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On the group of self-homotopy equivalences of principal S^3-bundles over spheres

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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Introduction

For any (based) space X, the set $\mathscr{E}(X)$ of all homotopy classes of homotopy equivalences of X to itself forms a group under the composition of maps. The group $\mathscr{E}(X)$ has been studied by several authors. In particular, in case when X is a principal S^3-bundle over S^n, the group $\mathscr{E}(X)$ is already known for $X=SU(3)$, $Sp(2)$ by [10], for $X=S^3\times S^n$ by [13] and for $X=E_{k\omega}$ by J. W. Rutter [11], where $E_{k\omega}$ is the principal S^3-bundle over S^7 with characteristic class $k\omega\in\pi_6(S^3)$, ω a generator of $\pi_6(S^3)=Z_{12}$.

The purpose of this note is to study groups $\mathscr{E}(X)$ for principal S^3-bundles over spheres. Our main result is stated as follows:

THEOREM 3.1. *Let E_f be the principal S^3-bundle over S^n ($n\geqq 5$) with characteristic class $f\in\pi_{n-1}(S^3)$. Assume that $\omega\circ S^3f\in f_*\pi_{n+2}(S^{n-1})$. Then we have the following exact sequence*:

$$0\rightarrow\pi_{n+3}(E_f)\rightarrow\mathscr{E}(E_f)\rightarrow\mathscr{E}(S^3\cup_f e^n)\rightarrow 1,$$

where $S^3\cup_f e^n$ is the mapping cone of f.

The group $\mathscr{E}(S^3\cup_f e^n)$ is given in [10, Th. 3.15] up to extension (see (2.2)), and the homotopy group $\pi_{n+3}(E_f)$ is studied for some f in §3.

Throughout this note, all spaces have base points, and all maps and homotopies preserve base points. For given spaces X and Y, we denote by $[X, Y]$ the set of (based) homotopy classes of maps of X to Y, and by the same letter a map $f: X\rightarrow Y$ and its homotopy class $f\in[X, Y]$.

§1. The homomorphism ϕ and its kernel

Throughout this note, let $f\in\pi_{n-1}(S^3)$ for $n\geqq 5$ be a given element, and let $X=E_f$ denote the principal S^3-bundle over S^n with characteristic class f and $K=S^3\cup_f e^n$ the mapping cone of f. Then by James-Whitehead [8], X has a cell structure given by

$$X = K \cup e^{n+3}.$$

Since $j_*: [K, K] \to [K, X]$ ($j: K \subset X$ is the inclusion) is bijective, the homomorphism

$$\phi: \mathscr{E}(X) \to \mathscr{E}(K)$$

is defined by the restriction on $\mathscr{E}(X)$ of the composition $[X, X] \xrightarrow{j^*} [K, X] \xleftarrow{j_*} [K, K]$.

In this section, we consider the kernel of ϕ. We define the coaction

$$\ell: X = K \cup e^{n+3} \to K \cup e^{n+3} \vee S^{n+3} = X \vee S^{n+3}$$

by shrinking the equator $S^{n+2} \times \{1/2\}$ of e^{n+3} to the base point. Since $\pi_{n+3}(S^3)$ and $\pi_{n+3}(S^n)$ for $n \geqq 5$ are finite groups (cf. [14, (4.2)]), $\pi_{n+3}(X)$ is a finite group by the exact sequence associated with the principal S^3-bundle X over S^n:

(1.1) $$S^3 \xrightarrow{i} X \xrightarrow{p} S^n.$$

Therefore, by the Blakers-Massey theorem and the exact sequence of the pair (X, K) we have

(1.2) $j_*: \pi_{n+3}(K) \to \pi_{n+3}(X)$ *is epimorphic.*

By Barcus-Barratt [1, Th. 6.1] and J. W. Rutter [11, Th. 2], we can define a homomorphism

