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Pseudo-coalescent classes of groups

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Introduction

Since Wielandt's fundamental paper [19] appeared, subnormality and coalescence in groups have been studied by many group-theorists, Baer, Gruenberg, Lennox, Robinson, Roseblade, Stonehewer and others in the last twenty years. In [18] Tôgô introduced two concepts of weak subnormality and weak ascendancy which are generalizations of subnormality and ascendancy. By using these concepts, we shall introduce the concepts of pseudo-coalescence, ascendant pseudo-coalescence, local pseudo-coalescence and locally ascendant pseudo-coalescence which are corresponding to coalescence, ascendant coalescence, local coalescence and locally ascendant coalescence respectively, and also the analogies of those in Lie algebras. We call a class $\mathfrak{X}$ of groups pseudo-coalescent (resp. ascendantly pseudo-coalescent) if in any group the join of any pair of a subnormal (resp. an ascendant) $\mathfrak{X}$-subgroup and a weakly subnormal (resp. a weakly ascendant) $\mathfrak{X}$-subgroup is always a weakly subnormal (resp. a weakly ascendant) $\mathfrak{X}$-subgroup. We also call a class $\mathfrak{X}$ of groups locally pseudo-coalescent (resp. locally ascendantly pseudo-coalescent) if whenever H is a subnormal (resp. an ascendant) $\mathfrak{X}$-subgroup and K is a weakly subnormal (resp. a weakly ascendant) $\mathfrak{X}$-subgroup of a group G every finite subset F of $J=\langle H, K\rangle$ is contained in some weakly subnormal (resp. weakly ascendant) $\mathfrak{X}$-subgroup X of G with $X \leq J$. Among the known coalescent classes are those of finite groups (Wielandt [19]), groups satisfying the maximal condition for subgroups (Baer [3]), finitely generated nilpotent groups (Baer [2]), groups satisfying the minimal condition for subnormal subgroups (Robinson [9] and Roseblade [12]), groups satisfying the maximal condition for subnormal subgroups (Roseblade [13]) and any subjunctive class of finitely generated groups (Roseblade and Stonehewer [14]). It is known that the following classes are ascendantly coalescent: $\{\mathrm{Q}, \mathrm{E}\}$-closed classes of groups satisfying the minimal condition for subnormal subgroups (Robinson [9]), $\{\mathrm{N}_0, \mathrm{S}\}$-closed classes of groups satisfying the maximal condition for subgroups (cf. Robinson [10]) and certain classes of finitely generated groups, e.g., the classes of finitely generated groups satisfying the maximal condition for ascendant subgroups (Hulse [5]). Among the known locally coalescent classes are the class of nilpotent groups (Baer [2]) and any subjunctive class (Roseblade and Stonehewer [14]). However, very little is known concerning locally ascendant coalescence. The purpose of this paper is to investigate the properties of weakly subnormal subgroups and weakly ascendant subgroups, and to show that several classes of groups are pseudo-coalescent, locally pseudo-coalescent, ascendantly pseudo-coalescent or locally ascendantly pseudo-coalescent. We shall also obtain many results corresponding to these for Lie algebras [6, 7, 15, 16].

In Section 2, we shall ask whether the join of a pair of a subnormal (resp. an ascendant) subgroup and a weakly subnormal (resp. a weakly ascendant) subgroup of a group is a weakly subnormal (resp. a weakly ascendant) subgroup. If H is an ascendant subgroup, K is a finitely generated weakly ascendant subgroup of a group and K normalizes H, then the join $\langle H, K\rangle$ is a weakly ascendant subgroup (Theorem 2.4). In Section 3, we shall investigate the concepts of pseudo-coalescence and ascendant pseudo-coalescence. Many of the known coalescent and ascendantly coalescent classes are pseudo-coalescent and ascendantly pseudo-coalescent (Theorem 3.3). In Section 4, we shall show that (1) corresponding to [7, Theorem 4.3] and [16, Theorem 3.2], for any classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{X}\leq\mathfrak{Y}\leq \mathrm{L}(\mathrm{sn})\mathfrak{X}$, $\mathfrak{X}$ is locally pseudo-coalescent (locally coalescent) if and only if so is $\mathfrak{Y}$, and (2) corresponding to [16, Theorem 4.2], for any classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{X}\leq\mathfrak{Y}\leq \mathrm{L}(\mathrm{asc})\mathfrak{X}$, $\mathfrak{X}$ is locally ascendantly pseudo-coalescent (locally ascendantly coalescent) if and only if so is $\mathfrak{Y}$ (Theorem 4.4). From the preceding results, we see that several known classes are locally coalescent or locally ascendantly coalescent (Theorem 4.5). In Section 5, we shall consider analogies of these for Lie algebras and obtain locally ascendantly pseudo-coalescent classes over fields of characteristic zero.

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1.

Throughout the paper we always consider not necessarily finite groups, and we denote by $\mathfrak{X}$ an arbitrary class of groups, unless otherwise specified.

Let G be a group and let X, Y be non-empty subsets of G; $\langle X\rangle$ is the subgroup generated by X; X^Y is the set of all the conjugates $x^y=y^{-1}xy$ $(x\in X, y\in Y)$. $[X, Y]_{\mathrm{set}}$ is the set of all the commutators $[x, y]=x^{-1}y^{-1}xy$ $(x\in X, y\in Y)$, $[X, {}_0Y]_{\mathrm{set}}=X$ and $[X, {}_{n+1}Y]_{\mathrm{set}}=[[X, {}_nY]_{\mathrm{set}}, Y]_{\mathrm{set}}$ $(n\geq 0)$. $[X, Y]=\langle [x, y] \mid x\in X, y\in Y\rangle$, $[X, {}_0Y]=\langle X\rangle$ and $[X, {}_{n+1}Y]=[[X, {}_nY], Y]$ $(n\geq 0)$.

We write $H\leq G$ if H is a subgroup of G. If a subgroup H of G is respectively normal, n-step subnormal, subnormal, ρ-step ascendant and ascendant in G, then we write $H\lhd G$, $H\lhd^n G$, H sn G, $H\lhd^\rho G$ and H asc G, where n is a non-negative integer and ρ is an ordinal.

For any ordinal λ, a subgroup H of G is said to be a λ-step weakly ascendant subgroup of G, denoted by $H\leq^\lambda G$, if there is an ascending series $(S_\alpha)_{\alpha\leq\lambda}$ of subsets of G such that

(1) $S_0=H$ and $S_\lambda=G$,

(2) $H^{S_{\alpha+1}}\subseteq S_\alpha$ for any ordinal $\alpha<\lambda$,

(3) $S_\beta=\bigcup_{\alpha<\beta}S_\alpha$ for any limit ordinal $\beta\leq\lambda$

(Tôgô [18]). Such a series $(S_\alpha)_{\alpha\leq\lambda}$ is called a weakly ascending series from H to G. H is called a weakly ascendant subgroup of G, denoted by H wasc G, if $H\leq^\lambda G$ for some ordinal λ. When $\lambda<\omega$, H is called a weakly subnormal subgroup of G, denoted by H wsn G. The weakly subnormal index of H in G, denoted by ws$(G: H)$, is the least integer $n\geq 0$ such that $H\leq^n G$. When Δ is any one of the relations $\leq$, sn, asc, wsn and wasc, we call H a Δ-subgroup of G if $H\Delta G$. $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{A}$, $\mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$ are respectively the classes of finite, finitely generated, abelian, nilpotent and solvable groups. $\mathfrak{G}^*$ is the class of finitely generated groups G such that $\langle x^G\rangle\in\mathfrak{G}$ for all $x\in G$. Max-Δ (resp. Min-Δ) is the class of groups satisfying the maximal (resp. minimal) condition for Δ-subgroups. Max-$\leq$ (resp. Min-$\leq$) is usually denoted by Max (resp. Min).

We use the closure operations S_n, Q, E and N_0. A group belonging to $\mathfrak{X}$ is called an $\mathfrak{X}$-group. $\mathfrak{X}$ is S_n-closed (resp. Q-closed) if a normal subgroup (resp. a homomorphic image) of an $\mathfrak{X}$-group is always an $\mathfrak{X}$-group. $\mathfrak{X}$ is E-closed if every extension of an $\mathfrak{X}$-group by an $\mathfrak{X}$-group belongs to $\mathfrak{X}$, and $\mathfrak{X}$ is N_0-closed if the product of two normal $\mathfrak{X}$-subgroups of a group always belongs to $\mathfrak{X}$.

