HIROSHIMA MATH. J.
14 (1984), 619–629

A remark on singular perturbation methods

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(Received May 15, 1984)

Abstract: We inspect singular perturbation methods, which Fife has introduced to deal with stationary problems of reaction-diffusion systems, and modify the main theorem in [1] into a more useful form.

1. Introduction

Reaction-diffusion systems arise in various fields; chemistry, ecology, population dynamics, morphogenesis, physiology and so on. One of interesting phenomena is that the systems often produce various spatial patterns of solutions. An important contribution to the mathematical illustration of such a phenomenon is made by Fife [1]. That is, under some assumptions, stationary solutions with boundary and interior transition layers are obtained constructively and rigorously by using singular perturbation techniques and matching arguments. His work itself is very attractive from a mathematical point of view. Moreover, it is recognized that his results play an important role for elucidating a complicated structure of (stationary) solution set of a type of reaction-diffusion systems (see, e.g., Mimura et al. [6], [7], Fujii et al. [3]). However, the arguments in [1] demand a hypothesis which, generally, is not expected to hold. In this paper, we intend to remove the hypothesis. In order to state our aim more precisely, we review his results.

Consider the following problem:

$$(1.1)\qquad \begin{cases} \varepsilon^2 u'' = f(u, v), & \\ & 0 < x < 1, \\ v'' = g(u, v), & \\ u(0) = \alpha_1, \quad u(1) = \alpha_2, \quad v(0) = \beta_1, \quad v(1) = \beta_2, & \end{cases}$$

where f, g are smooth functions, α_1, α_2, β_1, β_2 are given constants and $' = d/dx$. We make the following assumptions I$\sim$IV.

I. The equation $f(u, v) = 0$ has two distinct solutions $u = h_0(v)$, $u = h_1(v)$, for $v \in I_0$ and $v \in I_1$, respectively, where I_i are open overlapping intervals with $\beta_{i+1} \in I_i$ ($h_0(v) < h_1(v)$ on $I_0 \cap I_1$). On I_i,

$$f_u(h_i(v), v) > 0.$$

Let $J(v)=\int_{h_0(v)}^{h_1(v)} f(u, v)du$, defined on $I_0 \cap I_1$.

II. J has an isolated zero at some value $v^* \in I_0 \cap I_1$ and J changes sign as v passes through v^*. Furthermore,

$$\int_{h_0(v^*)}^{k} f(u, v^*)du > 0 \qquad \text{for} \quad k \in (h_0(v^*), h_1(v^*)).$$

Let

$$G(v) = \begin{cases} g(h_0(v), v), & v \in I_0 \cap \{v \leqq v^*\}, \\ g(h_1(v), v), & v \in I_1 \cap \{v > v^*\}. \end{cases}$$

III. The boundary value problem

$$V'' = G(V) \quad 0 < x < 1; \quad V(0) = \beta_1; \quad V(1) = \beta_2;$$

has a solution $V(x)$ with $V'(x) \neq 0$, such that $V(x^*)=v^*$ for some value $x^* \in (0, 1)$.

IV. For $i=1, 2$,

$$\int_{h_k(\beta_i)}^{t} f(u, \beta_i)du > 0$$

for $t \neq h_k(\beta_i)$ in the closed interval between $h_k(\beta_i)$ and α_i. Here $k=0$ if $V(x)<v^*$ for x in a neighborhood of the boundary point $i-1$, and $k=1$ if $V(x)>v^*$ in this neighborhood.

