\{|G([u/\varepsilon^2], X^{\varepsilon,M}_{[u/\varepsilon^2]}, \omega)|^4\}^{1/4}]E[|\Phi|^8]^{1/8}.$$

We can deal with $|X^{\varepsilon,M}_{[t\varepsilon^2]+1} - X^{\varepsilon,M}(t)|$ by the same way.

Next, we consider some modification of Theorem 2. Let us introduce the following conditions.

$\overline{\text{III}'}$) $\{\xi_j^{(i)}(n, \omega), j=1,\dots, d_i\}$ $(i=1, 2)$ are stationary vector processes and $E\{\xi_j^{(1)}(n, \omega)\}=0$.

There is a constant C independent of ε, n, for a $\delta>0$

$$E(|\xi_j^{(i)}(n, \omega)|^{(4+2\delta)}) \leqq C, \qquad i = 0, 1, j = 1,\dots, d_i$$

and

$$E(|F^{(3)}(n,\cdot,\varepsilon,\omega)|^{(2+\delta)}) \leqq C.$$

$\overline{\mathrm{IV}'})$ For $\alpha(n)$ defined in condition IV′), it holds

$$\textstyle\sum_{k=1}^{\infty} \alpha(n)^{\delta/(4+2\delta)} < \infty \qquad \text{for a} \quad \delta > 0.$$

THEOREM 2′. *Under Conditions* 0), *I′*), *II′*), $\overline{III'})$ $\overline{IV'})$ *and V′*), *the conclusion of Theorem* 2 *holds.*

The proof of Theorem 2′ is analogous to the one of Theorem 1′. We have to replace Lemmas 1 and 2 by discrete versions of Lemmas 3 and 4 in [13].

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