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Supersoluble Lie algebras

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Introduction

A Lie algebra is called *supersoluble* if it has an ascending series of ideals whose factors are of dimension ≤ 1. Many authors, especially Barnes [5] and Barnes and Newell [6], have presented some properties of finite-dimensional supersoluble Lie algebras. A group is said to be supersoluble (or hypercyclic) if it has an ascending normal series whose factors are cyclic. Some properties of finite supersoluble groups have been presented in [12]. In [2] and [4] Baer has investigated supersoluble groups and has established the close connection with hypercentral groups. The purpose of this paper is first to show the connection between supersoluble Lie algebras and hypercentral Lie algebras, secondly to generalize some properties of hypercentral Lie algebras to those of supersoluble Lie algebras, and thirdly to characterize supersoluble Lie algebras by the weak idealizer condition. We shall also investigate locally supersoluble Lie algebras.

In Section 1 we shall give basic properties of supersoluble Lie algebras. Baer [2, Proposition 2] has shown that the derived group of a supersoluble group is hypercentral. In Section 2 we shall show the Lie analogue of this and characterize the Hirsch-Plotkin radical of a supersoluble Lie algebra as the unique maximal hypercentral ideal. In Section 3 we shall give criteria for a supersoluble Lie algebra to be hypercentral and for a locally supersoluble Lie algebra to be locally nilpotent, by using the nonexistence of non-abelian 2-dimensional subalgebras. We shall also give a criterion for a locally finite Lie algebra over an algebraically closed field to be locally nilpotent. It is known [12, p. 7] that the product of two finite supersoluble normal subgroups of a group need not be supersoluble. We shall show in Section 4 that over a field of characteristic zero the sum of two supersoluble (resp. locally supersoluble) ideals of a Lie algebra is always supersoluble (resp. locally supersoluble). We shall also investigate coalescence. It has been shown that every infinite-dimensional hypercentral (resp. locally nilpotent) Lie algebra has an infinite-dimensional abelian ideal (resp. subalgebra) [1, Theorems 10.1.1 and 10.1.3]. We shall show in Section 5 that we may replace 'hypercentral' or 'nilpotent' by 'supersoluble' in the preceding assertion. Bear [4] characterized supersoluble groups and locally supersoluble groups by the weak normalizer condition. We shall consider its Lie analogue in Sections 6 and 7. Proofs are slightly different.

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1.

Throughout this paper we shall be concerned with Lie algebras over an arbitrary field Φ which are not necessarily finite-dimensional unless otherwise specified. We shall always denote by L a Lie algebra.

The notation will follow that of [1]. In particular $H\subseteq L$ (resp. $H\leq L$, $H\lhd L$, H si L, $H\lhd^n L$) indicates that H is a subset (resp. a subalgebra, an ideal, a subideal, an n-step subideal) of L. For A, $B\subseteq L$, A^B is the smallest subspace containing A which is B-invariant. $\zeta_\alpha(L)$ is the αth term of the upper central series of L. $\mathfrak{F}_1$ (resp. $\mathfrak{F}$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{Z}$, $\mathrm{L}\mathfrak{N}$, $\mathrm{L}\mathfrak{F}$) is the class of Lie algebras which are of dimension ≤ 1 (resp. finite-dimensional, abelian, nilpotent, soluble, hypercentral, locally nilpotent, locally finite). If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of Lie algebras, then $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras L having an ideal $I\in\mathfrak{X}$ such that $L/I\in\mathfrak{Y}$. If $L\in\mathfrak{X}\mathfrak{Y}$, then L is called an $\mathfrak{X}$-by-$\mathfrak{Y}$-algebra. $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ (resp. $\mathrm{E}(\lhd)\mathfrak{X}$) is the class of Lie algebras which have an ascending (resp. a finite) series of ideals whose factors belong to $\mathfrak{X}$. More precisely $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ (resp. $\mathrm{E}(\lhd)\mathfrak{X}$) if and only if there exist an ordinal (resp. a finite ordinal) α and a family $(L_\beta)_{\beta\leq\alpha}$ of ideals of L such that

(1) $L_0=0$, $L_\alpha=L$,

(2) $L_\lambda=\bigcup_{\beta<\lambda}L_\beta$ if λ is a limit ordinal,

(3) $L_\beta\lhd L_{\beta+1}$ if $\beta<\alpha$,

(4) $L_{\beta+1}/L_\beta\in\mathfrak{X}$ if $\beta<\alpha$.

In particular $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ is the class of supersoluble Lie algebras (or hyper-$\mathfrak{F}_1$ Lie algebras) and $\mathrm{E}(\lhd)\mathfrak{F}_1$ is the class of finite-dimensional supersoluble Lie algebras.

The next lemma characterizes $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$.

LEMMA 1.1. *Let $\mathfrak{X}$ be a* Q*-closed class of Lie algebras. Then $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ if and only if every non-zero homomorphic image of L has a non-zero $\mathfrak{X}$-ideal.*

PROOF. Let $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{X}$. Take an ascending $\mathfrak{X}$-series $(L_\beta)_{\beta\leq\alpha}$ of ideals of L and let I be a proper ideal of L. Then there exists an ordinal β minimal with respect to $L_\beta\not\leq I$. Clearly β is neither zero nor a limit ordinal. By the minimality of β we have $L_{\beta-1}\leq I$. Hence $(L_\beta+I)/I\cong L_\beta/(L_\beta\cap I)$ is a homomorphic image of an $\mathfrak{X}$-algebra $L_\beta/L_{\beta-1}$. Thus $(L_\beta+I)/I$ is a non-zero $\mathfrak{X}$-ideal of L/I.

Conversely suppose that every non-zero homomorphic image of L has a non-zero $\mathfrak{X}$-ideal. Put $L_0=0$. Now suppose that for a non-zero ordinal α we have constructed a well-ordered ascending sequence $(L_\beta)_{\beta<\alpha}$ of ideals of L such that $L_{\beta+1}/L_\beta\in\mathfrak{X}$ for $\beta+1<\alpha$. If α is a limit ordinal, then put $L_\alpha=\bigcup_{\beta<\alpha}L_\beta$.

Suppose that α is not a limit ordinal. If $L_{\alpha-1} \lneqq L$, then $L/L_{\alpha-1}$ has a non-zero $\mathfrak{X}$-ideal $L_\alpha/L_{\alpha-1}$. Thus we can construct an ascending $\mathfrak{X}$-series of ideals of L. This completes the proof.

COROLLARY 1.2. *If $\mathfrak{X}$ is a* Q*-closed class of Lie algebras, then* $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ *is* Q*-closed.*

PROOF. Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{X}$ and let M be a homomorphic image of L. Then a homomorphic image of M is also that of L. The statement follows from Lemma 1.1.

It is easy to show the following

LEMMA 1.3. *If $\mathfrak{X}$ is an* S*-closed class of Lie algebras, then* $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ *is* S*-closed.*

LEMMA 1.4. *Let $\mathfrak{X}$ be an* I*-closed class of Lie algebras and let I be a non-zero ideal of an* $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$*-algebra L. Then I contains a non-zero $\mathfrak{X}$-ideal of L.*

PROOF. Let $(L_\beta)_{\beta\leq\alpha}$ be an ascending $\mathfrak{X}$-series of ideals of L. Then there exists an ordinal β minimal with respect to $L_\beta \cap I \neq 0$. Clearly β is neither zero nor a limit ordinal. By the minimality of β we have $L_{\beta-1} \cap I = 0$. Hence we obtain

$$L_\beta \cap I \cong (L_\beta \cap I)/(L_{\beta-1} \cap I) \cong ((L_\beta \cap I) + L_{\beta-1})/L_{\beta-1} \lhd L_\beta/L_{\beta-1}.$$

Since $\mathrm{I}\mathfrak{X} = \mathfrak{X}$, we have $L_\beta \cap I \in \mathfrak{X}$. Hence $L_\beta \cap I$ is a non-zero $\mathfrak{X}$-ideal which is contained in I.

A class $\mathfrak{X}$ of Lie algebras is called D-closed if $L_\lambda \in \mathfrak{X}$ $(\lambda \in \Lambda)$ implies $\oplus_{\lambda\in\Lambda} L_\lambda \in \mathfrak{X}$.

LEMMA 1.5. *For any class $\mathfrak{X}$ of Lie algebras* $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ *is* D*-closed.*

PROOF. Let $(L_\beta)_{\beta<\alpha}$ be a well-ordered family of $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$-algebras and let $L = \oplus_{\beta<\alpha} L_\beta$. For $\beta<\alpha$ let $(M_{\beta,\gamma})_{\gamma\leq\delta(\beta)}$ be an ascending $\mathfrak{X}$-series of ideals of L_β. For $\beta<\alpha$ and $\gamma\leq\delta(\beta)$ put

$$N_{\beta,\gamma} = (\oplus_{\mu<\beta} L_\mu) \oplus M_{\beta,\gamma} \quad \text{and} \quad N_{\alpha,0} = L.$$

It is easy to see that $(N_{\beta,\gamma})$ is an ascending $\mathfrak{X}$-series of ideals of L.

