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Zeta function of Selberg's type for compact quotient of $SU(n, 1)$ $(n \geq 2)$

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§ 1. Introduction

Let R be a compact Riemann surface of genus greater than one. Let H be the upper half plane with the Poincaré metric. Then $R = H/\Gamma$ where Γ is a discrete torsion-free subgroup of $SL(2, \boldsymbol{R})$, acting freely on H via fractional linear transformations. In the well known paper [10], A. Selberg constructed a function Z_Γ associated with R for which the location and order of the zeros of Z_Γ gave us information about the topology of R and the spectrum of the Laplace-Beltrami operator on R. After that, in 1977, R. Gangolli showed how to attach a Selberg's type zeta function to a compact quotient of symmetric space of rank one in his paper [2].

By the way, these zeta functions can be thought of as providing information about the class one spectrum of G on $L^2(G/\Gamma)$, where G is a semisimple Lie group of real rank one. Namely, we decompose $L^2(G/\Gamma)$ into a direct sum of G-invariant irreducible subspaces and investigate those irreducible subspaces that contain a unique (up to scalar multiplication) K-invariant function. Here K is a maximal compact subgroup of G.

Let M be the centralizer in K of the split component of a minimal parabolic subgroup of G. Then the class one spectrum of G is contained in the representations induced from the trivial representation of M. D. Scott paid attention to this fact in [9]. Let ξ be an irreducible representation of M. As for $G = SL(2, \boldsymbol{C})$, he constructed a zeta function $Z_{\Gamma,\xi}$ which gave information about those principal series representations induced from ξ that appeared in the spectrum of G on $L^2(G/\Gamma)$.

In the present paper, we consider the analogues of those results when $G = SU(n, 1)$. That is, we construct the zeta functions $Z_{\Gamma,\tau}$ of Selberg's type for compact quotient of G, associated with the one dimensional representations τ of $K = U(n+1) \cap G$. The purpose of this paper is to show that these zeta functions have almost all the properties possessed by Selberg's one. Our main results are collected in Theorem 4.11.

In §2, we deal with preliminaries.

Making use of the trace formula, we will define the series $\eta_{\Gamma,\tau}$, the logarithmic derivative of our zeta function. On that occassion, we use the suitable function belonging to $\mathscr{C}^1(G, \tau)$ (see §2). That was the reason why we came to necessity of the characterization of $\mathscr{C}^1(G, \tau)$ under the τ-spherical Fourier transform. So we mention about this subject in §3. Also, we will apply this result to a certain function in $\mathscr{C}^1(G, \tau)$ and we shall obtain some consequence connected with the multiplicities of the discrete series representations in $L^2(G/\Gamma)$.

The first half of §4 is devoted to studying $\eta_{\Gamma,\tau}$. That is, we investigate the analytic continuation of $\eta_{\Gamma,\tau}$. The functional equation of $\eta_{\Gamma,\tau}$ is derived there. In the latter half, we define the zeta function and study its various properties which are derived from the first half of this section. Lastly we refer to the product expansion of $Z_{\Gamma,\tau}$.

§2. Preliminaries

Let $G=SU(n, 1)$ $(n \geq 2)$. Recall that $SU(n, 1)$ is the group of elements in $SL(n+1, \boldsymbol{C})$ leaving invariant the Hermitian form $\sum_{i=1}^{n} |z_i|^2 - |z_{n+1}|^2$. Let $\mathfrak{g}$ be the Lie algebra of G.

We take $K=U(n+1) \cap G$, the maximal compact subgroup of G. The Lie algebra of K is $\mathfrak{k}=\left\{\begin{bmatrix} X & 0 \\ 0 & y \end{bmatrix}\right\}$ with X an n by n skew hermitian matirx and y a complex number such that $\operatorname{tr}(X)+y=0$. If $\mathfrak{p}=\left\{\begin{bmatrix} 0 & Z \\ {}^t\bar{Z} & 0 \end{bmatrix}\right\}$ with Z an n-dimentional colume vector, then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ with respect to the involution θ. Here θ is given by $\theta(X)=-{}^t\bar{X}$ $(X \in \mathfrak{g})$. The subgroup T of diagonal matrices in K is a compact Cartan subgroup of G. The Lie algebra of T is denoted by $\mathfrak{t}$.

If $H_0=\begin{bmatrix} 0 & \cdots & 1 \\ \vdots & 0 & \vdots \\ 1 & \cdots & 0 \end{bmatrix} \in \mathfrak{g}$, then $\boldsymbol{R}H_0$ is a maximal abelian subalgebra of $\mathfrak{p}$. We denote this subalgebra by $\mathfrak{a}_{\mathfrak{p}}$.

Set

$$A_{\mathfrak{p}} = \left\{a_t=\exp tH_0 = \begin{bmatrix} \coth t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}; \quad t \in \boldsymbol{R}\right\}.$$

Let M be the centralizer of $A_{\mathfrak{p}}$ in K. Then $M=\left\{\begin{bmatrix} e^{i\theta} & & \\ & u & \\ & & e^{i\theta} \end{bmatrix}\right\}$ where $u \in U(n-1)$ and $e^{2i\theta}\det(u)=1$. Let $A_{\mathfrak{k}}$ be the subgroup of diagonal matrices in M. The Lie algebras of M and $A_{\mathfrak{k}}$ are written by $\mathfrak{m}$ and $\mathfrak{a}_{\mathfrak{k}}$ respectively. Then $A=A_{\mathfrak{k}}A_{\mathfrak{p}}$ is a Cartan subgroup of G and the pair (T, A) is a complete set (up to conjugacy) of Cartan subgroups of G.

Let $\mathfrak{a}$ be the Lie algebra of A. Namely,

$$\mathfrak{a} = \left\{ H = H(t, iu_1, \ldots, iu_n) = \begin{bmatrix} iu_1 & & & t \\ & iu_2 & & \\ & & \ddots & \\ & & & iu_n & \\ t & & & & iu_1 \end{bmatrix}; \quad \begin{matrix} \mathrm{tr}(H) = 0, \\ t \in \boldsymbol{R}, \quad u_j \in \boldsymbol{R} \end{matrix} \right\}.$$

The complexification $\mathfrak{a}_{\boldsymbol{C}}$ of $\mathfrak{a}$ consists of matrices of the same type as H with complex elements t and u_j. This is a Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{sl}(n+1, \boldsymbol{C})$, the complexification of $\mathfrak{g}$.

Let e_k be the linear function on $\mathfrak{a}_{\boldsymbol{C}}$ defined by

$$e_1(H) = iu_1 + t,$$

$$e_{n+1}(H) = iu_1 - t,$$

$$e_j(H) = iu_j \qquad (1 < j \le n).$$

Then

$$\Phi = \{\pm (e_i - e_{j+1});\ 1 \le i \le j \le n\}$$

is the root system of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{a}_{\boldsymbol{C}}$. We choose an ordering so that the positive roots are

$$\Phi^+ = \{\alpha_{i,j+1} = e_i - e_{j+1};\ 1 \le i \le j \le n\}.$$

Let

$$P_+ = \{\alpha \in \Phi^+;\ \alpha \not\equiv 0 \text{ on } \mathfrak{a}_{\mathfrak{p}}\}, \quad P_- = \{\alpha \in \Phi^+;\ \alpha \equiv 0 \text{ on } \mathfrak{a}_{\mathfrak{p}}\}.$$

Then we have

$$P_+ = \{\alpha_{1,j+1};\ 1 \le j < n\} \cup \{\alpha_{i,n+1};\ 1 < i \le n\} \cup \{\alpha_{1,n+1}\}.$$

Put $\rho = 2^{-1} \sum_{\alpha \in P_+} \alpha$. For $\alpha \in \Phi^+$, let X_α be a root vector belonging to α, and put $\mathfrak{n}_{\boldsymbol{C}} = \sum_{\alpha \in P_+} \boldsymbol{C} X_\alpha$. Then if $\mathfrak{n} = \mathfrak{n}_{\boldsymbol{C}} \cap \mathfrak{g}$, we have the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}$, $G = KA_{\mathfrak{p}}N$, where of course $N = \exp \mathfrak{n}$. For any subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we denote by $\mathfrak{l}^*$ the dual of $\mathfrak{l}$.

Let Σ be the set of restrictions to $\mathfrak{a}_{\mathfrak{p}}$ of elements of P_+. If 2β is the restriction of $\alpha_{1,n+1}$ to $\mathfrak{a}_{\mathfrak{p}}$, then the restrictions of all other elements of P_+ are β. Hence we have $\Sigma = \{\beta, 2\beta\}$. We note that $\beta(H_o) = 1$, and $\mathfrak{n} = \mathfrak{g}_\beta \oplus \mathfrak{g}_{2\beta}$. Here $\mathfrak{g}_\beta$ and $\mathfrak{g}_{2\beta}$ are given by

$$\mathfrak{g}_\beta = \left\{ \begin{bmatrix} 0 & {}^t\bar{X} & 0 \\ -X & 0 & X \\ 0 & {}^t\bar{X} & 0 \end{bmatrix};\ X \in \boldsymbol{C}^{n-1} \right\}, \quad \mathfrak{g}_{2\beta} = \left\{ \begin{bmatrix} \bar{y} & 0 & y \\ 0 & 0 & 0 \\ \bar{y} & 0 & y \end{bmatrix};\ \begin{matrix} \bar{y} = -y, \\ y \in \boldsymbol{C} \end{matrix} \right\}.$$

Throughout this paper, we will denote by ρ_o the number $\rho(H_o) = 2^{-1}\{2(n-1) + 2\} = n$.

