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Study of the behavior of logarithmic potentials by means of logarithmically thin sets

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1. Introduction and statement of results

Let R^n $(n \geqq 2)$ be the n-dimensional euclidean space. For a nonnegative (Radon) measure μ on R^n, we set

$$L\mu(x) = \int \log \frac{1}{|x-y|} d\mu(y)$$

if the integral exists at x. We note here that $L\mu$ is not identically $-\infty$ if and only if

$$\int \log (1+|y|) d\mu(y) < \infty. \tag{1}$$

Denote by $B(x, r)$ the open ball with center at x and radius r. For $E \subset B(0, 2)$, define

$$C(E) = \inf \mu(R^n),$$

where the infimum is taken over all nonnegative measures μ on R^n such that S_μ (the support of μ) $\subset B(0, 4)$ and

$$\int \log \frac{8}{|x-y|} d\mu(y) \geqq 1 \qquad \text{for every} \quad x \in E.$$

If $E \subset B(x^0, 2)$, then we set

$$C(E) = C(\{x-x^0;\ x \in E\}).$$

One notes here that this is well defined, i.e., independent of the choice of x^0.

Throughout this paper let k be a positive and nonincreasing function on the interval $(0, \infty)$ such that

$$k(r) \leqq Kk(2r) \qquad \text{for any} \quad r,\ 0 < r < 1,$$

where K is a positive constant independent of r. A set E in R^n is said to be k-logarithmically thin, or simply k-log thin, at $x^0 \in R^n$ if

$$\textstyle\sum_{j=1}^{\infty} k(2^{-j}) C(E'_j) < \infty,$$

where $E'_j = \{x \in B(x^0, 2) - B(x^0, 1);\ x^0 + 2^{-j}(x - x^0) \in E\}$. If $k(r) = \log r^{-1}$ for

r sufficiently small, then a set E which is k-log thin at x^0 is called simply logarithmically thin at x^0. Then the following result is well known (see [1; Theorem IX, 7] for $n=2$):

THEOREM A. *Let $x^0 \in R^n$ and μ be a nonnegative measure on R^n satisfying* (1).

(i) *There exists a set E in R^n which is logarithmically thin at x^0 and satisfies*

$$\lim_{x \to x^0, x \in R^n - E} L\mu(x) = L\mu(x^0).$$

(ii) *There exists a set E in R^n which is logarithmically thin at x^0 and satisfies*

$$\lim_{x \to x^0, x \in R^n - E} \left(\log \frac{1}{|x - x^0|}\right)^{-1} L\mu(x) = \mu(\{x^0\}).$$

Our first aim is to give a generalization of Theorem A.

THEOREM 1. *Let h be a nondecreasing and positive function on the interval* $(0, \infty)$ *such that $h(2r) \leqq M h(r)$ and*

$$\int_0^{1/2} \frac{dt}{h(t)(\log t^{-1})(r+t)} \leqq \frac{M}{h(r)} \tag{2}$$

for any r, $0<r<1$, where M is a positive constant independent of r. Let μ be a nonnegative measure on R^n satisfying (1),

$$\lim_{r \downarrow 0} h(r)(\log r^{-1})\mu(B(x^0, r)) = 0$$

and

$$\int \tilde{h}(|x^0 - y|) d\mu(y) < \infty,$$

where $\tilde{h}(0) = \infty$ and $\tilde{h}(r) = h(r)k(r)$ for $r > 0$. Then there exists a set E in R^n which is k-log thin at x^0 and satisfies

$$\lim_{x \to x^0, x \in R^n - E} h(|x - x^0|) L\mu(x) = 0.$$

REMARK 1. For $\delta > 0$, define

$$h_\delta(r) = \begin{cases} (\log r^{-1})^{-\delta} & \text{if } 0 < r \leqq 2^{-1}, \\ (\log 2)^{-\delta} & \text{if } r > 2^{-1}. \end{cases}$$

Then h_δ satisfies all the conditions on h in Theorem 1.

REMARK 2. If $h(r) = (\log r^{-1})^{-1}$ and $k(r) = \log r^{-1}$ for r sufficiently small, then Theorem 1 implies Theorem A, (ii).

Hereafter, when a positive function h on $(0, \infty)$ is given, we let $\tilde{h}$ be as in Theorem 1.

THEOREM 2. *Let h be a nonincreasing and positive function on the interval* $(0, \infty)$ *such that $rh(r)$ is nondecreasing on* $(0, \infty)$ *and* $\lim_{r\downarrow 0} rh(r)=0$. *Suppose furthermore $h(r)\log(r/s)\leqq M\tilde{h}(s)$ whenever $0<s<r\leqq 1$, where M is a positive constant independent of r and s. Let μ be a nonnegative measure on R^n satisfying* (1) *and*

$$\int \tilde{h}(|x^0-y|)d\mu(y) < \infty.$$

Then there exists a set E in R^n which is k-log thin at x^0 and satisfies

$$\lim_{x\to x^0, x\in R^n-E} h(|x^0-y|)\,[L\mu(x)-L\mu(x^0)] = 0.$$

REMARK. If $h(r)=1$ and $k(r)=\log r^{-1}$ for r sufficiently small, then Theorem 2 yields Theorem A, (i).

Fuglede [3] discussed fine differentiability properties of logarithmic potentials in the plane R^2. To state his result, we let $L(x)=\log(1/|x|)$ and set for a nonnegative integer m,

$$L_m(x, y) = L(x-y) - \sum_{|\lambda|\leqq m} \frac{(x-x^0)^\lambda}{\lambda!}\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](x^0-y),$$

where $\lambda=(\lambda_1,\dots, \lambda_n)$ is a multi-index with length $|\lambda|=\lambda_1+\cdots+\lambda_n$, $\lambda!=\lambda_1!\cdots\lambda_n!$, $x^\lambda=x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ and $(\partial/\partial x)^\lambda=(\partial/\partial x_1)^{\lambda_1}\cdots(\partial/\partial x_n)^{\lambda_n}$.

THEOREM B (cf. Fuglede [3; Notes 3]). *Let μ be a nonnegative measure on R^2 satisfying*

$$\int |x^0-y|^{-1}\log(2+|x^0-y|^{-1})d\mu(y) < \infty, \tag{3}$$

then there exists a set E in R^2 which is (logarithmically) thin at x^0 and satisfies

$$\lim_{x\to x^0, x\in R^n-E} |x-x^0|^{-1}\int L_1(x, y)d\mu(y) = 0. \tag{4}$$

For a proof of Theorem B, see Davie and Øksendal [2; Theorem 6]. Our second aim is to generalize Theorem B, and in fact to show, under a condition weaker than (3), that (4) holds for a set E which will be k-log thin at x^0 with an appropriate function k.

