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On the set of free homotopy classes and Brown's construction

Dedicated to Professor Nobuo Shimada on his 60th birthday

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Introduction

The purpose of this note is to demonstrate some simple facts about the set of free homotopy classes. An application will be found in the construction of G-CW approximations of G-spaces through Brown's construction.

Throughout this note, let $[A, B]$ denote the set of all *free homotopy classes* of continuous maps of A to B for any spaces A and B. Then, we have the following two theorems.

THEOREM 1. *Let X and Y be spaces and $f: X \to Y$ a continuous map. Suppose that X and Y are arcwise connected and*

$$(*) \qquad f_*: \pi_1(X, x) \longrightarrow \pi_1(Y, f(x)) \ (x \in X) \text{ is surjective.}$$

Then, $f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$ is injective or surjective if and only if $f_*: [S^n, X] \to [S^n, Y]$ is injective or surjective, respectively.

THEOREM 2. *Let $f: X \to Y$ be a continuous map and $N \geqq 1$. Then, the following three conditions are equivalent to each other*:

(1) *For any $x \in X$, the induced homomorphism ($n \geqq 1$) or map ($n = 0$)*

$$f_*: \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is bijective when $n < N$ and surjective when $n = N$.

(2) *For any* CW *complex K, the induced map*

$$f_*: [K, X] \longrightarrow [K, Y]$$

is bijective when $\dim K < N$ *and surjective when* $\dim K = N$.

(2)′ (2) *is valid for $K = *$ or S^n ($n \geqq 1$) and, in addition, f_* in* (2) *is surjective for $K = \vee_{\lambda \in \Lambda} S^1_\lambda$, the wedge of circles $S^1_\lambda = S^1$, where Λ is any set.*

Theorem 2 is a corollary to Theorem 1, because $(*)$ is a consequence of the last condition in (2)′. Here, we notice that Λ in (2)′ can be taken to be each conjugate class of $\pi_1(Y, f(x))$ (see Lemma 1.3), and to be the one-point-set when

$\pi_1(Y, f(x))$ is finite or nilpotent (see Proposition 3.1). So, we can restrict K in Theorem 2 to finite CW complexes under some finiteness conditions on the fundamental groups; but this is not the case in general. Such conditions and counter-examples will be given in §3 and §4, respectively.

Now, we present some results in the theory of G-spaces. Let G be a topological group. By a *G-space* X, we mean a space X together with a continuous G-action on X. For a subgroup H of G, the *H-stationary subspace* $\{x \in X: gx = x \text{ for every } g \in H\}$ is denoted by X^H. Let $\mathscr{F}$ be an *orbit type family* for G; $\mathscr{F}$ consists of subgroups of G, and $gHg^{-1} \in \mathscr{F}$ if $H \in \mathscr{F}$ and $g \in G$. A (not necessarily Hausdorff) G-CW complex K is called a *G-CW$_{\mathscr{F}}$ complex* if the isotropy subgroups of G-cells in K are contained in $\mathscr{F}$ (see [5]). Let $[\ ,\ ]_G$ denote the set of all *free G-homotopy classes* of G-maps. Then, an equivariant version of Theorem 2 is given by the following theorem, which is equivalent to Theorem 5.2*) of [4] when $\mathscr{F}$ consists of all closed subgroups of G.

THEOREM 3. *Let $f: X \to Y$ be a G-map between G-spaces and $N \geqq 1$. Then, the following four conditions are equivalent to each other:*

(1) *For any $H \in \mathscr{F}$, X^H is non-empty if and only if so is Y^H, and moreover, for any $x \in X^H$, the induced homomorphism ($n \geqq 1$) or map ($n = 0$)*

$$f_*: \pi_n(X^H, x) \longrightarrow \pi_n(Y^H, f(x))$$

is bijective when $n < N$ and surjective when $n = N$.

(2) *For any G-CW$_{\mathscr{F}}$ complex K, the induced map*

$$f_*: [K, X]_G \longrightarrow [K, Y]_G$$

is bijective when $\dim K/G < N$ and surjective when $\dim K/G = N$.

(2)′ (2) *is valid for $K = (G/H) \times L$ where $H \in \mathscr{F}$ and L is a CW complex with trivial G-action.*

(2)″ (2) *is valid for $K = G/H$ or $(G/H) \times S^n$ ($n \geqq 1$) and, in addition, f_* in* (2) *is surjective for $K = (G/H) \times \vee_{\lambda \in \Lambda} S^1_\lambda$ ($S^1_\lambda = S^1$), where $H \in \mathscr{F}$ and G acts trivially on the second factors.*

By using the construction of E. H. Brown [1] and by the above theorem, we have the following

THEOREM 4. *Let $\mathscr{F}$ be an orbit type family for G. Then, for any G-space X, there exists a pair of a G-CW$_{\mathscr{F}}$ complex $K_{\mathscr{F}}(X)$ and a G-map $\rho_X: K_{\mathscr{F}}(X) \to X$ such that*

$$(\rho_X)_*: \pi_n(K_{\mathscr{F}}(X)^H, v) \longrightarrow \pi_n(X^H, \rho_X(v)) \quad (n \geqq 0)$$