Any notations which are not explained in this paper may be found in [10, 11].

We first state some elementary properties of weakly ascendant subgroups.

LEMMA 1.1. *Let G be a group.*

(1) *If $H\leq^\lambda G$ and $K\leq G$, then $H\cap K\leq^\lambda K$.*

(2) *If $H\leq^\lambda G$ and $K\lhd G$, then $HK\leq^\lambda G$.*

(3) *Let f be a homomorphism of G onto a group $\bar{G}$. If $H\leq^\lambda G$, then $f(H)\leq^\lambda \bar{G}$. If $\bar{H}\leq^\lambda\bar{G}$, then $f^{-1}(\bar{H})\leq^\lambda G$.*

(4) *Let $(H_\alpha)_{\alpha\leq\sigma}$ be a tower of subgroups of G, indexed by ordinals $\alpha\leq\sigma$, such that H_α wasc $H_{\alpha+1}$ for $\alpha<\sigma$, $H_\lambda=\bigcup_{\alpha<\lambda}H_\alpha$ for limit ordinals $\lambda\leq\sigma$ and $H_\sigma=G$. Then H_0 wasc G. In particular if $H\leq^m K$ and $K\leq^n G$, then $H\leq^{m+n}G$.*

PROOF. (1) and (3) have been proved in [18].

(2) If $(S_\alpha)_{\alpha\leq\lambda}$ is a weakly ascending series from H to G, then $(S_\alpha K)_{\alpha\leq\lambda}$ is such a series from HK to G.

(4) We choose an ordinal ρ sufficiently large to ensure that $H_\alpha\leq^\rho H_{\alpha+1}$ for all $\alpha<\sigma$. Let $(S_{\alpha,\beta})_{\beta\leq\rho}$ be a weakly ascending series from H_α to $H_{\alpha+1}$. The set of all pairs (α, β) is well ordered by lexicographical order and one easily verifies that $(S_{\alpha,\beta})_{\alpha<\sigma,\beta\leq\rho}$, with the last term G added if necessary, is a weakly ascending series from H_0 to G.

LEMMA 1.2. *Let G be a group and H a subgroup of G. Let $n\geq 0$. Then*

(1) *H is n-step subnormal in G if and only if* $[G, {}_nH] \leq H$.

(2) *H is n-step weakly subnormal in G if and only if* $[G, {}_nH]_{\text{set}} \subseteq H$ ([18, Theorem 4]).

A proof of Lemma 1.2 (1) can be found in [10].

LEMMA 1.3. *Let G be a group and* H_λ, K_λ ($\lambda \in \Lambda$) *be subgroups of G. Let* $H = \bigcap_{\lambda \in \Lambda} H_\lambda$ *and* $K = \bigcap_{\lambda \in \Lambda} K_\lambda$. *If* H_λ wsn K_λ *and* $\text{ws}(K_\lambda: H_\lambda) \leq n$ *for all* $\lambda \in \Lambda$ *and some integer* $n \geq 0$, *then* H wsn K *and* $\text{ws}(K: H) \leq n$. *In particular if* $H_\lambda \leq^n G$ ($\lambda \in \Lambda$), *then* $H \leq^n G$.

PROOF. By hypothesis $H_\lambda \leq^n K_\lambda$ for each $\lambda \in \Lambda$. Therefore for all $\lambda \in \Lambda$

$$[K, {}_nH]_{\text{set}} \subseteq [K_\lambda, {}_nH_\lambda]_{\text{set}} \subseteq H_\lambda.$$

Then

$$[K, {}_nH]_{\text{set}} \subseteq \bigcap_{\lambda \leq \Lambda} H_\lambda = H$$

and so by Lemma 1.2 we have $H \leq^n K$.

In particular if H_i wsn G ($1 \leq i \leq r$), then $H = \bigcap_{i=1}^r H_i$ wsn G and $\text{ws}(G: H) \leq \max_i \text{ws}(G: H_i)$. However the interesection of an arbitrary collection of weakly subnormal subgroups is not necessarily weakly subnormal. Consider the infinite dihedral group $G = \langle a, b \mid bab = a^{-1}, b^2 = 1 \rangle$ and set $H_i = \langle a^{2^i}, b \rangle$. Then H_i is a weakly subnormal subgroup of G with $\text{ws}(G: H_i) = i$. But it is known that $H = \bigcap_{i \geq 0} H_i = \langle b \rangle$ and H is self-normalizing (cf. [10]). Hence H is not a weakly subnormal subgroup of G.

Tôgô [18] proved that weak subnormality coincides with subnormality in groups belonging to $\mathfrak{F}$, Min and E$\mathfrak{A}$. We have the following

PROPOSITION 1.4. *Let H be a subgroup of a group G. For* $m \leq 2$, *H is m-step weakly subnormal in G if and only if H is m-step subnormal in G.*

PROOF. If $m \leq 1$, the statement is trivial, and so let $m = 2$. Let $H = S_0 \subset S_1 \subset S_2 = G$ be a weakly ascending series from H to G such that $HS_1H = S_1$ and $S_1^{-1} = S_1$ (see [18, Lemma 1]). Since $H^G = H^{S_2} \subseteq S_1$, we see that

$$[G, H]_{\text{set}} \subseteq H^G H \subseteq S_1 H \subseteq S_1.$$

Then $[G, H] \leq \langle S_1 \rangle$. Since $H^{S_1} \subseteq S_0 = H$, it follows that

$$[[G, H], H]_{\text{set}} \subseteq H^{[G,H]} H \subseteq H^{\langle S_1 \rangle} H \subseteq H.$$

Therefore $[G, {}_2H] \leq H$ and so by Lemma 1.2 we have $H \lhd^2 G$.

REMARK. The statement for Lie algebras corresponding to this proposition dose not hold in general. In fact, let $L=\langle x, y, z\rangle$ be the 3-dimensional simple Lie algebra over a field of characteristic $\neq 2$ with multiplication

$$[x, z] = 2x, \quad [y, z] = -2y, \quad [x, y] = z.$$

Then $\langle x\rangle \leq^2 L$, but $\langle x\rangle$ is not a subideal of L (see [6]).

2.

In this section, we shall discuss by what condition the join of a pair of a subnormal (resp. ascendant) subgroup and a weakly subnormal (resp. weakly ascendant) subgroup is a weakly subnormal (resp. weakly ascedndant) subgroup.

We begin with the following lemma corresponding to [6, Lemma 2.3].

LEMMA 2.1. *Let $H \lhd^r G$, $K \leq^s G$ and $J=\langle H, K\rangle$. If H is normal in J, then $J=HK=KH\leq^{rs}G$.*

PROOF. Let $H=H_r \lhd H_{r-1} \lhd \cdots \lhd H_1 \lhd H_0 = G$ be the series of successive normal closures of H in G. By induction on i, we see that each H_i is normalized by K. In fact, if $i=0$ this is clear. So let $i>0$ and assume that $\langle H_{i-1}^K\rangle = H_{i-1}$. Then since $H \lhd J$,

$$\langle H_i^K\rangle = \langle\langle H^{H_{i-1}}\rangle^K\rangle = \langle\langle H^K\rangle^{H_{i-1}}\rangle = \langle H^{H_{i-1}}\rangle = H_i.$$

Hence $H_i \lhd KH_{i-1}$ $(i>0)$. Since $K\leq^s G$, $K\leq^s KH_{i-1}$. It follows from Lemma 1.1 (2) that $KH_i \leq^s KH_{i-1}$ $(i>0)$. By Lemma 1.1 (4) $J=HK=KH\leq^{rs}G$.

LEMMA 2.2. *Let H sn G and K wsn G. If $[H, K]$ sn G, then $J=\langle H, K\rangle$ wsn G.*

PROOF. Since $\langle H^K\rangle = H[H, K]$ and H normalizes $[H, K]$, by [8, Lemma 2.2] we see that $\langle H^K\rangle$ sn G. Since $J=\langle H^K\rangle K$ and K normalizes $\langle H^K\rangle$, it follows from Lemma 2.1 that J wsn G.