THEOREM 3. *Let h be a nonincreasing and positive function on the interval* $(0, \infty)$ *such that $rh(r)$ is nondecreasing on* $(0, \infty)$ *and* $\lim_{r\downarrow 0} rh(r)=0$. *Let μ be a nonnegative measure on R^n satisfying* (1) *and*

$$\int |x^0-y|^{-m}\tilde{h}(|x^0-y|)d\mu(y) < \infty,$$

for a positive integer m smaller than n. Then there exists a set E in R^n which is k-log thin at x^0 and satisfies

$$\lim_{x\to x^0, x\in R^n-E} |x-x^0|^{-m}h(|x-x^0|)\int L_m(x, y)d\mu(y) = 0.$$

REMARK. In case $n=2$ and $m=1$, if we take $h(r)\equiv 1$ and $k(r)=\log(2+r^{-1})$, then Theorem 3 coincides with Theorem B.

In case $m=n$, we shall establish the following result.

THEOREM 4. *Let μ be a nonnegative measure on R^n which satisfies* (1) *and the following two conditions*:

(a) $\lim_{r\downarrow 0} r^{-n}|\mu-a\Lambda_n|(B(x^0, r)) = 0$ for some a,
where Λ_n denotes the n-dimensional Lebesgue measure;

(b) $A_\lambda=\lim_{r\downarrow 0}\int_{R^n-B(x^0,r)}\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](x^0-y)d\mu(y)$ *exists and is finite for any λ with length n.*

Then there exists a set E in R^n which has the following properties:

(i) $\lim_{x\to x^0, x\in R^n-E} |x-x^0|^{-n}\left\{\int L_{n-1}(x, y)d\mu(y)-\sum_{|\lambda|=n}\frac{C_\lambda}{\lambda!}(x-x^0)^\lambda\right\}=0$;

(ii) $\lim_{j\to\infty} C(E'_j) = 0$,

where $C_\lambda=A_\lambda+aB_\lambda$ for $|\lambda|=n$ and B_λ will be defined later (*in Lemma* 4).

One may compare these theorems with fine and semi-fine differentiabilities of Riesz potentials investigated by Mizuta [6] and [7].

REMARK. If μ is a nonnegative measure on R^n with finite total mass, then (a) and (b) in Theorem 4 hold for almost every $x^0\in R^n$ (cf. [10; Chap. III, 4.1]).

We say that a set E in R^n is k-log semi-thin at x^0 if

$$\lim_{j\to\infty} k(2^{-j})C(E'_j) = 0.$$

The set E in Theorem 4 is k-log semi-thin at x^0 with $k\equiv 1$. The following theorem gives the behavior of logarithmic potentials in terms of k-log semi-thin sets.

THEOREM 5. *Let h be a nondecreasing and positive function on the interval* $(0, \infty)$ *such that* $\lim_{r\downarrow 0} h(r)=0$ *and*

$$\int_0^1 \frac{ds}{\tilde{h}(s)(r+s)} \leqq \frac{M}{h(r)} \qquad \textit{for} \quad r>0,$$

where M is a positive constant independent of r. Let m be a nonnegative integer and μ be a nonnegative measure on R^n satisfying (1) *and*

$$\lim_{r\downarrow 0} r^{-m}\tilde{h}(r)\mu(B(x^0, r)) = 0.$$

Then there exists a set E in R^n which is k-log semi-thin at x^0 and satisfies

$$\lim_{x\to x^0, x\in R^n - E} |x-x^0|^{-m}h(|x-x^0|)\int L_{m-1}(x, y)d\mu(y) = 0,$$

where $L_{-1}(x, y) = L(x-y)$.

In the final section we shall be concerned with the behavior at infinity of logarithmic potentials.

2. Proof of Theorem 1

We first prepare the following lemma, which will be used frequently.

LEMMA 1. *Let h be a positive Borel function on $(0, \infty)$ such that*

(5) $$h(s) \leqq Mh(r) \quad \textit{whenever} \quad 0 < r/2 \leqq s \leqq 2r \leqq 1,$$

where M is a positive constant independent of r and s. If μ is a nonnegative measure on R^n such that

$$\int \tilde{h}(|y|)d\mu(y) < \infty,$$

then there exists a set E in R^n which is k-log thin at 0 and satisfies

$$\lim_{x\to 0, x\in R^n - E} h(|x|)\int_{B(x,|x|/2)} \log\frac{|x|}{|x-y|}d\mu(y) = 0.$$

PROOF. Take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and $\sum_{j=1}^{\infty} a_j \int_{B_j} \tilde{h}(|y|)d\mu(y) < \infty$, where $B_j = B(0, 2^{-j+2}) - B(0, 2^{-j-1})$. Consider the sets

$$E_j = \left\{x\in A_j; \int_{B_j} \log\frac{|x|}{|x-y|}d\mu(y) \geqq h(2^{-j})^{-1}a_j^{-1}\right\}$$

for $j = 1, 2,\ldots$, and $E = \cup_{j=1}^{\infty} E_j$, where $A_j = B(0, 2^{-j+1}) - B(0, 2^{-j})$. By the assumption on h, one sees easily that

$$k(2^{-j})C(E_j') \leqq a_j h(2^{-j})k(2^{-j})\mu(B_j) \leqq \text{const.}\, a_j\int_{B_j}\tilde{h}(|y|)d\mu(y).$$

Hence E is k-log thin at 0. Furthermore,

$$\limsup_{x\to 0, x\in R^n - E} h(|x|)\int_{B(x,|x|/2)} \log\frac{|x|}{|x-y|}d\mu(y)$$

$$\leqq \text{const.} \limsup_{j\to\infty} \sup_{x\in A_j - E_j} h(2^{-j}) \int_{B_j} \log \frac{|x|}{|x-y|} d\mu(y)$$

$$\leqq \text{const.} \limsup_{j\to\infty} a_j^{-1} = 0,$$

and hence $\lim_{x\to 0, x\in R^n - E} h(|x|) \int_{B(x,|x|/2)} \log \frac{|x|}{|x-y|} d\mu(y) = 0.$

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. Without loss of generality, we may assume that x^0 is the origin 0. For a nonnegative measure μ on R^n satisfying (1), we write

$$L\mu(x) = \int_{\{y;|x-y|\geqq|x|/2\}} L(x-y)d\mu(y) + \int_{\{y;|x-y|<|x|/2\}} L(x-y)d\mu(y) = L'(x) + L''(x).$$