We define the constant π by

$$\pi = \sup \{|g_u(h_0(v), v)| : v \in I_0\} + \sup \{|g_u(h_1(v), v)| : v \in I_1\}.$$

THEOREM 1.1 (THEOREM 4.1 in [1]). *Under the above assumptions, there exist constants* π_0, ε_0, *such that if* $\pi<\pi_0$, *then there exists a family* ($u(x, \varepsilon)$, $v(x, \varepsilon)$) *of solutions of* (1.1), *defined for* $0<\varepsilon<\varepsilon_0$, *satisfying (for any* $\lambda>0$)

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} u(x, \varepsilon) &= h_0(V(x)) \quad \textit{uniformly for } x \in (\lambda, x^*-\lambda) \\ &= h_1(V(x)) \quad \textit{uniformly for } x \in (x^*+\lambda, 1-\lambda); \\ \lim_{\varepsilon \downarrow 0} v(x, \varepsilon) &= V(x) \quad \textit{uniformly for } x \in [0, 1]. \end{aligned}$$

When the system (1.1) describes a stationary problem for a reaction-diffusion process including autocatalytic reactions of a substance with concentration u, the assumptions I$\sim$IV are often satisfied (see, e.g., Fife [2], Mimura et al. [7]). These assumptions and the smallness of ε play a key role for the existence of solutions with properties in the above theorem. However, the smallness of π

seems unnecessary in showing the existence of such solutions. Indeed, it was introduced in [1] for the convenience of the proof. Moreover, the hypothesis, $\pi<\pi_0$ for some π_0, often makes us trouble when we elucidate global structure of solution set of (1.1). The main purpose of this paper is to show the claim of Theorem 1.1 for every π.

According to [1], the hypothesis that $\pi<\pi_0$ for small π_0 is used only in the proof of Theorem 3.1 (in [1]). Therefore, for our aim, it suffices to modify the theorem and its proof. In the next section, we give a modified theorem. In Section 3, we give some preliminaries. The final section is devoted to the proof.

By using a method based on arguments in this paper, the author has shown the existence of a stationary solution with the so-called transition corners to reaction-diffusion systems of competition type ([4]). He also has investigated the structure of solution set of the systems from a global bifurcation point of view ([5]).

2. Theorem

We make the following assumptions:

I′. There exists a function $h(v)$, defined for v in some interval Γ containing β_1 and β_2, which satisfies

$$f(h(v), v) = 0, \qquad f_u(h(v), v) > 0.$$

III′. The problem

$$V'' = g(h(V), V), \quad 0 < x < 1; \quad V(0) = \beta_1; \quad V(1) = \beta_2$$

has a solution $V(x)$ with range in Γ, such that $V'(x)\neq 0$.

IV′. The assumption IV holds with h_k replaced by h.

THEOREM 2.1. *Under the above assumptions I′, III′, IV′, there exists a constant ε_0 such that there exists a family $u(x, \varepsilon)$, $v(x, \varepsilon)$ of solutions (1.1), denned for $0<\varepsilon<\varepsilon_0$, satisfying*

(2.1) $\lim_{\varepsilon\downarrow 0} u(x, \varepsilon) = h(V(x))$ *uniformly for* $x\in(\kappa, 1-\kappa)$ *for every* $\kappa>0$;

(2.2) $\lim_{\varepsilon\downarrow 0} v(x, \varepsilon) = V(x)$ *uniformly in* $[0, 1]$;

(2.3) $\lim_{\varepsilon\downarrow 0} v_x(x, \varepsilon) = V'(x)$ *uniformly in* $[0, 1]$;

(2.4a) $\lim_{\varepsilon\downarrow 0} \varepsilon u_x(0, \varepsilon) = \mp \left(2\int_{h(\beta_1)}^{\alpha_1} f(u, \beta_1)du\right)^{1/2}$;

(2.4b) $\lim_{\varepsilon\downarrow 0} \varepsilon u_x(1, \varepsilon) = \pm \left(2\int_{h(\beta_2)}^{\alpha_2} f(u, \beta_2)du\right)^{1/2}$;

(2.5) $\quad |u(\cdot,\varepsilon)|_2^\varepsilon + |v(\cdot,\varepsilon)|_2$ *is bounded independently of* ε.

In (2.4) *the upper sign is chosen when the upper limit of integration surpasses the lower. For the norms* $|\cdot|_2^\varepsilon$, $|\cdot|_2$, *see* [1] (*or the next section*).