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*) We remark that a missing part of the proof of this theorem is covered by that in this note.

is bijective for any $H \in \mathscr{F}$ *and* $v \in K_{\mathscr{F}}(X)^H$. *Moreover, for any G-map* $f: X \to Y$, *there exists a G-cellular map* $K_{\mathscr{F}}(f): K_{\mathscr{F}}(X) \to K_{\mathscr{F}}(Y)$, *unique up to homotopy, such that* $\rho_Y \circ K_{\mathscr{F}}(f)$ *is G-homotopic to* $f \circ \rho_X$.

When $\mathscr{F}$ consists of all subgroups of G, $K_{\mathscr{F}}(X)$ is constructed more canonically in [5]. A variant of Brown's construction used in Hastings-Waner [2] also seems applicable to the proof of Theorem 3; but our construction given in §2 is much simpler. Besides, even when $G = \{e\}$, our construction which uses only the free homotopy classes is newly justified.

§1. Elementary study of free homotopy sets and proofs of Theorems 1, 2 and 3

We shall prove Theorem 1 by an elementary lemma. Let K and X be arcwise connected spaces with base points $v_0 \in K$ and $x_0 \in X$. Let $[K, v_0; X, x_0]$ denote the set of all *based* homotpy classes of (continuous) maps of (K, v_0) to (X, x_0). Then, we have the forgetful map

$$\psi: [K, v_0; X, x_0] \longrightarrow [K, X]$$

to the free homotopy set. Assume that K is a CW complex and v_0 is a vertex of K. Then, for any maps $f: (K, v_0) \to (X, x_0)$ and $\alpha: (I, \dot{I}) \to (X, x_0)$, we have a homotopy $f_t: K \to X$ with $f_0 = f$ and $f_t(v_0) = \alpha(t)$ $(t \in I)$, and denote $f_1: (K, v_0) \to (X, x_0)$ by $\alpha \cdot f$. The following lemma can be proved by a standard homotopy argument:

LEMMA 1.1. $\pi_1(X, x_0)$ *operates on* $[K, v_0; X, x_0]$ *by* $[\alpha] \cdot [f] = [\alpha \cdot f]$ *and the set* $[K, v_0; X, x_0]/\pi_1(X, x_0)$ *of all orbits is identified with* $[K, X]$ *by the forgetful map* ψ.

PROOF OF THEOREM 1. Consider the commutative diagram

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{f_*} & \pi_n(Y, f(x)) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ [S^n, X] & \xrightarrow{f_\#(=f_*)} & [S^n, Y] \end{array}$$

where ψ's are the forgetful maps and the lower f_* is denoted by $f_\#$ to distinguish it from the upper f_*.

Injectivity: Assume that $f_\#$ is injective. Take $g: (S^n, *) \to (X, x)$ with $f_*[g] = 0$ in $\pi_n(Y, f(x))$. Then, $f_\#[g] = 0$ in $[S^n, Y]$ and hence $[g] = 0$ in $[S^n, X]$. Since the orbit of 0 in $\pi_n(X, x)$ consists of 0 alone, we see that $[g] = 0$ in $\pi_n(X, x)$ by Lemma 1.1. Thus the group homomorphism f_* is injective.

Conversely, assume that f_* is injective. Let $g, g': S^n \to X$ be two maps such that $f_\#[g]=f_\#[g']$ in $[S^n, Y]$. We may assume that $g(*)=g'(*)=x$. By Lemma 1.1, there is a $\beta \in \pi_1(Y, f(x))$ with $\beta \cdot [f \circ g]=[f \circ g']$. Take an element $\alpha \in \pi_1(X, x)$ with $f_* \alpha=\beta$ by the assumption (*) in the theorem. Then, $f_*(\alpha \cdot [g])=f_*[g']$. So, $\alpha \cdot [g]=[g']$ by the assumption, which implies $[g]=[g']$ in $[S^n, X]$. Thus $f_\#$ is injective.

Surjectivity: If f_* is surjective, then so is $f_\#$ by Lemma 1.1.

Assume that $f_\#$ is surjective, and take any $h: (S^n, *) \to (Y, f(x))$. Then, there is a map $g: S^n \to X$ with $f_\#[g]=[h]$ in $[S^n, Y]$, where we may assume that $g(*)=x$. By Lemma 1.1, there is a $\beta \in \pi_1(Y, f(x))$ such that $\beta \cdot [f \circ g]=[h]$. Take $\alpha \in \pi_1(X, x)$ with $f_* \alpha=\beta$ by the assumption (*) in the theorem. Then $f_*(\alpha \cdot [g])=[h]$; and f_* is surjective. q. e. d.