THEOREM 2.3. *Let H sn G and K wsn G. If the derived subgroup G' of G belongs to* Max-sn, *then $J=\langle H, K\rangle$ wsn G.*

PROOF. We put $L=\langle H, H^{x_1},\dots, H^{x_n}\rangle$ with $x_1,\dots, x_n \in K$. Since H^{x_i} sn G $(i=1,\dots, n)$, by [8, Theorem 4.3] we have L sn G. We see that $\langle H, H^x\rangle = \langle H, [H, x]\rangle$ for $x\in K$. Consequently $L=\langle H, M\rangle$, where $M=\langle [H, x_1],\dots, [H, x_n]\rangle$. Since H normalizes $[H, x_i]$ for each i, it normalizes M, from which it follows that $M \lhd L$ and so M sn G. Then M sn G'. Since $G'\in$ Max-sn, we can find a subgroup M which is maximal of the above type. Then $M=[H, K]$, and hence $[H, K]$ sn G, which by the above lemma implies that J wsn G.

Next we shall show our main result in this section.

THEOREM 2.4. *If* $H \lhd^{\rho} G$, $K \leq^{\sigma} G$, $K \in \mathfrak{G}$ *and* K *normalizes* H, *then* $J = \langle H, K\rangle \leq^{\sigma(\rho+1)} G$.

To prove this theorem we need the following two lemmas corresponding to Lemma 2.1 and Corollary 2.2 in [5], but we omit the proof.

LEMMA 2.5. *If* K wasc G, $K \in \mathfrak{G}$, $P \leq G$ *and* $PK = KP$, *then for each finite subset* Y *of* PK *there exists a finitely generated subgroup* Y^* *of* P *such that*

$$\langle Y, K\rangle = (\langle Y, K\rangle \cap Y^*)K.$$

If further $K \in \mathfrak{G}^*$ *then* $\langle Y^K\rangle \in \mathfrak{G}$.

Let H and K be subgroups of a group G. H and K are permutable if $HK = KH$. The permutizer of K in H, denoted by $P_H(K)$, is the largest subgroup of H which permutes with K.

LEMMA 2.6. *Let* H asc G, K wasc G, $K \in \mathfrak{G}$ *and* $(H_\alpha)_{\alpha<\rho}$ *be any ascending series from* H *to* G. *Let* $P_\alpha = P_{H_\alpha}(K)$ *and* $N_\alpha = \bigcap_{k \in K} H_\alpha^k$. *Then* $N_\alpha \lhd P_{\alpha+1}$ *for all* $\alpha < \rho$, $\bigcup_{\alpha<\lambda} P_\alpha = P_\lambda$ *and* $\bigcup_{\alpha<\lambda} N_\alpha \lhd P_\lambda$ *for all limit ordinals* $\lambda \leq \rho$. *If further* $K \in \mathfrak{G}^* \cup (\mathfrak{G} \cap \text{Max-asc})$ *then* $\bigcup_{\alpha<\lambda} N_\alpha = N_\lambda$ *for all limit ordinals* $\lambda \leq \rho$.

PROOF OF THEOREM 2.4. Let $(H_\alpha)_{\alpha\leq\rho}$ be an ascending series from H to G and $N_\alpha = \bigcap_{k\in K} H_\alpha^k$. Then by Lemma 2.6 $N_\alpha \lhd N_{\alpha+1}K$ for all $\alpha < \rho$ and $\bigcup_{\alpha<\lambda} N_\alpha \lhd N_\lambda K$ for all limit ordinals $\lambda \leq \rho$. Since $K \leq^{\sigma} N_{\alpha+1}K$ and $K \leq^{\sigma} N_\lambda K$, by Lemma 1.1 (2) we see that $N_\alpha K \leq^{\sigma} N_{\alpha+1}K$ for all $\alpha < \rho$ and $(\bigcup_{\alpha<\lambda} N_\alpha)K \leq^{\sigma} N_\lambda K$ for all limit ordinals $\lambda \leq \rho$. Hence by Lemma 1.1 (4) $N_0 K \leq^{\sigma(\rho+1)} N_\rho K$. Since K normalizes H, $N_0 = H$. Clearly $N_\rho = G$. So we have $J = HK \leq^{\sigma(\rho+1)} G$.

Now we shall show the following result which is an analogue of [7, Proposition 2.6] for groups.

PROPOSITION 2.7. (1) *If* $H \lhd^2 G$ *and* $K \leq^{\sigma} G$, *then* $J = \langle H, K\rangle \leq^{\sigma 2} G$.

(2) *Let* $H = H_3 \lhd H_2 \lhd H_1 \lhd G$ *and let* H_2/H_3, H_1/H_2 *be groups of prime order. Then for any weakly subnormal (resp. weakly ascendant) subgroup* K *of* G, $J = \langle H, K\rangle$ *is a weakly subnormal (resp. weakly ascendant) subgroup of* G.

PROOF. (1) Let $H \lhd H_1 \lhd G$, where $H_1 = \langle H^G\rangle$. Then we have $H^x \lhd H_1$ for any $x \in K$. Therefore $\langle H^K\rangle \lhd H_1$, and hence $\langle H^K\rangle \lhd^2 G$. Thus we may assume that H is normalized by K. Then $H \lhd H_1 K$ and so by Lemma 1.1 (2) we have $J = HK \leq^{\sigma} H_1 K$. Also we have $H_1 K \leq^{\sigma} G$, since $H_1 \lhd G$. Therefore, by Lemma 1.1 (4) $J = HK \leq^{\sigma 2} G$. Thus (1) is proved.

(2) Let $K \leq^{\sigma} G$. Then by (1) we have

$$H_3 \lhd H_2 \lhd H_1 \cap \langle H_2, K\rangle \lhd \langle H_2, K\rangle \leq^{\sigma 2} G.$$

Since $H_3 \le H_2 \cap \langle H_3, K\rangle \le H_2$ and H_2/H_3 has prime order, we have

$$H_2 \le \langle H_3, K\rangle \quad \text{or} \quad H_3 = H_2 \cap \langle H_3, K\rangle .$$

In the first case, we have $J=\langle H_3, K\rangle=\langle H_2, K\rangle$ and so $J\le^{\sigma 2}G$. In the second case, we have $H_2\le\langle H_2, H_1 \cap \langle H_3, K\rangle\rangle \le H_1$. Since H_1/H_2 has prime order, we have

$$H_1 = \langle H_2, H_1 \cap \langle H_3, K\rangle\rangle \quad \text{or} \quad H_2 = \langle H_2, H_1 \cap \langle H_3, K\rangle\rangle .$$

If $H_1=\langle H_2, H_1 \cap \langle H_3, K\rangle\rangle$, then $H_3 \lhd H_1$ and so by (1) $J=\langle H_3, K\rangle \le^{\sigma 2}G$. If $H_2=\langle H_2, H_1 \cap \langle H_3, K\rangle\rangle$, then $H_2 \ge H_1 \cap \langle H_3, K\rangle$ and so $H_3 \le H_1 \cap \langle H_3, K\rangle \le H_2$. As H_2/H_3 has prime order, we have

$$H_2 = H_1 \cap \langle H_3, K\rangle \quad \text{or} \quad H_3 = H_1 \cap \langle H_3, K\rangle .$$

In the first case, we have $H_2\le\langle H_3, K\rangle$ and so $J=\langle H_3, K\rangle=\langle H_2, K\rangle \le^{\sigma 2}G$. In the second case, since $H_1 \lhd G$ we have $\langle H_3^K\rangle \le H_1 \cap \langle H_3, K\rangle = H_3$. As $H_3 \lhd H_2$, $H_3 \lhd \langle H_2, K\rangle$ and by Lemma 1.1 $J=\langle H_3, K\rangle \le^{\sigma}\langle H_2, K\rangle$. Hence $J\le^{\sigma 3}G$.

We can similarly prove the case that K is a weakly subnormal subgroup of G. Thus the proof is complete.

The following lemma is a generalization of [8, Lemma 4.5].