Let dt be the standard Lebesgue measure on $\boldsymbol{R}$. We take a Haar measure dh on $A_{\mathfrak{p}}$ by dt, when $h=a_t=\exp tH_o$. For any $\mu\in\mathfrak{a}_{\mathfrak{p}}^*$, we put $\nu=\nu(\mu)=\mu(H_o)$. Then ν is a parameter on $\mathfrak{a}_{\mathfrak{p}}^*$, and maps $\mathfrak{a}_{\mathfrak{p}}^*$ isomorphically onto $\boldsymbol{R}$. Let $d\nu$ be the Lebesgue measure on $\boldsymbol{R}$. Then $d\nu/2\pi$ is the measure on $\boldsymbol{R}$ dual to the measure dt on $\boldsymbol{R}$. We denote by $d\mu$ the measure on $\mathfrak{a}_{\mathfrak{p}}^*$ that we obtain from $d\nu/2\pi$. Then dh, $d\mu$ are dual in the sense of Fourier transforms.

Let dk and dm be the normalized Haar measures on K and M respectively. On N we fix a Haar measure normalized by the following condition: Let $\bar{n}=\theta(n^{-1})$ for $n\in N$, and for any $x\in G$, let $H(x)\in\mathfrak{a}_{\mathfrak{p}}$ be defined by $x=\kappa(x)\exp H(x)n(x)$, $\kappa(x)\in K$, $n(x)\in N$. The measure dn is to satisfy the condition $\int_N\exp(-2\rho(H(\bar{n}))dn=1$. Having fixed the above measures on K, $A_{\mathfrak{p}}$, N, we fix the Haar measure dx on G given by

$$dx = \exp 2\rho(\log h)dkdhdn.$$

For any subgroup L of G, let $\hat{L}$ be the set of equivalence classes of irreducible unitary representations of L. If $\nu\in\boldsymbol{C}(\simeq(\mathfrak{a}_{\mathfrak{p}})_{\boldsymbol{C}}^*)$ and $\xi\in\hat{M}$, let H_ξ denote the space of functions

$$f: K \longrightarrow E_\xi \qquad ((\xi,\, E_\xi)\in\xi),$$

$$f(km) = \xi(m)^{-1}f(k) \quad\text{and}\quad \int_K \|f(k)\|^2dk = \|f\|^2 < \infty.$$

If $f\in H_\xi$ let $f_\nu(ka_tn)=\exp(-(i\nu+\rho_o)t)f(k)$, $k\in K$, $t\in\boldsymbol{R}$, $n\in N$. Set $(\pi_{\xi,\nu}(x)f)(k)=f_\nu(x^{-1}k)$. Then $(\pi_{\xi,\nu}, H_\xi)$ is a representation of G. If $\nu\in\boldsymbol{R}$ $(\simeq\mathfrak{a}_{\mathfrak{p}}^*)$ then this representation is called a (unitary) principal series representation of G. On the other hand, for $\nu\in i\boldsymbol{R}$, the representation $\pi_{\xi,\nu}$ is called a complementary series representation of G whenever it is unitarizable. Such a representation appears when $\nu\in i[-\rho_o, \rho_o]$.

The unitary dual $\hat{T}$ of T can be identified with a lattice L_T in it^*. The set of regular elements will be denoted by L_T'. The Weyl group of G relative to T acts on L_T'. Let L_T^+ be a fundamental domain for this action. It is known that L_T^+ uniquely parameterizes the so-called discrete series representations of G, cf. [4].

If $f\in C_c(G)$, we define the Abel transform F_f by

$$F_f(ma_t) = \exp(t\rho_o)\int_{K\times N} f(kma_tnk^{-1})dndk \qquad (m\in M).$$

Let $\Theta_{\xi,\nu}=\Theta_{\pi_{\xi,\nu}}$ $(\xi\in\hat{M},\ \nu\in\boldsymbol{C})$ denote the character of $\pi_{\xi,\nu}$. Then it is known that

$$\Theta_{\xi,\nu}(f) = \int_M\int_{\boldsymbol{R}} F_f(ma_t)\operatorname{tr}\xi(m)\exp(it\nu)dtdm. \tag{2.1}$$

Applying the Fourier inversion formula and the Peter-Weyl theorem we have

$$F_f(ma_t) = (1/2\pi)\sum_{\xi\in\hat{M}}\int_{\mathbf{R}}\Theta_{\xi,\nu}(f)\exp(-it\nu)\operatorname{tr}\overline{\xi(m)}d\nu. \tag{2.2}$$

Now let Γ be a discrete torsion free subgroup of G such that G/Γ is compact. Then every element $\gamma\in\Gamma$ is conjugate in G to an element of the Cartan subgroup $A=A_{\mathfrak{t}}A_{\mathfrak{p}}$. Choose an element $h(\gamma)$ of A to which γ is conjugate, and let $h(\gamma)=h_{\mathfrak{t}}(\gamma)h_{\mathfrak{p}}(\gamma)$. We then define $u_\gamma=\beta(\log h_{\mathfrak{p}}(\gamma))$. Though u_γ will depend on the choice of $h(\gamma)$, its absolute value $|u_\gamma|$ depends only on γ.

An element $\gamma\in\Gamma$, $\gamma\neq e$ is called primitive if it can not be expressed as δ^n, for some $n>1$, $\delta\in\Gamma$. We denote the set of primitive elements of Γ by P_Γ. It is known that every $\gamma\neq e$ is equal to a positive power of a unique primitive element δ. The integer $j(\gamma)$ is defined by $\gamma=\delta^{j(\gamma)}$ [1].

Fix a G-invariant measure $d\dot{x}$ on G/Γ by requiring that for each $f\in C_c(G)$ we have

$$\int_G f(x)dx = \int_{G/\Gamma}\left(\sum_{\gamma\in\Gamma}f(x\gamma)\right)d\dot{x}.$$

We denote the volume of G/Γ in the invariant measure $d\dot{x}$ by $\operatorname{vol}(G/\Gamma)$.

Let C_Γ be the set of representatives in Γ for the Γ-conjugacy class of elements of Γ.

Let (T, E_T) be a finite dimensional unitary representation of Γ with character χ_T. Let $L^2(G/\Gamma, T)$ denote the set of functions $f\colon G\to E_T$ such that

$$f(x\gamma) = T(\gamma^{-1})f(x) \qquad \text{for all} \quad x\in G \quad \text{and} \quad \gamma\in\Gamma$$

and

$$\int_{G/\Gamma}\|f(x)\|_T^2 d\dot{x} < \infty$$

where $\|\cdot\|_T$ is the norm on E_T.

Because G/Γ is compact the left regular representation U of G on $L^2(G/\Gamma, T)$ splits into a direct sum of irreducible unitary representations of G and we can write

$$U = \sum_{\pi\in\hat{G}} m_\Gamma(\pi)\pi.$$

Here $m_\Gamma(\pi)=m_{\Gamma,T}(\pi)$ is the number of summands of U which lie in the class $\pi\in\hat{G}$.

In this paper, our chief tool is Selberg's trace formula. The notion of an admissible function (for the trace formula) is defined as usual, cf. [3], and one has the trace formula

$$\begin{aligned}\sum_{\pi\in\hat{G}} m_{\Gamma,T}(\pi)\Theta_\pi(f) &= \chi_T(e)\operatorname{vol}(G/\Gamma)f(e)\\ &\quad+\sum_{\gamma\in C_\Gamma-\{e\}}\chi_T(\gamma)\,|u_\gamma|\,j(\gamma)^{-1}C(h(\gamma))F_f(h(\gamma)).\end{aligned} \tag{2.3}$$

which was derived in [14]. Here $\Theta_\pi(f)$ stands for the character of $\pi \in \hat{G}$, and $C(h)$ is a positive function depending only on the structure of G. The number $C(h(\gamma))F_f(h(\gamma))$ depends only on the G-conjugacy class of γ. The value $C(h(\gamma))$ is given by

$$C(h(\gamma)) = \varepsilon(h(\gamma))\xi_\rho(h_{\mathfrak{p}}(\gamma))^{-1} \prod_{\alpha \in P_+} (1-\xi_\alpha(h(\gamma))^{-1})^{-1}.$$

Here, for any $\mu \in (\mathfrak{a}_{\mathfrak{p}})_C^*$, ξ_μ stands for the character of A defined by $\xi_\mu(h)=\exp\cdot\mu(\log h)$, and $\varepsilon(h)$ is, for $h \in A$, equal to the sign of $1-\xi_{\alpha_{1,n+1}}(h)^{-1}$.

Let $\mathfrak{D}$ be the universal enveloping algebra of G. Let $\sigma(x)=(2(n+1)\cdot\mathrm{tr}\cdot X^2)^{1/2}$, where $x=k\cdot\exp X$, $x \in \mathfrak{p}$ is the polar decomposition of $x \in G$.

For any one dimensional unitary representation τ of K, let

$$\phi^\tau(\nu, x) = \int_K \tau(k(xk))\overline{\tau(k)} \exp(-(i\mu+\rho)(H(xk))dk$$

where $\nu=\mu(H_o)$. We call $\phi^\tau(\nu, x)$ the τ-spherical function.

Let $\mathscr{C}^1(G, \tau)$ be the set of smooth function f on G for which

(I) $\nu_{D,r}(f)=\sup_{x\in G}\{(1+\sigma(x))^r\Xi^{-2}(x)|Df(x)|\}<+\infty$ for any $n \in \boldsymbol{Z}$ and any $D \in \mathfrak{D}$,

(II) $f(kxk')=\tau(k)f(x)\tau(k')$ for $k, k' \in K$.

Here $\Xi(x)$ is equal to the zonal spherical function $\phi^1(0, x)$.