(1.3) $$\lambda: \pi_{n+3}(X) = j_*\pi_{n+3}(K) \to \mathscr{E}(X) \quad \text{by} \quad \lambda(\alpha) = \nabla \circ (1 \vee \alpha) \circ \ell,$$

where $\alpha \in \pi_{n+3}(X)$, $\nabla: X \vee X \to X$ is the folding map and 1 is the class of the identity map of X; and since the attaching element $g \in \pi_{n+2}(K)$ of e^{n+3} in $X = K \cup e^{n+3}$ is of infinite order by [3, Th. 3.2], we have

(1.4) $$\mathrm{Im}\, \lambda = \mathrm{Ker}\,(\phi: \mathscr{E}(X) \to \mathscr{E}(K)).$$

Let $h: S^{n-1} \times S^3 \to S^3$ be the map defined by $h = (f \circ p_1) \cdot p_2$ where p_1 and p_2 are the projections and $\cdot$ is the canonical multiplication on S^3. Then by [7, (3.1)] and [3, (3.6)], we have

(1.5) $$Sg = i_* H((f \circ p_1) \cdot p_2) = i_* \gamma \circ S^4 f,$$

where $i: S^4 \subset SK$ is the inclusion, H is the Hopf construction and γ is the Hopf map $S^7 \to S^4$. Therefore

(1.6) $$SX = K_1 \cup_{i_1 \circ Sf} e^{n+1}, \quad K_1 = S^4 \cup_{\gamma \circ S^4 f} e^{n+4},$$

where $i_1: S^4 \subset K_1$ is the inclusion.

LEMMA 1.7. *Let* $S: \pi_{n+3}(X) \to \pi_{n+4}(SX)$ *be the suspension homomorphism. Then* Ker S *is generated by* $i_* \nu' \circ S^3 f \circ \eta_{n+2}$, *where* $\nu' \in \pi_6(S^3) = Z_{12}$ *is an element of order* 4 *and* $\eta_{n+2} \in \pi_{n+3}(S^{n+2}) = Z_2$ *is a generator.*

PROOF. Let $H_{Sf}: \pi_{n+6}(S^{n+1}) \to \pi_5(S^4)$ be the homomorphism defined by the composition:

$$\pi_{n+6}(S^{n+1}) \xrightarrow{H} \pi_{n+6}(S^{2n+1}) \xleftarrow{S^{n+1}} \pi_5(S^n) \xrightarrow{(Sf)_*} \pi_5(S^4),$$

where H is the generalized Hopf invariant of [15]. Let $Q: \pi_5(S^4) \to \pi_{n+5}(SK, S^4)$ be the homomorphism defined by $Q(\eta_4) = [u_{n+1}, \eta_4]$, where $SK = S^4 \cup_{Sf} e^{n+1}$, u_{n+1} is a generator of $\pi_{n+1}(SK, S^4) \cong \pi_{n+1}(S^{n+1}) = Z$ and [,] denotes the relative Whitehead product. Then by [4, Th. 2.1], we have the following exact sequence:

$$\pi_{n+6}(S^{n+1}) \xrightarrow{H_{Sf}} \pi_5(S^4) \xrightarrow{Q} \pi_{n+5}(SK, S^4) \longrightarrow \pi_{n+5}(S^{n+1}).$$

By [14, Table of $\pi_{n+k}(S^n)$, I], we have $\pi_{n+6}(S^{n+1}) = \pi_{n+5}(S^{n+1}) = 0$ for $n \geqq 6$, $\pi_{11}(S^6) = Z$ and $\pi_{10}(S^6) = 0$. Let $\Delta(\iota_{13})$ be the generator of $\pi_{11}(S^6)$. Then $H(\Delta(\iota_{13})) = \pm 2\iota_{11}$ by [14, Prop. 2.7]. Since $\pi_5(S^4) = Z_2$, we have $H_{Sf} = 0$: $\pi_{11}(S^6) \to \pi_5(S^4)$. Hence Q is an isomorphism in the above sequence for $n \geqq 5$ and we have

$$\pi_{n+5}(SK, S^4) \cong \pi_5(S^4) = Z_2, \tag{1.8}$$

which is generated by $[u_{n+1}, \eta_4]$.