REMARK. *The above theorem is the same as Theorem* 3.1 *in* [1] *except that it has no assumption on* $\pi' = \sup\{|g_u(h(v), v)| : v \in \Gamma\}$.

3. Preliminaries

For the notation of function spaces and their norms, we use the same one as in [1], say, C^ℓ is the Banach space of functions with derivatives up to order ℓ continuous on [0, 1] with the usual norm $|\cdot|_\ell$. C_0^ℓ is the subspace of functions in C^ℓ vanishing at $x=0, 1$. $|u|_2^\varepsilon = |u|_0 + \varepsilon|u'|_0 + \varepsilon^2|u''|_0$ and $C_{0,\varepsilon}^2$ is the Banach space of functions in C_0^2 endowed with the norm $|\cdot|_2^\varepsilon$. Let X, Y be Banach spaces. We denote by $B(X, Y)$ all of bounded linear operatos from X to Y, and by $\|\cdot\|_{B(X,Y)}$ the operator norm. We simply denote by const. various constants independent of ε.

We first note that the assumptions I′ and IV′ imply that there exist solutions $\prod z_0$, $\prod z_1$, with $|\prod z_i(\xi)| \leqq \text{const.} \exp[-\text{const.}\ \xi]$ $(i=0, 1)$, of

$$(3.1) \qquad \begin{cases} (d/d\xi)^2 \prod z_0 = f(h(V(0)) + \prod z_0, \ V(0)), \quad 0 < \xi < +\infty, \\ \prod z_0(0) = \alpha_1 - h(V(0)), \quad \prod z_0(+\infty) = 0, \end{cases}$$

$$(3.2) \qquad \begin{cases} (d/d\xi)^2 \prod z_1 = f(h(V(1)) + \prod z_1, \ V(1)), \quad 0 < \xi < +\infty, \\ \prod z_1(0) = \alpha_2 - h(V(1)), \quad \prod z_1(+\infty) = 0, \end{cases}$$

respectively. Let $z_0(x) = \prod z_0(x/\varepsilon)\zeta(x)$, $z_1(x) = \prod z_1((1-x)/\varepsilon)\zeta(1-x)$, where ζ is a C^∞-cutoff function with $\zeta(t) \equiv 1$ for $0 \leqq t \leqq 1/4$, $\zeta(t) \equiv 0$ for $t \geqq 1/2$. It is known from Lemma 2.1 in [1] that the operator $L_\varepsilon \in B(C_{0,\varepsilon}^2, C^0)$, defined by

$$(3.3) \qquad L_\varepsilon u \ \mathrm{R} = \varepsilon^2 u'' + f(h(V)) + z_0 + z_1, \ V)u \quad \text{for } u \in C_{0,\varepsilon}^2,$$

has an inverse L_ε^{-1} with $\|L_\varepsilon^{-1}\|_{B(C^0, C_{0,\varepsilon}^2)} \leqq \text{const.}$. We next note that the assumption III′ implies that the operator $M \in B(C_0^2, C^0)$, defined by

$$(3.4) \quad Mv = v'' + (g_u(h(V), V)h'(V) + g_v(h(V), V))v \quad \text{for } v \in C_0^2,$$

has a bounded inverse M^{-1}. Since M^{-1} is an integral operator with continuous kernel,

$$(3.5) \qquad |M^{-1}F|_0 \leqq \text{const.}\ |F|_{L_1},$$

where $|F|_{L_1} = \int_0^1 F(x)dx$. Let

$$v_1 = M^{-1}[g(h(V)+z_0+z_1, V)-g(h(V), V)]. \tag{3.6}$$

Then, v_1 belongs to C_0^2 and $|v|_2 \leqq \text{const.}$ Moreover, (3.5) yields

$$|v_1|_0 \leqq \text{const.}\,(|z_0|_{L_1}+|z_1|_{L_1}) \leqq \text{const.}\,\varepsilon, \tag{3.7}$$

where we used $|z_0(x)| \leqq \text{const.}\exp[-\text{const.}\,x/\varepsilon]$, $|z_1(x)| \leqq \text{const.}\ \exp[-\text{const.}\,(1-x)/\varepsilon]$.