It is known that $\mathscr{C}^1(G, \tau)$ is a Frécht space with $\nu_{D,r}$ as seminorms.

§3. The result of P. C. Trombi [11] and its applications

Let $\hat{K}_1$ be a subset of $\hat{K}$ consisting of one dimensional representations. For $k=\begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix} \in K$ ($u \in U(n)$, $w \in \boldsymbol{C}$ and $\det(u)w=1$) and $q \in \boldsymbol{Z}$, we define $\tau_q(k)=\det(u)^q$. Then $\hat{K}_1$ is parametrized by $\boldsymbol{Z}$.

In this section, we shall describe the result of Trombi concerning the characterization of $\mathscr{C}^1(G, \tau)$ under the τ-spherical Fourier transformation.

For each $\tau \in \hat{K}_1$, we define the one dimensional representation τ^M of M by restricting τ to M. It is known that the Plancherel measure $\mu_\tau(\nu)=\mu_\tau^M(\nu)$ at (τ^M, ν) is given by $\mu_\tau(\nu)=(c_\tau(\nu)c_\tau(-\nu))^{-1}$. Here $c_\tau(\nu)$ is given by

$$c_q(\nu) = c_{\tau_q}(\nu) = \frac{(n-1)!\Gamma(i\nu/2)\Gamma(i\nu/2+\frac{1}{2})}{\pi^{1/2}2^{1-n}\Gamma((n+i\nu+q)/2)\Gamma((n+i\nu-q)/2)}$$

when $\tau=\tau_q$. Here $\Gamma(\cdot)$ is the classical gamma function [7].

Let V^τ denote the following set:

$$V^\tau = \{\nu \in \boldsymbol{C};\ \nu = ir,\ r \leq 0 \text{ and } c_\tau(\nu) = 0\}.$$

If $\tau=\tau_q$, then we observe

$$V^{\tau} = \begin{cases} \emptyset \ (empty) & \text{if } \ |q| \leq n, \\ \{v_j = i(2j+n-|q|);\ 0 \leq j < (|q|-n)/2\} & \text{if } \ |q| > n. \end{cases}$$

Put $\mathscr{F}(\rho_o)=\{v \in \boldsymbol{C};\ |\mathrm{Im}\, v| \leq \rho_o = n\}$ and $V_o^{\tau} = V^{\tau} \cap \mathscr{F}(\rho_o)$.

For a moment, we consider the case $|q|>n$. Let $m=\min\{n, |q|-n\}$. Then we see that

$$V_o^{\tau_q} = \left\{ -ik;\ 1 \leq k \leq m, \begin{array}{l} k\text{: even if } |q| \equiv n \mod 2 \\ k\text{: odd if } |q| \not\equiv n \mod 2 \end{array} \right\}.$$

Let $\phi^{\tau}(v, x)$ $(x \in G)$ be the τ-spherical function corresponding to $v \in \boldsymbol{C}$. These τ-spherical functions possess the properties that $\phi^{\tau}(-v, x)=\phi^{\tau}(v, x)$ and $\phi^{\tau}(\bar{v}, x)=\overline{\phi^{\tau}(v, x)}$. Since $\phi^{\tau}(ik, x)$ are linearly independent over $\boldsymbol{C}$, we can choose $\alpha_j \in C_c^{\infty}(G, \tau_q)$ $(j \in iV_o^{\tau_q})$ such that

$$\int_G \alpha_j(x^{-1})\phi^{\tau}(ik, x)dx = \delta_{jk} \qquad (k \in iV_o^{\tau_q})$$

Suppose that $F(\xi, v)$ is a function defined on $\hat{M} \times \boldsymbol{C}$, differentiable on $\mathrm{Int}(\mathscr{F}(\rho_o))$ with respect to v. Let S denote the algebra of differential operators on $\boldsymbol{C}$. For each $u \in S$ and $\alpha \in \boldsymbol{R}$, let $\nu_{u,\alpha}(F)=\sup_{v \in \mathrm{Int}(\mathscr{F}(\rho_o)), \xi \in \hat{M}} |F(\xi, v; u)| (1+|v|)^{\alpha}$.

Let $\mathscr{C}_A^1(\hat{G}, \tau)$ be the linear space of all functions $F(\xi, v)$ on $\hat{M} \times \boldsymbol{C}$ which satisfy the following properties: (1) $F(\xi, v)\equiv 0$ if $\xi \not\leq \tau^M$, (2) $F(\xi, v)$ is holomorphic in $v \in \mathrm{Int}(\mathscr{F}(\rho_o))$, (3) $F(\xi, -v)=F(\xi, v)$ and (4) for all $u \in S$ and $\alpha \in \boldsymbol{R}$, $\nu_{u,\alpha}(F)$ are finite.

It is easy to see that $\mathscr{C}_A^1(\hat{G}, \tau)$ is a Fréchet space under the seminorms $\nu_{u,\alpha}$ $(u \in S, \alpha \in \boldsymbol{R})$.

Next, we set $L_T^+(\tau)=\{\lambda \in L_T^+;\ [\omega(\lambda)|_K : \tau]_K \neq 0\}$. Here $\omega(\lambda)$ $(\in \hat{G})$ is the discrete series representation corresponding to λ as in Section 2.

Let $\mathscr{C}_T^1(\hat{G}, \tau)$ denote the linear space of functions H on L_T^+ such that $H(\lambda)=0$ unless $\lambda \in L_T^+(\tau)$ and $\mu_{\alpha}(H)=\sup_{\lambda \in L_T^+}(1+\|\lambda\|)^{\alpha}|H(\lambda)|$ are finite for all $\alpha \in \boldsymbol{R}$. Here $\|\cdot\|$ denotes the norm introduced by the Killing form on $\mathfrak{g}$.

Topologize $\mathscr{C}_T^1(\hat{G}, \tau)$ by the seminorms μ_{α} $(\alpha \in \boldsymbol{R})$, we see that $\mathscr{C}_T^1(\hat{G}, \tau)$ is a Fréchet space.

If $|q| \leq n$, then we put $\mathscr{C}^1(\hat{G}, \tau_q)=\mathscr{C}_A^1(\hat{G}, \tau_q) \times \mathscr{C}_T^1(\hat{G}, \tau_q)$. On the other hand, if $|q|>n$, let $\mathscr{C}^1(\hat{G}, \tau_q)$ be the linear subspace of $\mathscr{C}_A^1(\hat{G}, \tau_q) \times \mathscr{C}_T^1(\hat{G}, \tau_q)$ of those functions $F=(F_A, F_T)$ which satisfy the following linear relation;

$$F_T(\lambda) = \textstyle\sum_{j \in iV_o^{\tau_q}} \Theta_{\omega(\lambda)}(\alpha_j) F_A(\tau_q^M, ij)$$

for all $\lambda \in L_T^+$ such that $\omega(\lambda) \notin \hat{G}^1$. Here $\hat{G}^1$ denotes the set of equivalence classes of irreducible unitary representations of G whose K-finite matrix coefficients belong to $L^1(G)$. Give $\mathscr{C}_A^1(\hat{G}, \tau) \times \mathscr{C}_T^1(\hat{G}, \tau)$ the product topology and $\mathscr{C}^1(\hat{G}, \tau)$ the induced topology. Then $\mathscr{C}^1(\hat{G}, \tau)$ is a closed subspace of product space. Hence it is a Fréchet space.

If $f \in \mathscr{C}^1(G, \tau)$, we define two maps as follows: $\mathscr{F}_A(f)(\xi, \nu) = \Theta_{\xi,\nu}(f)$ and $\mathscr{F}_T(f)(\lambda) = \Theta_{\omega(\lambda)}(f)$ for all $\xi \in \hat{M}$, $\nu \in \boldsymbol{C}$ and $\lambda \in L_T^+$.

PROPOSITION 3.1. (Trombi [11]) *There is a linear isomorphism* $\mathscr{C}^1(G, \tau)$ *onto* $\mathscr{C}^1(\hat{G}, \tau)$ *under the map* $\mathscr{F} = (\mathscr{F}_A, \mathscr{F}_T)$.

For the purpose of applying the trace formula for $f \in \mathscr{C}^1(G, \tau)$, we need the following fact mentioned in [1].

PROPOSITION 3.2. *The elements of* $\mathscr{C}^1(G, \tau)$ *are admissble.*

Making use of the above two propositions we obtain

PROPOSITION 3.3. *Let* $\omega(\lambda)$ $(\lambda \in L_T^+)$ *be the discrete series representation of* G *and* $d(\omega(\lambda))$ *its formal degree. Suppose that* $\omega(\lambda)$ *has a one dimensional* K*-type* τ_q *(i.e.* $[\omega(\lambda)|_K : \tau_q]_K \neq 0$*) for some* $q \in \boldsymbol{Z}$ *and* $\omega(\lambda) \in \hat{G}^1$. *Then we have*

$$m_{\Gamma,T}(\omega(\lambda)) = \chi_T(e) \operatorname{vol}(G/\Gamma) d(\omega(\lambda))$$

for our normalization of Haar measure.