Consider the following commutative diagram including the exact sequence of the triad (SX, SK, S^4):

$$\begin{array}{ccccccccc}
\pi_{n+6}(SX, S^4) & \to & \pi_{n+6}(SX, SK) & \to & \pi_{n+5}(SK, S^4) & \xrightarrow{j_*} & \pi_{n+5}(SX, S^4) & \to & \pi_{n+5}(SX, SK) \\
\uparrow{\scriptstyle j_{1*}} & & \downarrow{\scriptstyle \pi_*} & & & & \uparrow{\scriptstyle j_{1*}} & & \downarrow{\scriptstyle \pi_*} \\
\pi_{n+6}(K_1, S^4) & \xrightarrow{\pi_{1*}} & \pi_{n+6}(S^{n+4}) & & & & \pi_{n+5}(K_1, S^4) & \xrightarrow{\pi_{1*}} & \pi_{n+5}(S^{n+4}),
\end{array}$$

where $j_1: K_1 \subset SX$ is the inclusion given in (1.6), $\pi: SX \to SX/SK = S^{n+4}$ and $\pi_1: K_1 \to K_1/S^4 = S^{n+4}$ are the collapsing maps. We see that $\pi_{n+6}(S^{n+4}) = Z_2$, $\pi_{n+5}(S^{n+4}) = Z_2$, $\pi_{n+5}(SK, S^4) = Z_2$ by (1.8) and π_* and π_{1*} in the both squares are isomorphisms by the Blakers-Massey theorem. Therefore we have

(1.9) $\pi_{n+5}(SX, S^4) = Z_2 \oplus Z_2$ generated by $j_*[u_{n+1}, \eta_4]$ and $j_{1*}\tilde{\eta}_{n+4}$,

where $\tilde{\eta}_{n+4}$ is a coextension of η_{n+4}.

Consider the following commutative diagram:

$$(*) \qquad \begin{array}{ccccccc}
 & & \pi_{n+3}(S^3) & \xrightarrow{i_*} & \pi_{n+3}(X) & \xrightarrow{p_*} & \pi_{n+3}(S^n) \\
 & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} \\
\pi_{n+5}(SX, S^4) & \xrightarrow{\partial} & \pi_{n+4}(S^4) & \xrightarrow{i_*} & \pi_{n+4}(SX) & \xrightarrow{(Sp)_*} & \pi_{n+4}(S^{n+1}),
\end{array}$$

where the left homomorphism S is monomorphic by [14, Lemma 4.5] and the right homomorphism S is isomorphic for $n \geqq 5$. Here, we have

$$\begin{aligned} \partial j_*[u_{n+1}, \eta_4] &= -[\partial u_{n+1}, \eta_4] = [Sf, \eta_4] && \text{by [2, (3.5)]} \\ &= [\iota_4, \iota_4]\circ S^4 f\circ \eta_{n+3} && \text{by [15, (3.59)]} \\ &= (2\nu_4 - S\nu')\circ S^4 f\circ \eta_{n+3} && \text{by [14, (5.8)]} \\ &= S\nu'\circ S^4 f\circ \eta_{n+3}, \end{aligned}$$

and $\partial j_{1*}\tilde{\eta}_{n+4} = \gamma\circ S^4 f\circ \eta_{n+3}$ by the following commutative diagram:

$$\begin{array}{ccccc} \pi_{n+5}(SX, S^4) & \xrightarrow{\partial} & \pi_{n+4}(S^4) & \xleftarrow{(\gamma\circ S^4 f)_*} & \pi_{n+4}(S^{n+3}) \\ & \nwarrow^{j_{1*}} & \uparrow \partial_1 & & \cong \downarrow S \\ & & \pi_{n+5}(K_1, S^4) & \xrightarrow{\pi_{1*}} & \pi_{n+5}(S^{n+4}). \end{array}$$

Since $\pi_{n+4}(S^4) = S\pi_{n+3}(S^3) \oplus \gamma_* \pi_{n+4}(S^7)$ as is well known, (1.9) and these equalities show that

$$S\pi_{n+3}(S^3) \cap \partial\pi_{n+5}(SX, S^4) = \{S(\nu'\circ S^3 f\circ \eta_{n+2})\}.$$

Hence $\mathrm{Ker}(S: \pi_{n+3}(X) \to \pi_{n+4}(SX)) = \{i\circ\nu'\circ S^3 f\circ\eta_{n+2}\}$ by the diagram (*). q.e.d.