LEMMA 2.8. *Let $\mathfrak{X}$ be a $\{\mathrm{s}_n, \mathrm{N}_0\}$-closed class. Let H and K be $\mathfrak{X}$-subgroups of a group G and $J=\langle H, K\rangle$. If $H \lhd J$ and K sn J, then $J\in\mathfrak{X}$.*

PROOF. Let $K \lhd^r J$ and let $K=K_r \lhd K_{r-1} \lhd \cdots \lhd K_1 \lhd K_0 = J$ be the series of successive normal closures of K in J. If $r=0$ this is clear, and so we may assume that $r>0$. Using induction on $r-i$, we show that $K_i\in\mathfrak{X}$. If $r-i=0$, then $K_r=K\in\mathfrak{X}$. Assume that $r-i>0$ and $K_{i+1}\in\mathfrak{X}$. Since $J=HK$, $K_i=(H\cap K_i)K$ and so $K_i=(H \cap K_i)K_{i+1}$. As K_i sn J, $H\cap K_i$ sn H and $H\cap K_i \in \mathrm{s}_n\mathfrak{X}=\mathfrak{X}$. Since $H \lhd J$ and $K_i \le J$, $H\cap K_i \lhd K_i$. Hence $K_i=(H\cap K_i)K_{i+1}\in \mathrm{N}_0\mathfrak{X}=\mathfrak{X}$. In particular, we have $J=K_0\in\mathfrak{X}$.

3.

Throughout Sections 3 and 4 we always denote by Δ any one of the relations $\le$, sn, asc, wsn and wasc, and by Δ' any one of the realtions $\le$, asc and wasc. In this section we shall introduce and investigate the concepts of pseudo-coalescence. We say $\mathfrak{X}$ to be ascendantly pseudo-coalescent (resp. ascendantly coalescent) if whenever H is an ascendant $\mathfrak{X}$-subgroup and K is a weakly ascendant (resp. an ascendant) $\mathfrak{X}$-subgroup of a group G, their join $J=\langle H, K\rangle$ is a weakly ascendant (resp. an ascendant) $\mathfrak{X}$-subgroup of G. Pseudo-coalescent (resp.

coalescent) classes are similarly defined with 'subnormal' and 'weakly subnormal' (resp. 'subnormal') replacing 'ascendant' and 'weakly ascendant' (resp. 'ascendant') respectively.

The following lemma, which corresponds to [5, Lemma 3.1], is useful to show that some classes are pseudo-coalescent or ascendantly pseudo-coalescent.

LEMMA 3.1. *Let* $\mathfrak{X} \leq \mathfrak{G}$ *and let* $\mathfrak{X}$ *be coalescent (resp. ascendantly coalescent). Then* $\mathfrak{X}$ *is pseudo-coalescent (resp. ascendantly pseduo-coalescent) if following* (1) *or* (2) *is satisfied*:

(1) *The join of a pair of a subnormal (resp. an ascendant)* $\mathfrak{X}$*-subgroup and a weakly subnormal (resp. a weakly ascendant)* $\mathfrak{X}$*-subgroup of any group which are permutable is always a weakly subnormal (resp. a weakly ascendant)* $\mathfrak{X}$*-subgroup of the group.*

(2) $\mathfrak{X} \leq \mathfrak{G}^*$ *or* $\mathfrak{X} \leq$ Max-sn *(resp.* Max-asc*) and the join of a pair of a subnormal (resp. an ascendant)* $\mathfrak{X}$*-subgroup and a weakly subnormal (resp. a weakly ascendant)* $\mathfrak{X}$*-subgroup of a group such that the latter normalizes the former is always a weakly subnormal (resp. a weakly ascendant)* $\mathfrak{X}$*-subgroup of the group.*

PROOF. Suppose that (1) or (2) is satisfied and let H asc G, K wasc G, H, $K \in \mathfrak{X}$ and $J = \langle H, K \rangle$. Applying Lemma 2.5 with $P = \langle H^K \rangle$ and Y a finite set such that $H = \langle Y \rangle$, there exists a finitely generated subgroup Y^* of P such that

$$J = \langle Y, K \rangle \subseteq Y^* K.$$

Since $Y^* \in \mathfrak{G}$, there exist $k_1, \ldots, k_r \in K$ such that

$$Y^* \leq H^* = \langle H^{k_1}, \ldots, H^{k_r} \rangle$$

and so $J = H^* K$. Now H^{k_i} is an ascendant $\mathfrak{X}$-subgroup of G for all i and $\mathfrak{X}$ is ascendantly coalescent. Hence H^* is an ascendant $\mathfrak{X}$-subgroup of G. Thus in case (1) J is a weakly ascendant $\mathfrak{X}$-subgroup of G.

In case (2) if $\mathfrak{X} \leq \mathfrak{G}^*$ then by Lemma 2.5 $P = \langle H^K \rangle \in \mathfrak{G}$ and so we may take $H^* = P$. Also if $\mathfrak{X} \leq$ Max-asc then since $H^* \cap K$ asc $K \in$ Max-asc for all such subgroups H^* there exists such an H^* with $J = H^* K$ and $H^* \cap K = P \cap K$. But we have

$$P = (H^* K) \cap P = H^*(K \cap P) = H^*(H^* \cap K) = H^*.$$

Thus we may assume that $H^* \lhd J = H^* K$, whence J is a weakly ascendant $\mathfrak{X}$-subgroup of G.

We can similarly prove the case of pseudo-coalescence.

We note that $\mathfrak{G}^* \cap \mathfrak{N} = \mathfrak{G} \cap \mathfrak{N}$ since $\mathfrak{G} \cap \mathfrak{N} \leq$ Max (cf. [10, p. 20]). [5, Lemma 3.1] holds for subnormal subgroups instead of ascendant subgroups and also we may take Max-sn instead of Max-asc in this case. Then it is easily verified that $\mathfrak{X} \cap \mathfrak{G}^*$, $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-asc and $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-sn are coalescent if $\mathfrak{X} = \mathrm{N}_0\mathfrak{X}$. From this result, the following classes are coalescent:

$$\mathfrak{F},\quad \mathfrak{F} \cap \mathfrak{N},\quad \mathfrak{F} \cap \mathrm{E}\mathfrak{A},\quad \mathfrak{G}^*,\quad \mathfrak{G}^* \cap \mathfrak{N} = \mathfrak{G} \cap \mathfrak{N},\quad \mathfrak{G}^* \cap \mathrm{E}\mathfrak{A},$$
$$\mathfrak{G}^* \cap \text{Max-}\Delta,\quad \mathfrak{G}^* \cap \text{Min-}\Delta,\quad \mathfrak{G} \cap \text{Max-}\Delta,\quad \mathfrak{G} \cap \text{Max-asc} \cap \text{Min-}\Delta,$$
$$\mathfrak{G} \cap \text{Max-sn} \cap \text{Min-}\Delta,\quad \mathfrak{G} \cap \text{Max-asc} \cap \mathrm{E}\mathfrak{A},\quad \mathfrak{G} \cap \text{Max-sn} \cap \mathrm{E}\mathfrak{A}.$$

We can now state the following result.

THEOREM 3.2. *Let $\mathfrak{X}$ be a* {Q, E}*-closed class. Then*

(1) $\mathfrak{X} \cap \mathfrak{G}^*$ *is pseudo-coalescent and ascendantly pseudo-coalescent.*

(2) $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-asc *is pseudo-coalescent and ascendantly pseudo-coalescent.*

(3) $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-sn *is pseudo-coalescent.*

PROOF. (1) Let $H \operatorname{asc} G$, $K \operatorname{wasc} G$, $H, K \in \mathfrak{X} \cap \mathfrak{G}^*$ and $J = \langle H, K \rangle$. Since $\mathfrak{X}$ is {Q, E}-closed, we have $\mathfrak{X} = \mathrm{N}_0\mathfrak{X}$. Then by [5, Theorem A] $\mathfrak{X} \cap \mathfrak{G}^*$ is ascendantly coalescent. Hence by Lemma 3.1 we may suppose that $H \lhd J$. Then by Theorem 2.4 we obtain $J \operatorname{wasc} G$. Clearly $J \in \mathfrak{G}$ and if $x \in J$ then by Lemma 2.5 we have $\langle x^H \rangle \in \mathfrak{G}$ and also by Lemma 2.5 we have $\langle x^J \rangle = \langle \langle x^H \rangle^K \rangle \in \mathfrak{G}$. Thus $J \in \mathfrak{G}^*$. Since $\mathfrak{X} = \{\mathrm{Q}, \mathrm{E}\}\mathfrak{X}$, $J \in \mathrm{EQ}\mathfrak{X} = \mathfrak{X}$. Hence we have that $\mathfrak{X} \cap \mathfrak{G}^*$ is ascendantly pseudo-coalescent.

Pseudo-coalescence similarly follows from Lemmas 2.1 and 3.1.