Note here that $L'(x)$ is finite for any $x \neq 0$. Let

$$\varepsilon(\delta) = \sup_{0<r\leqq\delta} h(r)(\log r^{-1})\mu(B(0, r)).$$

Then by our assumption, $\lim_{\delta\downarrow 0} \varepsilon(\delta)=0$. If $x, y \in B(0, 1/4)$ and $|x-y|\geqq|x|/2>0$, then

$$0 < L(x-y) \leqq \text{const.} \log \frac{1}{|x|+|y|}.$$

By (2), $\lim_{r\downarrow 0} h(r)=0$. Hence we have again by (2),

$$\limsup_{x\to 0} h(|x|)|L'(x)| = \limsup_{x\to 0} h(|x|) \int_{B(0,\delta)-B(x,|x|/2)} L(x-y)d\mu(y)$$

$$\leqq \text{const.} \limsup_{x\to 0} h(|x|) \int_{B(0,\delta)} \log \frac{1}{|x|+|y|} d\mu(y)$$

$$\leqq \text{const.} \limsup_{x\to 0} h(|x|)\Big\{\mu(B(0, \delta)) \log (|x|+\delta)^{-1} + \int_0^\delta \mu(B(0, r))(|x|+r)^{-1}dr\Big\} \leqq \text{const.}\ \varepsilon(\delta)$$

for δ, $0<\delta<1/4$. This implies that $\lim_{x\to 0} h(|x|)L'(x)=0$. Since $\lim_{x\to 0} h(|x|)\cdot(\log|x|)\mu(B(x, |x|/2))=0$, with the aid of Lemma 1 we can find a set E in R^n which is k-log thin at 0 and satisfies

$$\lim_{x\to 0, x\in R^n - E} h(|x|)L''(x) = 0.$$

3. Proofs of Theorems 2 and 3

Before giving proofs of Theorems 2 and 3, we recall the next result.

LEMMA 2 (cf. [9; Lemma 4]). *If* $x, y \in B(0, 1)$ *and* $|x-y| \geqq |x|/2 > 0$, *then*

$$|L_m(x, y)| \leqq \text{const.} \min\left(1, \frac{|x|}{|y|}\right) \times \begin{cases} \log\left(2+\frac{|x|}{|y|}\right) & \textit{when} \quad m=0, \\ |x|^m |y|^{-m} & \textit{when} \quad m \geqq 1. \end{cases}$$

We shall give only a proof of Theorem 3, since Theorem 2 can be proved similarly by the use of Lemma 2.

PROOF OF THEOREM 3. We may assume that $x^0 = 0$. Let μ be a nonnegative measure on R^n satisfying (1) and

$$\int H(|y|)k(|y|)d\mu(y) < \infty,$$

where $H(r) = r^{-m}h(r)$ for $r > 0$. By the assumptions on h, H satisfies condition (5) with h replaced by H. We write

$$\begin{aligned} \int L_m(x, y)d\mu(y) &= \int_{R^n - B(0,2|x|)} L_m(x, y)d\mu(y) \\ &\quad + \int_{B(0,2|x|)-B(x,|x|/2)} L_m(x, y)d\mu(y) + \int_{B(x,|x|/2)} L_m(x, y)d\mu(y) \\ &= L'(x) + L''(x) + L'''(x). \end{aligned}$$

If $y \in R^n - B(0, 2|x|)$, then Lemma 2 implies that

$$|L_m(x, y)| \leqq \text{const.} |x|^{m+1}|y|^{-m-1},$$

so that Lebesgue's dominated convergence theorem gives

$$\begin{aligned} &\limsup_{x\to 0} |x|^{-m}h(|x|)|L'(x)| \\ &\quad \leqq \text{const.} \limsup_{x\to 0} |x|h(|x|) \int_{R^n - B(0,2|x|)} |y|^{-m-1}d\mu(y) \\ &\quad = \text{const.} \limsup_{x\to 0} |x|h(|x|) \int_{B(0,1)-B(0,2|x|)} |y|^{-m-1}d\mu(y) = 0 \end{aligned}$$

since $\lim_{r\downarrow 0} rh(r) = 0$ and $rh(r) \leqq k(1)^{-1}sh(s)k(s)$ for $0 < r < s < 1$.

If $y \in B(0, 2|x|)$ and $|x-y| \geqq |x|/2 > 0$, then Lemma 2 implies that

$$|L_m(x, y)| \leqq \text{const.} |x|^m|y|^{-m}.$$

Hence we obtain

$$\begin{aligned} &\limsup_{x\to 0} |x|^{-m}h(|x|)|L''(x)| \\ &\quad \leqq \text{const.} \limsup_{x\to 0} h(|x|) \int_{B(0,2|x|)} |y|^{-m}d\mu(y) = 0 \end{aligned}$$

since $h(r) \leqq h(s) \leqq 2h(2s) \leqq 2k(1)^{-1}h(2s)k(2s)$ whenever $0<s<r<1/2$.

As to L''', we note that

$$|x|^{-m}h(|x|)|L'''(x)| \leqq \text{const. } H(|x|)\int_{B(x,|x|/2)} \log\frac{|x|}{|x-y|}d\mu(y) + \text{const.} \int_{B(x,|x|/2)} H(|y|)d\mu(y).$$

The second term of the right hand side tends to zero as $x \to 0$ by the assumption. In view of Lemma 1, the first term of the right hand side tends to zero as $x \to 0$, $x \in R^n - E$, where E is k-log thin at 0. Thus the proof is complete.

REMARK 1. Theorem 3 is best possible as to the size of the exceptional set. In fact, if h and $\tilde{h}$ are as in Theorem 3 and E is a subset of R^n which is k-log thin at x^0, then one can find a nonnegative measure μ on R^n with compact support such that

$$\int |x^0-y|^{-m}\tilde{h}(|x^0-y|)d\mu(y) < \infty$$

and

$$\lim_{x \to x^0, x \in E} |x-x^0|^{-m}h(|x-x^0|)\int L_m(x, y)d\mu(y) = \infty.$$

REMARK 2. Let μ be a nonnegative measure on R^n satisfying (1) and let h be as in Theorem 3. If $\int |x^0-y|^{-m}h(|x^0-y|)d\mu(y)<\infty$ and there exist M, $r_0>0$ such that

$$h(|x-x^0|)\mu(B(x, r)) \leqq Mr^m$$

for any $x \in B(x^0, r_0)$ and any r, $0<r \leqq |x-x^0|/2$, then E appeared in Theorem 3 can be taken to be an empty set and $L\mu$ is m times differentiable at x^0.