By the preceding results we have

PROPOSITION 1.6. (1) *Let L be a Lie algebra. Then L is supersoluble if and only if every non-zero homomorphic image has a* 1*-dimensional ideal.*

(2) $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ *is* {Q, S}*-closed.*

(3) *Let L be a supersoluble Lie algebra and let I be a non-zero ideal of L. Then I contains a* 1*-dimensional ideal of L.*

(4) $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ *is* D*-closed.*

REMARK: A class $\mathfrak{X}$ of Lie algebras is called c-closed if $L_\lambda \in \mathfrak{X}$ ($\lambda \in \Lambda$) implies $\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda \in \mathfrak{X}$, where $\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda$ is the Cartesian sum of L_λ. By [1, p. 21] a {Q, S, C}-closed class is L-closed. We shall show later (Proposition 3.1) that $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ is not L-closed. Therefore $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ is not C-closed.

2.

In this section we shall show that the derived algebra of a supersoluble Lie algebra is hypercentral and investigate the Hirsch-Plotkin radical of a supersoluble Lie algebra.

A chief factor of L is a pair (H, K) of ideals of L such that $K \lneqq H$ and such that H/K is a minimal ideal of L/K. We denote a chief factor (H, K) by H/K.

LEMMA 2.1. *If H/K is a chief factor of a supersoluble Lie algebra L, then* $\dim H/K = 1$.

PROOF. By Proposition 1.6 (2) we have $L/K \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. By Proposition 1.6 (3) H/K contains a 1-dimensional ideal I/K of L/K. Since H/K is a minimal ideal of L/K, we have $H/K = I/K$. Therefore $\dim H/K = 1$.

As in [1, p. 244] let

$$\Psi(L) = \cap\, C_L(H/K)$$

where the intersection is taken over all chief factors H/K of L and $C_L(H/K)$ is the centralizer of H/K in L, that is, $C_L(H/K) = \{x \in L: [H, x] \subseteq K\}$. Clearly $\Psi(L) \lhd L$.

LEMMA 2.2. *If L is supersoluble, then $L^2 \leq \Psi(L)$.*

PROOF. Let H/K be a chief factor of L. By Lemma 2.1 $\dim H/K = 1$. Hence the derivation algebra of H/K is abelian. Therefore L^2 acts trivially on H/K. Thus $L^2 \leq \Psi(L)$.

LEMMA 2.3. *If L is supersoluble, then $\Psi(L)$ is hypercentral.*

PROOF. Let $(L_\beta)_{\beta\leq\alpha}$ be a strictly ascending $\mathfrak{F}_1$-series of ideals of L. Put $I = \Psi(L)$ and $I_\beta = I \cap L_\beta$. We shall prove by transfinite induction that $I_\beta \leq \zeta_\beta(I)$. Let $\beta > 0$ and assume that $I_\gamma \leq \zeta_\gamma(I)$ for $\gamma < \beta$. If β is a limit ordinal, then $I_\beta \leq \zeta_\beta(I)$. If β is not a limit ordinal, then $[L_\beta, I] \leq L_{\beta-1}$, since $L_\beta/L_{\beta-1}$ is a chief factor of L. Hence $I_\beta \leq \zeta_\beta(I)$. Thus $\Psi(L) \in \mathfrak{Z}$.

THEOREM 2.4. $\mathfrak{Z} \lneqq \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \lneqq \mathfrak{Z}\mathfrak{A}$.

PROOF. By Lemmas 2.2 and 2.3 we have $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \leq \mathfrak{Z}\mathfrak{A}$. Since the central series of a hypercentral Lie algebra can be refined to a series of ideals with 1-

dimensional factors, we have $\mathfrak{Z} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. The 2-dimensional non-abelian Lie algebra belongs to $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$, but it is not nilpotent. Therefore $\mathfrak{Z} \lneqq \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. Finally let V be an abelian Lie algebra with basis $\{e_1, e_2, \ldots\}$. Let σ be a derivation, called the upward shift on V, such that $e_n\sigma = e_{n+1}$ $(n \geq 1)$. Form the split extension $L = V \dotplus \langle\sigma\rangle$. Then $L \in \mathfrak{A}^2 \leq \mathfrak{Z}\mathfrak{A}$. It is easy to see that L has no 1-dimensional ideals. Thus $L \in \mathfrak{Z}\mathfrak{A} \setminus \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

REMARK: From [6, Lemma 2.4] it follows that $\mathrm{E}(\lhd)\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{N}\mathfrak{A}$ over an algebraically closed field. If Φ is not algebraically closed, then it follows from Proposition 3.7 and its remark that $\mathrm{E}(\lhd)\mathfrak{F}_1 \lneqq \mathfrak{F} \cap \mathfrak{N}\mathfrak{A}$.

It follows from [1, Lemma 8.1.1] that the derived algebra of a non-zero $\mathfrak{Z}\mathfrak{A}$-algebra is a proper subalgebra. Hence we have the following

COROLLARY 2.5. *If L is a non-zero supersoluble Lie algebra, then $L^2 \lneqq L$.*

LEMMA 2.6 ([8, Theorem 8]). $\mathfrak{E} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{F} = \mathfrak{E} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 = \mathrm{L}\mathfrak{N} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 = \mathfrak{Z}$, *where $\mathfrak{E}$ is the class of Engel algebras.*

The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal $\mathrm{L}\mathfrak{N}$-ideal of L (cf. [1, p. 113]).

THEOREM 2.7. *If L is a supersoluble Lie algebra, then $L^2 \leq \Psi(L) = \rho(L)$ and $\rho(L)$ is the unique maximal hypercentral ideal of L.*

PROOF. By Proposition 1.6 (2) and Lemma 2.6 we have $\rho(L) \in \mathfrak{Z}$. Since $\mathfrak{Z} \leq \mathrm{L}\mathfrak{N}$, we have that $\rho(L)$ is the unique maximal hypercentral ideal of L. By Lemmas 2.2 and 2.3 it is sufficient to prove that $\rho(L) \leq \Psi(L)$. Let H/K be a chief factor of L. We show that $[H, \rho(L)] \leq K$. By Lemma 2.1 it is sufficient to see that if I is a 1-dimensional ideal of a Lie algebra M and J is a locally nilpotent ideal of M, then $[I, J] = 0$. Let $I = \langle x\rangle$. Then for any $y \in J$ there exists $\alpha = \alpha(y) \in \Phi$ such that $[x, y] = \alpha x$. Since J is a locally nilpotent ideal of M, there exists a positive integer n such that $[x, {}_n y] = 0$. Hence $\alpha = 0$. Thus $[I, J] = 0$.

3.

The simplest example which is supersoluble but non-nilpotent is the 2-dimensional non-abelian Lie algebra. By using this we shall give criteria for a supersoluble Lie algebra to be hypercentral and for a locally supersoluble Lie algebra to be locally nilpotent. First we investigate the class of locally supersoluble Lie algebras.

PROPOSITION 3.1. $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \lneqq \mathrm{L}\mathrm{E}(\lhd)\mathfrak{F}_1 = \mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \leq \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$.

PROOF. By [10, Corollary 3.3] we easily obtain

$$\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \leq \mathrm{LE}(\lhd)\mathfrak{F}_1 = \mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \leq \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A}).$$

Assume that $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 = \mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. Let M be a McLain algebra $\mathscr{M}_\Phi(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is the field of rational numbers. Then

$$M \in \mathrm{L}\mathfrak{N}\backslash 0 \quad \text{and} \quad M = M^2$$

[11, p. 96]. By $\mathrm{L}\mathfrak{N} \leq \mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ and the assumption we have $M \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. By Corollary 2.5 $M^2 \lneqq M$. This is a contradiction. Thus we have $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \lneqq \mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

In the rest of this paper we mostly use the notation $\mathrm{LE}(\lhd)\mathfrak{F}_1$ for the class $\mathrm{LE}(\lhd)\mathfrak{F}_1 = \mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

Let H be a subalgebra of L and let Σ be a totally ordered set. Then H is called serial in L and denoted by $H \operatorname{ser} L$ if there exists a family $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ of subalgebras of L such that

(1) $H \leq \Lambda_\sigma$ and $H \leq V_\sigma$ for all σ,

(2) $L\backslash H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(3) $\Lambda_\tau \leq V_\sigma$ if $\tau < \sigma$,

(4) $V_\sigma \lhd \Lambda_\sigma$.

LEMMA 3.2. *Let L be locally finite. Then the following are equivalent*:

(1) *L is locally nilpotent.*

(2) *Every subalgebra of L is serial.*

(3) *Every 1-dimensional subalgebra of L is serial.*

PROOF. Let $L \in \mathrm{L}\mathfrak{N}$ and let H be a subalgebra of L. If F is a finite-dimensional subalgebra of L, then we have $H \cap F \operatorname{si} F$, since $L \in \mathrm{L}\mathfrak{N}$. By [1, Proposition 13.2.4] we have $H \operatorname{ser} L$. Thus (1) implies (2). It is trivial that (2) implies (3). Finally assume (3). Let H be a finitely generated subalgebra of L. Since $L \in \mathrm{L}\mathfrak{F}$, we have $H \in \mathfrak{F}$. By the assumption every 1-dimensional subalgebra of H is a subideal. Let $x \in H$ and let $\langle x \rangle \lhd^n H$. Then it is easy to see that $[H, {}_{n+1}x] = 0$. Hence $\operatorname{ad}_H x$ is nilpotent for any $x \in H$. By Engel's theorem we have $H \in \mathfrak{N}$. Hence $L \in \mathrm{L}\mathfrak{N}$. Thus conditions (1), (2) and (3) are equivalent.