PROOF. Let $\lambda_o \in L_T^+$. If $\omega(\lambda_o) \in \hat{G}^1$, then we need not consider the linear relation in the definition of $\mathscr{C}^1(\hat{G}, \tau)$. So we may take the element $F = (F_A, F_T)$ of $\mathscr{C}^1(\hat{G}, \tau)$ such that $F_A \equiv 0$, $F_T(\lambda_o) = 1$ and $F_T(\lambda) = 0$ for all $\lambda \neq \lambda_o$. Proposition 3.1 says that there is a function f in $\mathscr{C}^1(G, \tau)$ such that $\mathscr{F}(f) = F$. Applying the trace formula to this admissible function f, we get

$$\begin{aligned} m_{\Gamma,T}(\omega(\lambda_o))\Theta_{\omega(\lambda_o)}(f) &= \chi_T(e) \operatorname{vol}(G/\Gamma) f(e) \\ &= \chi_T(e) \operatorname{vol}(G/\Gamma) \textstyle\sum_{\lambda \in L_T^+} d(\omega(\lambda))\Theta_{\omega(\lambda)}(f) \\ &= \chi_T(e) \operatorname{vol}(G/\Gamma) d(\omega(\lambda_o))\Theta_{\omega(\lambda_o)}(f). \end{aligned}$$

REMARK 1. Applying Trombi's result in the general situation, namely, without restriction about the dimension of the representation of K, we obtain the same result as in Proposition 3.3 for any element of $\hat{G}^1$.

REMARK 2. Suppose that π is in $\hat{G}^1$. Then, as a consequence of the above consideration, we find that the quantity $m_{\Gamma,T}(\pi)/\chi_T(e)$ is independent of the choice of the finite dimensional unitary representation T of Γ.

§4. The Zeta function

In this section, we should like to define the logarithmic derivative $\eta_{\Gamma,T,\tau}$ of $Z_{\Gamma,T,\tau}$ and study its analytic continuation.

Let $\varepsilon_0>0$ be a fixed real number and let g be a real valued function in $C^\infty(\boldsymbol{R})$ such that (1) g is even, (2) g vanishes in some neighborhood of zero, (3) g is constant, equal to c for $|x|\geq\varepsilon_0$ and (4) $0\leq g\leq c$. Such functions surely exist. The value of c and ε_0 will be chosen conveniently later on.

We now put $\tilde{\varepsilon}_q(j)=\frac{1}{2}((-1)^{n+q+j}+1)$. For each $\tau_q\in\hat{K}_1$, we define a polynomial $P_q=P_{\tau_q}$ as follows:

$$P_q(v)=\begin{cases}1 & \text{if } |q|\leq n\\ \prod_{j=1}^{m}(v^2+j^2)\tilde{\varepsilon}_q(j) & \text{if } |q|>n,\end{cases}$$

where $m=\min\{n, |q|-n\}$.

Let D_q be a differential operator on $\boldsymbol{R}$ whose Fourier transform is P_q.

For any complex number s, define a function ${}_q\mathscr{G}_s$ on $MA_\mathfrak{p}$ by

$$(4.1)\qquad {}_q\mathscr{G}_s(ma_t)=\tau_q^M(m^{-1})D_q(g(|t|)\exp(\rho_o-s)|t|)\qquad(m\in M).$$

Since g vanishes in a neighborhood of zero, ${}_q\mathscr{G}_s$ is a smooth function on $MA_\mathfrak{p}$.

Let $H(r)=\int_0^\infty g'(x)\exp(irx)dx$ $(r\in\boldsymbol{C})$. Because of the properties of g, we see that g' is in $C_c^\infty(\boldsymbol{R})$ and $g'(x)=0$ if $|x|\geq\varepsilon_0$. Hence an application of the classical Paley-Wiener theorem gives us the following lemma as in [2].

LEMMA 4.1. *H is an entire function. Moreover, for any integers $n\geq1$ and $m\geq0$, we can find the constant $C_{m,n}>0$ such that we have the estimates*

$$|d^mH(r)/dr^m|\leq\begin{cases}C_{m,n}(|r|+1)^{-n} & \text{if } \operatorname{Im} r\geq0,\\ C_{m,n}(|r|+1)^{-n}\exp(\varepsilon_0|\operatorname{Im} r|) & \text{if } \operatorname{Im} r<0.\end{cases}$$

Using this function H, we can calculate the Fourier transform ${}_q\hat{\mathscr{G}}_s(\xi, v)$ of ${}_q\mathscr{G}_s$ at the character (χ_ξ, v) of $MA_\mathfrak{p}$.

LEMMA 4.2. *For* $\operatorname{Re}(s-2\rho_o)>0$, *we have*

$$(4.2)\qquad {}_q\hat{\mathscr{G}}_s(\xi, v)=\begin{cases}0 & \text{if } \xi\not\simeq\tau_q^M,\\ P_q(v)\left\{\dfrac{H(i(s-\rho_o)-v)}{s-\rho_o+iv}+\dfrac{H(i(s-\rho_o)+v)}{s-\rho_o-iv}\right\} & \text{if } \xi\simeq\tau_q^M.\end{cases}$$

The proof of this lemma is similar to that of Scott [9, p. 181]. So we omit it.

PROPOSITION 4.3. *Suppose that* $\operatorname{Re}s>2\rho_o$. *Then there exists a function*

${}_q g_s$ in $\mathscr{C}^1(G, \tau_q)$ such that $\Theta_{\xi,\nu}({}_q g_s)={}_q\hat{\mathscr{G}}_s(\xi, \nu)$ and $\Theta_{\omega(\lambda)}({}_q g_s)=0$ for all $\xi \in \hat{M}$ and $\lambda \in L_T^+$.

PROOF. It is clear that ${}_q\hat{\mathscr{G}}_s(\xi, \nu)=0$ if $\xi \not\simeq \tau_q^M$ and ${}_q\hat{\mathscr{G}}_s(\xi, -\nu)={}_q\hat{\mathscr{G}}_s(\xi, \nu)$. If $\operatorname{Re} s>2\rho_o$, then $(s-\rho_o \pm i\nu)^{-1}$ have all their derivatives bounded in a strip $|\operatorname{Im} \nu| \leq \rho_o+\varepsilon$ where $0<\varepsilon<\operatorname{Re}(s-2\rho_o)$. From Lemma 4.1 $P_q(\nu)H(i(s-\rho_o)\pm \nu)$ are holomorphic and rapidly decreasing functions of ν in any strip $|\operatorname{Im} \nu| \leq b$. Consequently ${}_q\hat{\mathscr{G}}_s$ is an element of $\mathscr{C}_A^1(\hat{G}, \tau_q)$.

Next, we shall show that $({}_q\hat{\mathscr{G}}_s, 0)$ is an element of $\mathscr{C}^1(\hat{G}, \tau_q)$. Since $0 \in \mathscr{C}_T^1(\hat{G}, \tau_q)$, it suffices to show that the linear relation holds for $({}_q\hat{\mathscr{G}}_s, 0)$. But we can easily check it directly as follows. In the case $|q| \leq n$, we see that $V_o^{\tau_q}=\emptyset$. Therefore we need not consider the linear relation in the definition of $\mathscr{C}^1(\hat{G}, \tau_q)$. Next, suppose that $|q|>n$. Then, since $P_q(ij)=0$ for all j $(1 \leq j \leq m)$, one finds that ${}_q\hat{\mathscr{G}}_s(\tau_q^M, ij)=0$. Hence it is clear that the linear relation holds.

On account of Proposition 3.1 we have the desired result in any case.

For each $\tau_q \in \hat{K}_1$, Proposition 4.3 and (2.2) say that

$$F_{{}_q g_s} = {}_q\mathscr{G}_s. \tag{4.3}$$

By the assumption on Γ, it is known that the numbers $\{|u_\gamma|;\ \gamma \in C_\Gamma-\{e\}\}$ are bounded away from zero [2]. If we choose and fix ε_0 so small that it is smaller than all of these values, we have

$$g(|u_\gamma|) = c \qquad (\gamma \in C_\Gamma-\{e\}). \tag{4.4}$$

If we restrict the function g to the region $\{t;\ |t| \geq \varepsilon_0\}$ on which $g(t)=c$ holds, then we are able to show that

$$D_q(g(t)\exp(\rho_o-s)|t|) = cP_q(i(\rho_o-s))\exp(\rho_o-s)|t|, \tag{4.5}$$

by the direct calculations.

Put $\pi_{q,\nu}=\pi_{\tau_q^M,\nu}$. If $q \equiv n \pmod 2$, $|q| \geq n$, then it is known that the representation $\pi_{q,0}$ is reducible. Moreover, for such q, we have $\pi_{q,0} \simeq \pi_{q,0}^+ \oplus \pi_{q,0}^-$. Here $\pi_{q,0}^+$ (resp. $\pi_{q,0}^-$) is so-called the limit of discrete series representation of G satisfying $[\pi_{q,0}^+|_K;\ \tau_q]=0$ for $q<0$ (resp. $[\pi_{q,0}^-|_K;\ \tau_q]=0$ for $q>0$) (cf. [5], [6]). Hence we see that

$$(*) \quad m_\Gamma(\pi_{q,0}^+)\Theta_{q,0}^+({}_q g_s) + m_\Gamma(\pi_{q,0}^-)\Theta_{q,0}^-({}_q g_s) = \begin{cases} m_\Gamma(\pi_{q,0}^+)\Theta_{q,0}^+({}_q g_s) & \text{if } q>0, \\ m_\Gamma(\pi_{q,0}^-)\Theta_{q\ 0}^-({}_q g_s) & \text{if } q<0. \end{cases}$$

So, with the idea of giving ourselves the least possible trouble, we make a change in the definition of $\pi_{q,0}$ to the following effect;

$$\pi_{q,0} = \begin{cases} \pi^+_{q,0} & \text{if} \quad q > 0, \\ \pi^-_{q,0} & \text{if} \quad q < 0. \end{cases}$$

Then we have

$$(*) = m_\Gamma(\pi_{q,0})\Theta_{q,0}({}_q g_s).$$

Remark that $\pi_{q,0}$ is irreducible if $q \not\equiv n \pmod 2$. Let $Q_q = \{\pi \in \hat{G};\ \pi \in L^2(G/\Gamma,\ T)$ and $\Theta_\pi({}_q g_s) \neq 0\}$. Define two subsets Q^1_q and Q^2_q of $\boldsymbol{C}$ by

$$Q^1_q = \{\lambda \in \boldsymbol{R}^+;\ \pi_{q,\lambda} \in \hat{G} \text{ and } \pi_{q,\lambda} \in L^2(G/\Gamma,\ T)\},$$
$$Q^2_q = \{_\lambda \in i\boldsymbol{R}^+ - \{0\};\ \pi_{q,\lambda} \in \hat{G} \text{ and } \pi_{q,\lambda} \in L^2(G/\Gamma,\ T)\}.$$

Since the definition of τ_q implies that $\pi_{q,\nu}$ is equivalent to $\pi_{q,-\nu}$, Q_q is parametrized by the set $Q^1_q \cup Q^2_q$ under the convention for the definition of $\pi_{q,0}$. Hereafter, we are looking on Q_q as $Q^1_q \cup Q^2_q$.