(2) Let $H \operatorname{asc} G$, $K \operatorname{wasc} G$, $H, K \in \mathfrak{X} \cap \mathfrak{G} \cap$ Max-asc and $J = \langle H, K \rangle$. Since $\mathfrak{X}$ is {Q, E}-closed, we have $\mathfrak{X} = \mathrm{N}_0\mathfrak{X}$. Therefore $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-asc is ascendantly coalescent by [5, Theorem B]. Then by Lemma 3.1 we may assume that $H \lhd J$. Hence by Theorem 2.4 we obtain $J \operatorname{wasc} G$. Also $J \in \mathrm{EQ}(\mathfrak{X} \cap \text{Max-asc}) = \mathfrak{X} \cap$ Max-asc and clearly $J \in \mathfrak{G}$. Therefore we see that $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-asc is ascendantly pseudo-coalescent.

The proof of pseudo-coalescence is similar.

(3) Let $H \operatorname{sn} G$, $K \operatorname{wsn} G$, $H, K \in \mathfrak{X} \cap \mathfrak{G} \cap$ Max-sn and $J = \langle H, K \rangle$. Since $\mathfrak{X}$ is N_0-closed, by the above remark we see that $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-sn is coalescent. Then by Lemma 3.1 we may assume that $H \lhd J$ and so by Lemma 2.1 $J \operatorname{wsn} G$. Clearly $J \in \mathfrak{G}$ and also $J \in \mathrm{EQ}(\mathfrak{X} \cap \text{Max-sn}) = \mathfrak{X} \cap$ Max-sn. Hence $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-sn is pseudo-coalescent.

THEOREM 3.3. (1) *The following classes are pseudo-coalescent and ascendantly pseduo-coalescent*:

$$\mathfrak{F},\quad \mathfrak{F} \cap \mathrm{E}\mathfrak{A},\quad \mathfrak{G}^*,\quad \mathfrak{G}^* \cap \mathrm{E}\mathfrak{A},\quad \mathfrak{G}^* \cap \text{Max-}\Delta,\quad \mathfrak{G}^* \cap \text{Min-}\Delta,$$

$\mathfrak{G} \cap$ Max-Δ', $\mathfrak{G} \cap$ Max-asc $\cap$ Min-Δ, $\mathfrak{G} \cap$ Max-asc $\cap$ E$\mathfrak{A}$.

(2) *The following classes are pseudo-coalescent*:

$\mathfrak{G} \cap$ Max-Δ, $\mathfrak{G} \cap$ Max-sn $\cap$ Min-Δ, $\mathfrak{G} \cap$ Max-sn $\cap$ E$\mathfrak{A}$.

(3) *The classes* $\mathfrak{F} \cap \mathfrak{N}$ *and* $\mathfrak{G}^* \cap \mathfrak{N} = \mathfrak{G} \cap \mathfrak{N}$ *are pseudo-coalescent.*

PROOF. (1) Since $\mathfrak{F}$, $\mathfrak{G}$, E$\mathfrak{A}$, Max-Δ and Min-Δ are {Q, E}-closed classes (cf. see [10, Lemma 1.31]), by (1) and (2) of Theorem 3.2 we see that these classes are all pseudo-coalescent and ascendantly pseudo-coalescent.

(2) By (3) of Theorem 3.2, we see that these classes are all pseudo-coalescent.

(3) Let $\mathfrak{X}$ be either of the classes $\mathfrak{F}$ and $\mathfrak{G}^*$ and let H sn G, K wsn G, H, $K \in \mathfrak{X} \cap \mathfrak{N}$ and $J = \langle H, K \rangle$. By (1) $\mathfrak{X}$ is pseudo-coalescent and so J is a weakly subnormal $\mathfrak{X}$-subgroup of G. Since $\mathfrak{X} \cap \mathfrak{N}$ is coalescent, by Lemma 3.1 we may assume that $H \lhd J$. Then $J/H \in \mathfrak{N}$, and therefore $J \in$ E$\mathfrak{A}$. By [18, Theorem 3], we see that K sn J. Since $\mathfrak{N}$ is $\{$S$_n$, N$_0\}$-closed, we have $J \in \mathfrak{N}$ by Lemma 2.8. Therefore $\mathfrak{X} \cap \mathfrak{N}$ is pseudo-coalescent.

REMARK. Any classes containing $\mathfrak{A}$, e.g. $\mathfrak{N}$ and E$\mathfrak{A}$, are neither pseudo-coalescent nor ascendantly pseudo-coalescent. In fact, it has been shown by Robinson [8, Theorem 6.1] that there exists a group G such that 1) G is the split extension of a group M by a group J; 2) $M \in \mathfrak{A}$, $J = \langle H, K \rangle$ where H, K are countably infinite abelian subgroups of G; 3) H, K sn G and $J = N_G(J)$. Then H sn G, K wsn G and H, $K \in \mathfrak{A}$. By 3), J is not a weakly subnormal and not a weakly ascendant subgroup of G.

4.

In this section we shall introduce and investigate the concepts corresponding to local coalescence. We say $\mathfrak{X}$ to be locally ascendantly coalescent if whenever H and K are ascendant $\mathfrak{X}$-subgroups of a group G every finite subset F of $J = \langle H, K \rangle$ is contained in some ascendant $\mathfrak{X}$-subgroup X of G with $X \leq J$. Locally coalescent classes [14] are similarly defined with 'subnormal' replacing 'ascendant'. Furthermore, we say $\mathfrak{X}$ to be locally pseudo-coalescent (resp. locally ascendantly pseudo-coalescent) if whenever H is a subnormal (resp. an ascendant) $\mathfrak{X}$-subgroup and K is a weakly subnormal (resp. a weakly ascendant) $\mathfrak{X}$-subgroup of a group G every finite subset F of $J = \langle H, K \rangle$ is contained in some weakly subnormal (resp. weakly ascendant) $\mathfrak{X}$-subgroup of G with $X \leq J$. Evidently any ascendantly coalescent class is locally ascendantly coalescent and any (ascendantly) pseudo-coalescent class is locally (ascendantly) pseudo-coalescent. In [14], Roseblade and Stonehewer proved that subjunctive classes, e.g. $\mathfrak{N}$, E$\mathfrak{A}$ and Max, are locally coalescent. We shall prove that some subclasses of L$\mathfrak{N}$ are locally coalescent.

L$(\Delta)\mathfrak{X}$ is the class of groups G such that every finite subset F of G lies in an

$\mathfrak{X}$-subgroup X of G with $X \Delta G$. $\mathrm{N}\mathfrak{X}$ (resp. $\acute{\mathrm{N}}\mathfrak{X}$) is the class of groups which are generated by their subnormal (resp. ascendant) $\mathfrak{X}$-subgroups. It is well known that N and $\acute{\mathrm{N}}$ are closure operations (see [10]) and also we easily see that $\mathrm{L}(\Delta)$ is a closure operation. Then

$$\mathrm{L}(\mathrm{sn})\mathfrak{X} \leq \mathrm{L}(\mathrm{asc})\mathfrak{X} \leq \mathrm{L}(\mathrm{wasc})\mathfrak{X} \leq \mathrm{L}\mathfrak{X},$$

$$\mathrm{L}(\mathrm{sn})\mathfrak{X} \leq \mathrm{L}(\mathrm{wsn})\mathfrak{X} \leq \mathrm{L}(\mathrm{wasc})\mathfrak{X},$$

$$\mathrm{L}(\mathrm{sn})\mathfrak{X} \leq \mathrm{N}\mathfrak{X} \quad \text{and} \quad \mathrm{L}(\mathrm{asc})\mathfrak{X} \leq \acute{\mathrm{N}}\mathfrak{X}.$$

We begin with the following

PROPOSITION 4.1. (1) *If* $\mathfrak{X}$ *is locally pseudo-coalescent* (*resp. locally ascendantly pseudo-coalescent, locally coalescent, locally ascendantly coalescent*), *then* $\mathfrak{X} \cap \mathfrak{G}$ *is pseudo-coalescent* (*resp. ascendantly pseudo-coalescent, coalescent, ascendantly coalescent*).

(2) *If* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are* S-*closed and locally pseudo-coalescent* (*resp. locally ascendantly pseudo-coalescent, locally coalescent, locally ascendantly coalescent*), *then so is* $\mathfrak{X} \cap \mathfrak{Y}$.