Now we can give criteria for a supersoluble Lie algebra to be hypercentral.

THEOREM 3.3. *Let L be a supersoluble Lie algebra. Then the following are equivalent*:

(1) *L is hypercentral.*

(2) *Every subalgebra of L is ascendant.*

(3) *Every subalgebra of L is serial.*

(4) *Every* 1-*dimensional subalgebra of L is ascendant.*
(5) *Every* 1-*dimensional subalgebra of L is serial.*
(6) *Every* 2-*dimensional subalgebra of L is abelian.*

PROOF. It is well known that (2) is a consequence of (1). Evidently we have the following implications:

$$(2) \Longrightarrow (3) \Longrightarrow (5), \quad (2) \Longrightarrow (4) \Longrightarrow (5).$$

Since every 2-dimensional non-abelian Lie algebra has a 1-dimensional subalgebra which is not a subideal, (5) implies (6). Finally assume (6). Suppose that $L \notin \mathfrak{Z}$ and so $\zeta_*(L) \lneqq L$. Put $\bar{L} = L/\zeta_*(L)$. In the rest of the proof we denote $a + \zeta_*(L)$ in $\bar{L}$ by $\bar{a}$. By Proposition 1.6 (1) $\bar{L}$ has a 1-dimensional ideal $\langle \bar{x} \rangle$ $(x \in L \setminus \zeta_*(L))$. Since $\zeta_1(\bar{L}) = 0$, there exists $y \in L \setminus \zeta_*(L)$ such that $[\bar{x}, \bar{y}] \neq 0$. Since $\langle \bar{x} \rangle \lhd \bar{L}$, we can find a non-zero scalar α such that $[\bar{x}, \bar{y}] = \alpha \bar{x}$. Replacing $(1/\alpha)\bar{y}$ by $\bar{y}$, we obtain

$$(*) \qquad [\bar{x}, \bar{y}] = \bar{x}.$$

Hence $[x, y] - x \in \zeta_*(L)$. By [8, Proposition 5] there exists a positive integer n such that $[[x, y] - x, {}_n y] = 0$. Put $z = [x, {}_n y]$. Then $[z, y] = z$. By $(*)$ $\bar{z} = [\bar{x}, {}_n \bar{y}] = \bar{x} \neq 0$. Thus $\langle y, z \rangle$ is a 2-dimensional non-abelian subalgebra of L. This is a contradiction. Therefore $L \in \mathfrak{Z}$. This completes the proof.

REMARK: By Proposition 3.1 and [7, Theorem 2.7] we can add the following conditions to those in Theorem 3.3:
(7) *Every subalgebra of L is weakly serial.*
(8) *Every* 1-*dimensional subalgebra of L is weakly serial.*

COROLLARY 3.4. *Let L be locally supersoluble. Then the following are equivalent*:
(1) *L is locally nilpotent.*
(2) *Every subalgebra of L is serial.*
(3) *Every* 1-*dimensional subalgebra of L is serial.*
(4) *Every* 2-*dimensional subalgebra of L is abelian.*

PROOF. By Proposition 3.1 and Lemma 3.2 we have the equivalence of (1), (2) and (3). Evidently (1) implies (4). Assume (4). Let H be a finitely generated subalgebra of L. Since $L \in \mathrm{LE}(\lhd)\mathfrak{F}_1$, we have $H \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Hence by Theorem 3.3 $H \in \mathfrak{N}$. Thus $L \in \mathrm{L}\mathfrak{N}$.

REMARK: By [7, Theorem 2.7] we can add the following conditions to those in Corollary 3.4:
(5) *Every subalgebra of L is weakly serial.*
(6) *Every* 1-*dimensional subalgebra of L is weakly serial.*

Over an algebraically closed field we can generalize Corollary 3.4.

LEMMA 3.5. *Let L be finite-dimensional over an algebraically closed field. Then L is nilpotent if and only if every 2-dimensional subalgebra of L is abelian.*

PROOF. The implication in one direction is evident. Assume that every 2-dimensional subalgebra of L is abelian. Let $x \in L$ and let α be an eigenvalue of $\mathrm{ad}_L x$. Let $y \in L \backslash 0$ such that $[y, x]=\alpha y$. It follows that $\alpha=0$. Hence $\mathrm{ad}_L x$ is a nilpotent endomorphism of L. By Engel's theorem we obtain $L \in \mathfrak{N}$.

By Lemmas 3.2 and 3.5 we have

THEOREM 3.6. *Let L be a locally finite Lie algebra over an algebraically closed field. Then the following are equivalent:*

(1) *L is locally nilpotent.*
(2) *Every subalgebra of L is serial.*
(3) *Every 1-dimensional subalgebra of L is serial.*
(4) *Every 2-dimensional subalgebra of L is abelian.*

The following example shows that we cannot remove the restriction on the base field in Theorem 3.6.

PROPOSITION 3.7. *If the base field Φ is not algebraically closed, then there exists $L \in (\mathfrak{F} \cap \mathfrak{A}^2) \backslash \mathfrak{N}$ in which every 2-dimensional subalgebra is abelian.*

PROOF. Since Φ is not algebraically closed, there exists a monic irreducible polynomial $f(t)$ of degree $n>1$. Put

$$f(t) = t^n + a_n t^{n-1} + \cdots + a_1 \qquad (a_i \in \Phi).$$

Let V be an abelian Lie algebra with basis $\{e_1, \dots, e_n\}$ and let x be a derivation such that

$$e_i x = e_{i+1} \ (i=1, \dots, n-1) \quad \text{and} \quad e_n x = -\sum_{i=1}^n a_i e_i.$$

Then it is not so difficult to see that the characteristic polynomial of x is $f(t)$. Let $L=V \dot{+} \langle x \rangle$ be the split extension of V by $\langle x \rangle$. Clearly $L \in \mathfrak{F} \cap \mathfrak{A}^2$. Since $f(t)$ is irreducible, we have $f(0) \neq 0$. Hence 0 is not an eigenvalue of $\mathrm{ad}_V x$. Thus $L \notin \mathfrak{N}$. Now suppose that there exists a 2-dimensional non-abelian subalgebra. Then there exist $u, v \in L \backslash 0$ such that $[u, v]=u$. Clearly $L^2 \leq V$. Hence $u \in V$. Put $v=w+\alpha x$ with $w \in V$ and $\alpha \in \Phi$. Then

$$[u, \alpha x] = [u, w+\alpha x] = [u, v] = u.$$

Since $u \neq 0$, we have $\alpha \neq 0$. Now put $u=\sum_{i=1}^n \beta_i e_i$ with $\beta_i \in \Phi$ $(i=1, \dots, n)$. Then

$$u = [u, \alpha x] = \alpha(\textstyle\sum_{i=1}^{n-1} \beta_i e_{i+1} - \beta_n \sum_{i=1}^{n} a_i e_i).$$

Hence we have

$$(*) \qquad \beta_1 = -\alpha a_1 \beta_n \quad \text{and} \quad \beta_i = \alpha\beta_{i-1} - \alpha a_i \beta_n \quad (i = 2,\dots, n).$$

If we add β_j multiplied by α^{n-j} $(j=1,\dots, n)$, we obtain

$$\beta_n(1+\alpha a_n+\alpha^2 a_{n-1}+\cdots+\alpha^n a_1) = 0.$$

Dividing by α^n, we obtain $\beta_n f(1/\alpha)=0$. Since $f(t)$ is irreducible, we have $f(1/\alpha)\neq 0$. Hence $\beta_n=0$. By $(*)$ we have

$$\beta_1 = \beta_2 = \cdots = \beta_n = 0.$$

This is a contradiction. Thus every 2-dimensional subalgebra of L is abelian. This completes the proof.

REMARKS: (1) By Theorem 3.3 the Lie algebra in Proposition 3.7 is not supersoluble.

(2) Let $L \in \mathrm{L}\mathfrak{F}$ over an algebraically closed field or $L \in \mathrm{LE}(\lhd)\mathfrak{F}_1$ over any field. If $\dim L \geq 2$, then L has a 2-dimensional subalgebra.

4.

A class $\mathfrak{X}$ of Lie algebras is called N_0-closed if whenever I and J are $\mathfrak{X}$-ideals of L then $I+J \in \mathfrak{X}$. In this section we shall show the N_0-closedness of $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ and $\mathrm{LE}(\lhd)\mathfrak{F}_1$ over a field of characteristic zero. First we give an example [1, Lemma 3.1.1] which suggests that we must restrict ourselves to fields of characteristic zero.