Now we define

$$A_q(s) = \textstyle\sum_{\lambda \in Q_q} m(q,\ \lambda)_q\hat{\mathscr{G}}_s(\tau^M_q,\ \lambda). \tag{4.6}$$

Here we put $m(q,\ \lambda) = m_\Gamma(\pi_{q,\lambda})$.

The following result is proved by Wallach [13].

PROPOSITION 4.4. *There is a real number α_o such that for any $\delta \in \hat{K}$ and all $\alpha > \alpha_o$*

$$\textstyle\sum_{\pi \in \hat{G}} [\pi|_K;\ \delta] m_\Gamma(\pi)(1 + |\pi(\Omega)|)^{-\alpha} < +\infty$$

where Ω is the Casimir operator on G.

Making use of Proposition 4.1 and 4.4, we obtain

PROPOSITION 4.5. *The function $A_q(s)$ has a meromorphic continuation to the whole complex plane. The poles of A_q occur at the points $s = \rho_o \pm i\lambda$ $(\lambda \in Q_q)$. These poles are all simple and the residue at $s = \rho_o \pm i\lambda$ is $m(q,\ \lambda)P_q(\lambda)H(0)$. Here, if $P_q(\lambda) = 0$, then we interpret that there is no pole at $s = \rho_o \pm i\lambda$.*

By Proposition 3.2, the function ${}_q g_s$ is admissible if $\operatorname{Re} s > \rho_o$. So we get, with the help of (4.3), (4.4) and (4.5),

$$\begin{aligned} A_q(s) &= \chi_T(e)\operatorname{vol}(G/\Gamma)_q g_s(e) \\ &+ cP_q(i(\rho_o - s)) \sum_{\gamma \in C_\Gamma - \{e\}} \chi_T(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma))\tau^M_q(h_t(\gamma)^{-1} \exp(\rho_o - s)|u_\gamma|. \end{aligned} \tag{4.7}$$

For $\operatorname{Re} s > 2\rho_o$, we define $\tilde{\eta}_{\Gamma,T,\tau_q}(s)$ by the second term on the rieght side of (4.7). For simplicity we put $\tilde{\eta}_q(s) = \tilde{\eta}_{\Gamma,T,\tau_q}(s)$.

Since ${}_q g_s$ is admissible, the sum is absolutely convergent and it is readily seen to be absolutely and uniformly convergent in any half plane $\operatorname{Re} s > 2\rho_o + \varepsilon$ with $\varepsilon > 0$. Hence $\tilde{\eta}_q(s)$ is holomorphic in the half plane $\operatorname{Re} s > 2\rho_o$.

We will now consider the term $\chi_T(e)\operatorname{vol}(G/\Gamma)_q g_s(e)$ and show that it is meromorphic with respect to s. By the Plancherel theorem, we have

$$(4.8) \qquad {}_q g_s(e) = (1/4\pi) \sum_{\xi \in \hat{M}} \int_{\mathbf{R}} \Theta_{\xi,\nu}({}_q g_s)\mu_\xi(\nu) d\nu$$
$$= (1/2\pi) \int_{\mathbf{R}} P_q(\nu) \frac{H(i(s-\rho_o)+\nu)}{s-\rho_o-i\nu} \mu_q(\nu) d\nu$$

by virtue of (4.3) and the evenness of the functions P_q and μ_q (see §3).

We now shift the integration into the complex plane by using rectangular contour as in [2]. The function μ_q is meromorphic in the upper half plane, and can only have simple poles which are listed below.

TABLE 1. $r_k = r_{q,k}$; *the pole of* μ_q $(k \in \mathbf{Z})$
$d_k = {}_{q,k}$; *the residue of* μ_q *at the pole* r_k
$\rho_o = n$

	r_k	d_k
$\lvert q\rvert \leq n$	$r_k = i(\rho_o + \lvert q\rvert + 2k)$ $(k \geq 0)$	$id_k = \frac{(-1)^n}{2^{2n-2}}(n+\lvert q\rvert+2k)\cdot \binom{n+\lvert q\rvert+k-1}{n-1}\binom{n+k-1}{n-1}$
$\lvert q\rvert > n$ $q \equiv n$ (mod 2)	$r_k = 2i(k+1)$ $(0 \leq k \leq (\lvert q\rvert-n)/2-1)$	$id_k = -\frac{k+1}{2^{2n-3}}\binom{(\lvert q\rvert+n)/2+k}{n-1}\cdot \binom{(\lvert q\rvert+n)/2-k-2}{n-1}$
	$r_{k+(\lvert q\rvert-n)/2}$ $(k \geq 0)$ $= i(\rho_o + \lvert q\rvert + 2k)$	$id_{k+(\lvert q\rvert-n)/2} = \left(id_k \textit{ in the case } \lvert q\rvert \leq n \right)$
$\lvert q\rvert > n$ $q \not\equiv n$ (mod 2)	$r_k = i(2k+1)$ $(0 \leq k \leq (\lvert q\rvert-n-1)/2$	$id_k = -\frac{k(k+\frac{1}{2})}{2^{2n-3}}\binom{(\lvert q\rvert+n-1)/2+k}{n-1}\cdot \binom{(\lvert q\rvert+n-1)/2-k-1}{n-1}$
	$r_{k+(\lvert q\rvert-n+1)/2}$ $(k \geq 0)$ $= i(\rho_o + \lvert q\rvert + 2k)$	$id_{k+(\lvert q\rvert-n+1)/2} = \left(id_k \textit{ in the case } \lvert q\rvert \leq n \right)$

The same argument of [2] shows that

$$(4.9)\qquad {}_q g_s(e)=i\sum_{k\geq 0}\frac{H(i(s-\rho_o)+r_k)}{s-\rho_o-ir_k}P_q(r_k)d_k \qquad (\mathrm{Re}\, s>2\rho_o)$$

by the residue theorem.

PROPOSITION 4.6. *The series on the right side of* (4.9) *covnerges absolutely and uniformly for s in any compact set disjoint from the numbers* $\{\rho_o+ir_k\}$, *and defines a meromorphic function of s in the whole complex plane. This function has simple poles at the points* $\rho_o+ir_k(k\geq 0,\ k\in \boldsymbol{Z})$ *and has the residue* $iH(0)P_q(r_k)d_k$ *at* $s=\rho_o+ir_k$.

The second assertion of Proposition 4.6 is proved by using Lemma 4.1. But since the proof is similar to that of [2, Proposition 2.6], we omit the proof.

Note that the value $i\chi_T(e)\,\mathrm{vol}\,(G/\Gamma)H(0)P_q(r_k)d_k$ is real since d_k is pure imaginary. As seen in [2], under our normalization of measure, it turns out that $\mathrm{vol}\,(G/\Gamma)$ is a rational multiple of the Eular-Poincaré characteristic E of the manifold $K\backslash G/\Gamma$. Also, Table 1 shows that id_k is a rational number and we are able to choose the denominator of the residue of the function $\chi_T(e)\,\mathrm{vol}\,(G/\Gamma)_q g_s(e)$ so that it depends only on G and not on k and q. Hence there is a positive integer $\kappa=\kappa(G)$ such that $i\,\mathrm{vol}\,(G/\Gamma)d_k=e_kE/\kappa$, where $e_k=e_{k,q}$ is an integer. Note that e_kE and id_k have the same sign.

Recall that, in defining $\tilde{\eta}_q(s)$ we had used a constant c, with $g(t)=c$ when $t\geq\varepsilon_0$. We now take κ for c. Then we see that $H(0)=\kappa$ and since $P_q(r_k)$ is an integer, the residues of the function $\chi_T(e)\,\mathrm{vol}\,(G/\Gamma)_q g_s(e)$ are all integers.

By means of Proposition 4.5, 4.6 and the definition of the function $\tilde{\eta}_q(s)$, we get the following proposition.

PROPOSITION 4.7. *For any* $\tau_q\in\hat{K}_1$, $\tilde{\eta}_q(s)$ *has meromorphic continuation to the whole complex plane, via the relation* $\tilde{\eta}_q(s)=A_q(s)-\chi_T(e)\ \mathrm{vol}\ (G/\Gamma)_q g_s(e)$. *The poles of* $\tilde{\eta}_q(s)$ *are all simple, and are as follows*:

Pole	*Residue*	
$s=\rho_o\pm i\lambda$	$\kappa m_\Gamma(q,\lambda)P_q(\lambda)$	$(\lambda\in Q_q)$
$s=\rho_o+ir_k$	$-e_kE\chi_T(e)P_q(r_k)$	$(k\geq 0,\ k\in \boldsymbol{Z})$.