REMARK 1.10. The kernel of the homomorphism $S: \pi_{n+3}(X) \to \pi_{n+4}(SX)$ is investigated by S. Sasao [12, Lemma 4.1] for S^m-bundles X over S^n with the condition $3 < m+1 < n < 2m-2$.

LEMMA 1.11. *Let $g \in \pi_{n+2}(K)$ be the attaching element of e^{n+3} in $X = K \cup e^{n+3}$. Then the induced homomorphism $(S^2 g)^*: [S^2 K, SX] \to \pi_{n+4}(SX)$ is trivial.*

PROOF. Consider the following commutative diagram which is obtained by (1.5):

$$\begin{array}{ccccc} \pi_5(S^4) & \xrightarrow{i_*} & \pi_5(SX) & \xleftarrow{i^*} & [S^2 K, SX] \\ \downarrow (S\gamma)^* & & \downarrow (S\gamma)^* & & \downarrow (S^2 g)^* \\ \pi_8(S^4) & \xrightarrow{i_*} & \pi_8(SX) & \xrightarrow{(S^5 f)^*} & \pi_{n+4}(SX), \end{array}$$

where the upper i_* is isomorphic for $n \geqq 6$ and is epimorphic for $n = 5$. Since $\eta_4\circ S\gamma = \eta_4\circ\nu_5 = S\nu'\circ\eta_7$ by [14, Lemma 5.4, Prop. 5.6 and (5.9)] and $\partial j_*[u_{n+1}, \eta_4] = S\nu'\circ S^4 f\circ\eta_{n+3}$ in the proof of Lemma 1.7, we have

$$(S^5 f)^* i_* (S\gamma)^* \eta_4 = i_* S\nu'\circ\eta_7\circ S^5 f = i_* S\nu'\circ S^4 f\circ\eta_{n+3} = i_* \partial j_*[u_{n+1}, \eta_4] = 0.$$

Therefore, by the above diagram,

$\mathrm{Im}\,(S^2g)^* \subset \mathrm{Im}\,(S^5f)^*(S\gamma)^* = \mathrm{Im}\,(S^5f)^*(S\gamma)^*i_* = \{(S^5f)^*i_*(S\gamma)^*\eta_4\} = 0.$ q.e.d.

PROPOSITION 1.12. *The kernel of* $\lambda: \pi_{n+3}(X) \to \mathscr{E}(X)$ *in* (1.3) *is contained in the subgroup generated by* $i_*\nu' \circ S^3 f \circ \eta_{n+2}$.

PROOF. For the suspended complex $SX = S^4 \cup e^{n+1} \cup e^{n+4}$, we define a homomorphism

$$\lambda_1: j_*\pi_{n+4}(SK) \to \mathscr{E}(SX) \quad \text{by} \quad \lambda_1(\alpha) = \nabla \circ (1 \vee \alpha) \circ \ell_1,$$

where $\alpha \in j_*\pi_{n+4}(SK)$ and $\ell_1: SX \to SX \vee S^{n+4}$ is the coaction defined by the similar way to ℓ. Then by (1.2) we have the commutative diagram

$$\begin{array}{ccc} \pi_{n+3}(X) = j_*\pi_{n+3}(K) & \xrightarrow{\lambda} & \mathscr{E}(X) \\ \downarrow S & & \downarrow S \\ \pi_{n+4}(SX) \supset j_*\pi_{n+4}(SK) & \xrightarrow{\lambda_1} & \mathscr{E}(SX), \end{array}$$

where $S: \mathscr{E}(X) \to \mathscr{E}(SX)$ is the suspension homomorphism. We notice that λ_1 coincides with the restriction of $\lambda_1': \pi_{n+4}(SX) \to [SX, SX]$ given by $\lambda_1'(\alpha) = 1 + \pi^*\alpha$, where $\pi: SX \to SX/SK = S^{n+4}$ is the collapsing map, $\pi^*: \pi_{n+4}(SX) \to [SX, SX]$ and $+$ is the comultiplication on SX. Then, by Lemma 1.11,