LEMMA 4.1. *Over a field of characteristic $p>0$ there exists a non-supersoluble Lie algebra which is a sum of two finite-dimensional supersoluble ideals.*

PROOF. Let V be an abelian Lie algebra with basis $\{e_0, e_1,\dots, e_{p-1}\}$. Let x and y be derivations of V such that

$$e_i x = e_{i+1} \quad \text{and} \quad e_i y = i e_{i-1} \qquad (i = 0,\dots, p-1)$$

where $e_{-1}=e_p=0$. Put $z=[x, y]$. Then z is the identity map of V. Let $L= V+\langle x, y\rangle$ be the split extension of V by $\langle x, y\rangle$. Put $L_1=V+\langle x, z\rangle$ and $L_2= V+\langle y, z\rangle$. Then $L_1, L_2 \lhd L$. L_1 has the following series of ideals of L_1:

$$\langle e_{p-1}\rangle \leq \langle e_{p-1}\rangle + \langle e_{p-2}\rangle \leq \cdots \leq V \leq V + \langle z\rangle \leq L_1.$$

Hence $L_1 \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Similarly we have $L_2 \in \mathrm{E}(\lhd)\mathfrak{F}_1$, since it has the following series:

$$\langle e_0\rangle \leq \langle e_0\rangle + \langle e_1\rangle \leq \cdots \leq V \leq V + \langle z\rangle \leq L_2.$$

Clearly $L=L_1+L_2$. Now $L^2=V+\langle z\rangle \notin \mathfrak{N}$. Hence by Theorem 2.4 we have $L \notin \mathrm{E}(\lhd)\mathfrak{F}_1$.

The following lemma, which is a Lie analogue of [4, Lemma J. 2], gives a criterion for a sum of two supersoluble ideals of a Lie algebra to be supersoluble.

LEMMA 4.2. *Let L be a sum of two supersoluble ideals. Then L is supersoluble if and only if L^2 is hypercentral.*

PROOF. By Theorem 2.4 we have one implication. Let I and J be supersoluble ideals of L such that $L=I+J$ and assume that $L^2 \in \mathfrak{Z}$. We may suppose that $L \neq 0$. Every homomorphic image of L is a sum of two supersoluble ideals and its derived algebra is hypercentral. Hence by Proposition 1.6 (1) it is sufficient to prove that L has a 1-dimensional ideal. Now we consider two cases.

Case 1: $L^2 \cap I \cap J = 0$. Clearly L/L^2, L/I and L/J are supersoluble. Hence by Proposition 1.6 (4) $L/L^2 \oplus L/I \oplus L/J$ is supersoluble. Since L is isomorphic to a subalgebra of $L/L^2 \oplus L/I \oplus L/J$, we have $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

Case 2: $L^2 \cap I \cap J \neq 0$. Since $L^2 \cap I \cap J \lhd L^2$ and $L^2 \in \mathfrak{Z}$, we have $\zeta_1(L^2) \cap I \cap J \neq 0$ (e.g., see [9, Lemma 3.1]). Put $K=\zeta_1(L^2) \cap I \cap J$. By Proposition 1.6 (3) K contains a 1-dimensional ideal of I. Hence there exists $x \in K \backslash 0$ such that $[x, y] \in \langle x\rangle$ for any $y \in I$. Put $V=\langle x^L\rangle$. Then $0 \neq V \leq K$. Let $y \in I$. Then $[x, y]=\alpha x$ for some $\alpha \in \Phi$. Let n be a non-negative integer and let $x_i \in L$ $(i=1,\dots, n)$. Then $[[x, x_1,\dots, x_n], y] = [[x, y], x_1,\dots, x_n] + \sum_{i=1}^{n} [x, x_1,\dots, x_{i-1}, [x_i, y], x_{i+1},\dots, x_n]$. Since $x \in K \leq \zeta_1(L^2) \lhd L$, we obtain

$$[[x, x_1,\dots, x_n], y] = \alpha[x, x_1,\dots, x_n].$$

Hence we have $[v, y]=\alpha v$ for any $v \in V$. Therefore

$$(*) \qquad [v, I] \subseteq \langle v\rangle \qquad \text{for any } v \in V.$$

Since $0 \neq V \lhd L$ and $V \leq K \leq J$, V is a non-zero ideal of J. Hence by Proposition 1.6 (3) V contains a 1-dimensional ideal U of J. By $(*)$ U is I-invariant. Since $L=I+J$, U is a 1-dimensional ideal of L. This completes the proof.

THEOREM 4.3. *Let L be a Lie algebra and let $I_1,\dots, I_n$ be finitely many supersoluble ideals of L such that $L=\sum_{i=1}^{n} I_i$. Then the following are equivalent*:

(1) *L is supersoluble.*
(2) *L^2 is hypercentral.*
(3) *L^2 is locally nilpotent.*
(4) *For $1 \leq i < j \leq n$ $[I_i, I_j]$ is locally nilpotent.*
(5) *For $1 \leq i < j \leq n$ $[I_i, I_j]$ is hypercentral.*

PROOF. By Theorem 2.4 we see that (1) implies (2). Since $\mathfrak{Z} \leq \mathrm{L}\mathfrak{N}$, (2) implies (3). It is clear that (3) implies (4). Since I_i is a supersoluble ideal of L, we have $[I_i, I_j] \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. Hence by Lemma 2.6 we see that (4) implies (5). We use induction on n to show that (5) implies (1). Let $n>1$ and assume that $\sum_{i=1}^{n-1} I_i \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. Then L is the sum of two supersoluble ideals $\sum_{i=1}^{n-1} I_i$ and I_n. By the N_0-closedness of $\mathfrak{Z}$ and $L^2 = \sum_{i=1}^{n} I_i^2 + \sum_{i<j} [I_i, I_j]$, we obtain $L^2 \in \mathfrak{Z}$. By Lemma 4.2 we have $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

REMARK: By Lemma 2.6 we may replace 'locally nilpotent' by 'an Engel algebra' in (3) and (4).

Modifying the proof of [8, Proposition 3] slightly, we have

PROPOSITION 4.4. *If I is a hypercentral ideal of a Lie algebra L and H is an ascendant supersoluble subalgebra of L, then $I+H$ is an ascendant supersoluble subalgebra of L.*

PROOF. Put $J=I+H$. We may assume that $J \neq 0$. As in the proof of Lemma 4.2 it is sufficient to show that J has a 1-dimensional ideal. If $I=0$, then $J=H \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. So we suppose that $I \neq 0$. Since $I \in \mathfrak{Z}$, we have $\zeta_1(I) \neq 0$. Let $(H_\beta)_{\beta \leq \alpha}$ be an ascending series from H to L. Then there exists an ordinal β minimal with respect to $H_\beta \cap \zeta_1(I) \neq 0$. Clearly β is not a limit ordinal. If $\beta=0$, then $0 \neq H \cap \zeta_1(I) \lhd H$ and hence by Proposition 1.6 (3) $H \cap \zeta_1(I)$ contains a 1-dimensional ideal K of H. Since $K \leq \zeta_1(I)$ and $J=H+I$, we have $K \lhd J$. If $\beta \neq 0$, then $H_{\beta-1} \cap \zeta_1(I)=0$. It follows that $[H_\beta \cap \zeta_1(I), I+H] \subseteq H_{\beta-1} \cap \zeta_1(I)=0$. Hence $0 \neq H_\beta \cap \zeta_1(I) \leq \zeta_1(J)$. Thus J has a 1-dimensional ideal.

The next corollary is a Lie analogue of [4, Lemma J.1].

COROLLARY 4.5. *A sum of a supersoluble ideal and a hypercentral ideal of L is supersoluble.*

In the rest of this section we assume that the base field Φ is of characteristic zero. It is known that the derived algebra of an $(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$-algebra is nilpotent. Hence it is easy to see that the derived algebra of an $\mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$-algebra is locally nilpotent.

THEOREM 4.6. *Over a field of characteristic zero $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ is N_0-closed.*

PROOF. Let L be a Lie algebra and let I and J be $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$-ideals of L such that $L=I+J$. By Proposition 3.1 I, $J \in \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$. By [1, Corollary 6.1.2] $L \in \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$. By the remark above $L^2 \in \mathrm{L}\mathfrak{N}$. Hence by Theorem 4.3 we have that $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

COROLLARY 4.7. *Over a field of characteristic zero $\mathrm{E}(\lhd)\mathfrak{F}_1$ is N_0-closed.*

A class $\mathfrak{X}$ of Lie algebras is called coalescent (resp. ascendantly coalescent) if in any Lie algebras the join of two $\mathfrak{X}$-subideals (resp. ascendant $\mathfrak{X}$-subalgebras) is always an $\mathfrak{X}$-subideal (reap. ascendant $\mathfrak{X}$-subalgebra). It is known that over a field of characteristic zero every $\{\mathrm{N}_0, \mathrm{I}\}$-closed subclass of $\mathfrak{F}$ is coalescent and ascendantly coalescent [1, Corollary 13.3.5]. By Proposition 1.6 (2) and Corollary 4.7 $\mathrm{E}(\lhd)\mathfrak{F}_1$ is $\{\mathrm{N}_0, \mathrm{I}\}$-closed. Hence we obtain

COROLLARY 4.8. *Over a field of characteristic zero* $\mathrm{E}(\lhd)\mathfrak{F}_1$ *is coalescent and ascendantly coalescent.*

By Corollary 4.7 and [1, Theorem 6.1.1] we have

THEOREM 4.9. *Over a field of characteristic zero* $\mathrm{LE}(\lhd)\mathfrak{F}_1$ *is* N_0*-closed.*

A class $\mathfrak{X}$ of Lie algebras is called locally coalescent if whenever H and K are $\mathfrak{X}$-subideals of L, to every finitely generated subalgebra C of $\langle H, K\rangle$ there corresponds an $\mathfrak{X}$-subideal X of L such that $C\leq X\leq\langle H, K\rangle$. By [1, Theorem 4.2.4] any complete (for the definition see [1, p. 85]) and $\{\mathrm{N}_0, \mathrm{I}\}$-closed subclass of $\mathrm{L}\mathfrak{F}$ is locally coalescent. As in [1, p. 85] it is seen that $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ and $\mathrm{LE}(\lhd)\mathfrak{F}_1$ are complete. Therefore we obtain

THEOREM 4.10. *Over a field of characteristic zero* $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ *and* $\mathrm{LE}(\lhd)\mathfrak{F}_1$ *are locally coalescent.*

5.