Let

$$\begin{cases} u(r, s, \varepsilon) = h(V+v_1+s) + z_0 + z_1 + r, \\ v(r, s, \varepsilon) = V+ v_1 + s, \end{cases} \tag{3.8}$$

where r and s are unknown functions belonging to $C_{0,\varepsilon}^2$ and C_0^2, respectively. Define a mapping T: $C_{0,\varepsilon}^2 \times C_0^2 \times (0, \varepsilon_0) \to C^0 \times C^0$ $(\varepsilon_0 > 0)$ by

$$T(t, \varepsilon) = \begin{bmatrix} R(r, s, \varepsilon) \\ S(r, s, \varepsilon) \end{bmatrix}, \tag{3.9}$$

where $t=(r, s)$,

$$R(r, s, \varepsilon) = \varepsilon^2 (d/dx)^2 u(r, s, \varepsilon) - f(u(r, s, \varepsilon), v(r, s, \varepsilon)), \tag{3.10}$$

$$S(r, s, \varepsilon) = (d/dx)^2 v(r, s, \varepsilon) - g(u(r, s, \varepsilon), v(r, s, \varepsilon)). \tag{3.11}$$

Since $u(0, 0, \varepsilon)(x)$ and $v(0, 0, \varepsilon)(x)$ satisfy the boundary conditions in (1.1), so do the functions $u(r, s, \varepsilon)(x)$ and $v(r, s, \varepsilon)(x)$. Hence $(u(x, \varepsilon), v(x, \varepsilon)) \equiv (u(r(\varepsilon), s(\varepsilon), \varepsilon)(x), v(r(\varepsilon), s(\varepsilon), \varepsilon)(x))$ is a solution of (1.1) if $t(\varepsilon)=(r(\varepsilon), s(\varepsilon))$ satisfies the equation $T(t, \varepsilon)=0$.

Thus, seeking a solution t of $T(t, \varepsilon)=0$, we obtain Theorem 2.1. It should be noted that the representation (3.8) of the solution is slightly different from (3.1) in [1]. This plays a key role in the proof of Theorem 2.1.

4. Proof of Theorem 2.1

We show the existence of a solution t of $T(t, \varepsilon)=0$ by using an implicit function theorem. We first show the following lemma.

LEMMA 4.1 (corresponding to LEMMA 3.3 in [1]).

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} |T(0, \varepsilon)|_0 &\equiv \lim_{\varepsilon\downarrow 0} (|R(0, 0, \varepsilon)|_0 + |S(0, 0, \varepsilon)|_0) \\ &= 0. \end{aligned} \tag{4.1}$$

PROOF. Since $V''=g(h(v), v)$ and $v_1 = M^{-1}[g(h(V)+z_0+z_1, V)-g(h(V), V)]$,

$$S(0, 0, \varepsilon) = -[g(h(V+v_1)+z_0+z_1, V+v_1)-g(h(V)+z_0+z_1, V)] \\ -[g_u(h(V), V)h'(V)+g_v(h(V), V)]v_1 \\ = -\int_0^1 H(\theta, z_0, z_1)d\theta v_1 - H(0, 0, 0)v_1,$$

where $H(\theta, z_0, z_1)=g_u(h(V+\theta v_1)+z_0+z_1, V+\theta v_1)h'(V+\theta v_1)+g_v(h(V+\theta v_1)+z_0+z_1, V+\theta v_1)$. Since $|H(\theta, z_0, z_1)|_0$, $|H(0, 0, 0)|_0 \leqq$ const., (3.7) implies

$$|S(0, 0, \varepsilon)|_0 \leqq \text{const.}\ \varepsilon. \tag{4.2}$$

Next, we estimate $R(0, 0, \varepsilon)$:

$$R(0, 0, \varepsilon) = \varepsilon^2[h(V+v_1)+z_0+z_1]'' + f(h(V+v_1)+z_0+z_1, V+v_1).$$