$$\lambda_1^{-1}(1) \subset \pi^{*-1}(0) = (S^2g)^*[S^2K, SX] = 0.$$

Hence the above diagram shows that

$$\begin{aligned} \lambda^{-1}(1) &\subset \lambda^{-1}(S^{-1}(1)) = S^{-1}(\lambda_1^{-1}(1)) \\ &= S^{-1}(0) = \{i \circ \nu' \circ S^3 f \circ \eta_{n+2}\} \quad \text{by Lemma 1.7.} \end{aligned}$$ q.e.d.

§ 2. The image of ϕ

In this section we consider the image of $\phi: \mathscr{E}(X) \to \mathscr{E}(K)$ defined in §1, where $X = K \cup_g e^{n+3}$, $g \in \pi_{n+2}(K)$. By [10, Lemma 2.2], we have

(2.1) $$\mathrm{Im}\,\phi = \{h \in \mathscr{E}(K): h \circ g = \varepsilon g \ (\varepsilon = \pm 1) \ \text{ in } \ \pi_{n+2}(K)\}.$$

Let $\ell_2: K = S^3 \cup e^n \to S^3 \cup e^n \vee S^n = K \vee S^n$ be the coaction defined by shrinking the equator $S^{n-1} \times \{1/2\}$ of e^n in $S^3 \cup e^n$ to the base point. Then we can define a homomorphism

$$\lambda_2: i_*\pi_n(S^3) \to \mathscr{E}(K) \quad \text{by} \quad \lambda_2(\alpha) = \nabla \circ (1 \vee \alpha) \circ \ell_2,$$

where $\alpha \in i_*\pi_n(S^3)$. Furthermore, let τ and ρ be the elements in $\mathscr{E}(K)$ such that the following diagrams are homotopy commutative, respectively:

$$\begin{array}{ccccc} S^3 & \longrightarrow & K & \longrightarrow & S^n \\ \downarrow{\scriptstyle -\iota_3} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle -\iota_n} \\ S^3 & \longrightarrow & K & \longrightarrow & S^n, \end{array} \qquad \begin{array}{ccccc} S^3 & \longrightarrow & K & \longrightarrow & S^n \\ \downarrow{\scriptstyle \iota_3} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle -\iota_n} \\ S^3 & \longrightarrow & K & \longrightarrow & S^n, \end{array}$$

where $S^3 \xrightarrow{i} K \xrightarrow{\pi} S^n$ is the cofibering of $K = S^3 \cup e^n$. Then, we have the following (2.2) by applying [10, Th. 3.15]:

(2.2) *For the cell complex $K = S^3 \cup_f e^n$ $(n \geqq 5)$, we have the exact sequence*

$$0 \to H_1 \to \mathscr{E}(K) \to Z_2 \to 1.$$

Here, by using $H = \pi_n(S^3)/\{f_*\pi_n(S^{n-1}) + (Sf)^*\pi_4(S^3)\}$, H_1 is given by

$$H_1 = H \quad if \quad 2f \neq 0; \; H_1 = D(H) \quad if \quad 2f = 0,$$

where $D(H)$ is the split extension

$$0 \to H \to D(H) \to Z_2 \to 1$$

acting $Z_2 = \{1, -1\}$ on H by $(-1) \cdot a = -a$ for $a \in H$. Furthermore, τ exists always, ρ exists only when $2f = 0$ and

$$(2.3) \quad \mathscr{E}(K) = \begin{cases} \{\lambda_2(\alpha) \circ \tau^\delta : \alpha \in i_*\pi_n(S^3),\ \delta = 0 \text{ or } 1\} & \text{if } \ 2f \neq 0, \\ \{\lambda_2(\alpha) \circ \tau^{\delta_1} \circ \rho^{\delta_2} : \alpha \in i_*\pi_n(S^3),\ \delta_k = 0 \text{ or } 1\ (k=1, 2)\} & \text{if } \ 2f = 0. \end{cases}$$

LEMMA 2.4. *The normal subgroup $\{\lambda_2(\alpha) : \alpha \in i_*\pi_n(S^3)\}$ of $\mathscr{E}(K)$ is contained in* Im ϕ *given in* (2.1).