In this section we shall show the existence of an infinite-dimensional abelian ideal (resp. subalgebra) in any infinite-dimensional supersoluble (resp. locally supersoluble) Lie algebra.

Let S be a subset of L. The centralizer $C_L(S)$ of S in L is the set $\{x\in L: [S, x]=0\}$. If $S\lhd L$, then $C_L(S)\lhd L$.

LEMMA 5.1. *If A is a maximal abelian ideal of a supersoluble Lie algebra, then* $C_L(A)=A$.

PROOF. Put $C=C_L(A)$. Assume that $A\lneqq C$. Let $(L_\beta)_{\beta\leq\alpha}$ be an ascending $\mathfrak{F}_1$-series of ideals of L. There exists an ordinal β minimal with respect to $C\cap L_\beta\nleq A$. Clearly β is neither 0 nor a limit ordinal. Hence $C\cap L_{\beta-1}\leq A$. We have

$$0\neq(C\cap L_\beta)/(C\cap L_{\beta-1})\cong((C\cap L_\beta)+L_{\beta-1})/L_{\beta-1}\leq L_\beta/L_{\beta-1}.$$

Hence $\dim(C\cap L_\beta)/(C\cap L_{\beta-1})=1$. Let $x\in(C\cap L_\beta)\backslash A$. It follows that $x\notin C\cap L_{\beta-1}$. Therefore $C\cap L_\beta=\langle x\rangle+(C\cap L_{\beta-1})$. Put $B=\langle x\rangle+A$. Since $x\in C=$

$C_L(A)$, we obtain $B \in \mathfrak{A}$. We have $B = \langle x \rangle + A = \langle x \rangle + (C \cap L_{\beta-1}) + A = (C \cap L_\beta) + A \lhd L$. Thus we obtained an abelian ideal B such that $A \lneqq B$. This is a contradiction.

By using the class Max-$\lhd\mathfrak{A}$ of Lie algebras which satisfy the maximal condition for abelian ideals, we have

THEOREM 5.2. $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \cap \text{Max-}\lhd\mathfrak{A} = \mathrm{E}(\lhd)\mathfrak{F}_1$.

PROOF. Evidently $\mathrm{E}(\lhd)\mathfrak{F}_1 \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \cap \text{Max-}\lhd\mathfrak{A}$. Conversely let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \cap \text{Max-}\lhd\mathfrak{A}$ and let A be a maximal abelian ideal of L. Since we may regard $L/C_L(A)$ as a subalgebra of $\mathrm{Der}\, A$ and $C_L(A) = A$ by Lemma 5.1, it is sufficient to prove that $A \in \mathfrak{F}$. Now let $(L_\beta)_{\beta \leq \alpha}$ be an ascending $\mathfrak{F}_1$-series of ideals of L. Clearly $\dim (A \cap L_{\beta+1})/(A \cap L_\beta) \leq 1$. Since $L \in \text{Max-}\lhd\mathfrak{A}$, we can find finitely many ordinals $\gamma(i)$ such that

$$\{A \cap L_\beta \colon 0 \leq \beta \leq \alpha\} = \{A \cap L_{\gamma(0)}, \dots, A \cap L_{\gamma(n)}\}$$

and

$$0 = A \cap L_{\gamma(0)} \lneqq A \cap L_{\gamma(1)} \lneqq \cdots \lneqq A \cap L_{\gamma(n)} = A.$$

For $i = 0, 1, \dots, n$ let $\beta(i)$ be an ordinal minimal with respect to $A \cap L_{\beta(i)} = A \cap L_{\gamma(i)}$. Clearly $\gamma(i-1) \lneqq \beta(i)$. Let $\gamma(i-1) \leq \delta \lneqq \beta(i)$. By the minimality of $\beta(i)$ we have that $A \cap L_{\gamma(i-1)} \leq A \cap L_\delta \lneqq A \cap L_{\beta(i)} = A \cap L_{\gamma(i)}$. Therefore $A \cap L_\delta = A \cap L_{\gamma(i-1)}$. It follows that $\beta(i)$ is not a limit ordinal and that $A \cap L_{\beta(i)-1} = A \cap L_{\gamma(i-1)}$. Hence $\dim (A \cap L_{\gamma(i)})/(A \cap L_{\gamma(i-1)}) = \dim (A \cap L_{\beta(i)})/(A \cap L_{\beta(i)-1}) = 1$. Thus $A = A \cap L_{\gamma(n)} \in \mathfrak{F}$. This completes the proof.

COROLLARY 5.3. *Every infinite-dimensional supersoluble Lie algebra has an infinite-dimensional abelian ideal.*

PROOF. Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \backslash \mathfrak{F}$. By Theorem 5.2 there exists a strictly increasing chain $(A_n)_{n=1}^\infty$ of abelian ideals of L. Then $\bigcup_{n=1}^\infty A_n$ is an infinite-dimensional abelian ideal of L.

THEOREM 5.4. *Every infinite-dimensional* $\acute{\mathrm{E}}\mathrm{LE}(\lhd)\mathfrak{F}_1$*-algebra has an infinite-dimensional abelian subalgebra, where* $\acute{\mathrm{E}}\mathrm{LE}(\lhd)\mathfrak{F}_1$ *is the class of Lie algebras which have an ascending* $\mathrm{LE}(\lhd)\mathfrak{F}_1$*-series.*

PROOF. Let $L \in \acute{\mathrm{E}}\mathrm{LE}(\lhd)\mathfrak{F}_1 \backslash \mathfrak{F}$ and let $(L_\beta)_{\beta \leq \alpha}$ be an ascending $\mathrm{LE}(\lhd)\mathfrak{F}_1$-series of L. There exists an ordinal β minimal with respect to $L_\beta \notin \mathfrak{F}$. Clearly $\beta \neq 0$. Put $M = L_\beta$. Suppose that β is not a limit ordinal. Then $L_{\beta-1} \in \mathfrak{F}$ and so $L_{\beta-1} \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$. By Theorem 2.4 it is easy to see that $\mathrm{LE}(\lhd)\mathfrak{F}_1 \leq (\mathrm{L}\mathfrak{N})\mathfrak{A}$. Put $\overline{M} = M/L_{\beta-1}$. Then $\overline{M} \in \mathrm{LE}(\lhd)\mathfrak{F}_1 \backslash \mathfrak{F}$. Hence $\overline{M}^2 \in \mathrm{L}\mathfrak{N}$. If $\overline{M}^2 \in \mathfrak{F}$, then

$M \in \mathrm{E}\mathfrak{A}\backslash\mathfrak{F}$. By [1, Theorem 10.1.1(b)] M has an infinite-dimensional abelian subalgebra. If $\overline{M}^2 \notin \mathfrak{F}$, then by [1, Theorem 10.1.3] $\overline{M}^2$ has an infinite-dimensional abelian subalgebra $H/L_{\beta-1}$. Since $L_{\beta-1} \in \mathrm{E}\mathfrak{A}$, H is an infinite-dimensional soluble subalgebra. By [1, Theorem 10.1.1(b)] H has an infinite-dimensional abelian subalgebra. Now suppose that β is a limit ordinal. Then by the minimality of β $L_\gamma \in \mathfrak{F}$ for any $\gamma<\beta$ and so $L_{\gamma+1}/L_\gamma \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Hence we can refine $(L_\gamma)_{\gamma\leq\beta}$ to an ascending $\mathfrak{A}$-series of M and so $M \in \acute{\mathrm{E}}\mathfrak{A}\backslash\mathfrak{F}$. [1, Theorem 10.1.1(e)] completes the proof.

6.

A group G satisfies the weak normalizer condition if for any $H \lneqq G$ there exists $x \in G\backslash H$ such that $\langle x^H \rangle \leq H_{\langle H,x\rangle}\langle x\rangle$, where $H_{\langle H,x\rangle}$ is the core of H in $\langle H, x\rangle$. Baer [4] investigated the role of the weak normalizer condition in supersoluble groups and locally supersoluble groups. We shall consider its Lie analogue in this and the next sections. We say that a Lie algebra L satisfies the *weak idealizer condition* if for any $H \lneqq L$ there exists $x \in L\backslash H$ such that $\langle x^H \rangle \leq H_{\langle H,x\rangle} + \langle x\rangle$. Here X_Y for a subalgebra X of a Lie algebra Y signifies the core of X in Y, that is, the largest ideal of Y which is contained in X. Equivalently L satisfies the weak idealizer condition if for any $H \lneqq L$ there exists $x \in L\backslash H$ such that $[x, H] \subseteq H_{\langle H,x\rangle} + \langle x\rangle$. It is easy to see that the idealizer condition implies the weak idealizer condition. We denote by WIC the class of Lie algebras which satisfy the weak idealizer condition. It is not so difficult to see the following

Lemma 6.1. *WIC is* Q*-closed.*

Lemma 6.2. $\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \leq \mathrm{WIC}$.