To prove Theorem 2, we notice the following lemma, where

$$\vee_\Lambda S^1 = \vee_{\lambda \in \Lambda} S^1_\lambda \ (S^1_\lambda = S^1), \qquad \prod_\Lambda \pi = \prod_{\lambda \in \Lambda} \pi_\lambda \quad (\pi_\lambda = \pi).$$

LEMMA 1.2. *For any set Λ, any map $f: X \to Y$ between arcwise connected spaces and $x \in X$, the induced map $f_\# \, (=f_*)$: $[\vee_\Lambda S^1, X] \to [\vee_\Lambda S^1, Y]$ can be identified with the map*

$$(\prod_\Lambda f_*)_\#: (\prod_\Lambda \pi)/\mathrm{ad}\, \pi \longrightarrow (\prod_\Lambda \pi')/\mathrm{ad}\, \pi'$$

induced from the product $\prod_\Lambda f_*$ of the induced homomorphism

$$f_*: \pi = \pi_1(X, x) \longrightarrow \pi' = \pi_1(Y, f(x)),$$

where $/\mathrm{ad}$ *denotes the set of orbits by the conjugation-action $\alpha \cdot (\alpha_\lambda)=(\alpha\alpha_\lambda\alpha^{-1})$.*

PROOF. $[\vee_\Lambda S^1, *; X, x]$ can be identified naturally with $\prod_\Lambda \pi$. Thus, the lemma follows immediately from Lemma 1.1. q. e. d.

LEMMA 1.3. *In Lemma 1.2, assume that $f_\#=(\prod_\Lambda f_*)_\#$ is surjective for any $\Lambda=\pi' \cdot \beta$, where $\pi' \cdot \beta=\{b\beta b^{-1}: b \in \pi'\}$ is the conjugate class of $\beta \in \pi'$. Then, $f_*: \pi \to \pi'$ is also surjective.*

PROOF. Take any $\beta \in \pi'$ and consider $\prod_\Lambda f_*: \prod_\Lambda \pi \to \prod_\Lambda \pi'$ *for* $\Lambda=\pi' \cdot \beta$. Then the assumption means that for any $(\beta_\lambda) \in \prod_\lambda \pi'$, some conjugate $b \cdot (\beta_\lambda)=(b\beta_\lambda b^{-1})$ $(b \in \pi')$ is contained in the image of $\prod_\Lambda f_*$. Now, take (β_λ) to be

$$\beta_\lambda = \lambda \qquad \text{for any} \quad \lambda \in \Lambda = \pi' \cdot \beta.$$

Then, $\beta=b\beta_{\lambda_0}b^{-1}$ for $\lambda_0=b^{-1}\beta b \in \Lambda$ and so $\beta \in \mathrm{Im} f_*$. Thus f_* is surjective. q. e. d.

PROOF OF THEOREM 2. The implication (1)⇒(2) is well-known in the theory of CW complexes. (2)′ is a special case of (2). (2)′ for $K=*$ implies (1) for $n=0$. Lemma 1.3 shows that the last condition in (2)′ implies the assumption (*) in Theorem 1. The implication (2)′⇒(1) now follows from Theorem 1.

q. e. d.

PROOF OF THEOREM 3. Let L be a CW complex with trivial G-action. Then, for any $H\in\mathscr{F}$, $K=(G/H)\times L$ is a G-CW$_{\mathscr{F}}$ complex and we can identify naturally as $K/G=L$ and $[K, Z]_G=[L, Z^H]$ for any G-space Z. So, (2)′ is a special case of (2), and Theorem 2 shows the equivalence of (1), (2)′ and (2)″. The implication (1)⇒(2) is due to a standard argument in the theory of G-CW complexes. q. e. d.

§2. Proof of Theorem 4 through Brown's construction

We shall construct $K_{\mathscr{F}}(X)$ in Theorem 4. Let $\mathscr{C}$ be the category of G-CW$_{\mathscr{F}}$ complexes and free G-homotopy classes of G-maps. The sum in this category stands for the disjoint union. Consider the *equalizer* $E(g_0, g_1)$ of two maps $g_0, g_1: A\to B$, defined to be the identification space

$$E(g_0, g_1)=A\times I+B/\sim \quad \text{with } (a, t)\sim g_t(a) \text{ for any } a\in A \text{ and } t\in\dot{I}.$$

If $A, B\in\mathscr{C}$ and g_0, g_1 are G-cellular, then $E(g_0, g_1)\in\mathscr{C}$.