Since the supports of $\zeta(x)$ and $\zeta(1-x)$ are disjoint,

$$f(h(V)+z_0+z_1, V) = \zeta(x)f(h(V)+z_0, V) \\ + \zeta(1-x)f(h(V)+z_1, V) \\ + (1-\zeta(x)-\zeta(1-x))f(h(V)+z_0+z_1, V).$$

From (3.1), (3.2) and the definitions of z_0 and z_1, we have

$$\begin{aligned} &R(0, 0, \varepsilon) \\ &= -[f(h(V+v_1)+z_0+z_1, V+v_1)-f(h(V)+z_0+z_1, V)] \\ &\quad + \zeta(x)[f(h(V(x))+\zeta(x)\prod z_0(x/\varepsilon), V(x)) \\ &\qquad - f(h(V(0))+\prod z_0(x/\varepsilon), V(0))] \\ &\quad + \zeta(1-x)[f(h(V(x))+\zeta(1-x)\prod z_1((1-x)/\varepsilon), V(x)) \\ &\qquad - f(h(V(1))+\prod z_1((1-x)/\varepsilon), V(1))] \\ &\quad + (1-\zeta(x)-\zeta(1-x))f(h(V)+z_0+z_1, V) \\ &\quad + \varepsilon N, \end{aligned} \tag{4.3}$$

where $N=\varepsilon[h(V+v_1)]''+2[\zeta'(x)(d\prod z_0/d\xi)(x/\varepsilon)+\zeta'(1-x)(d\prod z_1/d\xi)((1-x)/\varepsilon)]+\varepsilon[\zeta''(x)\prod z_0(x/\varepsilon)+\zeta''(1-x)\prod z_1((1-x)/\varepsilon]$. We can easily see that

$$|N|_0 \leqq \text{cont.}, \tag{4.4}$$

$$\begin{aligned} &|[f(h(V+v_1)+z_0+z_1, V+v_1)-f(h(V)+z_0+z_1, V)|_0 \\ &\quad \leqq \text{const.}\,|v|_0 \leqq \text{cons.}\,\varepsilon. \end{aligned} \tag{4.5}$$

Since $|z_0(x)| \leqq$ const. exp $[-\text{const.}\ x/\varepsilon]$, $|z_1(x)| \leqq$ const. exp $[-\text{const.}\,(1-x)/\varepsilon]$, $1-\zeta(x)-\zeta(1-x)=0$ for $x \in [0, 1/4] \cup [3/4, 1]$ and $f(h(V), V)=0$, we have

(4.6) $|(1-\zeta(x)-\zeta(1-x))f(h(V)+z_0+z_1,\ V)|_0 \leqq \text{const.}\exp[-\text{const.}/\varepsilon]$.

Denote the second and the third term in the right-hand side of (4.3) by $I_2(x)$ and $I_3(x)$, respectively. Since $I_2(0)=0$ and $\zeta(x)=1$ for $x\in[0,\ 1/4]$, we have

$$|I_2(x)| \leqq \text{const.}\,|x| \qquad \text{for} \quad x\in[0,\ 1/4]. \tag{4.7}$$

From the exponential decay of $\prod z_0$ and $f(h(V),\ V)=0$, we also have

$$|I_2(x)| \leqq \text{const.}\exp[-\text{const.}\ x/\varepsilon]. \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$|I_2|_0 \longrightarrow 0 \quad \text{as} \quad \varepsilon\downarrow 0. \tag{4.9}$$

Analogously,

$$|I_3|_0 \longrightarrow 0 \quad \text{as} \quad \varepsilon\downarrow 0. \tag{4.10}$$

(4.3)$\sim$(4.6), (4.9) and (4.10) yield

$$|R(0,\ 0,\ \varepsilon)|_0 \longrightarrow 0 \quad \text{as} \quad \varepsilon\downarrow 0. \tag{4.11}$$

(4.2) and (4.11) show Lemma 4.4.