(3) *If* $\mathfrak{X}$ *is* $\mathrm{L}(\mathrm{wsn})$-*closed* (*resp.* $\mathrm{L}(\mathrm{wasc})$-*closed*, $\mathrm{L}(\mathrm{sn})$-*closed*, $\mathrm{L}(\mathrm{asc})$-*closed*) *and locally pseudo-coalescent* (*resp. locally ascendantly pseudo-coalescent, locally coalescent, locally ascendantly coalescent*) *and* $\mathfrak{Y}$ *is pseudo-coalescent* (*resp. ascendantly pseudo-coalescent, coalescent, ascendantly coalescent*), *then* $\mathfrak{X} \cap \mathfrak{Y}$ *is pseudo-coalescent* (*resp. ascendantly pseudo-coalescent, coalescent, ascendantly coalescent*).

PROOF. (1) The proof is immediate.

(2) Assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are locally pseudo-coalescent. Let H (resp. K) be a subnormal (resp. a weakly subnormal) $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of a group G. Put $J = \langle H, K \rangle$ and let F be any finite subset of J. Then there exist a weakly subnormal $\mathfrak{X}$-subgroup X and a weakly subnormal $\mathfrak{Y}$-subgroup Y of G such that $F \subseteq X \leq J$ and $F \subseteq Y \leq J$. Since $\mathfrak{X}$ and $\mathfrak{Y}$ are S-closed, it follows that $X \cap Y$ is a weakly subnormal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of G with $F \subseteq X \cap Y \leq J$. Hence $\mathfrak{X} \cap \mathfrak{Y}$ is locally pseduo-coalescent.

The other cases are similarly proved.

(3) Assume that $\mathfrak{X}$ is $\mathrm{L}(\mathrm{wsn})$-closed and locally pseudo-coalescent and $\mathfrak{Y}$ is pseudo-coalescent. Let H (resp. K) be a subnormal (resp. a weakly subnormal) $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of a group G and put $J = \langle H, K \rangle$. Since $\mathfrak{Y}$ is pseudo-coalescent, J is a weakly subnormal $\mathfrak{Y}$-subgroup of G and so we may prove $J \in \mathfrak{X}$. For any finite subset F of J, there exists a weakly $\mathfrak{X}$-subgroup X of G such that $F \subseteq X \leq J$. Since X is a weakly subnormal $\mathfrak{X}$-subgroup of J, this implies that $J \in \mathrm{L}(\mathrm{wsn})\mathfrak{X} = \mathfrak{X}$. Therefore $\mathfrak{X} \cap \mathfrak{Y}$ is pseudo-coalescent.

The other cases follow in the same way.

PROPOSITION 4.2. (1) *If* $\mathfrak{X}$ *is locally coalescent, then* $L(sn)\mathfrak{X} = N\mathfrak{X} \leq L\mathfrak{X}$.

(2) *If* $\mathfrak{X}$ *is locally ascendantly coalescent, then* $L(asc)\mathfrak{X} = \acute{N}\mathfrak{X} \leq L\mathfrak{X}$.

PROOF. (1) The proof can be found in [14, p. 424].

(2) Let G be an $\acute{N}\mathfrak{X}$-group. If F is a finite subset of G, it follows that F is contained in a subgroup H generated by a finite number of ascendant $\mathfrak{X}$-subgroups of G. Since $\mathfrak{X}$ is locally ascedantly coalescent, we can find an ascendant $\mathfrak{X}$-subgroup X of G such that

$$F \subseteq X \leq H.$$

Therefore $G \in L(asc)\mathfrak{X}$ and so $\acute{N}\mathfrak{X} \leq L(asc)\mathfrak{X}$. Thus $L(asc)\mathfrak{X} = \acute{N}\mathfrak{X} \leq L\mathfrak{X}$.

THEOREM 4.3. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes such that* $\mathfrak{X} \leq \mathfrak{Y} \leq N\mathfrak{X}$ (*resp.* $\acute{N}\mathfrak{X}$). *Then the following are equivalent*:

(1) $\mathfrak{Y}$ *is locally coalescent* (*resp. locally ascendantly coalescent*).

(2) *For any finite number of subnormal* (*resp. ascendant*) $\mathfrak{X}$-*subgroups* $H_1, \dots, H_n$ *of a group G and for any finite subset F of* $\langle H_1, \dots, H_n \rangle$, *there exists a subnormal* (*resp. an ascendant*) $\mathfrak{Y}$-*subgroup X of G such that*

$$F \subseteq X \leq \langle H_1, \dots, H_n \rangle$$

PROOF. Assume that $\mathfrak{Y}$ is locally ascendantly coalescent. Then the statement in (2) holds obviously since $\mathfrak{X} \leq \mathfrak{Y}$. Conversely, assume that (2) holds for the locally ascendant case. Let H and K be ascendant $\mathfrak{Y}$-subgroups of a group G and F be any finite subset of $J = \langle H, K \rangle$. Since $\mathfrak{Y} \leq \acute{N}\mathfrak{X}$, it follows that F is contained in a subgroup M generated by a finite number of ascendant $\mathfrak{X}$-subgroups of H and a finite number of ascendant $\mathfrak{X}$-subgroups of K. Therefore by our assumption there exists an ascendant $\mathfrak{Y}$-subgroup X of G such that

$$F \subseteq X \leq M.$$

Since $M \leq J$, $F \subseteq X \leq J$. Therefore $\mathfrak{Y}$ is locally ascendantly coalescent.

We can similarly prove the locally coalescent case.

THEOREM 4.4. (1) *Let* $\mathfrak{X}$ and $\mathfrak{Y}$ *be classes such that* $\mathfrak{X} \leq \mathfrak{Y} \leq L(sn)\mathfrak{X}$. *Then* $\mathfrak{X}$ *is locally pseudo-coalescent* (*resp. locally coalescent*) *if and only if* $\mathfrak{Y}$ *is locally pseduo-coalescent* (*resp. locally coalescent*).

(2) *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes such that* $\mathfrak{X} \leq \mathfrak{Y} \leq L(asc)\mathfrak{X}$. *Then* $\mathfrak{X}$ *is locally ascendantly pseudo-coalescent* (*resp. locally ascendantly coalescent*) *if and only if* $\mathfrak{Y}$ *is locally ascendantly pseudo-coalescent* (*resp. locally ascendantly coalescent*).

PROOF. Here we only prove the case of local pseudo-coalescence, since the other cases are similarly proved. Assume that $\mathfrak{X}$ is locally pseudo-coalescent.

Let H (resp. K) be a subnormal (resp. weakly subnormal) $\mathfrak{Y}$-subgroup of a group G and $J=\langle H, K\rangle$. If F is a finite subset of J, then there exist finite sets $A\subseteq H$ and $B\subseteq K$ such that $F\subseteq\langle A, B\rangle\leq J$. Since H is an L(sn)$\mathfrak{X}$-subgroup, there exists a subnormal $\mathfrak{X}$-subgroup M of H containing A. Similarly there exists a subnormal $\mathfrak{X}$-subgroup N of K containing B. Then $F\subseteq\langle M, N\rangle$. Now M (resp. N) is a subnormal (resp. weakly subnormal) $\mathfrak{X}$-subgroup of G and $\mathfrak{X}$ is locally pseudo-coalescent. Therefore there exists a weakly subnormal $\mathfrak{X}$-subgroup X of G such that $F\subseteq X\leq\langle M, N\rangle$. Clearly X belongs to $\mathfrak{Y}$ and $F\subseteq X\leq J$. Hence $\mathfrak{Y}$ is locally pseudo-coalescent.

Conversely, assume that $\mathfrak{Y}$ is locally pseudo-coalescent. Let H (resp. K) be a subnormal (resp. weakly subnormal) $\mathfrak{X}$-subgroup of a group G. Then H (resp. K) is a subnormal (resp. weakly subnormal) $\mathfrak{Y}$-subgroup of G. Let F be any finite subset of $J=\langle H, K\rangle$. Then there exists a weakly subnormal $\mathfrak{Y}$-subgroup Y of G such that $F\subseteq Y\leq J$. Since $\mathfrak{Y}\leq$ L(sn)$\mathfrak{X}$, there exists a subnormal $\mathfrak{X}$-subgroup X of Y such that $F\subseteq X\leq J$. Therefore $\mathfrak{X}$ is locally pseudo-coalescent.