Choose one representative for each class of conjugate subgroups in $\mathscr{F}$ and put $\mathscr{F}'=\{\text{representatives}\}\subset\mathscr{F}$. Then,

$$\mathscr{C}'_0 = \{G/H, (G/H)\times S^n: H\in\mathscr{F}', n\geqq 1\} \quad \text{and}$$
$$\mathscr{C}'_1 = \mathscr{C}'_0 \cup \{(G/H)\times \vee_\Lambda S^1: H\in\mathscr{F}', \Lambda\subset \mathrm{Map}(S^1, X)\}$$

($\vee_\Lambda S^1=\vee_{\lambda\in\Lambda} S^1_\lambda$, $S^1_\lambda=S^1$) are small subcategories of $\mathscr{C}$. Let $\mathscr{C}_0$ (resp. $\mathscr{C}_1$) be a minimal subcategory which contains $\mathscr{C}'_0$ (resp. $\mathscr{C}'_1$) and is closed under the operation of taking finite sum and equalizer. Then, $\mathscr{C}_0$ and $\mathscr{C}_1$ are small, full subcategories of $\mathscr{C}$.

Now, we fix a G-space X and put $H(\cdot)=[\cdot, X]_G$. We see that $(\mathscr{C}, \mathscr{C}_0)$ is a homotopy category and H is a homotopy functor in the sense of E. H. Brown [1]. To construct $K_{\mathscr{F}}(X)\in\mathscr{C}$ in Theorem 4, we use Brown's construction given there.

If γ is anything and $Y\in\mathscr{C}$, $(Y, \gamma)\in\mathscr{C}$ will denote a copy of Y and $t_\gamma: (Y, \gamma)\to Y$ will be an identification. By induction on n, we define $K_n\in\mathscr{C}$ and $u_n\in H(K_n)$ so that

$$K_n\subset K_{n+1} \quad \text{and} \quad H(f_n)u_{n+1}=u_n,$$

where $f_n: K_n\to K_{n+1}$ is the inclusion. Put

$$K_0 = \sum (Y, u) \quad \text{and} \quad u_0 = \sum H(t_u)u \in H(K_0),$$

where the sum ranges over all $Y \in \mathscr{C}_1$ and all $u \in H(Y)$. Note that the choice of K_0 and u_0 in [1] is arbitrary. So, we specify them as above to get the following

LEMMA 2.1. *T_{u_0}: $[Y, K_0]_G \to H(Y)$ is surjective for any $Y \in \mathscr{C}_1$.*

Suppose that K_n and u_n ($n \geqq 0$) have been defined. Let $K_{n+1} \in \mathscr{C}$ be the equalizer of

$$\sum g_i \circ t_{(g_0, g_1)}: \sum (Y, (g_0, g_1)) \longrightarrow K_n \qquad \text{for} \quad i = 0, 1,$$

where the sum ranges over all $Y \in \mathscr{C}_0$ and all pairs of G-cellular maps g_0, g_1: $Y \to K_n$ such that g_0 is not freely G-homotopic to g_1 and $H(g_0)u_n = H(g_1)u_n$. Then, it is easy to see that there is a $u_{n+1} \in H(K_{n+1})$ with $H(f_n)u_{n+1} = u_n$.

From the way of the construction of K_n ($n \geqq 1$) together with Lemma 2.1 and $\mathscr{C}_0 \subset \mathscr{C}_1$, we see the following

LEMMA 2.2. *$\lim T_{u_n}$: $\lim [Y, K_n]_G \to H(Y)$ is bijective for any $Y \in \mathscr{C}_0$ and surjective for any $Y \in \mathscr{C}_1$.*

Let $K_{\mathscr{F}}(X) = \cup K_n$ be the direct limit and h_n: $K_n \to K_{\mathscr{F}}(X)$ the inclusion. Then, $K_{\mathscr{F}}(X) \in \mathscr{C}$ and there is a $u_X \in H(K_{\mathscr{F}}(X))$ such that $H(h_n)u_X = u_n$. Furthermore,

LEMMA 2.3. *T_{u_X}: $[Y, K_{\mathscr{F}}(X)]_G \to H(Y)$ is bijective for any $Y \in \mathscr{C}_0$ and surjective for any $Y \in \mathscr{C}_1$.*

In fact, $\lim T_{u_n}$ in lemma 2.2 is the composition of

$$\lim (h_n)_*: \lim [Y, K_n]_G \longrightarrow [Y, K_{\mathscr{F}}(X)]_G$$

and T_{u_X}; and $\lim (h_n)_*$ is bijective for any $Y \in \mathscr{C}_0$, because the image of Y or $Y \times I$ ($Y \in \mathscr{C}_0$) is contained in a finite G-$CW_{\mathscr{F}}$ subcomplex of $K_{\mathscr{F}}(X)$. Thus, Lemma 2.3 is a consequence of Lemma 2.2.

Take a G-map ρ_X: $K_{\mathscr{F}}(X) \to X$ representing $u_X \in H(K_{\mathscr{F}}(X)) = [K_{\mathscr{F}}(X), X]_G$. Then

$$(\rho_X)_* = T_{u_X}: [Y, K_{\mathscr{F}}(X)]_G \longrightarrow [Y, X]_G = H(Y),$$

which satisfies Lemma 2.3. So, the first half of Theorem 4 is a consequence of the implication $(2)'' \Rightarrow (1)$ in Theorem 3 by the definition of $\mathscr{C}_0$ and $\mathscr{C}_1$. The last half of Theorem 4 is clear by construction; and Theorem 4 is proved completely.