Here, if for some $\lambda\in Q_q$ *there is k such that* $\lambda=r_k$, *then we understand the residue at this pole is* $(\kappa m_\Gamma(q,r_k)-e_kE\chi_T(e))P_q(r_k)$. *Also, if* $\lambda=0$ *is in* Q_q, *the residue at this pole is* $2\kappa m_\Gamma(q,0)P_q(0)$. *Of course, if* $P_q(\mu)=0$, *then* $s=\rho_o\pm i\mu$ *is not a pole.*

To show that $\tilde{\eta}_q$ satisfies a functional equation, it is convenient to perform the change of variable $r=-i(s-\rho_o)$ and let $\bar{\eta}_q(r)=\tilde{\eta}_q(ir+\rho_o)=\tilde{\eta}_q(s)$. If $\operatorname{Re} s>2\rho_o$, that is $\operatorname{Im} r<-\rho_o$, then

$$(4.10)\qquad \bar{\eta}_q(r) = \kappa P_q(r) \sum_{\gamma\in C_\Gamma-\{e\}} \chi_T(\gamma)\,|u_\gamma|\,j(\gamma)^{-1}C(h(\gamma))\tau_q^M(h_t(\gamma))^{-1}\exp(-ir|u_\gamma|)$$

and the sum is absolutely and uniformly convergent in any half plane $\operatorname{Im} r<-\rho_o-\delta\ (\delta>0)$. By Proposition 4.7 we get

$$(4.11)\qquad \bar{\eta}_q(r) = -i\sum_{\lambda\in Q_q} m(q,\lambda)P_q(\lambda)\left\{\frac{H(-r-\lambda)}{r+\lambda}+\frac{H(-r+\lambda)}{r-\lambda}\right\}$$
$$-\chi_T(e)\operatorname{vol}(G/\Gamma)\sum_{k\geq 0}\frac{H(-r+r_k)}{r-r_k}P_q(r_k)d_k.$$

The residues of $\bar{\eta}_q(r)$ at $r=\pm\lambda\ (\lambda\in Q_q)$ (resp. $r=r_k\ k\geq 0$) are $-i\kappa m_\Gamma(q,\lambda)P_q(\lambda)$ (resp. $ie_kE\chi_T(e)P_q(r_k)$).

Now let

$$\Phi_q(t) = \kappa\chi_T(e)\operatorname{vol}(G/\Gamma)P_q(it)\mu_q(it).$$

In order to prove the functional equation, we need the following lemma.

LEMMA 4.8. *There is a sequence* $\{x_m\}\to\infty$ $(m\to\infty)$ *so that for any* $y\geq 0$ *there is a polynomial P such that*

$$\sup\{\bar{\eta}_q(r)P(|r|)^{-1};\ |\operatorname{Im} r|\leq y,\ \pm\operatorname{Re} r\in\{x_m\}\} < +\infty.$$

The proof of this lemma is a slight extention of that of [9, Proposition 4.14], making use of Proposition 4.4. We omit the proof.

PROPOSITION 4.9. *For* $\tau_q\in\hat{K}_1$, *we have a functional equation*:

$$(4.12)\qquad \tilde{\eta}_q(s)+\tilde{\eta}_q(2\rho_o-s)+\Phi_q(s-\rho_o)=0.$$

PROOF. Put $\bar{\Phi}_q(r)=\Phi_q(s-\rho_o)$. To prove (4.12), it suffices to show that the following equation holds:

$$(4.13)\qquad \bar{\eta}_q(r)+\bar{\eta}_q(-r)+\bar{\Phi}_q(r)=0.$$

Because of (4.11), the meromorphic function $\bar{\eta}_q(r)+\bar{\eta}_q(-r)$ has only simple poles at $r=\pm r_k$ with residues $\pm ie_kE\chi_T(e)P_q(r_k)$ respectively. On the other hand, the poles of $\bar{\Phi}_q(r)$ are at r_k and $-r_k$, and the residues are $-ie_kE\chi_T(e)P_q(r_k)$ and $ie_kE\chi_T(e)P_q(r_k)$ respectively. It follows that the left side of (4.13), say $\bar{q}_q(r)$, is an entire function. We will show that $\bar{q}_q(r)\equiv 0$.

Fix $\varepsilon>0$ and let b be an even holomorphic function that is rapidly decreasing in the strip $\{z;\ |\operatorname{Im} z|\leq\rho_o+2\varepsilon\}$. Let $y=\rho_o+\varepsilon$ and for any positive real number x such that $\pm x\notin Q_q$, let O_x be a rectangular contour in the complex r-plane with vertices $\pm x \pm iy$. Let E_x (resp. E_{-x}) be the side from $x-iy$ to $x+iy$ (resp. $-x+iy$ to $-x-iy$) and let B_x^+ (resp. B_x^-) be the side from $x+iy$ to $-x+iy$ (resp. $-x-iy$ to $x-iy$). Note that the function b is holomorphic inside of O_x. Thus by the residue theorem, we have

$$\begin{aligned}\int_{O_x} & b(r)\bar{\eta}_q(r)dr \\ &= 2\pi i\{i\chi_T(e)\textstyle\sum_{|r_k|\le\rho_o} e_k E P_q(r_k)b(r_k) \\ &\quad + \kappa \textstyle\sum_{\lambda\in Q_q\cap \mathrm{Int}(O_x)}(-i)\ m(q,\lambda)P_q(\lambda)(b(\lambda)+b(-\lambda))\}\end{aligned}$$

Put $O_\infty = \lim_{x\to\infty} O_x$. Then, since b is even we get

$$(4.14)\quad \begin{aligned}\int_{O_\infty} & b(r)\bar{\eta}_q(r)dr \\ &= -2\pi\chi_T(e)E\textstyle\sum_{|r_k|\le\rho_o} e_k P_q(r_k)b(r_k) + 4\pi\kappa\textstyle\sum_{\lambda\in Q_q} m(q,\lambda)P_q(\lambda)b(\lambda).\end{aligned}$$

On the other hand, the evenness of b and the relation $-\bar{\eta}_q(-r)=\bar{\eta}_q(r)+\bar{\Phi}_q(r)-\bar{q}_q(r)$ imply that

$$\begin{aligned}\int_{O_x} b(r)\bar{\eta}_q(r)dr &= 2\int_{B_x^-} b(r)\bar{\eta}_q(r)dr + \int_{B_x^-} b(r)\bar{\Phi}_q(r)dr \\ &-\int_{B_x^-} b(r)\bar{q}_q(r)dr + \int_{E_x} b(r)\bar{\eta}_q(r)dr + \int_{E_{-x}} b(r)\bar{\eta}_q(r)dr.\end{aligned}$$

Combinning Lemma 4.8 with the fact that b is rapidly decreasing, we conclude

$$\lim_{x\to\infty\ x\notin Q_q}\int_{E\pm x} b(r)\bar{\eta}_q(r)dr = 0.$$

Therefore we have

$$(4.15)\quad \begin{aligned}\int_{O_\infty} & b(r)\bar{\eta}_q(r)dr \\ &= 2\int_{L_{-y}} b(r)\bar{\eta}_q(r)dr + \int_{L_{-y}} b(r)\bar{\Phi}_q(r)dr - \int_{L_{-y}} b(r)\bar{q}_q(r)dr\end{aligned}$$

where L_{-y} denotes the line in the complex plane $t-iy$ as t goes from $-\infty$ to ∞.

From (4.14) and (4.15), we can write

$$(4.16)\quad \begin{aligned}-2^{-1}\chi_T(e)E & \textstyle\sum_{|r_k|\le\rho_o} e_k P_q(r_k)b(r_k) + \kappa\textstyle\sum_{\lambda\in Q_q} m(q,\lambda)P_q(\lambda)b(\lambda) \\ &= (1/2\pi)\left\{\int_{L_{-y}} b(r)\bar{\eta}_q(r)dr + 2^{-1}\int_{L_{-y}} b(r)\bar{\Phi}_q(r)dr\right. \\ &\qquad\qquad \left. - 2^{-1}\int_{L_{-y}} b(r)\bar{q}_q(r)dr\right\}.\end{aligned}$$

On the line L_{-y}, the series (4.10) that defines $\bar{\eta}_q$ converges absolutely and uniformly, so we have

$$(4.17)\quad (1/2\pi)\int_{L_{-y}} b(r)\bar{\eta}_q(r)dr$$
$$= \kappa \sum_{\gamma\in C_\Gamma-\{e\}} \chi_T(\gamma)\,|u_\gamma|\,j(\gamma)^{-1}C(h(\gamma))\tau_q^M(h_t(\gamma))^{-1}$$
$$\times (1/2\pi)\int_{L_{-y}} P_q(r)b(r)\exp(-ir|u_\gamma|)dr.$$

Since $P_q(r)b(r)\exp(-ir|u_\gamma|)$ is holomorphic and b is rapidly decreasing, we may shift the contour of integration to the real line. Hence we have

$$(4.18)\quad \int_{L_{-y}} P_q(r)b(r)\exp(-ir|u_\gamma|)dr = \int_{\boldsymbol{R}} P_q(r)b(r)\exp(-ir|u_\gamma|)dr.$$

Now we define a function B in $\mathscr{C}^1(\hat{G}, \tau_q)$ by

$$B(\xi, r) = \begin{cases} 0 & \text{if } \xi \not\simeq \tau_q^M, \\ P_q(r)b(r) & \text{if } \xi \simeq \tau_q^M. \end{cases}$$

Then applying Proposition 3.1, we see that there is a function f in $\mathscr{C}^1(G, \tau_q)$ such that $\mathscr{F}(f)=(B, 0)$. The Fourier inversion formula on MA_p implies

$$(4.19)\quad \tau_q^M(h_t(\gamma))^{-1}(1/2\pi)\int_{\boldsymbol{R}} P_q(r)b(r)\exp(-ir|u_\gamma|)dr$$
$$= (1/2\pi)\sum_{\xi\in\hat{M}}\int_{\boldsymbol{R}} \operatorname{tr}\xi(h_t(\gamma))B(\xi, r)\exp(-ir|u_\gamma|)dr$$
$$= F_f(h(\gamma)).$$

Since $\bar{\Phi}_q(r)$ is atempered function and b is rapidly decreasing, using the residue theorem again we may shift the contour of the integration to get

$$(4.20)\quad (1/4\pi)\int_{L_{-y}} b(r)\bar{\Phi}_q(r)dr$$
$$= (1/4\pi)\int_{\boldsymbol{R}} b(r)\bar{\Phi}_q(r)dr + (i/2)\{i\chi_T(e)E\sum_{|r_k|\le\rho_o} b(-r_k)e_kP_q(r_k)\}$$
$$= \kappa\chi_T(e)\operatorname{vol}(G/\Gamma)f(e) - (1/2)\chi_T(e)E\sum_{|r_k|\le\rho_o} b(r_k)e_kP_q(r_k).$$

The last equality is the Plancherel theorem.