We recall some classes of groups (cf. [10, §4]). A group is called a Gruenberg (resp. Baer) group if every cyclic subgroup is ascendant (resp. subnormal), equivalently if and only if every finitely generated subgroup is ascendant (resp. subnormal). A group G is called a Fitting group if G is a product of normal nilpotent subgroups, equivalently if and only if every element is contained in a normal nilpotent subgroup. $\mathfrak{Gr}$, $\mathfrak{B}$ and $\mathfrak{Ft}$ are respectively the classes of Gruenberg, Baer and Fitting groups. Furthermore $\mathfrak{D}$ (resp. $\mathfrak{D}'$) is the class of groups in which every subgroup is subnormal (resp. ascendant). Then $\mathfrak{D}'$ is the class of groups satisfying the normalizer condition. $\mathfrak{Z}$ is the class of hypercentral groups.

$\mathfrak{G}\cap\mathfrak{N}$ is coalescent, pseudo-coalescent and ascendantly coalescent (Theorem 3.3 and [5, Theorem A]). Therefore, by Proposition 4.2 we obtain $\mathfrak{Gr}=\acute{\mathrm{N}}\mathfrak{N}=$ L(asc)$(\mathfrak{G}\cap\mathfrak{N})$ and $\mathfrak{B}=\mathrm{N}\mathfrak{N}=$ L(sn)$(\mathfrak{G}\cap\mathfrak{N})$ (cf. [10, §4]). Hence, by Theorems 3.3 and 4.4 we have the following

THEOREM 4.5. (1) *The following classes are locally pseudo-coalescent and locally coalescent*:

$\mathfrak{N}$, $\mathfrak{D}$, $\mathfrak{F}t$, $\mathfrak{B}$, L(sn)$(\mathfrak{F}\cap\mathfrak{N})$, L(sn)$(\mathfrak{F}\cap \mathrm{E}\mathfrak{A})$, L(sn)$\mathfrak{F}$,
L(sn)$(\mathfrak{G}\cap\mathfrak{N})$, L(sn)$(\mathfrak{G}^*\cap \mathrm{E}\mathfrak{A})$, L(sn)$\mathfrak{G}^*$, L(sn)$(\mathfrak{G}^*\cap$ Max-$\Delta)$,
L(sn)$(\mathfrak{G}^*\cap$ Min-$\Delta)$, L(sn)$(\mathfrak{G}\cap$ Max-$\Delta)$, L(sn)$(\mathfrak{G}\cap$ Max-asc $\cap$ Min-$\Delta)$,
L(sn)$(\mathfrak{G}\cap$ Max-asc $\cap\,\mathfrak{N})$, L(sn)$(\mathfrak{G}\cap$ Max-asc $\cap\,\mathrm{E}\mathfrak{A})$,
L(sn)$(\mathfrak{G}\cap$ Max-sn $\cap$ Min-$\Delta)$, L(sn)$(\mathfrak{G}\cap$ Max-sn $\cap\,\mathfrak{N})$,
L(sn)$(\mathfrak{G}\cap$ Max-sn $\cap\,\mathrm{E}\mathfrak{A})$.

(2) *The following classes are locally ascendantly pseudo-coalescent and locally ascendantly coalescent*:

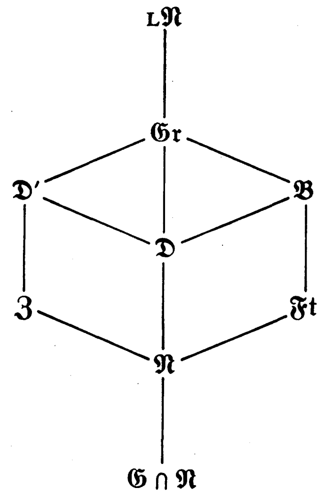
L(asc)$\mathfrak{F}$, L(asc)$(\mathfrak{F}\cap\mathrm{E}\mathfrak{A})$, L(asc)$\mathfrak{G}^*$, L(asc)$(\mathfrak{G}^*\cap\mathrm{E}\mathfrak{A})$,

$\text{L(asc)}(\mathfrak{G}^* \cap \text{Max-}\Delta)$, $\text{L(asc)}(\mathfrak{G}^* \cap \text{Min-}\Delta)$, $\text{L(asc)}(\mathfrak{G} \cap \text{Max-}\Delta')$,
$\text{L(asc)}(\mathfrak{G} \cap \text{Max-asc} \cap \text{Min-}\Delta)$, $\text{L(asc)}(\mathfrak{G} \cap \text{Max-asc} \cap \text{E}\mathfrak{A})$.

(3) *The following classes are locally ascendantly coalescent*:

$$\mathfrak{N},\quad \mathfrak{D},\quad \mathfrak{F}\mathfrak{t},\quad \mathfrak{B},\quad \mathfrak{Z},\quad \mathfrak{D}',\quad \mathfrak{G}\mathfrak{r}.$$

Obviously we have the following diagram of classes of groups:



(cf. [11, Chapter 6]). It is known that all these classes are distinct (see [11, Theorem 6.11]).

We generalize the 'only if' part in Theorem 4.4 in the following

PROPOSITION 4.6. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes such that* $\mathfrak{X} \leq \mathfrak{Y} \leq \text{L(wsn)}\mathfrak{X}$ (*resp.* $\text{L(wasc)}\mathfrak{X}$). *If* $\mathfrak{Y}$ *is locally pseudo-coalescent* (*resp. locally ascendantly pseudo-coalescent*), *then so is* $\mathfrak{X}$.

PROOF. Let H (resp. K) be a subnormal (resp. weakly subnormal) $\mathfrak{X}$-subgroup of a group G and $J = \langle H, K\rangle$. If F is any finite subset of J, then there exists a weakly subnormal $\mathfrak{Y}$-subgroup Y of G such that $F \subseteq Y \leq J$ since $\mathfrak{Y}$ is locally pseudo-coalescent. As $\mathfrak{Y} \leq \text{L(wsn)}\mathfrak{X}$, there exists a weakly subnormal $\mathfrak{X}$-subgroup X of Y such that $F \subseteq X \leq Y$. It follows that

$$X \text{ wsn } G, \quad X \in \mathfrak{X} \quad \text{and} \quad F \subseteq X \leq J.$$

Hence $\mathfrak{X}$ is locally pseudo-coalescent.

The other case can be proved in the same way.

5.

Throughout this section we employ the terminology and notations which were used in [1, 6, 7, 17]. The purpose of this section is to consider analogies of results in Sections 1–4 for Lie algebras. Let L be a Lie algebra over a field Φ. We recall that a subalgebra H of L is a λ-step weakly ascendant subalgebra of L, denoted by $H \leq^{\lambda} L$, if there exists an ascending chain $(M_{\alpha})_{\alpha \leq \lambda}$ of subspaces of L such that

(1) $M_0 = H$ and $M_{\lambda} = L$,

(2) $[M_{\alpha+1}, H] \subseteq M_{\alpha}$ for any ordinal $\alpha < \lambda$,

(3) $M_{\mu} = \cup_{\alpha < \mu} M_{\alpha}$ for any limit ordinal $\mu \leq \lambda$.

Then the chain $(M_{\alpha})_{\alpha \leq \lambda}$ is called a weakly ascending chain for H in L. H is a weakly ascendant subalgebra of L, denoted by H wasc L, if $H \leq^{\lambda} L$ for some ordinal λ. When $\lambda < \omega$, H is a weak subideal of L, denoted by H wsi L, which is a weak ideal of L in the sense of [6]. Let $\mathfrak{X}$ be a class of Lie algebras. We say $\mathfrak{X}$ to be ascendantly pseudo-coalescent if whenever H is an ascendant $\mathfrak{X}$-subalgebra and K is a weakly ascendant $\mathfrak{X}$-subalgebra of a Lie algebra L, their join $J = \langle H, K \rangle$ is a weakly ascendant $\mathfrak{X}$-subalgebra of L, and $\mathfrak{X}$ to be locally ascendantly pseudo-coalescent if whenever H is an ascendant $\mathfrak{X}$-subalgebra and K is a weakly ascendant $\mathfrak{X}$-subalgebra of a Lie algebra L every finite subset F of $J = \langle H, K \rangle$ is contained in some weakly ascendant $\mathfrak{X}$-subalgebra X of L with $X \leq J$. $\mathfrak{G}^*$ is the class of finitely generated Lie algebras L such that $\langle x^L \rangle \in \mathfrak{G}$ for all x in L and $\mathfrak{E}_*$ is the class of Lie algebras which can be generated by left Engel elements.