PROOF. Since $j_* g = \pm [u_n, \iota_3]$ by [3, Th. 3.2] for the generator u_n of $\pi_n(K, S^3) = Z$, we have $\ell_{2*} g = k_* g \pm [k_n, k_3]$ by [5, Lemma 5.4], where $k: S^3 \cup e^n \to S^3 \cup e^n \vee S^n$ and $k_r: S^r \to S^3 \cup e^n \vee S^n$ $(r = 3, n)$ are the inclusions. Therefore, for $\alpha = i_*\alpha' \in i_*\pi_n(S^3)$,

$$\begin{aligned} \lambda_2(\alpha) \circ g &= \nabla \circ (1 \vee \alpha) \circ \ell_2 \circ g = \nabla \circ (1 \vee \alpha) \circ (k \circ g \pm [k_n, k_3]) \\ &= \nabla \circ (1 \vee \alpha) \circ k \circ g \pm \nabla \circ (1 \vee \alpha) \circ [k_n, k_3] \\ &= g \pm [\nabla \circ (1 \vee \alpha) \circ k_n, \nabla \circ (1 \vee \alpha) \circ k_3] \\ &= g \pm [\alpha, i] = g \pm i_*[\alpha', \iota_3] = g. \end{aligned}$$

Hence we have $\{\lambda_2(\alpha) : \alpha \in i_*\pi_n(S^3)\} \subset$ Im ϕ. q.e.d.

By [3, (3.4)], we may regard X in (1.1) as the push-out

$$(2.5) \quad \begin{array}{ccc} S^{n-1} \times S^3 & \longrightarrow & CS^{n-1} \times S^3 \\ \downarrow{\scriptstyle (f \circ p_1) \cdot p_2} & & \downarrow \\ S^3 & \xrightarrow{\quad i \quad} & X. \end{array}$$

Then we have the following

LEMMA 2.6. (i) *If* $2f=0$, *then* ρ *in* (2.2) *can be taken in* $\operatorname{Im}\phi$ *of* (2.1).
(ii) *If* f *satisfies the assumption*

$$\omega\circ S^3f\in f_*\pi_{n+2}(S^{n-1}), \tag{2.7}$$

then τ *in* (2.2) *can be taken in* $\operatorname{Im}\phi$.

PROOF. (i) Since $2f=0$, the diagram

$$\begin{array}{ccc} S^{n-1}\times S^3 & \xrightarrow{(f\circ p_1)\cdot p_2} & S^3 \\ \Big\downarrow{\scriptstyle (-\iota_{n-1})\times\iota_3} & & \Big\downarrow{\scriptstyle \iota_3} \\ S^{n-1}\times S^3 & \xrightarrow[(f\circ p_1)\cdot p_2]{} & S^3 \end{array}$$

is homotopy commutative. Therefore from (2.5) we have an element $\bar\rho\in\mathscr{E}(X)$ such that $\bar\rho|K=\phi(\bar\rho)$ is an element ρ in (2.2).

(ii) Let ϕ: $S^3\times S^3\to S^3$ be the commutator defined by $\phi=p_2^{-1}\cdot p_1^{-1}\cdot p_2\cdot p_1$, where p_i is the projection. Then by [6, p. 176],

(2.8) $\pi_6(S^3)=Z_{12}$ *is generated by* ω *such that* $\omega_*\pi=\phi$,

where π: $S^3\times S^3\to S^3\times S^3/S^3\vee S^3=S^6$ is the collapsing map. By the assumption (2.7), there exists an element

$$\beta\in\pi_{n+2}(S^{n-1}) \quad \text{such that} \quad \omega_*S^3f=f_*\beta. \tag{2.9}$$