To prove this, assume that $x^0=0$. For the first assertion, in view of the proof of Theorem 3, it suffices to show that

$$\lim_{x \to 0} |x|^{-m}h(|x|)\int_{B(x,|x|/2)} \log\frac{|x|}{|x-y|}d\mu(y) = 0. \tag{6}$$

For $\delta>0$, set $\varepsilon(\delta)=\sup_{0<r \leqq \delta} r^{-m}h(r)\mu(B(0, r))$. If $0<\delta<|x|/2$, then

$$\begin{aligned} &|x|^{-m}h(|x|)\int_{B(x,|x|/2)} \log\frac{|x|}{|x-y|}d\mu(y) \\ &= |x|^{-m}h(|x|)\int_{B(x,\delta)} \log\frac{|x|}{|x-y|}d\mu(y) \\ &\quad + |x|^{-m}h(|x|)\int_{B(x,|x|/2)-B(x,\delta)} \log\frac{|x|}{|x-y|}d\mu(y) \end{aligned}$$

$$\leqq \text{const.} \left\{\left(\frac{\delta}{|x|}\right)^m \log\frac{|x|}{\delta} + |x|^{-m}h(|x|)\mu(B(0, 2|x|)) \log\frac{|x|}{\delta}\right\}$$

$$\leqq \text{const.} \left\{\left(\frac{\delta}{|x|}\right)^m + \varepsilon(2|x|)\right\} \log\frac{|x|}{\delta}.$$

Since $\lim_{x\to 0} \varepsilon(2|x|)=0$, for x sufficiently close to 0 we can choose $\delta>0$ so that

$$\log\frac{|x|}{\delta} = [\varepsilon(2|x|)+|x|]^{-1/2}.$$

Since $\lim_{x\to 0} (\delta/|x|)=0$, we derive (6).

To prove the second assertion, we first note that

$$\int |x-y|^{-m+1}d\mu(y) < \infty \qquad \text{for every} \quad x\in B(0, r_0),$$

and hence $L\mu$ is $m-1$ times differentiable at $x\in B(0, r_0)$ and

$$\left(\frac{\partial}{\partial x}\right)^\lambda L\mu(x) = \int\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](x-y)d\mu(y)$$

for any $x\in B(0, r_0)$ and any multi-index λ with $|\lambda|=m-1$. As in the proofs of Theorem 1 and Remark 4 in [6; Section 2], we can show that

$$\lim_{x\to 0} |x|^{-1}h(|x|)\{u_\lambda(x)-u_\lambda(0)-\textstyle\sum_{i=1}^n a_i x_i\} = 0,$$

where $x=(x_1,\dots, x_n)$ and $u_\lambda=(\partial/\partial x)^\lambda L\mu$ for a multi-index λ with length $m-1$ and $a_i=\int\left[\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y)d\mu(y)$. This implies that $L\mu$ is m times differentiable at 0.

4. Proof of Theorem 4

We first recall the following results.

LEMMA 3 (cf. [7; Lemma 1]). *Let μ be a nonnegative measure on R^n such that $\lim_{r\downarrow 0} r^{\alpha-n}\mu(B(0, r))=0$ for some real number α. Then the following statements hold:*

(i) *If $\beta < 0$, then $\lim_{r\downarrow 0} r^\beta \int_{B(0,r)} |y|^{\alpha-\beta-n}d\mu(y) = 0$;*

(ii) *If $n-\alpha+1>0$ and $\beta>0$, then*

$$\lim_{r\downarrow 0} r^\beta \int_{B(0,1)} (r+|y|)^{\alpha-\beta-n}d\mu(y) = 0.$$

LEMMA 4 (cf. [7; Lemma 4]). *Set $u(x)=\int_{B(x^0,1)} L(x-y)dy$. Then $u\in C^\infty(B(x^0, 1))$. Moreover, if λ is a multi-index with length n, then*

$$B_\lambda \equiv \left[\left(\frac{\partial}{\partial x}\right)^\lambda u\right](x^0) = \int_{\partial B(0,1)} y^{\lambda'}\left[\left(\frac{\partial}{\partial x}\right)^{\lambda''} L\right](y)dS(y),$$

where $\lambda=\lambda'+\lambda''$ *and* $|\lambda'|=1$.

Now we prove Theorem 4 by assuming that $x^0=0$. Let μ be a nonnegative measure on R^n satisfying (1), (a) and (b) with $x^0=0$. For $x\in B(0, 1/2)-\{0\}$, we write

$$|x|^{-n}\left\{\int L_{n-1}(x, y)d\mu(y) - \sum_{|\lambda|=n}\frac{C_\lambda}{\lambda!}x^\lambda\right\}$$
$$= |x|^{-n}\int_{R^n-B(0,1)} L_n(x, y)d\mu(y)$$
$$+ |x|^{-n}\int_{B(0,1)-B(0,2|x|)} L_n(x, y)d[\mu-aA_n](y)$$
$$- |x|^{-n}\sum_{0<|\lambda|\leqq n}\frac{x^\lambda}{\lambda!}\lim_{r\downarrow 0}\int_{B(0,2|x|)-B(0,r)}\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y)d[\mu-aA_n](y)$$
$$+ a|x|^{-n}\left\{\lim_{r\downarrow 0}\int_{B(0,1)-B(0,r)} L_n(x, y)dy - \sum_{|\lambda|=n}\frac{B_\lambda}{\lambda!}x^\lambda\right\}$$
$$+ |x|^{-n}\int_{B(0,2|x|)-B(x,|x|/2)} L_0(x, y)d[\mu-aA_n](y)$$
$$+ |x|^{-n}\int_{B(x,|x|/2)} L_0(x, y)d[\mu-aA_n](y)$$
$$= u_1(x) + u_2(x) - u_3(x) + au_4(x) + u_5(x) + u_6(x).$$

If $y\in R^n-B(0, 2|x|)$, then $|L_n(x, y)|\leqq$const. $|x|^{n+1}|y|^{-n-1}$ and hence

$$\lim_{x\to 0} u_1(x) = 0.$$

For simplicity, set $\nu=|\mu-aA_n|$. Then $\lim_{r\downarrow 0} r^{-n}\nu(B(0, r))=0$ by (a), and we have

$$\limsup_{x\to 0}|u_2(x)|\leqq \text{const.} \limsup_{x\to 0}|x|\int_{B(0,1)}(|x|+|y|)^{-n-1}d\nu(y) = 0$$

because of Lemma 3, (ii).

If $0<|\lambda|<n$, then Lemma 3, (i) yields

$$\limsup_{x\to 0}|x|^{|\lambda|-n}\int_{B(0,2|x|)}\left|\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y)\right|d\nu(y)$$
$$\leqq \text{const.} \limsup_{x\to 0}|x|^{|\lambda|-n}\int_{B(0,2|x|)}|y|^{-|\lambda|}d\nu(y) = 0.$$

If $|\lambda|=n$, then, by [5; Lemma 3.1], $\int_{B(0,r)-B(0,s)}\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y)dy=0$ for any

$r, s>0$. Hence by the definition of A_λ,

$$\lim_{x\to 0}\left\{\lim_{r\downarrow 0}\int_{B(0,2|x|)-B(0,r)}\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y)d[\mu-aA_n](y)\right\} = 0.$$

Therefore, $\lim_{x\to 0} u_3(x)=0$.