Next, we consider the linearized operator

$$T_t(0,\ \varepsilon) = \begin{bmatrix} R_r & R_s \\ S_r & S_s \end{bmatrix} : C^2_{0,\varepsilon} \times C^2_0 \longrightarrow C^0 \times C^0, \tag{4.12}$$

where
$$\begin{aligned} R_r w &= \varepsilon^2 w'' - f_u(h(V+v_1)+z_0+z_1,\ V+v_1)w,\\ R_s z &= \varepsilon^2[h'(V+v_1)z]'' + [f_u(h(V+v_1)+z_0+z_1,\ V+v_1)\\ &\qquad \times h'(V+v_1)+f_v(h(V+v_1)+z_0+z_1,\ V+v_1)]z,\\ S_r w &= -\,g_u(h(V+v_1)+z_0+z_1,\ V+v_1)w,\\ S_s z &= z'' - [g_u(h(V+v_1)+z_0+z_1,\ V+v_1)h'(V+v_1)\\ &\qquad +g_v(h(V+v_1)+z_0+z_1,\ V+v_1)]z, \end{aligned}$$

for any $w\in C^2_{0,\varepsilon}$, $z\in C^2_0$.

LEMMA 4.2. *Let ε be sufficiently small. The operators $R_r\in B(C^2_{0,\varepsilon},\ C^0)$, $S_s\in B(C^2_0,\ C^0)$ have inverses R_r^{-1} and S_s^{-1}, respectively.*
$\|R_r^{-1}\|_{B(C^0,C^2_{0,\varepsilon})}$ and $\|S_s^{-1}\|_{B(C_0,C^2_0)}$ are uniformly bounded for ε.

PROOF. R_r can be written in the form

$$R_r w = L_\varepsilon w - Bw,$$

where L_ε is as in (3.3) and $Bw=[f_u(h(V+v_1)+z_0+z_1, V+v_1)-f(h(V)+z_0+z_1, V)]w$. We can easily see that

$$\|B\|_{B(C_0,C_0)} \leqq \text{const.}\, |v_1|_0 \leqq \text{const.}\, \varepsilon,$$

and

$$\|L_\varepsilon^{-1}B\|_{B(C_0^{2,\varepsilon},C_0^{2,\varepsilon})} \leqq \|L_\varepsilon^{-1}B\|_{B(C^0,C_0^{2,\varepsilon})}$$

$$\leqq \|L_\varepsilon^{-1}\|_{B(C^0,C_0^{2,\varepsilon})}\|B\|_{B(C^0,C^0)} \leqq \text{const.}\, \varepsilon.$$

Hence, for small ε, $R_r^{-1}=(I-L_\varepsilon^{-1}B)^{-1}L_\varepsilon^{-1}$ exists and

$$\|R_r^{-1}\|_{B(C^0,C_0^{2,\varepsilon})} \leqq 2\|L_\varepsilon^{-1}\|_{B(C^0,C_0^{2,\varepsilon})},$$

which is uniformly bounded for ε.

S_s can be written in the form

$$S_s z = Mz - Cz,$$

where M is as in (3.4), $Cz=[g_u(h(V+v_1)+z_0+z_1, V+v_1)h'(V+v_1)+g_v(h(V+v_1)+z_0+z_1, V+v_1)-g_u(h(V), V)h'(V)-g_v(h(V), V)]z$. We can easily see that

$$|(Cz)(x)| \leqq \text{const.}\,(|v_1(x)|+|z_0(x)+z_1(x)|)\,|z|_0.$$

Since $|z_0(x)|\leqq$const. $\exp[-\text{const.}\, x/\varepsilon]$, $|z_1(x)|\leqq$const. $\exp[-\text{const.}\,(1-x)/\varepsilon]$, and (3.7) yield

$$\|M^{-1}C\|_{B(C^0,C^0)} \leqq \text{const.}\,(|v_1|_0+|z_0+z_1|_{L_1}) \leqq \text{const.}\, \varepsilon.$$