§ 3. Some finiteness conditions

In this section, we shall prove two propositions to give a condition that K in Theorem 2 can be restricted to finite CW complexes.

In the notations of Lemma 1.2, consider the induced homomorphism

$$\varphi = f_*: \pi = \pi_1(X, x) \longrightarrow \pi' = \pi_1(Y, f(x)) \quad (f: X \to Y,\ x \in X),$$

and the induced map $f_\#$ $(=f_*)$: $[\vee_\Lambda S^1, X] \to [\vee_\Lambda S^1, Y]$ identified with the map

$$(\prod_\Lambda \varphi)_\#: (\prod_\Lambda \pi)/\mathrm{ad}\,\pi \longrightarrow (\prod_\Lambda \pi')/\mathrm{ad}\,\pi'$$

induced from the product homomorphism $\prod_\Lambda \varphi$, where /ad denotes the set of orbits by the conjugation-action $\alpha \cdot (\alpha_\lambda) = (\alpha\alpha_\lambda\alpha^{-1})$. Then, we have the following proposition, where (sn) (resp. (bn)) means that
(sn) (resp. (bn)) $f_\# = (\prod_\Lambda \varphi)_\#$ *is surjective* (resp. *bijective*) *when* $|\Lambda| = n$.

PROPOSITION 3.1. (i) *When* π' (*resp.* π) *is finite or nilpotent,* (s1) (*resp.* (sn) *for all* n) *implies the assumption* $(*)$ *in Theorem* 1 *that* $f_* = \varphi$ *is surjective.*

(ii) *When* π *is nilpotent,* (b1) *and* (s2) *imply that* φ *is bijective.*

PROOF. (i) Put $\bar{\pi} = \mathrm{Im}\,\varphi \subset \pi'$. Then (s1) means that $\pi' = \{e\} \cup \bigcup_{\beta \in \pi'} \beta \cdot (\bar{\pi} - \{e\})$. So, when π' is finite, this implies that $|\pi'| \leqq 1 + (|\bar{\pi}| - 1)|\pi'/\bar{\pi}| = 1 + |\pi'| - |\pi'/\bar{\pi}|$ and $\pi' = \bar{\pi}$.

When π' is nilpotent, take the upper central series $\{e\} = Z'_0 \subset Z'_1 \subset \cdots \subset Z'_n = \pi'$. Let $\beta \in Z'_{i+1}$. Then, $b \cdot \beta \in \bar{\pi}$ for some $b \in \pi'$ by (s1), and $b \cdot \beta \equiv \beta \bmod Z'_i$ since $Z'_{i+1}/Z'_i = Z(\pi'/Z'_i)$. So, if $Z'_i \subset \bar{\pi}$, then $\beta \in \bar{\pi}$ and $Z'_{i+1} \subset \bar{\pi}$. Thus we see $Z'_i \subset \bar{\pi}$ by induction; and $\pi' = Z'_n = \bar{\pi}$.

Assume that (sn) holds for all n. Let $\beta \in \pi'$. Then $b \cdot (\{\beta\} \cup (\bar{\pi} - \{e\})) \subset \bar{\pi}$ for some $b \in \pi'$ by (s$|\bar{\pi}|$) when π is finite. This shows $\beta \in \bar{\pi}$ and $\pi' = \bar{\pi}$. Now consider the lower central series given by $\bar{\pi}_0 = \bar{\pi}$, $\bar{\pi}_{i+1} = [\bar{\pi}, \bar{\pi}_i]$ and $\pi'_0 = \pi'$, $\pi'_{i+1} = [\pi', \pi'_i]$. Then, for any $\beta_\lambda \in \pi'_{i(\lambda)}$ $(1 \leqq \lambda \leqq n)$, there is a $b \in \pi'$ with $b \cdot \beta_\lambda \in \bar{\pi}_{i(\lambda)}$. This is the assumption when $i = \max i(\lambda)$ is 0, and is proved by induction on i and by the definition of commutator subgroups. So, $\pi'_m = \{e\}$ if $\bar{\pi}_m = \{e\}$. Thus, when π is nilpotent, so is π' and we have $\pi' = \bar{\pi}$ as is shown already.