Since $u(x)\equiv\int_{B(0,1)} L(x-y)dy\in C^\infty(B(0,1))$ and

$$u_4(x) = |x|^{-n}\left\{u(x) - \sum_{|\lambda|\leqq n}\frac{x^\lambda}{\lambda!}\left[\left(\frac{\partial}{\partial x}\right)^\lambda u\right](0)\right\}$$

in view of Lemma 4, we see that $\lim_{x\to 0} u_4(x)=0$.

As to u_5, we obtain

$$|u_5(x)| \leqq \text{const.}\ |x|^{-n}\int_{B(0,2|x|)}\log\left(2+\frac{|x|}{|y|}\right)dv(y)$$

$$\leqq \text{const.}\ |x|^{1-n}\int_{B(0,2|x|)}|y|^{-1}dv(y),$$

which tends to zero as $x\to 0$ by Lemma 1, (i).

Applying the following Lemma 5 with $h(r)=r^{-n}$ and $k(r)=1$, we see that $u_6(x)$ tends to zero as $x\to 0$, $x\in R^n-E$, where E is a set in R^n satisfying (ii) of the theorem. The proof of Theorem 4 is now complete.

LEMMA 5. *Let h be a positive function on* $(0, \infty)$, *and define* $b_j=\sup\{h(r);\ 2^{-j}\leqq r<2^{-j+1}\}$. *If v is a nonnegative measure on R^n such that* $\lim_{j\to\infty} b_j k(2^{-j})v(B(0, 2^{-j+2})-B(0, 2^{-j-1}))=0$, *then there exists a set E in R^n which is k-log semi-thin at* 0 *and satisfies*

$$\lim_{x\to 0, x\in R^n-E} h(|x|)\int_{B(x,|x|/2)}\log\frac{|x|}{|x-y|}dv(y) = 0.$$

The proof is similar to that of Lemma 1.

REMARK 1. If $\lim_{j\to\infty} C(E_j')=0$, then we can find a nonnegative measure μ on R^n with compact support such that $\lim_{r\downarrow 0} r^{-n}\mu(B(0, r))=0$ and

$$\lim_{x\to 0, x\in E}|x|^{-n}\int L_{n-1}(x, y)d\mu(y) = \infty.$$

REMARK 2. Let μ be a nonnegative measure on R^n satisfying (1), (a), (b) and

(c) There exist $M, r_0>0$ such that $\mu(B(x, r))\leqq Mr^n$ for any $x\in B(x^0, r_0)$ and any $r\leqq r_0$.

Then the set E in Theorem 4 can be taken to be empty and, moreover, $L\mu$ is n

times differentiable at x^0.

This fact can be proved in the same way as in Remark 2 in Section 3.

5. Proof of Theorem 5

As before we assume that $x^0=0$. Let μ be a nonnegative measure on R^n satisfying (1) and

$$\lim_{r\downarrow 0} r^{-m}\tilde{h}(r)\mu(B(0, r)) = 0. \tag{7}$$

Define $\varepsilon(\delta)=\sup_{0<r\leqq\delta} r^{-m}\tilde{h}(r)\mu(B(0, r))$. By (7), $\lim_{\delta\downarrow 0}\varepsilon(\delta)=0$.

If $m=0$, then

$$\int_0^\delta \frac{\mu(B(0, s))}{r+s}\,ds \leqq M\varepsilon(\delta)[h(r)]^{-1}$$

whenever $0<r<\delta<1$, on account of the assumptions on h and $\tilde{h}$. Since $\int_r^\delta \mu(B(0, s))s^{-1}ds \geqq \mu(B(0, r))\log(\delta/r)$, it follows that $\limsup_{r\downarrow 0} h(r)(\log r^{-1})\mu(B(0, r)) \leqq M\varepsilon(\delta)$. Thus

$$\lim_{r\downarrow 0} h(r)(\log r^{-1})\mu(B(0, r)) = 0. \tag{8}$$

Then the case $m=0$ can be proved in the same way as in Theorem 1 by using Lemma 5 in place of Lemma 1.

Let $m\geqq 1$, and write

$$\int L_{m-1}(x, y)d\mu(y) = \int_{B(x,|x|/2)} L_{m-1}(x, y)d\mu(y) + \int_{R^n-B(x,|x|/2)} L_{m-1}(x, y)d\mu(y) = L'(x) + L''(x).$$

Since $\lim_{r\downarrow 0} r^{-m}h(2r)k(r)\mu(B(0, 4r))=0$ by (7), Lemma 5 implies that $|x|^{-m}\cdot h(|x|)L'(x)$ tends to zero as $x\to 0$ except for x in a set which is k-log semi-thin at 0. What remains is to prove that $|x|^{-m}h(|x|)L''(x)$ tends to zero as $x\to 0$. For this we deal only with the case $m=1$, because the case $m\geqq 2$ can be proved similarly.

Let $m=1$. By Lemma 2,

$$\begin{aligned}|x|^{-1}h(|x|)|L''(x)| &\leqq |x|^{-1}h(|x|)\int_{B(0,2|x|)} \log\left(2+\frac{|x|}{|y|}\right)d\mu(y)\\ &\quad + \text{const.}\, h(|x|)\int_{R^n-B(0,2|x|)} |y|^{-1}d\mu(y)\\ &= I_1(x) + \text{const.}\, I_2(x).\end{aligned}$$

Note that

$$I_1(x) \leqq \text{const.}\, |x|^{-1}h(|x|)\left\{\mu(B(0, 2|x|)) + \int_0^{2|x|} \mu(B(0, s))s^{-1}ds\right\}$$

$$\leqq \text{const.}\, |x|^{-1}h(|x|)\left\{\mu(B(0, 2|x|)) + \varepsilon(\delta)\int_0^{2|x|} \frac{ds}{\tilde{h}(s)}\right\}$$

$$\leqq \text{const.}\left\{|x|^{-1}h(|x|)\mu(B(0, 2|x|)) + \varepsilon(\delta)h(|x|)\int_0^{2|x|} \frac{ds}{\tilde{h}(s)(|x|+s)}\right\}$$

$$\leqq \text{const.}\, \{|2x|^{-1}\tilde{h}(2|x|)\mu(B(0, 2|x|)) + M\varepsilon(\delta)\}$$

whenever $0<2|x|<\delta$. Similarly,

$$I_2(x) \leqq h(|x|)\left\{\int_{R^n-B(0,\delta)} |y|^{-1}d\mu(y) + \delta^{-1}\mu(B(0, \delta)) + \varepsilon(\delta)\int_{2|x|}^{\delta} \frac{ds}{\tilde{h}(s)s}\right\}$$

$$\leqq h(|x|)\left\{\int_{R^n-B(0,\delta)} |y|^{-1}d\mu(y) + \delta^{-1}\mu(B(0, \delta))\right\} + 2M\varepsilon(\delta).$$

These yield that $\lim_{x\to 0} |x|^{-1}h(|x|)L''(x)=0$. Thus we conclude the proof of Theorem 5.