From the equalities (4.16)–(4.20) we obtain

$$(4.21)\quad (1/4\pi)\int_{L_{-y}} b(r)\bar{q}_q(r)dr$$
$$= \kappa\{\chi_T(e)\operatorname{vol}(G/\Gamma)f(e) + \textstyle\sum_{\gamma\in C_\Gamma-\{e\}} \chi_T(\gamma)\,|u_\gamma|\,j(\gamma)^{-1}C(h(\gamma))F_f(h(\gamma))$$
$$- \textstyle\sum_{\lambda\in Q_q} m(q,\lambda)b(\lambda)P_q(\lambda)\}\,.$$

Applying the trace formula to the admissible function f, it is clear that the right side of (4.21) is equal to zero. Therefore, by shifting the contour of integration from L_{-y} to $\boldsymbol{R}$, we get

$$(4.22)\qquad \int_{\boldsymbol{R}} b(r)\bar{q}_q(r) = 0.$$

Since b is arbitrary even holomorphic and rapidly decreasing function in the strip $|\operatorname{Im} r|\le\rho_o+2\varepsilon$, and $\bar{q}_q$ is an even function, one deduces from (4.22) that $\bar{q}_q(r)=0$ on $\boldsymbol{R}$. But $\bar{q}_q$ is entire, hence $\bar{q}_q\equiv 0$, and Proposition 4.9 is proved.

Now put

$$(4.23)\qquad \eta_q^o(s) = \tilde{\eta}_q(s)(P_q(i(s-\rho_o)))^{-1}$$

and

$$\phi_q^o(s) = -\,\Phi_q(s-\rho_o)(P_q(i(s-\rho_o)))^{-1}$$
$$= -\,\kappa\chi_T(e)\operatorname{vol}(G/\Gamma)\mu_q(i(s-\rho_o))\,.$$

Suppose that $|q|>n$. Then $P_q(i(s-\rho_o))=\prod_{j=1}^m\{-(s-\rho_o)^2+j^2\}\tilde{\varepsilon}_q(j)$. In this case, $s=\rho_o\pm j$ $(j\in iV_o^{\tau_q})$ is not a pole of $\tilde{\eta}_q(s)$ by means of Proposition 4.7. Thus $\eta_q^o(s)$ can have additional simple poles at $s=\rho_o\pm j$, $j\in iV_o^{\tau_q}$. Now let

$$r_q^{\pm}(j) = \operatorname{Res}_{s=\rho_o\pm j}\eta_q^o(s)\qquad (j\in iV_o^{\tau_q})\,.$$

Then the functional equation $\eta_q^o(s)+\eta_q^o(2\rho_o-s)=\phi_q^o(s)$ implies that

$$r_q^+(j) - r_q^-(j) = d_q(j)\qquad (j\in iV_o^{\tau_q})\,.$$

Here we put

$$d_q(j) = \operatorname{Res}_{s=\rho_o+j}\phi_q^o(s)\qquad (j\in iV_o^{\tau_q})\,.$$

We now define the following two functions:

$$(4.24)\qquad F_q(s) = \begin{cases} 0 & \text{if } |q|\le n,\\ \sum_{j=1}^m\left\{\dfrac{r_q^+(j)}{s-\rho_o-j} + \dfrac{r_q^-(j)}{s-\rho_o+j}\right\}\tilde{\varepsilon}_q(j) & \text{if } |q|>n,\end{cases}$$
$$G_q(s) = \begin{cases} 0 & \text{if } |q|\le n,\\ \sum_{j=1}^m\left\{\dfrac{d_q(j)}{s-\rho_o-j} - \dfrac{d_q(j)}{s-\rho_o+j}\right\}\tilde{\varepsilon}_q(j) & \text{if } |q|>n.\end{cases}$$

Moreover we define

$$\eta_q(s) = \eta_q^o(s) - F_q(s) \tag{4.25}$$

and

$$\phi_q(s) = \phi_q^o(s) - G_q(s).$$

Put $\tilde{Q}_q = \{\lambda \in Q_q;\ P_q(\lambda) \neq 0\}$. Note the fact that $Q_q^1 \subset \tilde{Q}_q$. We now summarize these observations.

PROPOSITION 4.10. *For $\tau_q \in \hat{K}_1$, η_q is a meromorphic function with simple poles. The (non-trivial) poles of η_q are located at $s = \rho_o \pm i\lambda$ ($\lambda \in \tilde{Q}_q$) with residues $\kappa m_\Gamma(q, \lambda)$ for any q. Apart from these poles, there exist a series of the (trivial) poles as follows:*

TABLE 2.

τ_q	*Pole*	*Residue* $(-\kappa id_k \chi_T(e)\ \mathrm{vol}(G/\Gamma))$
$\lvert q\rvert \leq 2n$	$-(\lvert q\rvert + 2k)$ $(k \geq 0)$	$id_k = \frac{(-1)^n}{2^{2n-2}}(n + \lvert q\rvert + 2k) \cdot \binom{n + \lvert q\rvert + k - 1}{n-1} \binom{n + k - 1}{n-1}$
$\lvert q\rvert > 2n$, $q \equiv n$ (mod 2)	$-(\lvert q\rvert + 2k)$ $(k \geq 0)$	*same* id_k *as in the case* $\lvert q\rvert \leq 2n$
	$n - 2(k+1)$ $([n/2] \leq k \leq (\lvert q\rvert - n)/2 - 1)$	$id_k = -\frac{k+1}{2^{2n-3}} \binom{(\lvert q\rvert + n)/2 + k}{n-1} \cdot \binom{(\lvert q\rvert + n)/2 - k - 2}{n-1}$
$\lvert q\rvert > 2n$, $q \not\equiv n$ (mod 2)	$-(\lvert q\rvert + 2k)$ $(k \geq 0)$	*same* id_k *as in the case* $\lvert q\rvert \leq 2n$
	$n - (2k+1)$ $([(n+1)/2] \leq k \leq (\lvert q\rvert - n - 1)/2)$	$id_k = -\frac{k(k+\frac{1}{2})}{2^{2n-3}} \binom{(\lvert q\rvert + n - 1)/2 + k}{n-1} \cdot \binom{(\lvert q\rvert + n - 1)/2 - k - 1}{n-1}$

The poles described above are the only poles of η_q.
Furthermore η_q satisfies the functional equation:

$$\eta_q(s) + \eta_q(2\rho_o - s) = \phi_q(s).$$

REMARK 1. If $q \neq 0$, then Table 2 shows that the trivial poles are all negative.

REMARK 2. If $q=0$ and $s=0$ is a non-trivial pole of $\eta_q(s)$ (that is $i\rho_o \in \tilde{Q}_q$), then we understand that the residue of the pole at this point is $\kappa m_\Gamma(0, i\rho_o) - \kappa i d_k \chi_T(e) \operatorname{vol}(G/\Gamma)$.

REMARK 3. If $0 \in \tilde{Q}_q$ then, of course, the residue at the point $s=\rho_o$ is $2\kappa m_\Gamma(q, 0)$.

Since the function $\eta_q(s)=\eta_{\Gamma,T,\tau_q}(s)$ has only simple poles with integer residues, we can find a meromorphic function $Z_q(s)=Z_{\Gamma,T,\tau_q}(s)$ such that $(d/ds)(\log Z_q(s))=\eta_q(s)$. The function Z_q will be defined up to multiplicative constant. Hence we can choose a point $s_o \in \boldsymbol{C}$ with $\operatorname{Re} s_o > 2\rho_o$ and a constant c_o which normalizes suitably Z_q such that

$$Z_q(s) = c_o \exp\left(\int_{s_o}^{s} \eta_q(z)dz\right).$$

We now come to our main result.

THEOREM 4.11. *For each $\tau_q \in \hat{K}_1$, the function Z_q has following properties.*

(A) *Z_q is holomorphic in a half plane $\operatorname{Re} s > 2\rho_o$ and has a meromorphic continuation to the whole complex plane.*

(B) *The following functional equation holds:*

$$Z_q(2\rho_o - s) = c_1 \exp\left(\int_{s_o}^{s} -\phi_q(z)dz\right) Z_q(s)$$

where $c_1 = Z_q(2\rho_o - s_o) Z_q(s_o)^{-1}$.