Denote by F the composition of maps:

$$F=\nabla\circ\{(-\iota_{n-1})\circ p_1\vee\beta\}\circ\ell\colon S^{n-1}\times S^3\xrightarrow{\ell}S^{n-1}\times S^3\vee S^{n+2}$$
$$\xrightarrow{(-\iota_{n-1})\circ p_1\vee\beta}S^{n-1}\vee S^{n-1}\xrightarrow{\nabla}S^{n-1},$$

where ℓ: $S^{n-1}\times S^3\to S^{n-1}\times S^3\vee S^{n+2}$ is the coaction defined by shrinking the equator $S^{n+1}\times\{1/2\}$ of e^{n+2} to the base point and p_1: $S^{n-1}\times S^3\to S^{n-1}$ is the projection. We see that $f\circ F=((-\iota_3)\circ f\circ p_1)\cdot(f\circ\beta\circ\pi)$, where π: $S^{n-1}\times S^3\to S^{n-1}\times S^3/S^{n-1}\vee S^3=S^{n+2}$ is the collapsing map. So,

$$\begin{aligned}
((f\circ p_1)\cdot p_2)\circ(F,(-\iota_3)\circ p_2) &= (f\circ F)\cdot((-\iota_3)\circ p_2)\\
&= ((-\iota_3)\circ f\circ p_1)\cdot(f\circ\beta\circ\pi)\cdot((-\iota_3)\circ p_2)\\
&= (f\circ\beta\circ\pi)\cdot((-\iota_3)\circ f\circ p_1)\cdot((-\iota_3)\circ p_2) \text{ by the similar way to [13, Lemma 6.5]}\\
&= (\omega\circ S^3f\circ\pi)\cdot((-\iota_3)\circ f\circ p_1)\cdot((-\iota_3)\circ p_2) \text{ by (2.9)}\\
&= (\phi\circ(f\times\iota_3))\cdot((-\iota_3)\circ f\circ p_1)\cdot((-\iota_3)\circ p_2) \text{ by (2.8)}
\end{aligned}$$

$= ((-\iota_3)\circ p_2)\cdot((-\iota_3)\circ f\circ p_1)\cdot p_2\cdot(f\circ p_1)\cdot((-\iota_3)\circ f\circ p_1)\cdot((-\iota_3)\circ p_2)$
$= (-\iota_3)\circ((f\circ p_1)\cdot p_2)$.

Thus we have the following homotopy commutative diagram:

$$\begin{array}{ccc} S^{n-1}\times S^3 & \xrightarrow{(f\circ p_1)\cdot p_2} & S^3 \\ \downarrow{\scriptstyle (F,\,(-\iota_3)\circ p_2)} & & \downarrow{\scriptstyle -\iota_3} \\ S^{n-1}\times S^3 & \xrightarrow[(f\circ p_1)\cdot p_2]{} & S^3. \end{array}$$

This diagram and (2.5) allow us to construct an element $\bar{\tau}\in\mathscr{E}(X)$ such that $\bar{\tau}|K=\phi(\bar{\tau})$ is an element τ in (2.2). q.e.d.

§3. Main theorem and examples

In this section we prove our main theorem and give some examples of $\mathscr{E}(X)$.

THEOREM 3.1. *Let $X=E_f$ be the principal S^3-bundle over S^n ($n\geqq 5$) with characteristic class $f\in\pi_{n-1}(S^3)$. Assume that $\omega\circ S^3f\in f_*\pi_{n+2}(S^{n-1})$ in (2.7). Then we have the following exact sequence*:

(3.2) $$0\to\pi_{n+3}(X)\to\mathscr{E}(X)\to\mathscr{E}(K)\to 1,$$

where $K=S^3\cup_f e^n$.

PROOF. If $\omega\circ S^3f\in f_*\pi_{n+2}(S^{n-1})$, then $\omega\circ S^3f=f\circ\beta$ for some $\beta\in\pi_{n+2}(S^{n-1})$ and we have $i_*\nu'\circ S^3f\circ\eta_{n+2}=i_*\omega\circ S^3f\circ\eta_{n+2}=i_*f\circ\beta\circ\eta_{n+2}=0$, since $i\circ f=0$. Therefore, by Proposition 1.12, the homomorphism λ: $\pi_{n+3}(X)\to\mathscr{E}(X)$ is monomorphic. Furthermore, by (2.3) and Lemmas 2.4 and 2.6, the homomorphism ϕ: $\mathscr{E}(X)\to\mathscr{E}(K)$ is epimorphic. Therefore, we have the exact sequence (3.2) by (1.4). q.e.d.