REMARK. The set E in Theorem 5 can be taken to satisfy

$$\text{(9)} \qquad \lim_{i\to\infty} H(2^{-i})k(2^{-i}) \sum_{j=i}^{\infty} \frac{C(E'_j)}{H(2^{-j+1})} = 0,$$

where $H(r)=r^{-m}h(r)$. In fact, take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j=\infty$, $\lim_{j\to\infty} a_jH(2^{-j+1})k(2^{-j+1})\mu(B(0, 2^{-j+2}))=0$ and

$$\sum_{j=i}^{\infty} a_j\mu(B_j) \leqq 2a_i \sum_{j=i}^{\infty} \mu(B_j) \qquad \text{for each} \quad i,$$

where B_j are defined as in the proof of Lemma 1; this is possible as will be shown in the Appendix. As in the proof of Lemma 1, define E_j with h replaced by H and $E=\bigcup_{j=1}^{\infty} E_j$. It is easy to see that E satisfies (9).

The next proposition shows that (9) gives a best possible condition as to the size of E, in case $H(2r)\leqq$const. $H(r)$.

PROPOSITION 1. *Let h be as in Theorem 5 and define H as above. If a set E in R^n satisfies* (9), *then there exists a nonnegative measure μ on R^n satisfying* (1), $\lim_{r\downarrow 0} H(r)k(r)\mu(B(0, r))=0$ *and*

$$\lim_{x\to 0, x\in E} H(2|x|)\int L_{m-1}(x, y)d\mu(y) = \infty.$$

PROOF. We assume that $C(E'_j)>0$ for each j. By definition of $C(\cdot)$, for each j we can find a nonnegative measure μ_j such that $\mu_j(R^n-B(0, 2^{-j+2}))=0$, $\mu_j(B(0, 2^{-j+2}))<2C(E'_j)$ and

$$\int \log \frac{2^{-j+3}}{|x-y|} d\mu_j(y) \geqq 1 \qquad \text{for every} \quad x \in E_j,$$

where $E_j = E \cap B(0, 2^{-j+1}) - B(0, 2^{-j})$. Take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$,

$$\lim_{i\to\infty} a_i H(2^{-i}) k(2^{-i}) \sum_{j=i}^{\infty} \frac{C(E'_j)}{H(2^{-j+1})} = 0$$

and

$$\sum_{j=i}^{\infty} a_j \frac{C(E'_j)}{H(2^{-j+1})} \leqq 2a_i \sum_{j=i}^{\infty} \frac{C(E'_j)}{H(2^{-j+1})} \qquad \text{for each } i$$

(see Lemma 6 in Appendix). Denote by μ'_j the restriction of μ_j to the set $B_j = B(0, 2^{-j+2}) - B(0, 2^{-j-1})$, and define a nonnegative measure μ by

$$\mu = \sum_{j=1}^{\infty} \frac{a_j}{H(2^{-j+1})} \mu'_j.$$

Let i be a positive integer. Then we see that

$$\begin{aligned} H(2^{-i})k(2^{-i})\mu(B(0, 2^{-i})) &\leqq H(2^{-i})k(2^{-i}) \sum_{j=i}^{\infty} \frac{a_j}{H(2^{-j+1})} \mu'_j(B_j) \\ &\leqq 4a_i H(2^{-i})k(2^{-i}) \sum_{j=i}^{\infty} \frac{C(E'_j)}{H(2^{-j+1})} \\ &\longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty, \end{aligned}$$

so that $\lim_{r\downarrow 0} H(r)k(r)\mu(B(0, r)) = 0$.

On the other hand, if $x \in E_j$, then

$$\begin{aligned} &H(2|x|) \int_{B_j} \log \frac{2^{-j+3}}{|x-y|} d\mu(y) \\ &\geqq 2^{-m} a_j \{1 - 4(\log 2)\mu_j(B(0, 2^{-j-1}))\} \longrightarrow \infty \quad \text{as} \quad j \longrightarrow \infty. \end{aligned}$$

Since $\lim_{r\downarrow 0} H(r)\mu(B(0, r)) = 0$,

$$\lim_{j\to\infty} \sup_{x\in E_j} H(2|x|) \int_{B_j - B(x,|x|/2)} \log \frac{|y|}{|x-y|} d\mu(y) = 0$$

and

$$\lim_{j\to\infty} \sup_{x\in E_j} H(2|x|) \sum_{1\leqq|\lambda|\leqq m-1} \int_{B(x,|x|/2)} \frac{x^\lambda}{\lambda!} \left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y) d\mu(y) = 0.$$

Hence it follows that

$$\lim_{x\to 0, x\in E} H(2|x|) \int_{B(x,|x|/2)} L_{m-1}(x, y) d\mu(y) = \infty$$

in case $m \geqq 1$. This also holds in case $m=0$ on account of (8). Noting that $h(2r)$ satisfies all the conditions on h in Theorem 5, we derive

$$\lim_{x \to 0} H(2|x|) \int_{R^n - B(x,|x|/2)} L_{m-1}(x, y) d\mu(y) = 0,$$

in view of the proof of Theorem 5. Thus $\lim_{x \to 0, x \in E} H(2|x|) \int L_{m-1}(x, y) d\mu(y) = \infty$.

By Theorem 5 we can establish the following result.

PROPOSITION 2. *Let h be as in Theorem 5, and μ be a nonnegative measure on R^n satisfying* (1). *Then the following statements are equivalent*:

(i) *There exists a set E in R^n which is logarithmically semi-thin at x^0 and satisfies*

$$\lim_{x \to x^0, x \in R^n - E} h(|x - x^0|) L\mu(x) = 0.$$

(ii) *There exists a sequence $\{x^{(j)}\}$ in R^n such that $x^{(j)} \to x^0$ as $j \to \infty$,* $\{|x^{(j)} - x^0| / |x^{(j+1)} - x^0|\}$ is bounded and

$$\lim_{j \to \infty} h(|x^{(j)} - x^0|) L\mu(x^{(j)}) = 0.$$

(iii) $\lim_{r \downarrow 0} h(r)(\log r^{-1}) \mu(B(x^0, r)) = 0$.

PROOF. Without loss of generality, we may assume that $x^0 = 0$. The implication (iii)$\to$(i) follows readily from Theorem 5.