Therefore, when ε is small, the problem $Mz-Cz=F(F\in C^0)$ has a solution $z=(I-M^{-1}C)M^{-1}F\in C^0$ with

$$|z|_0 \leqq 2\|M^{-1}\|_{B(C^0,C_0^2)}|F|_0.$$

Since $z=M^{-1}Cz-M^{-1}F$, using the above inequality, we have

$$|z|_2 \leqq \|M^{-1}\|_{B(C^0,C_0^2)}(|C|_0|z|_0+|F|_0) \leqq \text{const.}\, |F|_0.$$

This completes the proof.

LEMMA 4.3.

$$\|S_r\|_{B(C_0^{2,\varepsilon},C^0)} \leqq \textit{const.}, \tag{4.13}$$

$$\|R_s\|_{B(C_0^2,C^0)} \leqq \textit{const.}\ \varepsilon. \tag{4.14}$$

PROOF. (4.13) is obvious. We show (4.14). Let
$K(z_0, z_1)=f_u(h(V+v_1)+z_0+z_1, V+v_1)h'(V+v)+f_v(h(V+v_1)+z_0+z_1, V+v_1)$. Since $f(h(v), v)\equiv 0$ for all $v\in\Gamma$, differentiating with respect to v and substituting $v=V+v_1$, we have $K(0, 0)=0$. Thus, $K(z_0, z_1)=K(z_0, z_1)-K(0, 0)$ and

$$(4.15) \qquad |K(z_0, z_1)| \leqq \begin{cases} \text{const. exp. } [-\text{const. } x/\varepsilon], & 0 \leqq x \leqq 1/2, \\ \text{const. exp } [-\text{const. } (1-x)/\varepsilon], & 1/2 \leqq x \leqq 1. \end{cases}$$

Note that $z \in C_0^2$ satisfies

$$(4.16) \qquad |z(x)| \leqq \begin{cases} |z|_1 |x|, & 0 \leqq x \leqq 1/2, \\ |z|_1 |1-x|, & 1/2 \leqq x \leqq 1. \end{cases}$$

Since

$$|x| \exp [-\text{const. } x/\varepsilon], \ |1-x| \exp [-\text{const. } (1-x)/\varepsilon] \leqq \text{const. } \varepsilon,$$

we have from (4.15) and (4.16) that

$$|K(z_0, z_1)z|_0 \leqq \text{const. } \varepsilon |z|_1 \leqq \text{const. } \varepsilon |z|_2.$$

Since $\varepsilon^2 |[h(V+v_1)z]''|_0 \leqq \text{const. } \varepsilon^2 |z|_2$, we have

$$|R_s z|_0 = |\varepsilon^2 [h(V+v_1)z]'' + K(z_0, z_1)z|_0 \leqq \text{const. } \varepsilon |z|_2.$$

This shows (4.14).

COROLLARY 4.4 (corresponding to LEMMA 3.2 in [1]). *Let ε be sufficiently small. The operator $T_t(0, \varepsilon) \in B(C_{0,\varepsilon}^2 \times C_0^2, C^0 \times C^0)$ has an inverse which is bounded independently of ε.*

PROOF. Consider the problem

$$T_t(0, \varepsilon) q \equiv \begin{bmatrix} R_r & R_s \\ S_r & S_s \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

where $q = (w, z)$ is unknown and F_1, F_2 are given functions in C^0. This can be written in the form

$$(4.17a) \qquad w = - R_r^{-1} R_s z + R_r^{-1} F_1,$$

$$(4.17b) \qquad z = - S_s^{-1} S_r w + S_s^{-1} F_2.$$

Substituting (4.17b) into (4.17b), we have

$$(4.18) \qquad w = R_r^{-1} R_s S_s^{-1} S_r w - R_r^{-1} R_s S_s^{-1} F_2 + R_r^{-1} F_1.$$