By using the above theorem and (2.2), we give some examples of $\mathscr{E}(E_f)$. For the calculations, we use several results on the homotopy groups of spheres. The main reference is Toda's book [14].

In case when $f=\eta_3\in\pi_4(S^3)$, $k\omega\in\pi_6(S^3)$ or $0\in\pi_{n-1}(S^3)$, we can see that f satisfies the assumption (2.7). Therefore we obtain exact sequences (3.2) for such f, which are already known for $E_{\eta_3}=SU(3)$, $E_\omega=Sp(2)$ by [10], for $E_0=S^3\times S^n$ by [13], and for $E_{k\omega}$ ($0\leqq k\leqq 6$) by J. W. Rutter [11]. The group structure of $\mathscr{E}(E_f)$ is also given in each case except for $E_{6\omega}$.

EXAMPLE 3.3. *Let $\nu'\circ\eta_6\in\pi_7(S^3)=Z_2$ be the generator. Then we have the following exact sequence*:

$$0\to Z_{24}\oplus Z_2\to\mathscr{E}(E_{\nu'\circ\eta_6})\to Z_2\oplus Z_2\to 1.$$

PROOF. Since $\omega\circ S^3(\nu'\circ\eta_6)=\nu'\circ 2\nu_6\circ\eta_9=0$ in $\pi_{10}(S^3)$ by [14, (5.5)], we have an exact sequence (3.2) for $f=\nu'\circ\eta_6$. In general, let $n\geqq 6$. Then $\pi_{n+4}(S^n)=0$ by [14, Table of $\pi_{n+k}(S^n)$, I] and we have the exact sequence of the principal S^3-bundle X over S^n in (1.1):

$$0\longrightarrow \pi_{n+3}(S^3)\xrightarrow{i_*}\pi_{n+3}(X)\xrightarrow{p_*}\pi_{n+3}(S^n)\xrightarrow{\partial}\pi_{n+2}(S^3)\longrightarrow\cdots,$$

where $\pi_{n+3}(S^n)=Z_{24}$ generated by ω_n and $\partial(\omega_n)=f\circ\omega_{n-1}$ by [9, (2.2)]. Let $n=8$ in the above sequence and $f=\nu'\circ\eta_6$. Then we have an exact sequence

$$0\to Z_2\to\pi_{11}(E_f)\to Z_{24}\to 0,$$

since $f\circ\omega_7=\nu'\circ\eta_6\circ\omega_7=0$, and $\{\nu'\circ\eta_6,\ \omega_7,\ 8\iota_{10}\}\supset\nu'\circ\{\eta_6,\ \omega_7,\ 8\iota_{10}\}\equiv 0$ modulo $(\nu'\circ\eta_6)_*\pi_{11}(S^7)+8\pi_{11}(S^3)=0$. Therefore, by [9, Th. 2.1], $\pi_{11}(E_f)=Z_{24}\oplus Z_2$. For $f=\nu'\circ\eta_6$, we can easily see that H in (2.2) is 0 and $\mathscr{E}(S^3\cup_f e^8)=Z_2\oplus Z_2$ by [10]. Hence we have the required result. q.e.d.

EXAMPLE 3.4. *Let $f=\nu'\circ\eta_6^2\in\pi_8(S^3)=Z_2$ be the generator. Then we have the following exact sequences*:

$$0\to Z_2\oplus Z_2\oplus Z_{24}\to\mathscr{E}(E_f)\to G\to 1,$$
$$0\to D(Z_3)\to G\to Z_2\to 1.$$

EXAMPLE 3.5. *Let $f=\alpha_1(3)\circ\alpha_1(6)\in\pi_9(S^3)=Z_3$ be the generator. Then we have the following exact sequence*:

$$0\to Z_2\oplus Z_4\oplus Z_{72}\to\mathscr{E}(E_f)\to Z_{30}\to 1.$$

These last two examples are obtained by the similar way to Example 3.3.

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