Proof. Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ and let $H \lneqq L$. Then $H_L \lneqq L$. By Proposition 1.6 (1) L/H_L contains a 1-dimensional ideal, say, $(\langle x\rangle + H_L)/H_L$. If $x \in H$, then $\langle x\rangle + H_L \leq H_L$. Therefore $x \notin H$. Furthermore we have

$$\langle x^H \rangle \leq \langle x^L \rangle \leq \langle x\rangle + H_L \leq \langle x\rangle + H_{\langle H,x\rangle}.$$

This completes the proof.

The following is a Lie analogue of [4, p. 112, Excursus on finite supersoluble groups].

Proposition 6.3. *Let L be a finite-dimensional Lie algebra. Then the following are equivalent:*

(1) *L is supersoluble.*

(2) *L satisfies the weak idealizer condition.*

(3) *If M is a maximal subalgebra of L, then there exists $x \in L\backslash M$ such*

that $\langle x^L \rangle \leq M_L + \langle x \rangle$.

PROOF. By Lemma 6.2 (1) implies (2). It is evident that (2) implies (3). Finally assume (3). Note that the property (3) is inherited by homomorphic images. We use induction on $n = \dim L$. If $n = 1$, then there is nothing to prove. Assume that $n > 1$ and that the result is true for $\dim L < n$. If there exists a maximal subalgebra M with $M_L = 0$, then by (3) we can find a 1-dimensional ideal I of L. By induction hypothesis we have $L/I \in \mathrm{E}(\lhd)\mathfrak{F}_1$, whence $L \in \mathrm{E}(\lhd)\mathfrak{F}_1$. We may assume that $M_L \neq 0$ for any maximal subalgebra M of L. If $\cap M_L = 0$, where the intersection is taken over all the maximal subalgebras of L, then there exist finitely many maximal subalgebras M_i of L such that $(M_1)_L \cap \cdots \cap (M_n)_L = 0$. By induction hypothesis $L/(M_i)_L \in \mathrm{E}(\lhd)\mathfrak{F}_1$. By Proposition 1.6 (4) $\oplus_{i=1}^n L/(M_i)_L \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Since we may regard L as a subalgebra of $\oplus_{i=1}^n L/(M_i)_L$, we obtain $L \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Finally let $I = \cap M_L \neq 0$. By induction hypothesis $L/I \in \mathrm{E}(\lhd)\mathfrak{F}_1$. [5, Theorem 6] completes the proof.

LEMMA 6.4. *If the weak idealizer condition and the maximal condition on soluble subalgebras are satisfied by L, then L is finite-dimensional supersoluble.*

PROOF. Let M be a maximal soluble subalgebra of L. Suppose that $M \lneqq L$. Then by the weak idealizer condition there exists $x \in L \backslash M$ such that $\langle x^M \rangle \leq M_{\langle M, x\rangle} + \langle x \rangle$. Since $M \in \mathrm{E}\mathfrak{A}$, we have $M_{\langle M, x \rangle} + \langle x \rangle \in \mathrm{E}\mathfrak{A}$. Hence $\langle x^M \rangle \in \mathrm{E}\mathfrak{A}$ and so $\langle M, x \rangle = M + \langle x^M \rangle \in \mathrm{E}\mathfrak{A}$, which contradicts the maximality of M. Hence $L = M \in \mathrm{E}\mathfrak{A}$. Therefore $L \in \mathrm{Max} \cap \mathrm{E}\mathfrak{A} = \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$, where Max is the class of Lie algebras which satisfy the maximal condition on subalgebras. By Proposition 6.3 we have $L \in \mathrm{E}(\lhd)\mathfrak{F}_1$.

We denote by Max-$\mathrm{E}\mathfrak{A}$ the class of Lie algebras which satisfy the maximal condition on soluble subalgebras. Then we have the following

COROLLARY 6.5. $\text{Max-}\mathrm{E}\mathfrak{A} \cap \mathrm{WIC} = \mathrm{Max} \cap \mathrm{WIC} = \mathfrak{F} \cap \mathrm{WIC} = \mathrm{E}(\lhd)\mathfrak{F}_1$.

PROOF. By Lemma 6.2 $\mathrm{E}(\lhd)\mathfrak{F}_1 \leq \mathfrak{F} \cap \mathrm{WIC}$. It is easy to see that

$$\mathfrak{F} \cap \mathrm{WIC} \leq \mathrm{Max} \cap \mathrm{WIC} \leq \text{Max-}\mathrm{E}\mathfrak{A} \cap \mathrm{WIC}.$$

By Lemma 6.4 $\text{Max-}\mathrm{E}\mathfrak{A} \cap \mathrm{WIC} \leq \mathrm{E}(\lhd)\mathfrak{F}_1$.

LEMMA 6.6. $\mathfrak{A}\mathfrak{F}_1 \cap \mathrm{WIC} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

PROOF. Let $L \in \mathfrak{A}\mathfrak{F}_1 \cap \mathrm{WIC}$ and let E be a non-zero homomorphic image of L. By Proposition 1.6 (1) it is sufficient to see that E has a 1-dimensional ideal. Evidently $E \in \mathfrak{A}\mathfrak{F}_1 \cap \mathrm{WIC}$. Hence there exist an abelian ideal A of E and $x \in E$ such that $E = A + \langle x \rangle$. We may suppose that $A, \langle x \rangle \lneqq E$. Since $E \in \mathrm{WIC}$, there exists $y \in E \backslash \langle x \rangle$ such that $[x, y] \in \langle x \rangle_{\langle x, y \rangle} + \langle y \rangle$. Hence $\langle x, y \rangle$ is 2-

dimensional. By the modular law we obtain $\langle x, y\rangle = (\langle x, y\rangle \cap A) + \langle x\rangle$. Hence $\langle x, y\rangle \cap A$ is 1-dimensional. Put $\langle x, y\rangle \cap A = \langle a\rangle$ with $a \in A \backslash 0$. We have $[a, A] = 0$ and $[a, x] \in A \cap \langle x, y\rangle = \langle a\rangle$. Thus $\langle a\rangle$ is a 1-dimensional ideal of E.

Now we consider the following condition:

(C) *Every finitely generated subalgebra of L satisfies the weak idealizer condition.*

Modifying the proof of [4, p. 115, (5)], we have the next lemma, which is a generalization of its Lie analogue.

Lemma 6.7. *Let* $L \in \mathfrak{A}\mathrm{E}(\lhd)\mathfrak{F}_1$. *If L satisfies the condition* (C), *then L is supersoluble.*

Proof. Let E be a non-zero homomorphic image of L. By Proposition 1.6 (1) it is sufficient to show that E has a 1-dimensional ideal. Clearly $E \in \mathfrak{A}\mathrm{E}(\lhd)\mathfrak{F}_1$ and so there exists an abelian ideal A of E such that $E/A \in \mathrm{E}(\lhd)\mathfrak{F}_1$. We may suppose that $A \neq 0$. Since $E/A \in \mathrm{E}(\lhd)\mathfrak{F}_1$, there exist ideals A_i of E such that

$$A = A_0,\ A_{i+1}/A_i \in \mathfrak{F}_1 \quad \text{and} \quad A_n = E.$$

We prove by induction on i that A contains a non-zero $\mathfrak{F}$-ideal of A_i. This is certainly true for $i=0$, since $A_0 = A \neq 0$ is abelian. Now assume that $i<n$ and that I is a non-zero $\mathfrak{F}$-ideal of A_i with $I \leq A$. Since $A_{i+1}/A_i \in \mathfrak{F}_1$, there exists $x \in A_{i+1}$ such that $A_{i+1} = A_i + \langle x\rangle$. Put $J = I^{\langle x\rangle} = \sum_{m\geq 0} [I, {}_m x]$. Then $J \leq A$. By induction on m and

$$[[I, {}_m x], A_i] \subseteq [[I, {}_{m-1}x], A_i, x] + [[I, {}_{m-1}x], [x, A_i]],$$

we obtain $J \lhd A_{i+1}$. Since $J \leq A \in \mathfrak{A}$, we have $J \in \mathfrak{A}$. Hence $\langle I, x\rangle = \langle I^{\langle x\rangle}, x\rangle = J + \langle x\rangle \in \mathfrak{A}\mathfrak{F}_1$. By hypothesis it is easy to see that E satisfies the condition (C). Since $I \in \mathfrak{F}$, we have $\langle I, x\rangle \in \mathfrak{G}$. Hence $\langle I, x\rangle \in \mathfrak{A}\mathfrak{F}_1 \cap \mathrm{WIC} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ by Lemma 6.6. By Proposition 3.1 $\langle I, x\rangle \in \mathfrak{G} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \leq \mathfrak{F}$. Hence $J \in \mathfrak{F}$. Thus A contains a non-zero $\mathfrak{F}$-ideal of E. In particular every non-zero homomorphic image of L has a non-zero $\mathfrak{F}$-ideal. Hence by Lemma 1.1 we have $E \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}$. By [10, Corollary 3.3] $E \in \mathrm{L}\mathfrak{F}$. Now let K be a non-zero $\mathfrak{F}$-ideal of E with $K \leq A$. Since $E/A \in \mathrm{E}(\lhd)\mathfrak{F}_1$, there exist finitely many elements $x_1,\dots, x_k$ of E such that $E = \langle A, x_1,\dots, x_k\rangle$. Since $E \in \mathrm{L}\mathfrak{F}$, we have $\langle K, x_1,\dots, x_k\rangle \in \mathfrak{F}$. Put $H = \langle K, x_1,\dots, x_k\rangle$. Since E satisfies the condition (C), we have $H \in \mathfrak{F} \cap \mathrm{WIC}$. Hence by Corollary 6.5 $H \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Since $0 \neq K \lhd H$, we see by Proposition 1.6 (3) that K contains a 1-dimensional ideal N of H. Hence $[N, x_i] \subseteq N$ for $1 \leq i \leq k$. Since $N \leq K \leq A \in \mathfrak{A}$, we have $[N, A] = 0$. Thus N is a 1-dimensional ideal of $E = \langle A, x_1,\dots, x_k\rangle$. This completes the proof.