Since Lemmas 4.2, 4.3 yield

$$\|R_r^{-1} R_s S_s^{-1} S_r\|_{B(C_{0,\varepsilon}^2, C_{0,\varepsilon}^2)} \leqq \|R_r^{-1}\|_{B(C^0, C_{0,\varepsilon}^2)} \|R_s\|_{B(C_0^2, C^0)} \times \|S_s^{-1}\|_{B(C^0, C_0^2)} \|S_r\|_{B(C_{0,\varepsilon}^2, C^0)} \leqq \text{const. } \varepsilon,$$

the operator $(I-R_r^{-1}R_sS_s^{-1}S_r)^{-1}\in B(C_{0,\varepsilon}^2,\ C_{0,\varepsilon}^2)$ is well-defined for small ε so that

$$\|(I-R_r^{-1}R_sS_s^{-1}S^r)^{-1}\|_{B(C_{0,\varepsilon}^2,C_{0,\varepsilon}^2)}\leqq 2.$$

Hence, (4.18) has a solution w such that

$$\begin{aligned}(4.19)\qquad |w|_2^\varepsilon &= |(I-R_r^{-1}R_sS_s^{-1}S_r)^{-1}(R_r^{-1}F_1-R_r^{-1}R_sS_s^{-1}F_2)|_2^\varepsilon\\ &\leqq 2\|R_r^{-1}\|_{B(C^0,C_{0,\varepsilon}^2)}(|F_1|_0+\|R_s\|_{B(C_0^2,C^0)}\|S_s^{-1}\|_{B(C^0,C_0^2)}|F_2|_0)\\ &\leqq \text{const.}\,(|F_1|_0+|F_2|_0)\qquad \text{(from Lemmas 4.2, 4.3).}\end{aligned}$$

Substituting w into (4.17b) and using Lemmas 4.2, 4.3, we have

$$(4.20)\qquad |z|_2\leqq \text{const.}\,(|F_1|_0+|F_2|_0).$$

(4.19) and (4.20) show the corollary.

From Lemma 4.1, Corollary 4.4 and the implicit function theorem (Theorem 3.4 in [1]), it follows that, for small ε, there exists a solution $t(\varepsilon)=(r(\varepsilon),\ s(\varepsilon))$ of $T(t,\ \varepsilon)=0$ such that

$$(4.21)\qquad |r(\varepsilon)|_2^\varepsilon+|s(\varepsilon)|_2\longrightarrow 0\quad \text{as}\quad \varepsilon\downarrow 0.$$

Hence, $(u(x,\ \varepsilon),\ v(x,\ \varepsilon))\equiv(u(r(\varepsilon),\ s(\varepsilon),\ \varepsilon)(x),\ v((r(\varepsilon),\ s(\varepsilon),\ \varepsilon)(x))$ is a solution of (1.1). Note that v_1 in (3.6) satisfies

$$(4.22)\qquad |v_1|_1\leqq \text{const.}\,\varepsilon^{1/2}.$$

Indeed, we can show (4.22) by using (3.7), the fact $|v_1|_2\leqq$const. and an interpolation. In the same way as in [1], we can show from (3.8), (4.21) and (4.22) that $(u(x,\ \varepsilon),\ v(x,\ \varepsilon))$ satisfies (2.1)$\sim$(2.5). Thus the proof of Theorem 2.1 is completed.

Added in proof: After this paper had been received, the author was informed that van Harten and Vader-Burger[8] delt with the same problem in another way.

Acknowledgements. The author would like to thank Professor M. Mimura who suggested him to write the paper. He would also like to thank Professor Y. Hosono for stimulating discussions.

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