(i)$\to$(ii): Let $B = B(0, 1)$. Then $B'_j = B(0, 2) - B(0, 1)$ and $\lim_{j \to \infty} jC(B'_j - E'_j) = \infty$. Hence we can find a sequence $\{x^{(j)}\}$ such that $x^{(j)} \in B(0, 2^{-j+1}) - B(0, 2^{-j}) - E$ for large j. This sequence satisfies all the conditions in (ii).

(ii)$\to$(iii): Let $\{x^{(j)}\}$ be a sequence in (ii). Then one notes that

$$\begin{aligned}
& h(|x^{(j)}|)\left(\log \frac{1}{|x^{(j)}|}\right) \mu(B(0, |x^{(j)}|)) \\
& \leqq h(|x^{(j)}|) \int_{B(0,|x^{(j)}|)} \log \frac{2}{|x^{(j)} - y|} d\mu(y) \\
& \leqq h(|x^{(j)}|) \int_{B(0,1)} \log \frac{2}{|x^{(j)} - y|} d\mu(y) \\
& \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty.
\end{aligned}$$

Take $M > 1$ such that $|x^{(j)}| \leqq M |x^{(j+1)}|$ for each j. Then $(0, M|x^{(1)}|] \subset \bigcup_{j=1}^{\infty} [M^{-1}|x^{(j)}|, M|x^{(j)}|]$. If $M^{-1}|x^{(j)}| \leqq r \leqq M|x^{(j)}| < 1$, then

$$\begin{aligned}
& h(M^{-1}r)(\log M r^{-1}) \mu(B(0, M^{-1}r)) \\
& \qquad \leqq \text{const.}\, h(|x^{(j)}|)\left(\log \frac{1}{|x^{(j)}|}\right) \mu(B(0, |x^{(j)}|)) \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty,
\end{aligned}$$

from which (iii) follows readily. The proof in now complete.

For similar results on semi-fine limits of Riesz potentials, see Mizuta [8; Theorems 2 and 2′].

REMARK. Let $\tilde{h}(r)$ be nonincreasing on the interval (0, 1) and define

$$E = \{x \in R^n;\ \limsup_{r\downarrow 0} \tilde{h}(r)\mu(B(x, r)) > 0\}$$

for a nonnegative measure μ on R^n. If $\mu(E)=0$, then $\Lambda_{\tilde{h}^{-1}}(E)=0$, where $\Lambda_{\tilde{h}^{-1}}$ denotes the Hausdorff measure with respect to the measure function $\tilde{h}^{-1}$; in particular, if μ is absolutely continuous with respect to the n-dimensional Lebesgue measure and $\lim_{r\downarrow 0} r^n\tilde{h}(r)=0$, then $\Lambda_{\tilde{h}^{-1}}(E)=0$.

5. Logarithmic potentials of functions in L^p

For a nonnegative measurable function f on R^n such that

$$\int [\log(1+|y|)]f(y)dy < \infty, \tag{10}$$

we define

$$Lf(x) = \int L(x-y)f(y)dy.$$

If in addition $f \in L^p(R^n)$, $p>1$, then Lf is continuous on R^n.

PROPOSITION3 . *Let m be a positive integer smaller than n, and f be a nonnegative function in $L^p(R^n)$ satisfying* (10). *Then there exists a set E in R^n such that $B_{n-m,p}(E)=0$ and for any $x^0 \in R^n - E$,*

$$\lim_{x\to x^0} |x-x^0|^{-m} \int L_m(x, y)f(y)dy = 0. \tag{11}$$

Here $B_{\alpha,p}$ denotes the Bessel capacity of index (α, p) (see [4]).

PROOF OF PROPOSITION 3. Consider the sets

$$E_1 = \left\{x;\ \int |x-y|^{-m}f(y)dy=\infty\right\},$$

$$E_2 = \left\{x;\ \limsup_{r\downarrow 0} r^{(n-m)p-n}\int_{B(x,r)} f(y)^p dy > 0\right\}.$$

Then, in view of [4; Theorem 21], $B_{n-m,p}(E_1 \cup E_2)=0$. We have only to show that for $x^0 \in R^n - E_1 - E_2$,

$$\lim_{x\to x^0} |x-x^0|^{-m} \int_{B(x,|x-x^0|/2)} \log\frac{|x-x^0|}{|x-y|} f(y)dy = 0$$

(see the proof of Remark 2 in Section 3). For this, without loss of generality, we may assume that $x^0=0$. By Hölder's inequality,

$$
\begin{aligned}
&|x|^{-m}\int_{B(x,|x|/2)} \log\frac{|x|}{|x-y|} f(y)dy \\
&\leqq |x|^{-m}\left\{\int_{B(x,|x|/2)}\left(\log\frac{|x|}{|x-y|}\right)^{p'} dy\right\}^{1/p'}\left\{\int_{B(x,|x|/2)} f(y)^p dy\right\}^{1/p} \\
&\leqq \text{const.}\left\{|x|^{(n-m)p-n}\int_{B(0,2|x|)} f(y)^p dy\right\}^{1/p},
\end{aligned}
$$

which tends to zero as $x\to 0$, where $1/p+1/p'=1$.

In the same way we can prove the next result (see also [7; Theorem 3 and its corollary]).

PROPOSITION 4. *If f is as above, then* (11) *with $m=n$ holds for almost every $x^0\in R^n$.*

6. Fine limits at infinity of logarithmic potentials

We say that a set E in R^n is logarithmically thin at infinity if $E^*=\{x/|x|^2;\ x\in E\}$ is logarithmically thin at 0. Then it is easy to see that E is logarithmically thin at infinity if and only if

$$\textstyle\sum_{j=1}^{\infty} jC(E'_{-j}) < \infty, \quad E'_{-j} = \{x\in B(0,2) - B(0,1);\ 2^j x\in E\}.$$

By inversion we can establish the next result.

THEOREM A′. *Let μ be a nonnegative measure on R^n satisfying* (1). *Then the following statements hold*:

(i) *There exists a set E in R^n which is logarithmically thin at infinity and satisfies*

$$\lim_{|x|\to\infty,\, x\in R^n-E}\,[L\mu(x)+\mu(R^n)\log|x|] = 0.$$

(ii) *There exists a set E in R^n which is logarithmically thin at infinity and satisfies*

$$\lim_{|x|\to\infty,\, x\in R^n-E}\,[\log|x|]^{-1}\int \tilde{L}_0(x, y)d\mu(y) = -\mu(R^n),$$

where $\tilde{L}_0(x, y)=L(x-y)$ if $|y|\leqq 1$ and $\tilde{L}_0(x, y)=L(x-y)-L(y)$ if $|y|>1$.