(ii) φ is injective by (b1) and we regard φ as the inclusion. Take the upper central series $\{e\} = Z_0 \subset Z_1 \subset \cdots \subset Z_n = \pi$. Then, we see by induction that Z_i is a normal subgroup of π'; and so is $\pi = Z_n$ and $\pi' = \pi$ by (s1). In fact, take any $\alpha \in Z_{i+1}$ and $\beta \in \pi'$. Then $b' \cdot (\alpha, \beta) \in \pi \times \pi$ for some $b' \in \pi'$ by (s2), and so $b' \cdot \alpha = a \cdot \alpha$ for some $a \in \pi$ by (b1). Thus $b \cdot (\alpha, \beta) = (\alpha, \alpha_1)$ where $b = a^{-1}b' \in \pi'$ and $\alpha_1 \in \pi$. So, $b \cdot (\beta \cdot \alpha) = (\alpha_1 b) \cdot \alpha = \alpha_1 \cdot \alpha \equiv \alpha \bmod Z_i$ since $Z_{i+1}/Z_i = Z(\pi/Z_i)$, and $\beta \cdot \alpha \equiv b^{-1} \cdot \alpha = \alpha \bmod Z_i$ by inductive assumption. Hence $\beta \cdot \alpha \in Z_{i+1}$ and

Z_{i+1} is normal in π', as desired. q. e. d.

We now consider the following finiteness condition (**) for any group π:

(**) *There exists a finite subset A of π such that $Z(A)\cdot\alpha=\{a\alpha a^{-1}: a\in Z(A)\}$ is finite for any $\alpha\in\pi$.* ($Z(A)$ is centralizer of A.)

EXAMPLE 3.2. *π satisfies (**), when*

(1) *π is a FC-group, i.e., each conjugate class $\pi\cdot\alpha$ of $\alpha\in\pi$ consists of finite elements (e.g., π is abelian or finite), or*

(2) *π is finitely generated group or a free group.*

In fact, any FC-group π satisfies (**) by taking the empty set for A. If π is generated by a finite set A, then $Z(A)=Z(\pi)$. If π is free and $A=\{a_1, a_2\}$ $(a_1\neq a_2)$ is a subset of a system of free generators of π, then $Z(A)=\{e\}$. So, $Z(A)\cdot\alpha=\{\alpha\}$ in these cases.

PROPOSITION 3.3. *When π or π' satisfies (**), $\varphi=f_*$ is bijective if* (bn) *holds for all n.*

By the proof of Theorem 2, we have the following

COROLLARY 3.4. *In cases of Propositions* 3.1 *and* 3.3, *Theorem* 2 *is valid by restricting K to finite* CW *complexes.*

In Proposition 3.3, φ is injective by (b1) (see the proof of Theorem 1), and we regard φ: $\pi\subset\pi'$ as the inclusion hereafter. When $\Lambda\subset\pi$, we denote by $d_\Lambda=(d_\lambda)\in\prod_\Lambda\pi$ the element with $d_\lambda=\lambda$ for any $\lambda\in\Lambda$. Then, $\alpha\cdot d_\Lambda=d_\Lambda$ means $\alpha\in Z(\Lambda)$ when $\alpha\in\pi$ and $\alpha\in Z(\Lambda, \pi')=\{\beta\in\pi': \beta\lambda=\lambda\beta$ for any $\lambda\in\Lambda\}$ (the centralizer of Λ in π') when $\alpha\in\pi'$, respectively.

LEMMA 3.5. *Assume that* (bn) *holds, and let A and B be finite sets with $A\subset\pi$, $|A|=n$ and $|B|=m-n$, and $\beta_B\in\prod_B\pi'$ be any element.*

(i) *If* (sm) *holds, then there exists $\alpha_B\in(\prod_B\pi)\cap Z(A, \pi')\cdot\beta_B$.*

(ii) *If* (bm) *holds in addition, then $Z(A)\cdot\alpha_B=(\prod_B\pi)\cap Z(A, \pi')\cdot\beta_B$.*

PROOF. (i) For $(d_A, \beta_B)\in\prod_A\pi'\times\prod_B\pi'$, there is a $(x_A, x_B)\in(\prod_A\pi\times\prod_B\pi)\cap\pi'\cdot(d_A, \beta_B)$ by (sm), and so $x_A=\alpha\cdot d_A$ for some $\alpha\in\pi$ by (bn) since $d_A\in\prod_A\pi$. Thus, $\alpha_B=\alpha^{-1}\cdot x_B\in\prod_B\pi$ and $(d_A, \alpha_B)=\beta\cdot(d_A, \beta_B)$ for some $\beta\in\pi'$. This means that $\beta\in Z(A, \pi')$ and (i).

(ii) If $\alpha'_B\in(\prod_B\pi)\cap Z(A, \pi')\cdot\beta_B$ in addition, then $(d_A, \alpha'_B)\in\pi'\cdot(d_A, \beta_B)$ and so $(d_A, \alpha'_B)=\alpha'\cdot(d_A, \alpha_B)$ for some $\alpha'\in\pi$ by (bm). This means $\alpha'\in Z(A)$ and $\alpha'_B\in Z(A)\cdot\alpha_B$. q. e. d.