(C) *Z_q satisfies a sort of modified Riemann hypothesis. Namely, the non-trivial zeros of Z_q lie on the line $\{s \in \boldsymbol{C}; \operatorname{Re} s = \rho_o\}$ except for the finite ones. These finite exceptional values are, provided that they exist, all real and lie in the interval $[0, 2\rho_o]$ symmetrically about ρ_o. The corresponding representations are all in the complementary series. Moreover, the order of the non-trivial zeros of Z_q at $s=\rho_o \pm i\lambda$ ($\lambda \in \tilde{Q}_q$, $\lambda \neq 0$) is $\kappa m_\Gamma(q, \lambda)$. If $0 \in \tilde{Q}_q$ then the order of the zero at the point $s=\rho_o$ is $2\kappa m_\Gamma(q, 0)$. If $q=0$ then the point $s=0$ is somewhat special* (*see* (E bis)).

(D) *Z_q has the trivial zeros and poles at $s=\rho_o + i r_{k,q}$ with the order $|e_{k,q} E \cdot \chi_T(e)| = |\kappa i d_{k,q} \chi_T(e) \operatorname{vol}(G/\Gamma)|$* (*see* Proposition 4.10) *listed below.*

TABLE 3.

	τ_q	$\rho_o+ir_{k,q}$	*zero or pole*
n: *odd*	$q=0$	$-2k \quad (k\geq 1)$	*zero*
	$q\neq 0$	$-(\lvert q\rvert+2k) \quad (k\geq 0)$	
n: *even*	$q=0$	$-2k \quad (k\geq 1)$	*pole*
	$0<\lvert q\rvert\leq 2n$	$-(\lvert q\rvert+2k) \quad (k\geq 0)$	
	$\lvert q\rvert>2n$	$-(\lvert q\rvert+2k) \quad (k\geq 0)$	*pole*
	q: *even*	$n-2(k+1) \quad (n/2\leq k\leq(\lvert q\rvert-n)/2-1)$	*zero*
	$\lvert q\rvert>2n$	$-(\lvert q\rvert+2k) \quad (k\geq 0)$	*pole*
	q: *odd*	$n-(2k+1) \quad (n/2\leq k\leq(\lvert q\rvert-n-1)/2)$	*zero*

(E bis) *Suppose that* $q=0$. *If* n *is odd, then* $\eta_q(s)$ *has the zero at* $s=0$ *with the order* $\kappa m_\Gamma(0, i\rho_o)-e_{0,0}E\chi_T(e)$. *On the other hand, if* n *is even, then* $\eta_q(s)$ *has the zero (resp. pole) at* $s=0$ *if and only if the sign of the number* $\kappa m_\Gamma(0, i\rho_o)-e_{0,0}E\chi_T(e)$ *is positive (resp. negative). One way or the other, the order of the zero or pole at this point is* $|\kappa m_\Gamma(0, i\rho_o)-e_{0,0}E\chi_T(e)|$.

(F) *Enumerate the roots in* P_+ *as* $\alpha_1,\dots,\alpha_t$. *Let* L *be the semi lattice in* $\mathfrak{a}_C^*$ *defined by* $L=\{\sum_{i=1}^t m_i\alpha_i;\ m_i\geq 0,\ m_i\in \boldsymbol{Z}\}$. *For* $\lambda\in L$, *define* m_λ *to the number of distinct ordered* t – *tuples* $(m_1,\dots, m_t)$ *such that* $\lambda=\sum_{i=1}^t m_i\alpha_i$. *For any* $\gamma\in\Gamma$, $\gamma\neq e$, *we now further demand that* $h(\gamma)=h_{\mathfrak{p}}(\gamma)h_{\mathfrak{t}}(\gamma)$ *be chosen so that* $h_{\mathfrak{p}}(\gamma)$ *lies in* $A_+=\{a_t;\ t>0\}$. *We now put*

$$f_q(s) = \exp\int_{s_o}^{s}(-F_q(z))dz,$$

for $\mathrm{Re}\, s>2\rho_o$. *Since the residues* $r_q^{\pm}(j)$ *at the poles* $z=\rho_o\pm j$ $(1\leq j\leq \min\{n, |q|-n\})$ *of the meromorphic function* $F_q(z)$ *need not be integers*, $f_q(s)$ *is only well defined in* $\boldsymbol{C}\backslash(-\infty, 0]$. *Therefore we take and fix a particular path in the half plane* $\mathrm{Re}\, s>2\rho_o$, *when the above integral is interpreted as a contour integral. With these understood, the function* Z_q *has an infinite product representation in the half plane* $\mathrm{Re}\, s>2\rho_o$, *that is, there is a non-zero constant* C *such that*

$$Z_q(s)=Cf_q(s)\prod_{\delta\in P_\Gamma}\prod_{\lambda\in L}(\det(I-T(\delta)\tau_q^M(h_{\mathfrak{t}}(\delta))^{-1}\xi_\lambda(h(\delta))^{-1}\exp(-su_\delta))^{m_\lambda\kappa}.$$

Here I *is identity matrix and* det *means determinant.*

Proof. The assertions (A) and (B) follow from the definition of Z_q and the functional equation of η_q. Also, Proposition 4.10 implies (C). As to the judgement of the trivial zero or pole, we will make use of the results of Table 2. If the sign of the number $-e_{k,q}E$ (or $-id_{k,q}$) is positive (resp. negative), then Z_q has zero (resp. pole). This implies the properties (D) and (E bis).

The proof of (F) proceeds from the formula

$$(4.26)\quad \eta_q(s)=\kappa \sum_{\gamma\in C_\Gamma-\{e\}} \chi_T(\gamma)\,|u_\gamma|\,j(\gamma)^{-1}C(h(\gamma))\tau_q^M(h_{\mathfrak{k}}(\gamma))^{-1}\exp(\rho_o-s)|u_\gamma|-F_q(s)$$

for the logarithmic derivative of Z_q, valid for $\operatorname{Re} s>2\rho_o$. Beacuse of our special choice of $h(\gamma)$, we see that $\varepsilon(h(\gamma))=1$, and $u_\gamma>0$, and we find that

$$C(h(\gamma)) = \xi_\rho(h_{\mathfrak{p}}(\gamma))^{-1}\prod_{\alpha\in P_+}(1-\xi_\alpha(h(\gamma))^{-1})^{-1}.$$

Thus (4.26) can be written as

$$(4.27)\quad \begin{aligned}&(d/ds)\log Z_q(s)\\ &= \kappa\sum_{\delta\in P_\Gamma}\sum_{j\geq 1}\{\chi_T(\delta^j)u_\delta\prod_{\alpha\in P_+}(1-\xi_\alpha(h(\delta))^{-j})^{-1}\tau_q^M(h_{\mathfrak{k}}(\delta))^{-j}\\ &\quad\times\exp(-sju_\delta\}-F_q(s).\end{aligned}$$

Now expand $(1-\xi_\alpha(h(\delta))^{-j})^{-1}$ as a power series,

$$\sum_{m\geq 0}\xi_\alpha(h(\delta))^{-jm}.$$

This series converges because $\xi_\alpha(h_{\mathfrak{p}}(\delta))^{-1}<1$ by our choice of $h(\delta)$. Next multiply together these series for the various $\alpha\in P_+$, then we find that the product

$$\prod_{\alpha\in P_+}(1-\xi_\alpha(h(\delta))^{-j})^{-1} = \sum_{\lambda\in L} m_\lambda\xi_\lambda(h(\delta))^{-j}.$$

Therefore (4.27) becomes, with a rearrangement,

$$(4.28)\quad \begin{aligned}(d/ds)\log Z_q(s) &= \kappa\sum_{\delta\in P_\Gamma}\sum_{\alpha\in L}\sum_{j\geq 1}\\ &\quad u_\delta m_\lambda\chi_T(\delta^j)\xi_\lambda(h(\delta))^{-j}\tau_q^M(h_{\mathfrak{k}}(\delta))^{-j}\exp(-sju_\delta)-F_q(s).\end{aligned}$$

If $\varepsilon_1(\delta)$, $\varepsilon_2(\delta),\dots,\varepsilon_d(\delta)$ are the eigenvalues of $T(\delta)$, then

$$\chi_T(\delta^j) = \sum_{i=1}^d(\varepsilon_i(\delta))^j.$$

Hence we can write

$$\begin{aligned}(d/ds)\log Z_q(s) &= \kappa\sum_{i=1}^d\sum_{\delta\in P_\Gamma}\sum_{\lambda\in L} m_\lambda u_\delta\\ &\quad\times\sum_{j\geq 1}\varepsilon_i(\delta)^j\xi_\lambda(h(\delta))^{-j}\tau_q^M(h_{\mathfrak{k}}(\delta))^{-j}\exp(-sju_\delta)-F_q(s)\\ &= \kappa\sum_{i=1}^d\sum_{\delta\in P_\Gamma}\sum_{\lambda\in L} m_\lambda u_\delta\\ &\quad\times\frac{\varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1}\tau_q^M(h_{\mathfrak{k}}(\delta))^{-1}\exp(-su_\delta)}{1-\varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1}\tau_q^M(h_{\mathfrak{k}}(\delta))^{-1}\exp(-su_\delta)}-F_q(s).\end{aligned}$$

These manipulations are valid because of the absolute convergence of the series (4.26) for $\eta_q(s)$. Integrating this logarithmic derivative, we find that

$$Z_q(s)=Cf_q(s)\prod_{i=1}^{d}\prod_{\delta\in P_\Gamma}\prod_{\lambda\in L}(1-\varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1}$$
$$\times\tau_q^M(h_t(\delta))^{-1}\operatorname{exe}(-su_\delta))^{m_\lambda\kappa}$$

where $C\neq 0$. This is exactly the assertion of (F).

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