We have the following lemma which corresponds to Lemma 1.1.

Lemma 5.1. *Let L be a Lie algebra over a field Φ.*

(1) *If $H \leq^{\lambda} L$ and $K \leq L$, then $H \cap K \leq^{\lambda} K$.*

(2) *Let f be a homomorphism of L onto a Lie algebra $\bar{L}$. If $H \leq^{\lambda} L$, then $f(H) \leq^{\lambda} \bar{L}$. If $\bar{H} \leq^{\lambda} \bar{L}$, then $f^{-1}(\bar{H}) \leq^{\lambda} L$.*

(3) *Let $H \leq^{\lambda} L$ and $K \lhd L$. Then $H + K \leq^{\lambda} L$.*

(4) *Let $(H_{\alpha})_{\alpha \leq \sigma}$ be a tower of subalgebras of L, indexed by ordinals $\alpha \leq \sigma$, such that H_{α} wasc $H_{\alpha+1}$ for $\alpha < \sigma$, $H_{\lambda} = \cup_{\alpha < \lambda} H_{\alpha}$ for limit ordinals $\lambda \leq \sigma$, and $H_{\sigma} = L$. Then H_0 wasc L.*

(1), (2) and (3) have been proved in [17].

Proposition 5.2. (1) *If $H \lhd^2 L$ and $K \leq^{\sigma} L$, then $J = \langle H, K \rangle \leq^{\sigma 2} L$.*

(2) *Let $H = H_3 \lhd H_2 \lhd H_1 \lhd L$ and let H_2/H_3, H_1/H_2 be at most 1-dimensional. Then for any weakly ascendant subalgebra K of L, $J = \langle H, K \rangle$ is a weakly ascendant subalgebra of L.*

This corresponds to Proposition 2.7 and is a generalization of [7, Proposition 2.6].

We note that [1, Lemma 5.1] holds for weakly ascendant subalgebras instead of ascendant subalgebras and also [1, Lemma 6.1] holds for 'subideal' and 'coalescence' instead of 'ascendant subalgebra' and 'ascendant coalescence' respectively. By [4, Theorem 2.5] and [17, Lemma 4], we can show the corresponding Lie-theoretic statement of Lemma 3.1.

LEMMA 5.3. *Over fields of characteristic zero, let* $\mathfrak{X} \leq \mathfrak{G} \cap \mathfrak{E}_*$ *and* $\mathfrak{X}$ *be a coalescent (resp. an ascendantly coalescent) class. Then* $\mathfrak{X}$ *is pseudo-coalescent (resp. ascendantly pseudo-coalescent) if the following* (1) *or* (2) *holds*:

(1) *The join of a pair of an* $\mathfrak{X}$*-subideal (resp. an ascendant* $\mathfrak{X}$*-subalgebra) and a weak* $\mathfrak{X}$*-subideal (resp. a weakly ascendant* $\mathfrak{X}$*-subalgebra) of any Lie algebra which are permutable is always a weak* $\mathfrak{X}$*-subideal (resp. a weakly ascendant* $\mathfrak{X}$*-subalgebra) of the Lie algebra.*

(2) $\mathfrak{X} \leq \mathfrak{G}^*$ *or* $\mathfrak{X} \leq$ Max-si *(resp.* Max-asc*) and the join of a pair of an* $\mathfrak{X}$*-subideal (resp. an ascendant* $\mathfrak{X}$*-subalgebra) and a weak* $\mathfrak{X}$*-subideal (resp. a weakly ascendant* $\mathfrak{X}$*-subalgebra) of a Lie algebra such that the latter idealizes the former is always a weak* $\mathfrak{X}$*-subideal (resp. a weakly ascendant* $\mathfrak{X}$*-subalgebra) of the Lie algebra.*

We remark that if H and K are weakly ascendant $\mathfrak{E}_*$-subalgebras of a Lie algebra L then the join $J=\langle H, K\rangle$ belongs to $\mathfrak{E}_*$, and if H and K are permutable and weakly ascendant $\mathfrak{G}^*$-subalgebras of L then the join $J=\langle H, K\rangle$ belongs to $\mathfrak{G}^*$. Then, by Lemma 5.3, [1, Theorem 6.2] and [4, Theorem 2.5] we have the following

THEOREM 5.4. *Over fields of characteristic zero, let* $\mathfrak{X}=\{\text{Q}, \text{E}\}\mathfrak{X}$. *Then the classes* $\mathfrak{X} \cap \mathfrak{G}^* \cap \mathfrak{E}_*$ *and* $\mathfrak{X} \cap \mathfrak{G} \cap$ Max-asc $\cap\, \mathfrak{E}_*$ *are pseudo-coalescent and ascendantly pseudo-coalescent. In particular* $\mathfrak{G}^* \cap \mathfrak{E}_*$, Max $\cap\, \mathfrak{E}_*$, $\mathfrak{G}^{\text{I}} \cap \mathfrak{E}_*$ *and* $\mathfrak{G} \cap$ Max-asc $\cap\, \mathfrak{E}_*$ *are pseudo-coalescent and ascendantly pseudo-coalescent.*

$\mathfrak{N}$ is not E-closed, but by using the ascendant coalescence of $\mathfrak{F} \cap \mathfrak{N}$, Lemma 5.3 and [6, Lemma 3.1] we can prove that over fields of characteristic zero $\mathfrak{F} \cap \mathfrak{N}$ is ascendantly pseudo-coalescent.

Next, concerning the local coalescence we have the following result which corresponds to Proposition 4.1.

PROPOSITION 5.5. (1) *If* $\mathfrak{X}$ *is locally ascendantly pseudo-coalescent, then* $\mathfrak{X} \cap \mathfrak{G}$ *is ascendantly pseudo-coalescent.*

(2) *If* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are* S*-closed and locally ascendantly pseudo-coalescent, then so is* $\mathfrak{X} \cap \mathfrak{Y}$.

(3) *If* $\mathfrak{X}$ *is* L(wasc)*-closed and locally ascendantly pseudo-coalescent and* $\mathfrak{Y}$ *is ascendantly pseudo-coalescent, then* $\mathfrak{X} \cap \mathfrak{Y}$ *is ascendantly pseudo-coalescent.*

THEOREM 5.6. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes such that* $\mathfrak{X} \leq \mathfrak{Y} \leq$ L(asc)$\mathfrak{X}$. *Then* $\mathfrak{X}$ *is locally ascendantly pseudo-coalescent if and only if* $\mathfrak{Y}$ *is locally ascendantly* *pseudo-coalescent.*

This corresponds to Theorem 4.4 (2).

Over any fields of characteristic zero,

$$\mathfrak{F} \cap \mathfrak{N} < \mathfrak{N} < \mathfrak{F}\mathfrak{t} < \mathfrak{B} < \mathfrak{G}\mathfrak{r} < \mathrm{L}\mathfrak{N},$$

$$\mathfrak{N} \leq \mathfrak{D} < \mathfrak{D}' < \mathfrak{G}\mathfrak{r},$$

$$\mathfrak{N} < \mathfrak{Z} \leq \mathfrak{D}'$$

(cf. [16, §4]). By the ascendant coalescence of $\mathfrak{F} \cap \mathfrak{N}$, we see that $\mathrm{L}(\mathrm{asc})(\mathfrak{F} \cap \mathfrak{N}) = \mathfrak{G}\mathfrak{r}$. Therefore, from the ascendant pseudo-coalescence of $\mathfrak{F} \cap \mathfrak{N}$ and the above theorem we obtain the following locally ascendantly pseudo-coalescent classes over fields of characteristic zero:

$$\mathfrak{N}, \quad \mathfrak{F}\mathfrak{t}, \quad \mathfrak{B}, \quad \mathfrak{D}, \quad \mathfrak{D}', \quad \mathfrak{Z}, \quad \mathfrak{G}\mathfrak{r}.$$

The result corresponding to Proposition 4.6 is the following

PROPOSITION 5.7. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes such that $\mathfrak{X} \leq \mathfrak{Y} \leq \mathrm{L}(\mathrm{wasc})\mathfrak{X}$. If $\mathfrak{Y}$ is locally ascendantly pseudo-coalescent, then so is $\mathfrak{X}$.*

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