REMARK 1. Let μ be a nonnegative measure on R^n. Then $\int|\tilde{L}_0(x, y)|d\mu(y)<\infty$ for almost every x if and only if $\int(1+|y|)^{-1}d\mu(y)<\infty$ on account of [9; Lemma 4].

REMARK 2. Let μ be a nonnegative measure on R^n such that $L\mu(0)$ is finite, and define μ^* by setting $\mu^*(A^*)=\mu(A)$ for $A\subset R^n$, where $A^*=\{x^*=x/|x|^2;\ x\in A\}$. Then

$$L\mu^*(x^*) = L\mu(x) + \mu(R^n)\log|x| - L\mu(0).$$

We say that a set E in R^n is logarithmically semi-thin at infinity if $\lim_{j\to\infty} jC(E'_{-j})=0$. By Proposition 2 we have the following result.

PROPOSITION 2′. *Let h be a nonincreasing and positive function on* $(0, \infty)$ *such that*

$$\int_2^\infty \frac{dt}{h(t)(\log t)t(t+r)} \leqq \text{const.}\,\frac{1}{rh(r)} \qquad \textit{for any} \quad r>1.$$

Let μ be a nonnegative measure on R^n with finite total mass. Then the following statements are equivalent:

(i) *There exists a set E in R^n which is logarithmically semi-thin at infinity such that*

$$\lim_{|x|\to\infty, x\in R^n-E} h(|x|)\left\{\int \tilde{L}_0(x, y)d\mu(y)+\mu(R^n)\log|x|\right\} = 0.$$

(ii) *There exists a sequence $\{x^{(j)}\}$ in R^n such that* $\lim_{j\to\infty}|x^{(j)}|=\infty$, $\{|x^{(j+1)}|/|x^{(j)}|\}$ *is bounded and*

$$\lim_{j\to\infty} h(|x^{(j)}|)\left\{\int \tilde{L}_0(x^{(j)}, y)d\mu(y)+\mu(R^n)\log|x^{(j)}|\right\} = 0.$$

(iii) $\lim_{r\downarrow 0} h(r)(\log r)\mu(R^n-B(0, r)) = 0.$

Theorems 1 and 2 can be reformulated similarly; but we do not go into detail.

Finally, corresponding to Theorems 3 and 5, we give generalizations of Theorems 1 and 2 in [9].

THEOREM 3′. *Let h and k^* be nondecreasing positive functions on* $(0, \infty)$ *such that*

(a) $r^{-1}h(r)$ *is nonincreasing on* $(0, \infty)$ *and* $\lim_{r\to\infty} r^{-1}h(r)=0$;

(b) $k^*(2r)\leqq Mk^*(r)$ *for* $r>0$;

(c) $\dfrac{s}{r}\log\dfrac{r}{s} \leqq M\dfrac{\tilde{h}(r)}{h(r)}$ *whenever* $0<s<r$,

where $\tilde{h}=hk^*$ and M is a positive constant independent of r and s. Let μ be a nonnegative measure on R^n satisfying

$$\int |y|^{-m-1}\tilde{h}(|y|)d\mu(y) < \infty$$

for a nonnegative integer m. Then there exists a set E in R^n having the following properties:

(i) $\lim_{|x|\to\infty, x\in R^n-E} |x|^{-m-1}h(|x|)\int \tilde{L}_m(x, y)d\mu(y) = 0$;

(ii) $\sum_{j=1}^{\infty} k^*(2^j)C(E'_{-j}) < \infty$.

Here $\tilde{L}_m(x, y) = L(x-y)$ if $|y|<1$ and $\tilde{L}_m(x, y) = L(x-y) - \sum_{|\lambda|\leq m} \frac{x^\lambda}{\lambda!}\left[\left(\frac{\partial}{\partial x}\right)^\lambda L\right](-y)$ *if* $|y|\geq 1$.

THEOREM 5′. *Let h and k^* be as above. Assume further that*

(d) $\int_1^\infty \frac{dt}{\tilde{h}(t)(t+r)} \leq \frac{M}{h(r)}$ *for* $r>1$,

where M is a positive constant independent of r. If μ is a nonnegative measure on R^n satisfying $\lim_{r\downarrow 0} r^{-m-1}\tilde{h}(r)\mu(B(0, r))=0$ for a nonnegative integer m, then there exists a set E in R^n having (i) *of Theorem* 3′ *and*

(ii)′ $\lim_{j\to\infty} k^*(2^j)C(E'_{-j})=0$.

Appendix

Here we prove the next elementary fact.

LEMMA 6. *Let $\{b_j\}$, $\{c_j\}$ be sequences of positive numbers such that $\lim_{j\to\infty} b_j=\infty$ and $\sum_{j=1}^{\infty} c_j<\infty$. Then there exists a sequence $\{a_j\}$ of positive numbers such that $a_j\leq b_j$ for each j, $\lim_{j\to\infty} a_j=\infty$ and*

$$\sum_{j=k}^{\infty} a_j c_j \leq 2a_k \sum_{j=k}^{\infty} c_j \qquad \textit{for each} \quad k.$$

PROOF. We may assume that $b_j\leq b_{j+1}\leq pb_j$ for each j, where $1<p<2$. For given $q>0$ we can find a sequence $\{k_i\}$ of nonnegative integers such that $k_0=0$, $k_1=1$, $k_i<k_{i+1}$ for $i=1, 2,\dots$ and

$$\sum_{j=k_{i+1}+1}^{\infty} c_j \leq q \sum_{j=k_i+1}^{k_{i+1}} c_j \qquad \text{for} \quad i = 1, 2,\dots.$$

Define $a_j=b_i$ if $k_i<j\leq k_{i+1}$. For $k_i<k\leq k_{i+1}$ we have

$$\begin{aligned}\sum_{j=k}^{\infty} a_j c_j &= \sum_{j=k}^{k_{i+1}} a_j c_j + \sum_{\ell=i}^{\infty} \left(\sum_{j=k_{\ell+1}+1}^{k_{\ell+2}} a_j c_j\right) \\ &= b_i \sum_{j=k}^{k_{i+1}} c_j + \sum_{\ell=i}^{\infty} \left(b_{\ell+1} \sum_{j=k_{\ell+1}+1}^{k_{\ell+2}} c_j\right) \\ &\leq b_i \sum_{j=k}^{k_{i+1}} c_j + \left(\sum_{\ell=i}^{\infty} (pq)^{\ell-i}\right) b_{i+1} \sum_{j=k_{i+1}+1}^{k_{i+2}} c_j \\ &\leq \frac{p}{1-pq} b_i \sum_{j=k}^{\infty} c_j = \frac{p}{1-pq} a_k \sum_{j=k}^{\infty} c_j,\end{aligned}$$

if $pq<1$. Hence if q is chosen sufficiently small, then $\{a_j\}$ satisfies all the conditions in the lemma.

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