PROOF OF PROPOSITION 3.3. Assume that π' satisfies (**) by a finite subset B of π'. Then, there is a $(\alpha_b)\in(\prod_B\pi)\cap\pi'\cdot d_B$ by (s$|B|$). So, $A=\{\alpha_b: b\in B\}\subset\pi$

satisfies $A=\beta_0\cdot B$ for some $\beta_0\in\pi'$. Take any $\beta\in\pi'$. Then $B'=Z(A,\pi')\cdot\beta=\beta_0\cdot(Z(B)\cdot(\beta_0^{-1}\cdot\beta))$ is finite by (**). By (b$|A|$), (s$(|A|+|B'|)$) and Lemma 3.5 (i), we have $b\cdot d_{B'}\in\prod_{B'}\pi$ for some $b\in Z(A,\pi')$. So, for $b'=b^{-1}\cdot\beta\in B'$, we see that $\beta=b\cdot b'=b\cdot d_{b'}\in\pi$; and $\pi'=\pi$.

Assume now that π satisfies (**) by a finite subset A of π. Take any $\beta\in\pi'$. Then there is an $\alpha\in\pi\cap Z(A,\pi')\cdot\beta$ by (b$|A|$), (s$(|A|+1)$) and Lemma 3.5 (i). Put $A'=Z(A)\cdot\alpha$ which is a finite subset of π by (**). Take again $\alpha'\in\pi$ with $\alpha'=b\cdot\beta$ for some $b\in Z(A,\pi')\cap Z(A',\pi')$ by (b$(|A|+|A'|)$) and (s$(|A|+|A|+1)$). Then, $\alpha'\in Z(A)\cdot\alpha=A'$ by (b$(|A|+1)$) and Lemma 3.5 (ii). So, $\beta=b^{-1}\cdot\alpha'=\alpha'\in\pi$; and $\pi'=\pi$. q.e.d.

§4. Counter-examples

In this section, we shall show that Proposition 3.3 and Corollary 3.4 do not hold in general without any assumption on π or π', that is, K in Theorem 2 cannot be restricted to finite CW complexes.

Counter-examples are given by using the *infinte symmetric group* $S_\infty=\bigcup_{n\in N}S_n$, where N is the set of positive integers and S_n is the symmetric group of n letters $\{1,2,\dots,n\}$. Any element $\sigma\in S_\infty$ is a bijection $\sigma\colon N\to N$ such that $m(\sigma)=\{n\in N\colon\sigma(n)\neq n\}$ is a finite subset of N.

PROPOSITION 4.1. *For any injection $\varphi\colon N\to N$, let $\bar\varphi\colon S_\infty\to S_\infty$ be the homomorphism defined by*

$$\bar\varphi\sigma\,|\,N-\varphi N=\mathrm{id},\quad \bar\varphi\sigma\,|\,\varphi N=\varphi\circ\sigma\circ\varphi^{-1}\quad(\sigma\in S_\infty).$$

Then the induced map $(\prod_\Lambda\bar\varphi)_\#$ of $(\prod_\Lambda S_\infty)/\mathrm{ad}\,S_\infty$ to itself is bijective for any finite set Λ.

COROLLARY 4.2 *Let X be the Eilenberg-MacLane complex $K(S_\infty,1)$, and $f\colon X\to X$ be the map such that $f_*=\bar\varphi$ on $\pi_1(X)=S_\infty$. Then, the induced map $f_\#$ $(=f_*)$ of the free homotopy set $[K,X]$ to itself is bijective for any finite* CW *complex K.*

$\bar\varphi$ is not surjective unless φ is surjective. So, these give counter-examples.

We see that $f_\#$ in Corollary 4.2 is bijective for the 1-skeleton K^1 of any finite CW complex K by Proposition 4.1 and Lemma 1.2, and so for K by a standard homotopy argument because $\pi_n(X)=0$ for $n\geqq 2$.

PROOF OF PROPOSITION 4.1. By the definition of $\bar\varphi$, it is clear that $\bar\varphi$ is injective and that $\sigma\in\mathrm{Im}\,\bar\varphi$ if and only if $m(\sigma)\subset\varphi N$ for $\sigma\in S_\infty$.

Let Λ be a finite set. Take any $(\sigma_\lambda)\in\prod_\Lambda S_\infty$ and put $M=\bigcup_{\lambda\in\Lambda}m(\sigma_\lambda)$. Then, M is a finite subset of N and there exists a $\sigma\in S_\infty$ such that $\sigma(M)\subset\varphi N$. So,

$m(\sigma\sigma_\lambda\sigma^{-1})\subset\varphi N$ and $\sigma\sigma_\lambda\sigma^{-1}\in\operatorname{Im}\bar{\varphi}$ for any $\lambda\in\Lambda$. Thus, $(\prod_\Lambda\bar{\varphi})_\#$ is surjective.