THEOREM 6.8. *A Lie algebra L is locally supersoluble if and only if*

(1) *L belongs to $\mathrm{L}\acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$ and*

(2) *L satisfies the condition* (C).

PROOF. The implication in one direction is clear. Assume (1) and (2). Let H be a finitely generated subalgebra of L. Clearly $H \in \acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$ and H satisfies the condition (C). Let $(H_\beta)_{\beta \leq \alpha}$ be an ascending $\mathfrak{A}$-series of ideals of H. We can find an ordinal β minimal with respect to $H/H_\beta \in \mathfrak{F}$. By Lemma 6.1 and Corollary 6.5 $H/H_\beta \in \mathfrak{F} \cap \mathrm{WIC} = \mathrm{E}(\triangleleft)\mathfrak{F}_1$. Assume $\beta \neq 0$. If β is not a limit ordinal, we have $H/H_{\beta-1} \in \mathfrak{G} \cap \mathfrak{A}\mathrm{E}(\triangleleft)\mathfrak{F}_1$. By Lemma 6.7 $H/H_{\beta-1} \in \mathfrak{G} \cap \acute{\mathrm{E}}(\triangleleft)\mathfrak{F}_1$ and so $H/H_{\beta-1} \in \mathrm{E}(\triangleleft)\mathfrak{F}_1$ by Proposition 3.1. This contradicts the minimality of β. We may assume that β is a limit ordinal. By [10, Lemma 3.1] there exist finitely many elements $x_1, \ldots, x_n$ of H_β such that $H_\beta = \sum_{i=1}^n \langle x_i^H \rangle$. Since β is a limit ordinal, there exists an ordinal $\gamma < \beta$ such that $x_i \in H_\gamma$ for $i = 1, \ldots, n$ and so $H_\beta = H_\gamma$. This is another contradiction. Hence $\beta = 0$ and so $H \in \mathrm{E}(\triangleleft)\mathfrak{F}_1$. Thus $L \in \mathrm{LE}(\triangleleft)\mathfrak{F}_1$.

COROLLARY 6.9. *L is finite-dimensional supersoluble if and only if*

(1) *L belongs to $\mathfrak{G} \cap \acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$ and*

(2) *L satisfies the condition* (C).

PROPOSITION 6.10. *Let L be a finitely generated Lie algebra which satisfies the condition* (C). *Then the following are equivalent*:

(1) *L is finite-dimensional supersoluble.*

(2) *L is supersoluble-by-supersoluble.*

(3) *L is supersoluble-by-soluble.*

(4) *L is hypercentral-by-soluble.*

(5) *L belongs to $\mathfrak{Z}\acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$.*

PROOF. Clearly (1) implies (2) and (4) implies (5). Since $\mathfrak{G} \cap \acute{\mathrm{E}}(\triangleleft)\mathfrak{F}_1 \leq \mathrm{E}(\triangleleft)\mathfrak{F}_1 \leq \mathrm{E}\mathfrak{A}$, we have that (2) implies (3). By Theorem 2.4 it is easy to see that (3) implies (4). Finally assume (5). Since every term of the upper central series of a Lie algebra is a characteristic ideal, we have $\mathfrak{Z}\acute{\mathrm{E}}(\triangleleft)\mathfrak{A} \leq \acute{\mathrm{E}}(\triangleleft)\mathfrak{A}$. Hence by Corollary 6.9 $L \in \mathrm{E}(\triangleleft)\mathfrak{F}_1$.

THEOREM 6.11. *A Lie algebra L is supersoluble if and only if*

(1) *L satisfies the weak idealizer condition,*

(2) *L satisfies the condition* (C) *and*

(3) *every locally supersoluble subalgebra of L is supersoluble.*

PROOF. The implication in one direction is clear. Assume (1), (2) and (3). By Zorn's lemma there exists a maximal locally supersoluble subalgebra M of L. By (3) $M \in \acute{\mathrm{E}}(\triangleleft)\mathfrak{F}_1$. Assume by way of contradiction that $M \lneqq L$. By (1) we can find $x \in L \backslash M$ such that $[x, M] \subseteq M_{\langle M,x\rangle} + \langle x\rangle$. Put $H = \langle M, x\rangle = M + \langle x\rangle$ and $I = M_{\langle M,x\rangle}$. Since $[x, M] \subseteq I + \langle x\rangle$, we have $(\langle x\rangle + I)/I \lhd H/I$. It follows that

$$H/(\langle x\rangle + I) = (M + \langle x\rangle)/(\langle x\rangle + I) \cong M/(M \cap (\langle x\rangle + I)) \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1.$$

Hence $H/I \in \mathfrak{F}_1 \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 = \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. Let K be a finitely generated subalgebra of H. Then $K/(K \cap I) \cong (K + I)/I \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. Since $I \leq M \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$, $K \cap I$ is a supersoluble ideal of K. Hence $K \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1 \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$. By (2) and Proposition 6.10 $K \in \mathrm{E}(\lhd)\mathfrak{F}_1$. Hence $H \in \mathrm{LE}(\lhd)\mathfrak{F}_1$. This contradicts the maximality of M, since $M \lneqq H$. Thus $L = M \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$.

7.

In Theorem 6.8 we have given a characterization of locally supersoluble Lie algebras. In this section we shall give further characterizations of locally supersoluble Lie algebras.

For a subalgebra H of L the idealizer $I_L(H)$ of H in L is the set $\{x \in L: [H, x] \subseteq H\}$.

LEMMA 7.1. *Let $\mathfrak{X}$ be a non-trivial $\{\mathrm{s}, \mathrm{E}\}$-closed class of Lie algebras and let L be a Lie algebra such that $\langle x^{\langle y\rangle}\rangle \in \mathfrak{G}$ for any $x, y \in L$. If H is an $\mathrm{L}\mathfrak{X}$-subalgebra of L and $z \in I_L(H)$, then $H + \langle z\rangle \in \mathrm{L}\mathfrak{X}$.*

PROOF. Let A be a finitely generated subalgebra of $H + \langle z\rangle$. Then there exist finitely many elements x_i of H such that $A \leq \langle x_1, \dots, x_n, z\rangle$. Put $B = \langle x_1, \dots, x_n, z\rangle$ and $I = \langle \langle x_i\rangle^{\langle z\rangle}: i = 1, \dots, n\rangle$. Since $z \in I_L(H)$, we have $I \leq H$. Evidently $I \lhd B$. Hence $B = I + \langle z\rangle$. By hypothesis $I \in \mathfrak{G}$. Since $H \in \mathrm{L}\mathfrak{X}$, we have $I \in \mathfrak{X}$. Hence $B \in \mathfrak{X}\mathfrak{F}_1 \leq \mathrm{E}\mathfrak{X} = \mathfrak{X}$ and so $A \in \mathrm{s}\mathfrak{X} = \mathfrak{X}$. Thus $H + \langle z\rangle \in \mathrm{L}\mathfrak{X}$.

LEMMA 7.2. *Let L be a Lie algebra which has ideals A, B and a subalgebra H with the following properties:*

(1) A, H and L/B are locally finite and

(2) $B \leq A \cap H$ and $L = A + H$.

Then L is locally finite.