Now, assume that (σ_λ), $(\sigma'_\lambda)\in\prod_\Lambda S_\infty$ are contained in $\operatorname{Im}\prod_\Lambda\bar{\varphi}$ and $(\sigma'_\lambda)=\sigma\cdot(\sigma_\lambda)$ for some $\sigma\in S_\infty$. When $m(\sigma)\not\subset\varphi N$, take $n\in m(\sigma)-\varphi N$ and put $\sigma'=(nn')\sigma$, where (nn') is the transposition of n and $n'=\sigma(n)$ $(\neq n)$. Then,

$$m(\sigma')\subset m(\sigma)-\{n\},\quad (\sigma'_\lambda)=\sigma'\cdot(\sigma_\lambda).$$

In fact, the first one is clear. Since σ_λ, $\sigma'_\lambda\in\operatorname{Im}\bar{\varphi}$ and $n\notin\varphi N$, n and n' are fixed by $\sigma'_\lambda=\sigma\sigma_\lambda\sigma^{-1}$, which shows the second one. Since $m(\sigma)$ is finite, the repeating use of this process shows that $(\sigma'_\lambda)=\tau\cdot(\sigma_\lambda)$ for some $\tau\in S_\infty$ with $m(\tau)\subset\varphi N$, i.e., $\tau\in\operatorname{Im}\bar{\varphi}$. Thus, $(\prod_\Lambda\bar{\varphi})_\#$ is injective. q.e.d.

In the end of this section, we note the following counter-example, which is given by T. Ohkawa before we obtain Corollary 4.2, where spaces are assumed to be arcwise connected CW complexes.

REMARK 4.3. (T. Ohkawa). *For any based map $f: X\to Y$, we can construct $X_\infty\supset X$, $Y_\infty\supset Y$ and an extension $f_\infty: X_\infty\to Y_\infty$ of f with the following properties*:

(1) $f_{\infty *}: [K, X_\infty]\to[K, Y_\infty]$ *is bijective for any finite* CW *complex K.*

(2) *For the induced homomorphisms* $\pi_1(X)\xrightarrow{f_*}\pi_1(Y)\xrightarrow{i_*}\pi_1(Y_\infty)\xleftarrow{f_{\infty *}}\pi_1(X_\infty)$ $(i: Y\subset Y_\infty)$, i_* *is injective and* $\operatorname{Im}f_{\infty *}\cap\operatorname{Im}i_*=\operatorname{Im}(i_*\circ f_*)$, (*and so* $f_{\infty *}$ *is not surjective when* f_* *is not surjective*).

The construction is done by modifying the one given in §2 so as to satisfy (2), and is sketched as follows: Let $X_1=X\vee\bigvee(K, h)$, Y_1 be the equalizer of

$$\bigvee h\circ t_h,\ j\text{ (the inclusion)}: \bigvee(K, h)\longrightarrow Y'=Y\vee\bigvee(K, h),$$

and $f_1: X_1\to Y'\subset Y_1$ be defined by f and the identity map, where the wedge ranges over all finite CW complexes K with base points and all based maps $h: K\to Y$ (up to free homotopy). Furthermore, let X_2, Y_2 be the equalizers of

$$\sum g_i\circ t_{(g_0,g_1)}: \sum(K, (g_0, g_1))\longrightarrow X_1\quad (i=0, 1),$$
$$\sum f_1\circ g_i\circ t_{(g_0,g_1)}: \sum(K, (g_0, g_1))\longrightarrow Y_1\quad (i=0, 1),$$

respectively, and $f_2: X_2\to Y_2$ be defined by the identity map and f_1, where the sum (disjoint union) ranges over all K of above and all pairs of based maps g_0, g_1: $K\to X_1$ such that $f_1\circ g_0$ is freely homotopic to $f_1\circ g_1$. Let X_n, Y_n and $f_n: X_n\to Y_n$ be defined inductively by the first or second construction according to n is odd or even. Then, $X_\infty=\bigcup X_n$, $Y_\infty=\bigcup Y_n$ and $f_\infty=\bigcup f_n: X_\infty\to Y_\infty$ are the desired ones. In fact, (1) is clear. (2) is seen by the following result:

Let $E=E(g_0, g_1)$ be the equalizer of based maps g_0, g_1: $A\to B$ and consider

$$\pi_1(A)\xrightarrow{g_{i*}}\pi_1(B)\xrightarrow{i_{1*}}\pi_1(E)\xleftarrow{i_{2*}}\pi_1(S^1)\quad (i=0, 1)$$

where i_1 and i_2: $S^1 = * \times I/\sim \subset E$ are the inclusions. Then, the isomorphism

$$\pi_1(E) \cong \pi_1(B) * \pi_1(S^1) / \langle (g_{0*}\alpha)^{-1} s(g_{1*}\alpha)s^{-1} : \alpha \in \pi_1(A) \rangle$$

is induced from i_{1*} and i_{2*}, where $s \in \pi_1(S^1)$ is a generator, (which is shown by using van Kampen's theorem). Furthermore, if $\mathrm{Ker}\, g_{0*} = \mathrm{Ker}\, g_{1*}$, then i_{1*} is injective and the right hand side is an HNN-extension which satisfies the Normal Form Theorem (cf., e.g., [3, Ch. IV, Th. 2.1]).

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