PROOF. Let K be a finitely generated subalgebra of L. Since $L = A + H$, there exist finitely generated subalgebras A_1 of A and H_1 of H such that $K \leq \langle A_1, H_1\rangle$. Since $H \in \mathrm{L}\mathfrak{F}$, we have $H_1 \in \mathfrak{F}$. Put $M = \langle A_1, H_1\rangle$. Since $L/B \in \mathrm{L}\mathfrak{F}$, we have $(M + B)/B \in \mathfrak{F}$. We have

$$(M + B)/B \ \cap\ A/B = ((M \cap A) + B)/B \cong (M \cap A)/(M \cap B),$$

whence $(M \cap A)/(M \cap B) \in \mathfrak{F}$. Hence there exists an $\mathfrak{F}$-subalgebra A_2 of $M \cap A$ such that $M \cap A = A_2 + (M \cap B)$, and so by the modular law we obtain $B + A_2 =$

$B+(M\cap A)=(B+M)\cap A$. Hence $B+A_2\lhd B+M$. Since A_1 is a finitely generated subalgebra of $B+A_2$, there exists a finitely generated subalgebra B_1 of B such that $A_1\leq\langle B_1, A_2\rangle$. Hence we have

$$(*)\qquad K\leq\langle B_1, A_2, H_1\rangle .$$

Since A_2, $H_1\in\mathfrak{F}$, we have that $[A_2, H_1]$ is finite-dimensional. Since $H_1\leq M\leq B+M$ and $B+A_2\lhd B+M$, we have $[A_2, H_1]\subseteq B+A_2$. Hence there exists a finitely generated subalgebra B_2 of B such that $[A_2, H_1]\subseteq\langle B_2, A_2\rangle$. Put $B_3=B\cap\langle B_1, B_2, H_1\rangle$. Note that $B_3\lhd\langle B_1, B_2, H_1\rangle\in\mathfrak{G}$. Since B_1, $B_2\leq B\leq H$, we have $\langle B_1, B_2, H_1\rangle\leq H$. Hence $B_3\in\mathfrak{F}$ and so $\langle B_3, A_2\rangle\in\mathfrak{G}$. Since $B_3\leq B\leq A$, $A_2\leq A$ and $A\in\mathrm{L}\mathfrak{F}$, we have $\langle B_3, A_2\rangle\in\mathfrak{F}$. Since $B_3\lhd\langle B_1, B_2, H_1\rangle$, we have $[B_3, H_1]\subseteq\langle B_3, A_2\rangle$. Also we have $[A_2, H_1]\subseteq\langle B_2, A_2\rangle\leq\langle B_3, A_2\rangle$. Hence $\langle B_3, A_2\rangle\lhd\langle B_3, A_2, H_1\rangle$ and so $\langle B_3, A_2, H_1\rangle=\langle B_3, A_2\rangle+H_1\in\mathfrak{F}$. By $(*)$ and $B_1\leq B_3$ we obtain $K\leq\langle B_3, A_2, H_1\rangle$ and so $K\in\mathfrak{F}$. Thus $L\in\mathrm{L}\mathfrak{F}$.

Remark: The proof of Lemma 7.2 is a modification of that of [3, p. 351, Satz 1], in which it has been shown that if G is a group which has normal subgroups N, D and a subgroup U with the properties

(1) N, U and G/D belong to LMax and

(2) $D\leq N\cap U$ and $G=NU$,

then G belongs to LMax. In Lie algebras everything goes well if Max is replaced by $\mathfrak{F}$.

Theorem 7.3. *Let L be a Lie algebra. Then L is locally supersoluble if and only if L satisfies the condition* (C) *and one of the following conditions*:

(1) *$\langle x^{\langle y\rangle}\rangle$ is finitely generated for any x, $y\in L$.*

(2) *If H is an* LMax-*subalgebra of L and $x\in I_L(H)$, then $H+\langle x\rangle$ belongs to* LMax.

(3) *If H is a locally finite subalgebra of L and $x\in I_L(H)$, then $H+\langle x\rangle$ is locally finite.*

(4) *If H is a locally supersoluble subalgebra of L and $x\in I_L(H)$, then $H+\langle x\rangle$ is locally supersoluble.*

Proof. If $L\in\mathrm{LE}(\lhd)\mathfrak{F}_1$, then by Lemma 6.2 and Proposition 3.1 we have that L satisfies the conditions (C) and (1). By Lemma 7.1 we have that (1) implies (2). In the rest of the proof we suppose that L satisfies the condition (C). Assume (2). Let H be a locally finite subalgebra of L and let $x\in I_L(H)$. Since $\mathrm{L}\mathfrak{F}\leq\mathrm{L}\mathrm{Max}$, we have $H+\langle x\rangle\in\mathrm{L}\mathrm{Max}$. Hence $H+\langle x\rangle\in\mathrm{L}(\mathrm{Max}\cap\mathrm{WIC})=\mathrm{LE}(\lhd)\mathfrak{F}_1\leq\mathrm{L}\mathfrak{F}$ by Corollary 6.5 and Proposition 3.1. Thus (2) implies (3). Similarly we have the equivalence of (3) and (4). Finally assume (3). Let K be a finitely generated subalgebra of L and let M be a maximal locally finite subalgebra of K. Assume by way of contradiction that $M\lneqq K$. Since $K\in\mathrm{WIC}$, there exists $x\in K\backslash M$

such that $[x, M]\subseteq M_{\langle M,x\rangle}+\langle x\rangle$. Put $H=M_{\langle M,x\rangle}$, $N=\langle M, x\rangle$ and $I=H+\langle x\rangle$. Clearly H and I are ideals of N, $H\leq M\cap I$ and $N=I+M$. Since $M\in \mathrm{L}\mathfrak{F}$, we have $H\in \mathrm{L}\mathfrak{F}$. By (3) $I\in \mathrm{L}\mathfrak{F}$. Since I/H is a 1-dimensional ideal of N/H and $N/I=(M+I)/I\in \mathrm{L}\mathfrak{F}$, we have $N/H\in \mathfrak{F}_1\mathrm{L}\mathfrak{F}\leq \mathrm{L}\mathfrak{F}$. Hence by Lemma 7.2 we obtain $N\in \mathrm{L}\mathfrak{F}$. This contradicts the maximality of M. Therefore $M=K\in \mathfrak{G}\cap \mathrm{L}\mathfrak{F}=\mathfrak{F}$. Thus $L\in \mathrm{L}(\mathfrak{F}\cap \mathrm{WIC})=\mathrm{LE}(\lhd)\mathfrak{F}_1$ by Corollary 6.5. This completes the proof.

Now we consider the following condition for a Lie algebra L, which is a Lie analogue of [4, p. 121], to characterize locally supersoluble Lie algebras:

(D) *If H is a proper subalgebra of L and if I is an ideal of L such that $L=I+H$, then there exists $x\in I\backslash H$ such that $\langle x^H\rangle\leq H_{\langle H,x\rangle}+\langle x\rangle$.*

With the choice $I=L$, we see that the condition (D) implies the weak idealizer condition.

LEMMA 7.4. *Every supersoluble Lie algebra satisfies the condition* (D).

PROOF. Let $L\in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$, $H\lneqq L$ and $I\lhd L$ such that $L=I+H$. Put $J=(I\cap H)_L$. Since $H\lneqq L$, we have $J\lneqq L$. Clearly $I\nleq H$ and so $J\lneqq I$. Hence I/J is a non-zero ideal of L/J. By Proposition 1.6 (3) I/J contains a 1-dimensional ideal K/J of L/J. Let $x\in K$ such that $K=J+\langle x\rangle$. If $x\in H$, then $K\leq I\cap H$ and so $K=J$, which is a contradiction. Hence $x\in I\backslash H$. Clearly $J\leq H_{\langle H,x\rangle}$. Therefore we obtain

$$\langle x^H\rangle \leq \langle x^L\rangle \leq K = J + \langle x\rangle \leq H_{\langle H,x\rangle} + \langle x\rangle .$$

THEOREM 7.5. *Let L be a Lie algebra. Then L is locally supersoluble if and only if every finitely generated subalgebra of L satisfies the condition* (D).

PROOF. By Lemma 7.4 we have one implication. Suppose that every finitely generated subalgebra of L satisfies the condition (D). By Theorem 7.3 and the argument before Lemma 7.4 it is sufficient to see that if H is a locally finite subalgebra of L and $x\in I_L(H)$, then $H+\langle x\rangle\in \mathrm{L}\mathfrak{F}$. Put $K=H+\langle x\rangle$ and let G be a finitely generated subalgebra of K. We may suppose $x\in G$. Hence by the modular law $G=(G\cap H)+\langle x\rangle$. Assume by way of contradiction that $G\notin \mathrm{L}\mathfrak{F}$. Let M be a maximal locally finite subalgebra of G which contains x. Clearly $M\lneqq G$, $G=(G\cap H)+M$ and $G\cap H\lhd G$. Hence by (D) we can find $y\in (G\cap H)\backslash M$ such that $\langle y^M\rangle\leq M_{\langle M,y\rangle}+\langle y\rangle$. Put $J=\langle M, y\rangle$, $A=M_{\langle M,y\rangle}+\langle y\rangle$ and $B=M_{\langle M,y\rangle}$. Then A, $B\lhd J$, $B\leq A\cap M$ and $J=M+\langle y\rangle=A+M$. Since $M\in \mathrm{L}\mathfrak{F}$, we have $B\in \mathrm{L}\mathfrak{F}$. Since $\langle y^M\rangle\leq A$, we have $A=B+\langle y^M\rangle$. Since $y\in H\lhd K$, we have $\langle y^M\rangle\leq H\in \mathrm{L}\mathfrak{F}$. Hence by the N_0-closedness of $\mathrm{L}\mathfrak{F}$ [1, Corollary 6.1.2] $A\in \mathrm{L}\mathfrak{F}$. Clearly $A/B\in \mathfrak{F}_1$ and $J/A\in \mathrm{L}\mathfrak{F}$. Hence $J/B\in \mathfrak{F}_1\mathrm{L}\mathfrak{F}\leq \mathrm{L}\mathfrak{F}$. Therefore by Lemma 7.2 we have $J\in \mathrm{L}\mathfrak{F}$. This contradicts the maximality of M. Hence $G\in \mathfrak{G}\cap \mathrm{L}\mathfrak{F}=\mathfrak{F}$. Thus $K\in \mathrm{L}\mathfrak{F}$. This completes the proof.

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